ON A FAMILY OF LIE ALGEBRAS OF CHARACTERISTIC p

BY

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Introduction. We study a family of Lie algebras of characteristic p which are defined as subalgebras of the derivation algebra of the group algebra of an elementary p-group. In particular we show that simple Lie algebras of dimensions \( m(p^n - 1), mp^n, p^n - 2 \), where \( m \) and \( n \) are arbitrary integers such that \( 1 \leq m < n \), and where \( p > 2 \) only for the dimensions \( p^n \) and \( p^n - 2 \), are associated with this family. The algebras studied by M. S. Frank [2] are included in our family, but those of dimension \( m(p^n - 1) \) in general appear to be new.

Since this paper was written, the paper of A. A. Albert and M. S. Frank [1] has been published. The relation between the algebras studied in [1] and those in this paper will be mentioned in §9, although it is not thoroughly clarified yet.

1. Definition of the family \( \mathfrak{g} \). Let \( \Phi \) be an algebraically closed field of characteristic \( p > 0 \), and \( \mathfrak{A} \) the group algebra over \( \Phi \) of an abelian group \( \mathfrak{G} \) of type \( (p, p, \ldots, p) \) and order \( p^n \). Let \( D_0, \ldots, D_m \) be derivations of \( \mathfrak{A} \) such that \( D_i \circ D_j = 0 \) for all \( i, j \), and let \( a_0, \ldots, a_m \in \mathfrak{A} \) be such that

\[
D_ia_j = D_ja_i \quad (i, j = 0, 1, \ldots, m).
\]

Consider the set \( \mathfrak{g} = \mathfrak{g}(D_i, a_i) \) of all derivations of the form \( D = f_0D_0 + \cdots + f_mD_m \), where \( f_i \in \mathfrak{A} \) satisfy \( \sum D_if_i = \sum a_if_i \). By an elementary computation, we see easily that \( \mathfrak{g} \) is a subalgebra of the derivation algebra \( \mathfrak{A} \) of \( \mathfrak{A} \). (The case when \( m + 1 = n, a_0 = \cdots = a_m = 0, D_i = \partial/\partial g_i \), where \( g_0, \ldots, g_m \) is a set of independent generators of the group \( \mathfrak{G} \), was considered by M. S. Frank [2], and the case \( m + 1 = n, a_1 = 1, D_i = \partial/\partial g_i \), by A. A. Albert and M. S. Frank [1].)

In this paper, we study the family \( \mathfrak{g} \) of algebras \( \mathfrak{g}(D_i, a_i) \), where \( D_0, \ldots, D_m \) satisfy the following conditions:

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(2) By a derivation \( D \) of an algebra \( \mathfrak{A} \) over a field \( \Phi \) we mean a linear mapping of \( \mathfrak{A} \), regarded as a vector space over \( \Phi \), into itself such that \( D(fg) = (Df)g + f(Dg) \) for all \( f, g \) in \( \mathfrak{A} \). If \( D_1, D_2 \) are derivations of \( \mathfrak{A} \), then \( D_1 \circ D_2 = D_1D_2 - D_2D_1 \) is easily seen to be a derivation of \( \mathfrak{A} \). The totality of derivations of \( \mathfrak{A} \) forms a Lie algebra over \( \Phi \) with the ordinary addition and the multiplication \( \circ \). It is called the derivation algebra of \( \mathfrak{A} \).
(1.0.2) \( D_i \circ D_j = 0 \) for all \( i, j \);

(1.0.3) \( \sum f_i D_i = 0 \), where \( f_i \in \mathcal{A} \), implies \( f_i = 0 \) for all \( i \);

(1.0.4) \( D_i f = 0 \) for all \( i \) implies \( f \in \Phi \);

(1.0.5) If \( f \in \mathcal{A} \) is such that \( D_i f = \lambda_i f \), where \( \lambda_i \in \Phi \), for all \( i \), then \( f = 0 \) or \( f \) is a unit in \( \mathcal{A} \).

The elements \( a_0, \ldots, a_m \) of \( \mathcal{A} \) will be always assumed to be chosen such that (1.0.1) is satisfied. An ordered set \( (D_0, \ldots, D_m) \) of derivations of \( \mathcal{A} \) will be called a \textit{semi-system} if (1.0.2)-(1.0.4) are satisfied, and a \textit{system} if (1.0.2)-(1.0.5) are satisfied\(^{(3)}\). Since we fix \( m > 0 \) throughout this paper, a semi-system or a system \( (D_0, \ldots, D_m) \) will usually be denoted by the notation \( (D_i) \).

It is shown in \([4]\) that \( m < n \) must hold for a system. The following lemma is also shown in \([4]\):

\textbf{Lemma 1.1.} For a system \( (D_i) \), if \( f \) and \( a_i \in \mathcal{A} \) are such that \( D_i f = a_i f \) for all \( i \), then \( f = 0 \) or \( f \) is a unit in \( \mathcal{A} \).

\section{Equivalent systems.} Two semi-systems \( (D_i) \) and \( (D'_i) \) are said to be \textit{equivalent} if there exist \( c, y \in \mathcal{A} \) such that

\begin{equation}
(2.0.1) \quad D'_i = \sum_{s=0}^{m} c_{is} D_s \end{equation}

for \( i = 0, \ldots, m \), and such that \( \det (c_{ij}) \) is a unit in \( \mathcal{A} \). From the properties (1.0.2)-(1.0.3) for \( (D'_i) \) it follows easily that

\begin{equation}
(2.0.2) \quad D'_i c_{jk} = D'_j c_{ik}
\end{equation}

for all \( i, j, \text{ and } k \).

\textbf{Lemma 2.1.} A semi-system equivalent to a system is a system.

\textbf{Proof.} Let \( (D_i) \) be a semi-system equivalent to a system \( (D'_i) \), and let the relation (2.0.1) hold. Suppose \( f \in \mathcal{A} \) and \( \lambda_i \in \Phi \) are such that \( D_i f = \lambda_i f \) for all \( i \). Then (2.0.1) yields \( D'_i f = (\sum c_{is} \lambda_s) f \) for all \( i \). Then from Lemma 1.1 it follows that \( f = 0 \) or \( f \) is a unit in \( \mathcal{A} \). Therefore \( (D'_i) \) is a system.

Let \( (D_i) \) and \( (D'_i) \) be equivalent systems related by (2.0.1). Let \( (c_{ij})^{-1} = (c'_{ij}) \). Then \( D_i = \sum c'_{is} D'_s \), and \( \sum f_i D_i = \sum f'_i D'_i \), where \( f'_i = \sum f c'_{is} \). It may be readily verified that \( \sum D'_i f'_i = \sum a'_i f'_i \), where

\begin{equation}
(2.2.1) \quad a'_i = \sum (a_s c_{is} - D_s c_{is}), \quad i = 0, 1, \ldots, n.
\end{equation}

Thus we may state

\textit{(3) Semi-system and system} in this paper may be called in the language of \([4]\) "orthogonal system satisfying (1.0.4)" and "orthogonal system satisfying (1.0.4)-(1.0.5)," respectively.
Theorem 2.2. If the system \((D_1)\) is given by (2.0.1), then \(q(D_i, a_i) = q(D_1', a_1')\), where \(a_1'\) are given by (2.2.1).

The following lemma is useful in changing the formula (2.2.1).

Lemma 2.3. Let \((D_i)\) be a system, and let \(a_{ij} \in \mathbb{R}\) be such that \(D_i a_{jk} = D_j a_{ik}\) for all \(i, j, k = 0, 1, \ldots, m\). Let \(\bar{a}_{ij}\) be the cofactor of \(a_{ij}\) in the determinant of the \((m+1) \times (m+1)\) matrix \((a_{ij})\). Then \(\sum_{s=0}^{m} D_s \bar{a}_{is} = 0\) for all \(i\).

Proof. For simplicity we assume that \(i = 0\). The other cases may be proved similarly. Since

\[
det (a_{ij}) = \sum \epsilon(s_0s_1 \cdots s_m) a_{s_0s_1} \cdots a_{s_m},
\]

where \(\epsilon(s_0s_1 \cdots s_m)\) denotes \(+1\) if the permutation

\[
\pi = \left(\begin{array}{ccc}
0 & 1 & \cdots \\
s_0 & s_1 & \cdots & s_m
\end{array}\right)
\]

is even, \(-1\) if \(\pi\) is odd, therefore

\[
\bar{a}_{0s} = \sum' \epsilon(s_1 \cdots s_m) a_{s_1} \cdots a_{s_m},
\]

where the summation \(\sum'\) runs over all permutations \(\pi\) such that \(s_0 = s\). Since \(D_s\) is a derivation, we have

\[
\sum D_s \bar{a}_{0s} = \sum \epsilon(s_1 \cdots s_m) \left[D_s a_{s_1} a_{s_2} \cdots a_{s_m} + a_{s_1} D_s a_{s_2} \cdots a_{s_m} + \cdots\right],
\]

where the summation on the right runs over all permutations

\[
\pi = \left(\begin{array}{ccc}
0 & 1 & \cdots \\
\bar{s} & s_1 & \cdots & s_m
\end{array}\right).
\]

By hypothesis \(D_s a_{s_1} = D_{\bar{s}} a_{s_1}\). Since \(\epsilon(s_1 \cdots s_m) = -\epsilon(s_1 s \cdots s_m)\), the two terms \(\epsilon(s_1 \cdots s_m) (D_s a_{s_1}) a_{s_2} \cdots a_{s_m}\) and \(\epsilon(s_1 s \cdots s_m) (D_{\bar{s}} a_{s_1}) a_{s_2} \cdots a_{s_m}\) cancel each other. Similarly all the other terms are divided into such pairs. Thus we see that \(\sum D_s \bar{a}_{0s} = 0\). Similarly \(\sum D_s \bar{a}_{is} = 0\) for all \(i\). Thus Lemma 2.3 is proved.

Using Lemma 2.3, we can change (2.2.1) into a more convenient form. We set \(a_{ij} = c_{ij}\). Then the formula corresponding to (2.0.2) shows that \(a_{ij}\) satisfy the condition of Lemma 2.3. Let \(f = \det (c_{ij})\). Then \(\bar{a}_{ij} = f c_{ij}\). Hence by Theorem 2.2 we have \(\sum D_s (f c_{is}) = 0\) for all \(i\). Therefore \(f \sum D_s c_{is} + \sum c_{is} D_s f = 0\), and we obtain

\[
(2.3.1) \quad \sum D_s c_{is} + \sum c_{is} (f^{-1} D_s f) = 0.
\]

From (2.3.1) and (2.2.1) we see that

\[
(2.3.2) \quad a_i' = \sum_s c_{is} (a_s + f^{-1} D_s f), \quad f = \det (a_{ij})^{-1}, \quad \text{for all } i.
\]
3. **Principal systems.** A system \((D_i)\) is called *principal* if \(D_f \in \Phi\) for all \(i\) implies \(f \in \Phi\). Elements \(g_1, \ldots, g_n \in \mathcal{A}\) are said to form a set of *principal generators* of \(\mathcal{A}\) if \(g_1^p = 1\) for all \(i\) and if the \(p^n\) elements \(g_i^1 \cdots g_i^n\), where \(0 \leq u_i < p\), \(g_i^0 = 1\), form a basis of \(\mathcal{A}\) over \(\Phi\). The following Lemmas 3.1 and 3.2 are proved in [4].

**Lemma 3.1.** Any system is equivalent to a principal system.

**Lemma 3.2.** For any principal system \((D_i)\), there exists a set of principal generators \(g_1, \ldots, g_n\) of \(\mathcal{A}\) such that

\[
D_i = \sum_{i=1}^{n} \alpha_{ij} G_j
\]

for all \(i\), where \(\alpha_{ij} \in \Phi\) and where \(G_i = g_i \partial / \partial g_i\) are derivations of \(\mathcal{A}\) such that \(G_i G_j = \delta_{ij}\) for all \(i, j\) and \(\delta_{ij}\) is the Kronecker delta. The principal generators \((g_i)\) will be said to belong to the principal system \((D_i)\).

From (1.0.3)—(1.0.4) we see easily that the \(\alpha_{ij}\) in (3.2.1) must satisfy (3.2.2)—(3.2.3) below:

(i) If \(u_1, \ldots, u_n\) are integers such that \(\sum_{i} \alpha_{ij} u_j = 0\) for all \(i\), then

\[
u_i \equiv 0 \pmod{p}
\]

for all \(i\);

(2.2) If \(\xi_0, \ldots, \xi_m \in \Phi\) are such that \(\sum_{i=0}^{m} \xi_i \alpha_{ii} = 0\) for all \(i\), then \(\xi_i = 0\) for all \(i\).

Conversely if elements \(\alpha_{ij} \in \Phi\) satisfy (3.2.2)—(3.2.3) and if \(D_i\) are defined by (3.2.1) with an arbitrary set of principal generators \(g_1, \ldots, g_n\) of \(\mathcal{A}\), then \((D_i)\) is a system, as is proved in §9 of [4]. We shall now show that the system \((D_i)\) is principal. Let \(D_f \in \Phi\) for all \(i\), where \(f = \sum_{u} \gamma_u g^u\), \(\gamma \in \Phi\). Then \(\gamma u(e_i : u) = 0\) for all \(u \not= 0\) and \(i\), and hence by (3.2.4) we have \(\gamma u = 0\) for all \(u \not= 0\). Therefore \(f \in \Phi\), and hence \((D_i)\) is shown to be principal.

For any integers \(m\) and \(n\) such that \(0 \leq m < n\), there exist \(\alpha_{ij} \in \Phi\) such that (3.2.2)—(3.2.3) hold, since \(\Phi\) is assumed to be algebraically closed and hence infinite.

Suppose that the system \((D_i)\) is given by (3.2.1). Consider the \((m+1)\)-dimensional vector space \(\mathcal{B}\) over \(\Phi\) consisting of all \((m+1)\)-tuples \(x = (\xi_0, \ldots, \xi_m)\), \(\xi \in \Phi\), and also the \(n\)-dimensional vector space \(\mathcal{B}^\ast\) over \(\Phi\) consisting of all \(n\)-tuples \(u = (u_1, \ldots, u_n)\), \(u \in \Phi\). Let \(\mathcal{B}\) be the subset of \(\mathcal{B}^\ast\) consisting of all \(u\) such that \(u_i \in GF(p) \subseteq \Phi\) for \(i = 1, 2, \ldots, n\). Define a bilinear function \(x \cdot u\), where \(x \in \mathcal{B}\), \(u \in \mathcal{B}^\ast\), with values in \(\Phi\) by setting \(x \cdot u = \sum_{i,j} \xi_i \alpha_{ij} u_j\). Then (3.2.2) and (3.2.3) are equivalent to (3.2.4) and (3.2.5), below, respectively:
(3.2.4) If \( x \cdot u = 0 \) for all \( x \in \mathbb{R} \) and if \( u \in \mathfrak{B} \) then \( u = 0 \);

(3.2.5) \( x \cdot u = 0 \) for all \( u \in \mathfrak{B} \) implies \( x = 0 \).

Suppose now that \( g_1, \ldots, g_n \) are principal generators belonging to the principal system \((D_i)\). For \( u = (u_1, \ldots, u_n) \in \mathfrak{B} \) we shall write \( g^u = g_1^{u_1} \cdots g_n^{u_n} \).

Let \( e_i \in \mathbb{R} \) be a vector whose \((i+1)\)th component is 1 and whose other components are all 0. Then \( D_i g^u = (e_i \cdot u) g^u \), and, more generally,

\[
(3.2.6) \quad (\xi_0 D_0 + \cdots + \xi_m D_m) g^u = (x \cdot u) g^u, \text{ where } x = (\xi_0, \ldots, \xi_m) \in \mathbb{R}.
\]

The notations introduced in this section will be preserved in what follows.

4. **Type and dimension of \( \mathfrak{L} \).** For a derivation \( D \) and an element \( a \in \mathfrak{A} \), we define a linear mapping \( D - a \) of \( \mathfrak{A} \), regarded as a vector space over \( \Phi \), into itself by \( (D - a)f = Df - af \). Then the condition \( D_i a_j = D_j a_i \) is equivalent to saying that the linear mappings \( D_i - a_i \) and \( D_j - a_j \) are commutative. Therefore, if \( \mathfrak{L} = \mathfrak{L}(D_i; a_i) \subset \mathfrak{F} \), then there exist a nonzero element \( b \in \mathfrak{A} \) and \( \alpha_i \in \Phi \) such that

\[
(4.0.1) \quad (D_i - a_i)b = \alpha_i b
\]

for all \( i \in \mathbb{N} \) will be called a proper element of \( (D_i; a_i) \) and \( (\alpha_0, \ldots, \alpha_m) \) proper root belonging to \( b \).

**Lemma 4.1.** If \((D_i)\) is a principal system and if \( b \) is a proper element of \((D_i; a_i)\), then \( b \) is a unit in \( \mathfrak{A} \) and all the other proper elements of \((D_i; a_i)\) are, up to a constant factor, of the form \( bg^u \), where \( g_1, \ldots, g_n \) is any fixed set of principal generators of \( \mathfrak{A} \) belonging to \( (D_i) \) and where \( u \) runs over \( \mathfrak{B} \). If \( (\alpha_0, \ldots, \alpha_m) \) is the proper root belonging to \( b \), then \( (\alpha_0 - (e_0 \cdot u), \ldots, \alpha_m - (e_m \cdot u)) \) is the proper root belonging to \( bg^u \).

**Proof.** The fact that \( b \) is a unit follows immediately from Lemma 1.1, since \( D_i b = (a_i - \alpha_i) b \) for all \( i \).

Let \( D_i b' = (a_i - \alpha'_i) b' \) for all \( i \). Then \( D_i(b^{-1}b') = (a_i - \alpha'_i) b^{-1}b' \). We may suppose that \( b^{-1}b' = \sum u \in \mathfrak{B} \gamma_u g^u \), where \( \gamma_u \in \Phi \). Then \( (e_i \cdot u) \gamma_u = (\alpha_i - \alpha'_i) \gamma_u \) for all \( i \). Therefore if \( \gamma_u \neq 0 \) then \( e_i \cdot u = \alpha_i - \alpha'_i \) for all \( i \). Furthermore, if \( \gamma_u 
eq 0 \) then \( e_i \cdot u = \alpha_i - \alpha'_i = e_i \cdot u' \), and hence \( (e_i \cdot u - u') = 0 \) for all \( i \). Hence we have \( u = u' \). Therefore, any proper element of \((D_i; a_i)\), is, up to a constant factor, of the form \( bg^u \), and the proper root belonging to \( bg^u \) is \( (\alpha_0 - (e_0 \cdot u), \ldots, \alpha_m - (e_m \cdot u)) \).

It is easily seen that \( bg^u \) is a proper element of \((D_i; a_i)\) for any \( u \in \mathfrak{B} \). Thus Lemma 4.1 is proved.

By Theorem 2.2 and Lemma 3.1, every \( \mathfrak{L} \subset \mathfrak{F} \) can be expressed as \( \mathfrak{L} = \mathfrak{L}(D_i; a_i) \) with some principal system \((D_i)\). If there exists a proper element \( b \) of \((D_i; a_i)\) such that the proper root belonging to \( b \) is zero, i.e. \( \alpha_i = 0 \) for all \( i \), then we shall say that \( \mathfrak{L} \) is of **type I**. Otherwise, \( \mathfrak{L} \) is said to be of **type II**. We will show that the above definition of the type of \( \mathfrak{L} = \mathfrak{L}(D_i; a_i) \) is independent.
of the principal system \((D_i)\) used to form \(\mathfrak{g}\). This will be done by computing the dimension of \(\mathfrak{g}\) over \(\Phi\) as follows.

Let \(b\) be a proper element of \(\mathfrak{g} = \mathfrak{g}(D_i; a_i)\) and let \((\alpha_0, \ldots, \alpha_m)\) be the proper root belonging to \(b\). Since \(b\) is a unit in \(\mathfrak{g}\), every element \(D \in \mathfrak{g}\) can be written in the form \(D = b \sum f_i D_i\), with \(f_i \in \mathfrak{g}\). An elementary computation shows that the condition \(\sum D_i (bf_i) = \sum a_i bf_i\) is equivalent to \(\sum D_i f_i = \sum \alpha_i f_i\). Hence we have \(\mathfrak{g}(D_i; a_i) = b \mathfrak{g}(D_i; a_i)\) where \(b \mathfrak{g} = \{ bD \mid D \in \mathfrak{g} \}\). In particular, \(\dim \mathfrak{g}(D_i; a_i) = \dim \mathfrak{g}(D_i; a_i)\). Suppose now that \((D_i)\) is a principal system and the \(g_1, \ldots, g_n\) form a set of principal generators belonging to \((D_i)\).

Consider \(D = \sum f_i D_i \in \mathfrak{g}(D_i; a_i)\). We may write \(f_i = \sum u \in \mathfrak{g} \phi_i u g^u\), where \(\phi_i u \in \Phi\). Then the condition \(\sum D_i f_i = \sum \alpha_i f_i\) is easily seen to be equivalent to

\[
\sum_i (e_i \cdot u) \phi_i u = \sum_i \alpha_i \phi_i u, \quad \text{(for all } u).\]

We set \(D_u = g^u \sum_i \phi_i u D_i\). Then \(D = \sum D_u\), \(D_u \in \mathfrak{g}(D_i; a_i)\). Thus the vector space \(\mathfrak{g}(D_i; a_i)\) over \(\Phi\) is a direct sum of the vector spaces \(\mathfrak{g}_u\), \(u \in \mathfrak{g}\), where \(\mathfrak{g}_u\) consists of elements of the form \(g^u \sum_i \xi_i D_i\), \(\xi_i \in \Phi\). Now \(g^u \sum_i \xi_i D_i \in \mathfrak{g}_u\) if and only if

\[
\sum_i (e_i \cdot u) \xi_i = \sum_i \alpha_i \xi_i. \tag{4.2.1}\]

Suppose that \(\mathfrak{g} = \mathfrak{g}(D_i; a_i)\) is of type I. Then we may assume \(\alpha_i = 0\) for all \(i\). From (3.2.4) and (4.2.1) it follows easily that \(\dim \mathfrak{g}_u = m\) for \(u \neq 0\) and that \(\dim \mathfrak{g}_0 = m+1\). Hence \(\dim \mathfrak{g} = mp^n+1\).

Suppose that \(\mathfrak{g} = \mathfrak{g}(D_i; a_i)\) is of type II. By (3.2.5), we may set \(\alpha_i = e_i \cdot k\), where \(k \in \mathfrak{g}\). Then by Lemma 4.1 we see that

\[
((e_0 \cdot k - u), \ldots, (e_m \cdot k - u)) \neq 0 \tag{4.2.2}\]

for all \(u \in \mathfrak{g}\). Now (4.2.1) can be expressed in the form \((x \cdot k - u) = 0\), where \(x = (\xi_0, \ldots, \xi_m)\). Therefore, because of (4.2.2), we have \(\dim \mathfrak{g}_u = m\) for all \(u \in \mathfrak{g}\). Hence \(\dim \mathfrak{g} = mp^n\). Thus we have proved

**Theorem 4.2.** If \(\mathfrak{g}\) is of type I, then \(\dim \mathfrak{g} = mp^n+1\). If \(\mathfrak{g}\) is of type II, then \(\dim \mathfrak{g} = mp^n\).

5. **Another characterization of \(\mathfrak{g}\).** Let \(\mathfrak{g} = \mathfrak{g}(D_i; a_i) \subseteq \mathfrak{g}\) be defined by a principal system \((D_i)\). Let \(b\) be a proper element, and \((\alpha_0, \ldots, \alpha_m)\) the proper root belonging to \(b\). We set \(\alpha_i = e_i \cdot k\), \(k \in \mathfrak{g}\), as before. (If \(L\) is of type I, then, by Lemma 4.1, we may take \(k\) in \(\mathfrak{g}\).) It was shown in the course of the proof of Theorem 4.2 that \(\mathfrak{g}\) is spanned by the elements of the form \(bg^u(\sum \xi_i D_i)\), where \((x \cdot u - k) = 0\), \(x = (\xi_0, \ldots, \xi_m)\).

Introduce the symbol \((x, u) = bg^u(\sum \xi_i D_i)\). Then:

(4) The idea of considering the case \(k \neq 0\) for algebras of type I will become clear when the reader reaches §7.
\begin{align*}
(5.0.1) \ & \mathcal{L} \text{ consists of elements of the form } \sum_{u \in \mathfrak{B}} (x_u, u), \text{ where } (x_u \cdot u - k) = 0 \text{ for all } u \in \mathfrak{B}; \\
(5.0.2) \ & \sum (x_u, u) = \sum (y_u, u) \text{ if and only if } x_u = y_u \text{ for all } u \in \mathfrak{B}; \\
(5.0.3) \ & \lambda \sum (x_u, u) = \sum (\lambda x_u, u) \text{ if } \lambda \in \Phi; \\
(5.0.4) \ & \sum (x_u, u) + \sum (y_u, u) = \sum (x_u + y_u, u); \\
(5.0.5) \ & (x, u) \circ (y, v) = \sum_{w \in \mathfrak{B}} \beta_w((x \cdot v + w)y - (y \cdot u + w)x, u + v + w). 
\end{align*}

The coefficients $\beta_w$ in (5.0.5) are those in the representation $b = \sum_{w \in \mathfrak{B}} \beta_w g^w$. Note that $\sum \beta_w \neq 0$ since $b$ is a unit in $\mathfrak{A}$. Note also that $(x \cdot u - k) = (y \cdot v - k) = 0$ implies $[((x \cdot v + w)y - (y \cdot u + w)x) \cdot (u + v + w - k)) = 0$. Conversely if we start with a bilinear function $x \cdot u$, $x \in \mathfrak{B}$, $u \in \mathfrak{B}$, satisfying (3.2.4)-(3.2.5), an element $k \in \mathfrak{B}$, and arbitrary elements $\beta_u \in \Phi$, then by (5.0.1)-(5.0.5) we can define an algebra $\mathfrak{L}$ over $\Phi$. It can be easily verified that the multiplication $\circ$ is skew-symmetric and satisfies the Jacobi-identity. Therefore $\mathfrak{L}$ is a Lie algebra. If $\sum_{w \in \mathfrak{B}} \beta_w \neq 0$ then $\mathfrak{L}$ is isomorphic to an algebra in $\mathfrak{B}$. This can be seen as follows: Let $g_1, \ldots, g_n$ be a set of principal generators of $\mathfrak{A}$. We define linear mappings $D_i$, $0 \leq i \leq m$, by $D_i g^u = (e_i \cdot u) g^u$. It is easily verified that $D_i$ are derivations of $\mathfrak{A}$, and it appears that $(D_0, \ldots, D_m)$ is a system. If $b = \sum_{w \in \mathfrak{B}} \beta_w g^u$, then $\sum \beta_u = 0$ implies that $b$ is a unit in $\mathfrak{B}$. Set $a_i = b^{-1} D_i b + e_i \cdot k$ for all $i$. Then $D_i a_i = D_i a_i$, and we have $\mathfrak{L} \sim_{\mathfrak{L}} (D_i; a_i)$, where $(x, u)$ corresponds to $b g^u \sum \xi_i D_i$, $x = (\xi_0, \ldots, \xi_m)$.

In the above formulation (5.0.1)-(5.0.5), $\mathfrak{L}$ is of type I if and only if there exists $k' \in \mathfrak{B}$ such that $x \cdot k = x \cdot k'$ for all $x \in \mathfrak{B}$.

Suppose that $\mathfrak{L}$ is of type I. Then we may assume $k \in \mathfrak{B}$. Consider the first derived algebra $\mathfrak{L}'$ of $\mathfrak{L}$. In the right hand side of (5.0.5), if $u + v + w = k$, then for $x \in \mathfrak{L}$ and $y \in \mathfrak{L}$, $(x \cdot v + w) = - (x \cdot u - k) = 0$, $(y \cdot u + w) = - (y \cdot v - k) = 0$. Therefore, if $\sum (x_u, u) \in \mathfrak{L}'$ then $x_k = 0$. Thus we have proved

**Theorem 5.1.** If the algebra $\mathfrak{L} \in \mathfrak{B}$ is of type I, then $\mathfrak{L}'$ is contained, as an ideal, in the subalgebra of $\mathfrak{L}$ consisting of all $\sum (x_u, u) \in \mathfrak{L}$ such that $x_k = 0$. In particular, $\dim \mathfrak{L}' \leq m(\phi^n - 1)$.

Consider now the special case where $m = 1$, $0 \neq k \in \mathfrak{B}$, $\beta_0 = 1$, $\beta_w = 0$ for all $w \neq 0$. If $\mathfrak{L}$ is of type I, and if $\sum (x_u, u) \in \mathfrak{L}'$ then $x_k = 0$, so that (5.0.5) becomes

\begin{equation}
(5.2.1) \quad (x, u) \circ (y, v) = ((x \cdot v)y - (y \cdot u)x, u + v).
\end{equation}

Suppose $u + v = 2k$. Then $(x \cdot u - k) = (y \cdot u - k) = 0$. Therefore, if $u \neq k$, $x \neq 0$, then $y = \lambda x$ with $\lambda \in \Phi$ since $m = 1$. Hence

\begin{equation}
(x \cdot v)y - (y \cdot u)x = \lambda (x \cdot v)x - \lambda (x \cdot u)x = 0.
\end{equation}

Thus we see that if $\sum (x_u, u) \in \mathfrak{L}''$, the second derived algebra of $\mathfrak{L}$, then $x_k = x_{2k} = 0$. In other words, $\mathfrak{L}''$ is contained, as an ideal, in the subalgebra con-
consisting of all $\sum (x_u, u) \in \mathcal{G}$ such that $x_k = x_{2k} = 0$. In particular, $\dim \mathcal{G}'' \leq p^n - 2$. Later we shall see that $\mathcal{G}''$ is simple and of dimension $p^n - 2$, provided $p \neq 2$.

6. Reduction theorems. We define a subfamily $\mathcal{G}_c$ of $\mathcal{G}$ as follows: $\mathcal{G} \in \mathcal{G}_c$ if and only if there exists a principal system $(D_i)$ and elements $\lambda_i \in \Phi$ such that $\mathcal{G} = \mathcal{G}(D_i; \lambda_i)$. As we shall see later, algebras in $\mathcal{G}_c$ can be discussed fairly easily. It is an open question whether $\mathcal{G} = \mathcal{G}_c$ or not.

**Theorem 6.1.** Let $\mathcal{G} = \mathcal{G}(D_i; \lambda_i)$ be defined by a principal system $(D_i)$. Then $\mathcal{G} \in \mathcal{G}_c$ if and only if there exists $c_0, \ldots, c_m \in \mathcal{A}$ and $\lambda_0, \ldots, \lambda_m \in \Phi$ such that $f = \det (\delta_{ij} + D_i c_j)$ is a unit in $\mathcal{A}$ and such that

$$
i_i = -f^{-1} D_i f + D_i (\sum \lambda_i c_i) + \lambda_i \text{ for all } i = 0, \ldots, m.$$

For the proof of Theorem 6.1 we need the following

**Lemma 6.2.** Suppose $(D_i)$ is a principal system. If $h_0, \ldots, h_m \in \mathcal{A}$ are such that $D_i h_j = D_j h_i$ for all $i, j$, then there exist $h \in \mathcal{A}$ and $\gamma_0, \ldots, \gamma_m \in \Phi$ such that $h_i = D_i h + \gamma_i$ for all $i$.

**Proof of Lemma 6.2.** Let $g_1, \ldots, g_n$ be a set of principal generators of $\mathcal{A}$ belonging to $(D_i)$, and let $h_i = \sum u \in B \eta_{iu} g^n$, $\eta_{iu} \in \Phi$. Then $D_i h_j = D_j h_i$ implies $(e_i \cdot u) \eta_{iu} = (e_j \cdot u) \eta_{iu}$ for all $u \in B$. From (3.2.4) we have $((e_0 \cdot u), \ldots, (e_m \cdot u)) \neq 0$ if $u \neq 0$. Hence there exists $\rho_u \in \Phi$, for all $u \neq 0$, such that $\eta_{iu} = (e_i \cdot u) \rho_u$ for all $i$. Put $h = \sum u \in B \rho_u g^n$, $\gamma_i = \eta_{i0}$. Then $h_i = D_i h + \gamma_i$ for all $i$, as required.

**Proof of Theorem 6.1.** Suppose $\mathcal{G} \in \mathcal{G}_c$. Then there exist a principal system $(D'_i)$ equivalent to $(D_i)$ and a $\lambda_i \in \Phi$ such that $\mathcal{G} = \mathcal{G}(D'_i; \lambda_i)$. Let $(D_i)$ and $(D'_i)$ be related as in (2.0.1). Then (2.3.2) yields

$$
(6.1.2) \quad \lambda_i = \sum c_{is} (a_s + f^{-1} D_i f), \quad f = \det (c_{ij}).
$$

By a formula corresponding to (2.0.2) and Lemma 6.2, we see that there exist $c_i \in \mathcal{A}$ and $\gamma_{ij} \in \Phi$ such that

$$
(6.1.3) \quad c'_{ij} = D_i c_j + \gamma_{ij}, \quad i, j = 0, \ldots, m,
$$

where $\gamma_{ij}$ are uniquely determined by $c'_{ij}$, since $(D_i)$ is principal. We shall show that $\det (\gamma_{ij}) \neq 0$. Suppose $\xi_i \in \Phi$ are such that $\sum_{i=0}^m \gamma_{is} \xi_s = 0$ for all $i$. Then (6.1.3) yields $\sum c_i \xi_i = D_i c$, where $c = \sum c_i \xi_i$, and hence $D'_i c = \xi_i \in \Phi$ for all $i$. Since $(D'_i)$ is principal, we have $c \in \Phi$, and hence $\xi_i = 0$ for all $i$. Thus $\det (\gamma_{ij}) \neq 0$ is proved. Let $(\gamma'_{ij})$ be the inverse matrix of $(\gamma_{ij})$, and let $\lambda_i = \sum \gamma_{is} \lambda_s$, $c_i = \sum c_{ij} \lambda_j$, $\tilde{f} = \det (D_i \tilde{e}_j + \delta_{ij})$, $\gamma = \det (\gamma_{ij})$. Then $\tilde{f} = f \gamma$, and from (6.1.2) and (6.1.3) we have easily $a_i = -f^{-1} D_i f + D_i (\sum \lambda_i \tilde{e}_i) + \tilde{\lambda}_i$ for all $i$.

Suppose conversely, that there exist $c_i \in \mathcal{A}$ and $\lambda_i \in \Phi$ such that $f = \det (D_i c_j + \delta_{ij})$ is a unit in $\mathcal{A}$ and such that (6.1.1) holds. We set $c'_{ij} = D_i c_j + \delta_{ij}$, $(c_{ij}) = (c'_{ij})^{-1}$, $D'_i = \sum c_{ij} D_i$. First, we shall show that $(D'_i)$ is a system. Since $(D_i)$ is already a system, by Lemma 2.1 it is sufficient to show that $D'_i \circ D'_j = 0$ for all $i, j$. Since $D_i = \sum c'_{ij} D'_i$ for all $i$, we have
0 = D_i \circ D_j = \sum_{s,t} (c_{ij}' D_{ij}' D_i' - \sum_{s,t} (c_{ij}' D_{ij}' D_i')D_i' + \sum_{s,t} c_{ij}' c_{ij}' (D_i' \circ D_j')
\sum_{s,t} [((D_i c_{ij}'))D_i' - (D_j c_{ij}')(D_i') + \sum_{s,t} c_{ij}' c_{ij}' (D_i' \circ D_i')].

Now D_i c_{ij}' = D_j c_{ij}' for all i, j, t, so that \sum_{s,t} c_{ij}' c_{ij}' (D_i' \circ D_j') = 0 for all i, j. Finally since det (c_{ij}') is a unit in \mathfrak{A}, we have D_i' \circ D_j' = 0 for all i and j. Thus (D_i') is proved to be a system. We shall show that (D_i') is principal. Suppose D_i f = \xi_i \in \Phi for all i. Then D_i = \sum_{s,t} c_{ij}' D_i' implies D_i (f - \sum \xi_s c_s) = \xi_i \in \Phi for all i. Since (D_i) is principal we have f - \sum \xi_s c_s = \Phi, \xi_i = 0 for all i, and hence f \in \Phi. Thus (D_i') is a principal system. The fact that \mathfrak{G} = \mathfrak{G}(D_i'; \lambda_i) follows easily from (6.1.1) and (2.3.2), and Theorem 6.1 is proved.

Define a subfamily \mathfrak{S}_0 of \mathfrak{G} as follows: \mathfrak{S} \in \mathfrak{S}_0 if and only if there exists a principal system (D_i) such that \mathfrak{S} = \mathfrak{S}(D_i'; 0). Clearly every algebra in \mathfrak{S}_0 is of type I. Later we shall show that the first derived algebras \mathfrak{S}' of \mathfrak{G} in \mathfrak{S}_0 are simple for any prime \rho > 0. The following theorem may be proved just like Theorem 6.1.

Theorem 6.3. Let \mathfrak{S} = \mathfrak{S}(D_i'; a_i) be defined by a principal system (D_i). Then \mathfrak{S} \in \mathfrak{S}_0 if and only if there exist c_0, \ldots, c_m \in \mathfrak{A} such that f = det (D_i c_i + \delta_{ij}) is a unit in \mathfrak{A} and such that a_i = -f^{-1} D_i f for all i.

Let (D_i) be a principal system, and (g_1, \ldots, g_n) a set of principal generators belonging to (D_i). For convenience an element h \in \mathfrak{A} will be called "unitary" with respect to (D_i) if \eta_0 in the expression h = \sum_{u \in \mathfrak{G}} \eta_u g_u^u, \eta_u \in \Phi, is not zero. This property does not depend on the choice of principal generators belonging to (D_i).

Corollary 6.4. Let (D_i) be a principal system, and let f be a unit in \mathfrak{A} which is unitary with respect to (D_i). Then \mathfrak{S}(D_i'; -f^{-1}D_i f) \in \mathfrak{S}_0.

Proof. In view of Theorem 6.3 it is sufficient to show that there exist c_0, \ldots, c_m \in \mathfrak{A} such that f = \gamma det (D_i c_i + \delta_{ij}) with a nonzero element \gamma in \Phi.

It was proved in §9 of [4] that for any principal system (D_i), there exist elements \alpha_i \in \Phi such that the derivation D = \sum \alpha_i D_i satisfy the condition:

\theta_i = 0 \implies h \in \Phi.

Let (g_1, \ldots, g_n) be a set of principal generators belonging to (D_i), and f = \gamma_0 det (D_i c_i + \delta_{ij}) with a nonzero element \gamma in \Phi. By hypothesis. Put c = \gamma_0^{-1} \sum_{u \neq 0} \gamma_u \delta_u^{-1} g_u, c_i = \alpha_i c. Then f = \gamma_0 (1 + D c), and we have det (D_i c_i + \delta_{ij}) = 1 + D_i c_i = 1 + D c, and hence f = \gamma_0 det (D_i c_i + \delta_{ij}). Thus Corollary 6.4 is proved.

7. Some lemmas. Algebras in \mathfrak{S}_0 are those obtained by setting b = \sum \beta_{uw} g_u^w = 1 in the characterization (5.0.1)-(5.0.5), and will be considered in this section and the one following. For our purposes, however, it is more convenient to consider the algebra \mathfrak{G} which is defined as follows: Assuming always that
\[ \beta_0 = 1, \beta_w = 0 \text{ for } w \neq 0 \text{ in } (5.0.1)-(5.0.5), \text{ then} \]

(i) if \( \mathfrak{g} \in \mathfrak{g}_c \) is of type II, then we set \( \mathfrak{g} = \mathfrak{g} \);

(ii) if \( \mathfrak{g} \in \mathfrak{g}_c \) is of type I and if either \( m > 1 \) or \( k = 0 \), then we set \( \mathfrak{g} \) to be the algebra consisting of all \( \sum (x_u, u) \in \mathfrak{g} \) such that \( x_k = 0 \);

(iii) if \( \mathfrak{g} \in \mathfrak{g}_c \) is of type I, if \( m = 1 \), and if \( k \neq 0 \), then we set \( \mathfrak{g} \) to be the algebra consisting of all \( \sum (x_u, u) \in \mathfrak{g} \) such that \( x_k = x_{2k} = 0 \).

We shall assume \( p \neq 2 \) in case (iii) and also in case (i) if \( m = 1 \). With this assumption we shall prove that \( \mathfrak{g} \) is simple. Then we see from the result in §5 that \( \mathfrak{g} \) in case (i), \( \mathfrak{g}' \) in case (ii), and \( \mathfrak{g}'' \) in case (iii) are simple and of dimensions \( mp^n \), \( m(p^n-1) \), and \( p^n-2 \), respectively. In this section we shall prepare for the proof of the simplicity of \( \mathfrak{g} \).

**Lemma 7.1.** If nonzero elements \( u, v \in \mathfrak{g} \) are such that \( xu = 0 \), where \( x \in \mathfrak{g} \), implies \( x-v = 0 \), and vice versa, then there exists a nonzero \( x \in \mathfrak{g} \) such that \( x-u = x-v \) for all \( x \in \mathfrak{g} \).

**Proof.** There exist \( \alpha_{ij} \in \Phi \) such that \( x-u = \sum_{i=0}^{m} \sum_{j=1}^{n} \xi_i \alpha_{ij} x_j \), where \( x = (\xi_0, \ldots, \xi_m) \), \( u = (u_1, \ldots, u_n) \). Set \( \beta_i = \sum_j \alpha_{ij} x_j \). Then our hypothesis implies that \( \xi_0 \beta_0 + \cdots + \xi_m \beta_m = 0 \) if and only if \( \xi_0 \gamma_0 + \cdots + \xi_m \gamma_m = 0 \). Therefore, there exists a nonzero \( \lambda \in \Phi \) such that \( \beta_i = \lambda \gamma_i \) for all \( i \), so that \( x-u = \lambda x-v \) for all \( x \in \mathfrak{g} \).

An element \( (x, u) \in \mathfrak{g} \) will be called a \( u \)-term or simply a term. Let \( \mathfrak{g} \) be a nonzero ideal of \( \mathfrak{g} \), and let \( A = \left( x_i, u_i \right) \), where \( x_i \neq 0 \), \( i=1, \ldots, r \), and where \( u_1, \ldots, u_r \) are distinct, be a nonzero element in \( \mathfrak{g} \) such that the number \( r \) of nonzero terms is as small as possible. Such an element \( A \) will be called a minimal element in \( \mathfrak{g} \).

**Lemma 7.2.** Suppose \( k \neq 0 \). If \( A = \sum (x_i, u_i) \) is a minimal element in an ideal \( \mathfrak{g} \neq 0 \), then, for any distinct \( i \) and \( j \leq r \) there exists a nonzero \( \lambda \in \Phi \) such that \( x \cdot (u_j-u_i) = \lambda x \cdot k \) for all \( x \in \mathfrak{g} \).

**Proof.** By Lemma 7.1, it is sufficient to show that \( y \cdot k = 0 \) implies \( y \cdot (u_i-u_j) = 0 \). Consider \( A' = A \circ (y, 0) = \sum_{i=1}^{r} (y \cdot u_i) x_i, u_i \). Since \( A' \in \mathfrak{g} \), \( A' - (y \cdot u_j) A \) is also in \( \mathfrak{g} \) and has less than \( r \) nonzero terms. Hence \( A' = (y \cdot u_j) A \), from which it follows that \( (y \cdot u_i) x_i - (y \cdot u_j) x_i = 0 \). Therefore \( y \cdot (u_i-u_j) = 0 \).

**Lemma 7.3.** Suppose \( k = 0 \). If \( A = \sum (x_i, u_i) \) is a minimal element in \( \mathfrak{g} \), then, for any \( i \) and \( j \), there exists a nonzero \( \lambda \in \Phi \) such that \( x \cdot u_i = \lambda x \cdot u_j \) for all \( x \in \mathfrak{g} \).

**Proof.** By Lemma 7.1, it is sufficient to show that \( y \cdot u_i = 0 \) if and only if \( y \cdot u_i = 0 \). Let \( y \cdot u_i = 0 \). Then \( A' = A \circ (y, -u_i) \in \mathfrak{g} \), and \( A' \) contains less than \( r \) terms, so that \( A' = 0 \). Therefore

\[
(x_i \cdot u_i) y + (y \cdot u_i) x_i = 0
\]

for all \( i \). Since \( x_i \cdot u_i = 0 \), (7.3.1) yields \( (x_i \cdot u_i)(y \cdot u_i) = 0 \). Suppose \( y \cdot u_i \neq 0 \).
Then \( x_i \cdot u_i = 0 \), and hence (7.3.1) yields \( y \cdot u_i = 0 \), a contradiction. Thus \( y \cdot u_i = 0 \), and Lemma 7.3 is proved.

**Lemma 7.4.** If \( A = \sum(x_i, u_i) \) is a minimal element in \( \mathfrak{S} \), then \( x_i \cdot u_j = 0 \) for any \( i \neq j \).

**Proof.** Since \( A \circ (x_i, u_i) \) contains less than \( r \) terms, we have \( (x_j, u_j) \circ (x_i, u_i) = 0 \) for any \( i \) and \( j \). Hence

\[
(x_i \cdot u_j)x_j - (x_j \cdot u_i)x_i = 0. 
\]

Therefore \( (x_i \cdot u_j)(x_j \cdot u_i) - (x_j \cdot u_i)(x_i \cdot u_j) = 0 \). Suppose \( x_i \cdot u_j \neq 0 \). Then (7.4.1) yields

\[
x_j \cdot (u_j - u_i) = 0. 
\]

If \( k = 0 \) then Lemma 7.4 follows immediately from Lemma 7.3. Hence we assume \( k \neq 0 \). Then by Lemma 7.2 there exists \( \lambda \neq 0 \) such that \( x_j \cdot (u_j - u_i) = \lambda x_j \cdot k \). Therefore (7.4.2) gives \( x_j \cdot k = 0 \), and hence \( x_j \cdot u_j = 0 \). Then by (7.4.2) we have \( x_j \cdot u_i = 0 \). But then (7.4.1) yields \( x_i \cdot u_j = 0 \), since \( x_j \neq 0 \). This is a contradiction, and Lemma 7.4 is proved.

**Lemma 7.5.** If \( r > 1 \) for a minimal element in \( \mathfrak{S} \), then \( \mathfrak{S} \) contains a minimal element \( \sum(x_i, u_i) \) such that \( u_1 \neq 0 \), \( u_2 \neq 0 \).

**Proof.** If \( k = 0 \), then every \( u_i \neq 0 \), and hence the lemma is clear. Suppose that \( k \neq 0 \), \( u_1 \neq 0 \), \( u_2 = 0 \). Since \( u_2 \neq 0 \), there exists \( v \in \mathfrak{S} \) such that \( x_2 \cdot v \neq 0 \). If \( u_1 + v = 0 \) then \( x_2 \cdot v = -x_2 \cdot u_1 = 0 \) by Lemma 7.4, which is a contradiction. Hence

\[
u_1 + v \neq 0, \quad v \neq 0.
\]

There exists a nonzero element \( y \in \mathfrak{R} \) such that \( y \cdot (v - k) = 0 \). Consider \( A' = A \circ (y, v) \in \mathfrak{S} \). Then \( A' = \sum(x'_i, u'_i) \) contains a term \( ((x_2 \cdot v)y, v) \neq 0 \). Therefore \( A' \) is a minimal element, and \( u'_1 = u_1 + v \neq 0 \), \( u'_2 = v \neq 0 \) by (7.5.1).

**Lemma 7.6.** Suppose \( m > 1 \). If \( A = \sum(x_i, u_i) \) is a minimal element in \( \mathfrak{S} \), and if \( u_i \neq 0 \) for some \( i \), then \( x_j \cdot k = 0 \) for all \( j \neq i \).

**Proof.** The subspace \( \mathfrak{R}' \) of \( \mathfrak{R} \) consisting of all \( x' \) such that \( x' \cdot u_i = 0 \) is of dimension \( m > 1 \). Hence there exists \( y \in \mathfrak{R}' \) such that \( y \) and \( x_j \) are linearly independent. The element \( A' = A \circ (y, k - u_i) \) is in \( \mathfrak{S} \) and contains less than \( r \) terms. Hence \( A' = 0 \), and we have \( (x_j, u_i) \circ (y, k - u_i) = (x_j \cdot (k - u_i))y - (y \cdot u_i)x_j = 0 \) for \( j \neq i \). Since \( y \) and \( x_j \) are linearly independent, we have \( x_j \cdot (k - u_i) = 0 \). Then, by Lemma 7.4, we have \( x_j \cdot k = 0 \), as required.

**Lemma 7.7.** Suppose \( m = 1 \), \( p > 2 \), \( k = 0 \). If \( \sum_{i=1}^{r}(x_i, u_i) \) is a minimal element in \( \mathfrak{S} \), and if \( r > 1 \), then \( x_i \cdot k = 0 \) for all \( i \).

**Proof.** We may assume \( i = 1 \). We have \( x_1 \cdot (u_1 - k) = 0 \), and \( x_1 \cdot u_2 = 0 \) by
Lemma 7.4. Hence $x_1 \cdot (u_1 - u_2 - k) = 0$. On the other hand, there exists a non-
zero $\lambda \in \Phi$ such that

$$x \cdot (u_1 - u_2) = \lambda x \cdot k$$

for all $x \in R$. By setting $x = x_1$ in (7.7.1), we have $(\lambda - 1) x_1 \cdot k = 0$. If $\lambda \neq 1$ then

$$x_1 \cdot k = 0,$$

as required. Suppose $\lambda = 1$. Then by (7.7.1) we have $x \cdot (u_1 - u_2) = x \cdot k$ for all $x \in R$. Therefore $\mathfrak{g}$ is of type $I$, and we may assume $u_1 - u_2 = k$.

Hence $u_2 \neq 0$, and we have $x_2 \cdot (u_2 + k) = 0$, $x_2 \cdot (u_2 - k) = 0$. Since $p \neq 2$, we have

$$x_2 \cdot u_2 = x_2 \cdot k = 0.$$ 

By Lemma 7.4, $x_1 \cdot u_2 = 0$. Now the subspace $R'$ consisting of all $x'$ such that $x' \cdot u_2 = 0$ is of dimension $m = 1$, since $0 \neq u_2 \in \mathfrak{g}$. Hence $x_1 = \mu x_2$ with some $\mu \in \mathfrak{g}$. Then $x_1 \cdot k = \mu x_2 \cdot k = 0$, as required.

**Lemma 7.8.** If $A = \sum_{i=1}^{r} (x_i, u_i)$, $x_i \neq 0$, is a minimal element in a nonzero ideal $\mathfrak{g}$ in $R$, where $p$ is assumed $\neq 2$ if both of $k \neq 0$ and $m = 1$ hold, then $r = 1$.

**Proof.** Suppose $r > 1$. We shall derive a contradiction.

First consider the case $k \neq 0$. By Lemma 7.5, we may assume that $u_i \neq 0$, $u_2 \neq 0$. Then, by Lemmas 7.6 and 7.7, we have $x_i \cdot u_i = x_i \cdot k = 0$ for all $i = 1, \ldots, r$. Since $x_1 \neq 0$, there exists an element $v \in \mathfrak{g}$ with $x_1 \cdot v \neq 0$. Then $x_1 \cdot (v - k) \neq 0$, since $x_1 \cdot k = 0$. The subspaces $R' = \{x' \mid x' \cdot (v - k) = 0\}$ and $R'' = \{x'' \mid x'' \cdot k = 0\}$ are both of dimension $m$. Since $x_1 \in R'$, $x_1 \in R''$ we have $R' \neq R''$. Let $y \in R'$, $y \in R''$. Then $y \cdot (v - k) = 0$, $y \cdot k \neq 0$, and also $u_i + v \neq 0$ for all $i$. Since

$$A' = A \circ (y, v) = \sum ((x_i \cdot v) y - (y \cdot u_i) x_i, u_i + v)$$

is a minimal element, by Lemmas 7.6 and 7.7, we have $(x_i \cdot v)(y \cdot k) - (y \cdot u_i)(x_i \cdot k) = 0$ for all $i$. Since $x_i \cdot k = 0$, $y \cdot k \neq 0$, we have $x_i \cdot v = 0$ for all $i = 1, \ldots, r$, a contradiction. Therefore $r = 1$, as required.

Next consider the case $k = 0$. Choose $v \in \mathfrak{g}$, as before, such that $x_1 \cdot v \neq 0$, and $y \in R$ such that $y \cdot v = 0$, $y \cdot u_i \neq 0$. Consider $A'$ given by (7.8.1). By Lemma 7.4, we have $(x_i \cdot v) y - (y \cdot u_i) x_i \cdot (u_i + v) = 0$ for all $i$, and hence $(x_1 \cdot v)(y \cdot u_i) = (y \cdot u_i)(x_1 \cdot v)$, which yields $y \cdot (u_i - u_1) = 0$, since $x_1 \cdot v \neq 0$. By Lemma 7.3, there exists a nonzero $\lambda \in \Phi$ such that $y \cdot u_i = \lambda y \cdot u_1$. Then $(\lambda - 1)(y \cdot u_i) = 0$. Since $y \cdot u_i \neq 0$, $\lambda = 1$. Then $x \cdot u_i = x \cdot u_1$ for all $x \in R$, and hence $u_i = u_1$, $r = 1$. Thus Lemma 7.8 is proved.

In the following, we shall denote by $\mathfrak{R}(u)$, where $u \in \bar{\mathfrak{g}}$, the subspace $\mathfrak{R}' = \{x' \mid x' \cdot u = 0\}$ of $R$, provided there exists at least one element $x \in \mathfrak{R}$ such that $x \cdot u \neq 0$. Note that $\mathfrak{R}(u)$, if it exists, is always of dimension $m$. If $\mathfrak{g}$ is of type $II$ and if $u \in \mathfrak{g}$ then by (4.2.2) there exists $x \in \mathfrak{R}$ such that $x \cdot (u - k) \neq 0$, and hence we can always define $\mathfrak{R}(u - k)$.

**Lemma 7.9.** If $0 \neq (x, u) \in \mathfrak{g}$, an ideal of $\bar{\mathfrak{g}}$, and if $x \in \mathfrak{R}(v - k)$, $x \in \mathfrak{R}(v - 2k)$, then all $v$-terms are contained in $\mathfrak{g}$.

**Proof.** Since $x \in \mathfrak{R}(v - 2k)$, we have $v - u \neq k$. Let $y_1, \ldots, y_m$ be a basis
of $R(v-u-k)$. Then $(z_i, v) = (x, u) \circ (y_i, v-u) \in \mathfrak{I}$, where $z_i = (x \cdot v - u)y_i - (y_i \cdot u)x$. It is sufficient to show that $z_1, \ldots, z_m$ are linearly independent. Suppose $\sum \lambda_i z_i = 0$ with $\lambda_i \in \Phi$. Then

$$(7.9.1) \quad (x \cdot v - u) \sum \lambda_i y_i - (\sum \lambda_i y_i \cdot u) x = 0.$$ 

Since $y_i \in R(v-u-k)$, $(7.9.1)$ yields $(\sum \lambda_i y_i \cdot u)(x \cdot v - u - k) = 0$. However, $(x \cdot v - u - k) = (x \cdot v - 2k) \neq 0$. Hence $\sum \lambda_i y_i \cdot u = 0$. Then $(7.9.1)$ gives $\sum \lambda_i y_i = 0$, because $(x \cdot v - u) = (x \cdot v - k) \neq 0$, and since $y_1, \ldots, y_m$ are linearly independent, $\lambda_i = 0, i = 1, \ldots, m$.

Lemma 7.10. If all $u$-terms are contained in $\mathfrak{I}$ and if $R(u-k) \neq R(v-k)$, $R(u-k) \neq R(v-2k)$, then all $v$-terms are contained in $\mathfrak{I}$.

Proof. By Lemma 7.9, it is sufficient to show that there exists $x \in R(u-k)$ such that $x \in R(v-k), x \in R(v-2k)$. Suppose that every $x \in R(u-k)$ is either in $R(v-k)$ or in $R(v-2k)$. Let $x_i \in R(u-k)$ be such that $x_i \in R(v-ik), i = 1, 2$. Then $x_1 \in R(v-2k)$ and $x_2 \in R(v-k)$. Then $x = x_1 + x_2 \in R(v-ik), i = 1, 2$, and $x \in R(u-k)$.

Lemma 7.11. Suppose $k \neq 0$. If $0 \neq (x, 0) \in \mathfrak{I}$ and if $x \cdot v \neq 0$, then all $v$-terms are contained in $\mathfrak{I}$. If all $0$-terms are contained in $\mathfrak{I}$ and if $R(k) \neq R(v)$ then all $v$-terms are contained in $\mathfrak{I}$.

Proof. Lemma 7.11 follows immediately from Lemmas 7.9 and 7.10, since $x \cdot k = 0$.

Lemma 7.12. Suppose $p \neq 2$. If $0 \neq x \in R$ then there exists $u \in \mathfrak{B}$ such that $x \in R(u-k), x \in R(u-2k)$.

Proof. If $x \cdot (u' - k) = 0$ for all $u' \in \mathfrak{B}$, then $x \cdot u' = 0$ for all $u' \in \mathfrak{B}$, and hence $x = 0$. Therefore there exists $u' \in \mathfrak{B}$ such that $x \cdot (u' - k) \neq 0$. If $x \cdot (u' - 2k) = 0$, then $u = u'$ is the required element. Suppose $x \cdot (u' - 2k) = 0$. Then $x \cdot (u' - k) = x \cdot k \neq 0$. Hence $k \neq 0$ and $u = 0$ is the required element of $\mathfrak{B}$, since $x \cdot 2k \neq 0$ follows from $p \neq 2$.

Lemma 7.13. Suppose that $k \neq 0$ and that $p > 2$ if $m = 1$. Then all $0$-terms are contained in any ideal $\mathfrak{I} \neq 0$ of $\overline{R}$.

Proof. First consider the case $p \neq 2$. By Lemma 7.8 there exists a nonzero element $(x', u')$ in $\mathfrak{I}$. Since $x' \neq 0$, by Lemma 7.12 there exists $u \in \mathfrak{B}$ such that $x' \in R(u-k), x' \in R(u-2k)$. Then, by Lemma 7.9, all $u$-terms are contained in $\mathfrak{I}$. Let $0 \neq x \in R(u-k)$. Then, again by Lemma 7.12, there exists $v \in \mathfrak{B}$ such that $x \in R(v-ik), i = 1, 2$. Thus by Lemma 7.9 all $v$-terms are in $\mathfrak{I}$, and clearly $R(u-k) \neq R(v-k)$. Now $R(-k) = R(-2k)$, since $p \neq 2$. Since $R(u-k) \neq R(v-k)$, we see that either $R(u-k)$ or $R(v-k)$ is different from $R(-k) = R(-2k)$. Then by Lemma 7.10 all $0$-terms are contained in $\mathfrak{I}$.

Next consider the case $p = 2, m > 1$. Let $0 \neq (x, u) \in \mathfrak{I}$. If $x \cdot k = 0$ then take
v \in \mathfrak{B} \text{ such that } x \cdot v \neq 0. \text{ Hence } x \cdot (v - k) \neq 0. \text{ Since } \mathfrak{R}(k) \text{ and } \mathfrak{R}(v - k) \text{ are different and both of dimension } m, \text{ there exists } y \in \mathfrak{R}(v - k) \text{ such that } y \notin \mathfrak{R}(k). \text{ Consider } (x', u + v) = (x, u) \circ (y, v) = ((x \cdot v)y - (y \cdot u)x, u + v). \text{ Then } (x', u + v) \in \mathfrak{F}, \text{ and } x' \cdot k = ((x \cdot v)y - (y \cdot u)x) \cdot k = (x \cdot v)(y \cdot k) \neq 0. \text{ Therefore we may assume that there exists a nonzero element } (x, u) \text{ in } \mathfrak{F} \text{ such that } x \cdot k \neq 0. \text{ Let } x_1 = x, x_2, \cdots, x_m \text{ be a basis of } \mathfrak{R}(u - k). \text{ Put } (y_1, 0) = (x_1, u) \circ (x_1, u). \text{ Then } (y_1, 0) \in \mathfrak{F} \text{ and } y_1 = (x_1 \cdot k)x_1 - (x_1 \cdot k)x_1. \text{ Since } x_1 \cdot k \neq 0, \text{ the elements } y_2, \cdots, y_m \text{ form a basis of } \mathfrak{R}(u - k) \cap \mathfrak{R}(k). \text{ Set } y_2 = y. \text{ Then there exists } v \in \mathfrak{B} \text{ such that } y \cdot v \neq 0. \text{ Since } y \cdot k = 0, \text{ we have } y \cdot (v - k) \neq 0. \text{ Since } \mathfrak{R}(k) \neq \mathfrak{R}(v - k), \text{ there exists } z \in \mathfrak{R}(v - k) \text{ such that } z \notin \mathfrak{R}(k). \text{ Now } (y, 0) \circ (z, v) = ((y \cdot v)z, v) \in \mathfrak{F}. \text{ Since } y \cdot v \neq 0, \text{ we have } (z, v) \in \mathfrak{F}. \text{ Now } z \cdot k \neq 0 \text{ implies, as before, that } (z', 0) \in \mathfrak{F} \text{ for any } z' \in \mathfrak{R}(v - k) \cap \mathfrak{R}(k). \text{ We have } (y, 0) \in \mathfrak{F} \text{ with } y \in \mathfrak{B} \text{ such that } y \cdot (v - k) \neq 0. \text{ Since } y \in \mathfrak{R}(v - k) \cap \mathfrak{R}(k) \text{ is of dimension } m - 1, \text{ we see that all 0-terms are contained in } \mathfrak{F}.

8. Simplicity of \( \mathfrak{Q} \). We are now ready to prove the following

**Theorem 8.1.** If \( \mathfrak{Q} \in \mathfrak{F}_0 \), then the first derived algebra \( \mathfrak{Q}' \) is simple for any prime \( p > 0 \). \( \mathfrak{Q}' \) is of dimension \( m(p^n - 1) \), where \( 1 \leq m < n \).

**Proof.** If \( \mathfrak{Q} \in \mathfrak{F}_0 \) then \( \mathfrak{Q} \) belongs to the case (ii) of \( \S 7 \) with \( k = 0 \). Therefore, by Theorem 5.1, it is sufficient to show that \( \mathfrak{Q} \) is simple for this case.

Let \( \mathfrak{F} \) be a nonzero ideal of \( \mathfrak{Q} \). By Lemma 7.8, \( \mathfrak{F} \) contains an element of the form \((x, u)\neq 0\). Since \( x \neq 0 \) there exists \( v \in \mathfrak{B} \) such that \( x \cdot v \neq 0 \). Then by Lemma 7.9 all \( v \)-terms are contained in \( \mathfrak{F} \). Now, let nonzero \( w \in \mathfrak{B} \) be such that \( x \cdot w = 0 \). Since \( x \cdot v \neq 0 \), we have \( \mathfrak{R}(w) \neq \mathfrak{R}(v) \). Hence there exists \( y \in \mathfrak{R}(v) \) such that \( y \notin \mathfrak{R}(w) \). Since \( y, v \) is a \( v \)-term, we have \( (y, v) \in \mathfrak{F} \). Then, by Lemma 7.9, \( y \in \mathfrak{R}(w) \) implies that all \( w \)-terms are contained in \( \mathfrak{F} \). Therefore \( \mathfrak{F} = \mathfrak{Q} \), and hence \( \mathfrak{Q} = \mathfrak{Q}' \) is simple.

In the following, we shall denote by \( \mathfrak{F}_I \) and \( \mathfrak{F}_{II} \), the subfamilies of \( \mathfrak{F} \) consisting of all algebras of types I and II respectively. Then \( \mathfrak{F}_0 \subset \mathfrak{F}_I \). Let \( \mathfrak{F}_I - \mathfrak{F}_0 \) be the set-theoretical difference of \( \mathfrak{F}_I \) and \( \mathfrak{F}_0 \).

**Theorem 8.2.** If \( m > 1 \) then the first derived algebra \( \mathfrak{Q}' \) of any algebra \( \mathfrak{Q} \) in \( \mathfrak{F}_I \cap (\mathfrak{F}_I - \mathfrak{F}_0) \) is simple and of dimension \( m(p^n - 1) \), where \( 1 < m < n \), for any prime \( p > 0 \).

**Proof.** As in the proof of Theorem 8.1, it is sufficient to show that \( \mathfrak{Q} \) is simple for the case (ii) of \( \S 7 \) when \( k \neq 0 \).

Let \( \mathfrak{F} \) be a nonzero ideal of \( \mathfrak{Q} \). By Lemma 7.13, all \( 0 \)-terms are contained in \( \mathfrak{F} \). Hence by Lemma 7.11, if \( \mathfrak{R}(u) \neq \mathfrak{R}(k) \) then all \( u \)-terms are contained in \( \mathfrak{F} \).

Suppose that \( \mathfrak{R}(u) = \mathfrak{R}(k) \), with \( u \neq k, 2k \). Then \( \mathfrak{R}(u - k) = \mathfrak{R}(u - 2k) = \mathfrak{R}(k) \). Let \( 0 \neq x \in \mathfrak{R}(k), x \cdot v \neq 0, v \in \mathfrak{B} \). Then \( \mathfrak{R}(k) \neq \mathfrak{R}(v) \) and hence by Lemma 7.11 all \( v \)-terms are contained in \( \mathfrak{F} \). We have \( x \cdot (v - k) = x \cdot (v - 2k) = x \cdot v \neq 0 \). Hence \( \mathfrak{R}(v - k) \neq \mathfrak{R}(u - k) = \mathfrak{R}(u - 2k) \). Then by Lemma 7.10 all \( u \)-terms are contained in \( \mathfrak{F} \).
Suppose now \( u = 2k \neq 0 \). Then \( p \neq 2 \). Choose \( v \in \mathfrak{B} \) such that \( \mathfrak{R}(v) \neq \mathfrak{R}(k) \). Then \( \mathfrak{R}(2k-v) \neq \mathfrak{R}(k) \). Therefore by Lemma 7.11 all \( v \)-terms and all \( 2k-v \) terms are contained in \( \mathfrak{J} \). Let \( x_1, \ldots, x_m \) be a basis of \( \mathfrak{R}(v-k) \), and let \( x_1 \cdot k \neq 0 \). We set \( (y_i, 2k) = (x_i, v) \circ (x_i, 2k-v) \). Then \( (y_i, 2k) \in \mathfrak{J} \) and \( y_2, \ldots, y_m \) are linearly independent. Hence \( (y, 2k) \in \mathfrak{J} \) for any \( y \in \mathfrak{R}(v-k) \cap \mathfrak{R}(k) \).

Let \( 0 \neq y \in \mathfrak{R}(v-k) \cap \mathfrak{R}(k) \), which is possible since \( m > 1 \), and let \( y \cdot v \neq 0 \). Then \( \mathfrak{R}(y') \neq \mathfrak{R}(k) \), and as before \( (y', 2k) \in \mathfrak{J} \) for any \( y' \in \mathfrak{R}(v'-k) \cap \mathfrak{R}(k) \).

Since \( y \in \mathfrak{R}(v'-k) \cap \mathfrak{R}(k) \), all \( 2k \)-terms are contained in \( \mathfrak{J} \). Thus \( \mathfrak{J} = \mathfrak{B} \), which proves the simplicity of \( \mathfrak{B} = \mathfrak{B}' \).

The following two theorems may be proved similarly.

**Theorem 8.3.** Suppose \( m = 1 \), \( p > 2 \). Then the second derived algebra \( \mathfrak{B}'' \) of any algebra \( \mathfrak{B} \) in \( \mathfrak{B} \cap (\mathfrak{B}_1 - \mathfrak{B}_0) \) is simple and of dimension \( p^n - 2 \), where \( n > 1 \).

**Theorem 8.4.** Suppose \( p > 2 \) if \( m = 1 \). Then any algebra \( \mathfrak{B} \) in \( \mathfrak{B} \cap \mathfrak{B}_1 \) is simple and of dimension \( mp^n \), where \( 1 \leq m < n \).

9. **Remarks.** Let \( g_1, \ldots, g_n \) be a set of principal generators of \( \mathfrak{A} \). The algebra considered by M. S. Frank [2] is obtained as \( \mathfrak{B} = \mathfrak{B}(D_1, \ldots, D_n; a_1, \ldots, a_n) \) by setting \( D_i = \partial/\partial g_i, a_i = \cdots = a_n = 0 \). Put \( D'_i = g_i \partial/\partial g_i \). Then \( (D'_i) \) is a principal system equivalent to \( (D_i) \), and \( \mathfrak{B}(D_i; 0) = \mathfrak{B}(D'_i; a'_i) \), where \( a'_1 = \cdots = a'_n = -1 \), as is easily seen from (2.2.3). Put \( k = (-1, \ldots, -1) \in \mathfrak{B} \). Then \( a'_i = e_i \cdot k \) for all \( i \). Hence \( \mathfrak{B} \) falls into the family considered in Theorem 8.2. \( \mathfrak{B} \) is simple and of dimension \( (n-1)(p^n - 1) \) if \( n > 2 \).

The algebra denoted by the notation \( \mathfrak{B}_n \) in [1] is obtained as \( \mathfrak{B}(D_i; a_i) \) by setting \( D_i = \partial/\partial g_i, a_i = 1 \) for \( i = 1, 2, \ldots, n \). Set \( D'_i = g_i \partial/\partial g_i \) as before. Then (2.2.3) yields \( a'_i = g_i - 1 \). Suppose that \( \mathfrak{B} = \mathfrak{B}(D'_i, a'_i) \) is of type I. Then there exists a nonzero \( b \in \mathfrak{A} \) such that \( (D'_i - a'_i)b = 0 \) for all \( i \), from which it follows easily that \( \partial(bg_i)/\partial g_i = bg_i \) for all \( i \). Hence we have \( bg_i = 0, b = 0 \), a contradiction. Thus \( \mathfrak{B}_n \) is of type II, and hence of dimension \( (n-1)p^n \). The authors have been unable to decide whether or not \( \mathfrak{B} \in \mathfrak{B}_n \). If \( \mathfrak{B} \in \mathfrak{B}_n \) then \( \mathfrak{B} \) will fall into the family considered in Theorem 8.4.

Consider now any simple algebra \( \mathfrak{B} \) of dimension \( p^n - 1 \) obtained by setting \( m = 1 \) in our Theorem 8.1. It is spanned by elements of the form \( g^u(\xi_0 D_0 + \xi_1 D_1) \), where \( g_1, \ldots, g_n \) is a set of principal generators belonging to the principal system \( (D_0, D_1) \) and where \( \xi_0, \xi_1 \in \Phi \) are such that \( \xi_0 D_0 g^x + \xi_1 D_1 g^x = 0 \). Therefore we may take as a basis of \( \mathfrak{B} \) the elements of the form \( e_u = (D_1 g^u) D_0 - (D_0 g^u) D_1 \), \( u \) running over all elements \( \neq 0 \) in \( \mathfrak{B} \). Set

\[
D_1 g^u = \phi_1(u)g^u, \quad i = 0, 1; \quad \phi(u, v) = \phi_1(u)\phi_0(v) - \phi_0(u)\phi_1(v).
\]

Then it is easily seen that \( e_u \circ e_v = \phi(u, v)e_{u+v} \) for all \( u \) and \( v \). The function \( \phi(u, v) \) is a skew-symmetric bilinear form with respect to \( u \) and \( v \). Therefore the algebra \( \mathfrak{B} \) becomes a special case of the algebras considered in Theorem 11 of [1] if \( \phi(u, v) \) satisfies the condition:
(9.0.1) \( \phi(u, v) = 0 \) if and only if \( u \) and \( v \) are linearly dependent over \( GF(p) \).

However, an arbitrary principal system \((D_0, D_1)\), which can be used to define a simple algebra of dimension \( p^n - 1 \) as in Theorem 8.1, does not always satisfy the condition (9.0.1).

Similar remarks may be made about the connection between simple algebras of dimension \( p^n - 2 \) given in our Theorem 8.3 and those in Theorem 12 of [1].

References


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