SEMI-DIRECT PRODUCTS WITH AMPLE HOMOMORPHISMS

BY
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1. Introduction. If \( G = HK \) where \( H \) is a normal subgroup of the group \( G \) and where \( K \) is a subgroup of \( G \) with the trivial intersection with \( H \), then \( G \) is said to be a semi-direct product of \( H \) and \( K \) or a splitting extension of \( H \) by \( K \). In a recent paper, D. G. Higman [5] has considered the problem of the existence of \( H \) when finite \( G \) and \( K \) are given and has extended the classical results. Following Cartan [2], Malcev [7] has restated the definition of this product in terms of automorphisms of \( H \). Specifically, one considers the semi-direct product as an ordered triple \( G = (H, K; \phi) \) where \( \phi \) is a homomorphism from \( K \) into the automorphism group of \( H \); \( \phi: k \rightarrow \phi_k \). Special types of semi-direct products are the holomorphs and the dihedral groups. The former, for the case \( H \) abelian, are discussed in a paper of Mills [8]. The automorphism groups of the holomorphs of characteristically simple groups are treated in a paper of Gol’fand [3].

In this paper, some properties of semi-direct products are found and connections with the results in the literature are given. In §2, two special elements of \( \text{Hom} ((H, K; \phi), \phi(K)) \) are constructed. It is proved that the extensions of a group \( K \) by a group of automorphisms of a group \( H \) determine extensions of \( K \) by the related relative holomorph, extensions which prove to be semi-direct products of \( H \) by the corresponding extensions of \( K \).

A necessary and sufficient condition is found, in §3, for a pair of homomorphisms, one on \( H \) into \( \phi(K) \), the other on \( K \) into \( \phi(K) \), to be compounded to a homomorphism of \( G \) into \( \phi(K) \), where \( G = (H, K; \phi) \). If \( H \) is of class 2 and if \( \mathfrak{Z}(H) \) is the group of inner automorphisms of \( H \), then the natural map \( \theta \) on \( H \) onto \( \mathfrak{Z}(H) \) and each of the power maps \( \pi_a: \alpha \rightarrow \alpha^a \) of the abelian group \( \mathfrak{Z}(H) \) can be compounded to a homomorphism of the relative holomorph [9] of \( \mathfrak{Z}(H) \) over \( H \) to \( \mathfrak{Z}(H) \). Continuing in §4, we show that for the holomorph if the natural map \( \theta \) on \( H \) onto \( \mathfrak{Z}(H) \) can be compounded with an inner automorphism of the automorphism group of \( H \) generated by an inner automorphism of \( H \), then \( H \) must be nilpotent of class 3.

In §5, two products are constructed from group inclusions. The ascending central series is determined for these, in one case in terms of repeated commutator quotients [1]. It is shown that each endomorphism of the auto-

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morphism group of $H$ leads to an extension $W$ of $H$ with normal subgroup $H_0$ anti-isomorphic to $H$ such that $W/H_0 \cong W/H$. Automorphisms of $H$ are extended, in §6, to inner automorphisms of $(H, K; \phi)$, and a fundamental homomorphism of the latter is constructed onto a semi-direct product of the group of inner automorphisms of $(H, K; \phi)$ generated by the elements of $H$ and of the group of such automorphisms generated by the elements of $K$.

The automorphisms of $K$ which can be extended to inner automorphisms of $(H, K; \phi)$ turn out (§7) to be the inner automorphisms of $K$. If the ascending central series ends with the center and if $\phi(K) = \mathfrak{Z}(H)$, then $\mathfrak{Z}(K)$ can be injected into $\mathfrak{Z}(H, K; \phi)$.

Those automorphisms of $G$ which induce (§8) a pair of automorphisms, one on $H$, the other on $K$, form a subgroup $\mathfrak{S}$. If $H$ is of class 2 and if $\phi \in \text{Hom} (K, \mathfrak{Z}(H))$, then $\mathfrak{Z}(H)$ has an isomorphic image which is a direct summand of a normal subgroup of $\mathfrak{S}$.

If $\mathfrak{S}$ is a group of automorphisms of $H$ which includes the inner automorphisms and is in the centralizer of the group of normal automorphisms, then the normal automorphism group of the relative holomorph of $\mathfrak{S}$ over $H$ turns out to be a semi-direct product. For a class 2 group $H$, construct the centrally normal automorphisms, all those automorphisms which induce the identity on the center and on the inner automorphism group. Then construct the relative holomorph over $H$ of this group of automorphisms, and form the centrally normal automorphism group of this relative holomorph. By modifying the proof of the result on the holomorph of $\mathfrak{S}$ above, we can show that our new group of centrally normal automorphisms splits into a direct sum of two groups, one of which lies in the $\mathfrak{S}$ of the relative holomorph.

The ascending central series is determined, in §9, for the case where $\phi$ is \textit{ample} in the sense that $\phi(K) \subset \mathfrak{Z}(H)$. The members of this series and the related quotient groups turn out to be semi-direct products related to the ascending central series of $H$ and of $K$ and to the repeated commutator quotients of kernels of appropriate mappings. Even if $\phi$ is not ample, one can readily construct the first two members of the ascending central series. $G$ is of class 2 if and only if $H$ and $K$ are of class 2 and each automorphism in $\phi(K)$ is a special type of normal automorphism of $H$.

A short discussion of the derivative occurs in §10. The factor-commutator group of $G$ is the direct product of $H$ modulo a subgroup of modified commutators and of the factor-commutator group of $K$.

Normalizers and centralizers of $H$ and $K$ are found in §11. The largest normal subgroup of $G$ included in $K$ is the kernel of $\phi$. The normalizer tower of $K$ in $G$ is constructed by using a modification of the normalizer tower of the subgroup of universal fixed points (under the $\phi_k$) of $H$ in $H$. Again the commutator quotients will be indispensable in the consideration of this tower. The normalizer modulo the centralizer of $K$ in $G$ is isomorphic to $\mathfrak{Z}(K)$, also a property of a direct summand.
As for notation, $H \triangleleft G$ shall mean that $H$ is a normal subgroup of the group $G$. The element $e$ is used for the unity of every group, and $(e)$ is the one-element subgroup which it generates. If $A$ is a subset of $G$, then $\{A\}$ is to be the subgroup of $G$ generated by the elements of $A$. $\mathfrak{A}(H)$, $\mathfrak{Y}(H)$ and $\mathfrak{G}(H)$ are to be, respectively, the group of automorphisms, the group of inner automorphisms and the set of endomorphisms of the group $H$. $\text{Hom}(A, B)$ is to be the set of homomorphisms on the group $A$ into the group $B$. For a homomorphism $\alpha$, we shall denote the kernel by $\text{kern } \alpha$. $\iota$ will be reserved for the trivial automorphism or for an injection of a group $A$ into a group $B$ (isomorphism into). For an abelian group $H$, the mappings $\pi_n$ mentioned above are endomorphisms. We write $\pi_{-1} = \omega$. For $\alpha, \beta \in \text{Hom}(A, B)$, $\alpha + \beta$ shall denote the mapping (not necessarily a homomorphism) given by $(\alpha + \beta)(x) = \alpha(x)\beta(x)$. As usual, $\otimes$ is to denote cartesian product, while $\oplus$ will indicate direct sum (product). $\cong$ means that there is an isomorphism from, say, left to right. The members of $\mathfrak{Y}(H)$ are the $\theta_h$, where $\theta_h(x) = x^h = hxh^{-1}$. $\nu$ will be reserved for the trivial map: $\nu(a) = e$ for every $a \in A$. $\nu \in \text{Hom}(A, B)$. The $Z_i(G)$ are to be the members of the ascending central series [9] of $G$. $Z_1(G)$ is the center, and $Z_1(G/Z_n(G)) \cong Z_{n+1}(G)/Z_n(G)$. $G$ is said to be (nilpotent) of class $n$ (or to be $n$-nilpotent) if $G = Z_n(G)$. $\mathfrak{Z}_i(G)$, the set of normal automorphisms [9] of $G$, is the centralizer of $\mathfrak{Y}(G)$ in $\mathfrak{G}(G)$. Let $\mathfrak{A}(A; A/B)$ be the subset of all those automorphisms of $A$ which induce $\iota$ on $A/B$. Then $\mathfrak{Z}_1(G) = \mathfrak{A}(G; G/Z_1(G))$. Let $\mathfrak{G}_n(G) = \mathfrak{G}(G; G/Z_n(G))$ [4]. Occasionally, if $B$ is a non-normal subgroup of $A$, we shall write $\mathfrak{A}(A; A \bmod B)$ for the set of automorphisms $\alpha$ of $A \bmod B$ for the set of automorphisms $\alpha$ of $A \bmod B$ where $x^{-1}\alpha(x) \in B$ for every $x \in A$. $\mathfrak{A}(H; B)$ is to be the set of automorphisms of $H$ which induce the identity on the subgroup $B$ of $H$. $H'$ is to be the commutator subgroup $[H, H] = D(H)$ of $H$ with generators all $[h_1, h_2] = h_1h_2h_1^{-1}h_2^{-1}$ with $h_1, h_2 \in H$. $H'' = D(2)(H) = D(D(H)), \ldots, D(n)(H) = D(D(\cdots D(H)))$, are the members of the derived series of $H$ [9]. $\mathfrak{C}(A, B)$ is to be the centralizer of $A$ in $B$ where $A$ is a subgroup of the group $B$. $\mathfrak{C}(n)(A, B) = \mathfrak{C}(\mathfrak{C}(n-1)(A, B), B)$ where $\mathfrak{C}(1)(A, B) = \mathfrak{C}(A, B)$. Replace $\mathfrak{C}$ by $\mathfrak{R}$ and the word centralizer by the word normalizer in the last two sentences. For a normal subgroup $A$ and for any subgroup $B$ of $G$, $A \triangleleft B$ (= $A \triangleleft \alpha B$) is to be the set of all $g \in G$ such that $[g, b] \in A$ for every $b \in B$. $A \triangleleft B$ is called [1] the commutator quotient of $A$ by $B$ in $G$. It is a subgroup of $G$, normal if $B$ is normal. The formally defined $K \triangleleft \alpha K$ is the subgroup $\mathfrak{R}(K, G)$ even though $K$ need not be normal. A subgroup $A$ of $B$ is said to be $\alpha$-admissible for $\alpha \in \mathfrak{C}(B)$ if $\alpha(a) \in A$ for every $a \in A$. The numbering of items starts anew with each section.

2. Mappings into $\phi(K)$. Let $G^*$ be a group, and let $H$ and $K$ be subgroups of $G^*$ such that (1) $G^* = \{H, K\}$, (2) $H \cap K = (e)$ and (3) $H \triangleleft G$. Then each element of $G^*$ has a unique representation in the form $hk$, $h \in H$, $k \in K$, and $(hk)(h'k') = (h(kh'k^{-1}))(kk')$, where $h' \in H, k' \in K$ [3]. Alternately, let $H$ and $K$ be groups, and let $\phi$ be in $\text{Hom}(K, \mathfrak{A}(H))$. We write $\phi_a$ instead of
\(\phi(u)\) for \(u \in K\). \(\phi_{uv} = \phi_u \phi_v\) where mappings are written to the left. \(\phi(K)\) is to be the range of \(\phi\) in \(\mathfrak{A}(H)\) and is a subgroup thereof. Let us also write \(\text{kern } \phi = f(\phi) = f\) For ordered pairs from the cartesian product \(H \times K\), define multiplication by \((h, k)(h', k') = (h\phi_k(h'), kk')\) [7]. The element \((e, e)\) is the obvious multiplicative identity, and \((\phi_{k-1}(h^{-1}), k^{-1})(h, k) = (e, e)\). The associative law for multiplication is quickly verifiable so that \(G^f\), \(H \times K\) with the given multiplication, is a group. We write \(G = (H, K; \phi)\) and call this group the semi-direct product of \(H\) with respect to \(K\) via \(\phi\) [7]. A short argument shows that \(G^f\) is a splitting extension, i.e., an extension of \(H\) by \(K\) with a retractable factor system [9]; and conversely, every such extension is a semi-direct product. This formulation of the semi-direct product goes back to Cartan [2]. If \(K\) is a subgroup of \(\mathfrak{A}(H)\), and if \(\phi = \iota\), the injection of \(K\) into \(\mathfrak{A}(H)\), then \((H, K; \iota)\) is the relative holomorph of \(K\) over \(H\) [9]. In particular, \((H, \mathfrak{A}(H); \iota)\) is the holomorph \(\mathfrak{S}(H)\) of \(H\).

\(G^f\) has a subgroup \(H^* \cong H\) and a subgroup \(K^* \cong K\), where \(H^*\) consists of all \((h, e)\), and \(K^*\) is the set of all \((e, k)\). It is immediate that \((1) G^f = \{H^*, K^*\}\), that \((2) H^* \cap K^* = (e)\) and that \((3) H^* \triangle \Delta G\). Conversely, if a group \(G^*\) has subgroups \(H\) and \(K\) satisfying (1) and (2), and (3), we can define

\[
\phi \in \text{Hom } (K, \mathfrak{A}(H))
\]

by letting \(\phi_k\) be that automorphism of the normal subgroup \(H\) which is induced by the inner automorphism \(\theta_k \in \mathfrak{A}(G)\). Then \(G^f = (H, K; \phi) \cong G^*\) \(= H^* \times K^*\) under the mapping given by \((h, k) \rightarrow hk\). For this reason we shall write \(G = (H, K; \phi)\) in what follows. Thus, in a semi-direct product, \(\phi_k\) is the relativization of \(\theta_{(e, k)}\) to \(H\). It is well known [9] that every automorphism of \(H\) can be extended to an inner automorphism of \(\mathfrak{S}(H)\), so that we have verified the obvious generalization that every element of \(\phi(K)\) can be extended to an element of \(\mathfrak{S}(H, K; \phi)\). Since \(\theta_{(h,k)}(x, e) = (\theta_h \phi_k(x), e)\) for every \(x \in H\), \(\theta_{(h,k)}\) relativizes to some \(\phi_k\), if and only if \(\theta_h \in \phi(K)\). Let us say that \(\phi\) is ample if \(\phi(K) \supseteq \mathfrak{S}(H)\) and define \(B_1 = B_1(G)\) as the set of all \((h, k)\) with \(\theta_{h^{-1}} = \phi_k\). Note that if \(\phi\) is ample, then the mapping \(\gamma_1\) on \(G = (H, K; \phi)\) onto \(\phi(K)\) given by \(\gamma_1(h, k)\) \(= \theta_h \phi_k\) is a homomorphism, and \(\text{kern } \gamma_1 = B_1(G)\).

Let \(F_1 = F_1(H)\) be the set of all \(x \in H\) for which \(\phi_y(x) = x\) for every \(y \in K\). \(F_1(H)\) is a subgroup of \(H\). If \(\phi(K) \subseteq \mathfrak{S}_1(H)\), then \(F_1 \Delta H\), and the latter normal inclusion is equivalent, in any case, to \(\phi(K) \subseteq \mathfrak{A}(H; H/\mathfrak{C}(F_1(H), H))\); for if \(h \in H\), \(f \in F_1(H)\) and if \(y \in K\), then \(F_1(H) \Delta H\) implies that \(\phi_y(hfh^{-1}) = \phi_y(h)/\phi_y(h^{-1}) = hfh^{-1}\), or \(h^{-1} \phi_y(h) \in \mathfrak{C}(F_1(H), H)\), and conversely. Now \((h, k) \in Z_1(G)\) if and only if \((h, k)(x, y) = (x, y)(h, k)\) for every \((x, y) \in G\). Equivalently, \(h \phi_k(x) = x \phi_y(h)\) and \(k \in Z_1(K)\). Taking \(x = e\), we find \(h \in F_1\) so that \(\phi_k(x) = \theta_{h^{-1}}(x)\). Conversely, if \(h \in F_1\), \(k \in Z_1(K)\) and if \((h, k) \in B_1\), then \((h, k) \in Z_1(G)\). We have proved that \(Z_1(H, K; \phi) = B_1(H, K; \phi) \cap (F_1(H) = Z_1(G)\).

\(\phi\) See §1 for notation.
Let \( \theta \) be the natural homomorphism of a group \( A \) onto \( \mathfrak{H}(A) \) given by \( \theta(a) = \theta_a \). Then for \( \phi \) ample and \( G = (H, K; \phi) \), \( \gamma_2 \) defined by \( \gamma_2(\theta(h, k)) = \theta_{\phi(k)} \) is an onto mapping in \( \text{Hom}(\mathfrak{H}(G), \phi(K)) \) such that \( \gamma_2 = \gamma_2 \theta \) and \( \text{ker} \gamma_2 = \theta(B_1) \), where \( G = (H, K; \phi) \).

If \( G \) is the relative holomorph of a group \( K \) over a group \( H \) such that \( K \supset \mathfrak{H}(H) \), then \( \gamma_2 \) is onto \( K \) and has the form \( \gamma_2(h, k) = \theta_{\phi(k)}. \) In particular, there exists a homomorphism \( \gamma_1 \) of \( \mathfrak{H}(G) \) onto \( \mathfrak{H}(H) \). Let \( \gamma_1 \) on \( G = (H, K; \phi) \) into \( \mathfrak{H}(H) \) be given by \( \gamma_1(h, k) = (h, \phi_k) \). \( \gamma_3 \in \text{Hom}(G, (H, \phi(K); \iota)) \). But there exists \( \gamma'_1 \in \text{Hom}((H, \phi(K); \iota), \phi(K)) \) which is onto \( \phi(K) \), where \( \gamma'_1(h, \phi_k) = \theta_{\phi(k)} \), as we saw just above, provided that \( \phi \) is ample. Clearly, \( \gamma_1 = \gamma'_1 \gamma_3 \gamma_2 \). In what follows, let it be understood that \( \phi \) is ample. Let \( \gamma_4 \) be the natural map on \( G \) onto \( K \) given by \( \gamma_4(h, k) = k \). \( \gamma_4 \) induces an isomorphism: \( G/H \cong K \). Then \( \gamma_5 = \phi \gamma_4 \) is on \( G \) onto \( \phi(K) \). \( \gamma_5 = \gamma_1 \gamma_3 \) where \( \gamma'_1 \) is the \( \gamma_1 \) of \( (H, \phi(K); \iota) \). Now both \( \gamma_1 \) and \( \gamma_5 \) are on \( G \) onto \( \phi(K) \). \( \gamma_1(h, k) = \theta_{\phi(k)} \), and \( \gamma_5(h, k) = \phi_k \). Hence \( \gamma_1 = \gamma_5 \) if and only if \( H \) is abelian. Equivalently, \( \gamma'_1 = \gamma'_5 \).

Let \( \gamma_6 \) be the natural map of \( \phi(K) \) onto \( \phi(K)/\mathfrak{H}(K) \), where \( \phi \) is ample. Then \( \gamma_6 \gamma_1 = \gamma_6 \gamma_5 \). Let \( B_5 = \ker \gamma_5. B_5 \cong H \oplus \mathfrak{T}. \) It is easy to show that \( B_1 \subseteq B_5, B_3 \subseteq B_1 \) and \( B_1 = B_5 \) are equivalent conditions, and these, in turn, are equivalent to \( H \) being abelian. The only case where \( \gamma_1 = \gamma_5 \) (or \( \gamma_6 = \alpha \gamma_1 \)) with \( \alpha \in \mathfrak{H}(\phi(K)) \) is that with \( \alpha = 1 \), whence \( H \) is abelian. \( B_1 \cap B_5 \cong \mathfrak{Z}_1(H) \oplus \mathfrak{T} \) so that \( B_1 \cap B_5 = (e) \) if and only if \( G \) is a relative holomorph over a centerless group \( H \) of an extension of the inner automorphism group of \( H \). In particular, if \( H \) is a complete group \([9] \), then in \( \mathfrak{H}(H), B_1 \cap B_5 = (e) \).

The mapping \( \gamma_6 \) can be used to prove

**Lemma 1.** Let \( H \) and \( K_0 \) be groups, and let \( \mathfrak{U}_0 \) be a subgroup of \( \mathfrak{U}(H) \). Then each extension of \( K_0 \) by \( \mathfrak{U}_0 \) determines an extension of \( K_0 \) by the relative homomorph of \( \mathfrak{U}_0 \) over \( H \) to a semi-direct product of \( H \) by the extension of \( K_0 \).

**Proof.** Let \( K/K_0 \cong \mathfrak{U}_0. \) Define \( \phi \in \text{Hom}(K, \mathfrak{U}_0) \) by \( \phi(k) = \phi_k = "kK_0" \) where \( "kK_0" \) is that element of \( \mathfrak{U}_0 \) which corresponds to \( kK_0 \subseteq K/K_0 \) under the given isomorphism. Let \( G = (H, K; \phi) \). Let \( \mathfrak{T}^* \) be the set of all \( (e, k) \in G, k \in \ker \phi. \mathfrak{T}^* \cong K_0. \) For \( \gamma_3 \) on \( G \) onto \( (H, \phi(K); \iota) \) defined by \( \gamma_3(h, k) = (h, \phi_k) \), \( \ker \gamma_3 = \mathfrak{T}^* \). Since \( \phi(K) = \mathfrak{U}_0 \) and since \( \mathfrak{T}^* \cong K_0 \), there exists a group \( G_0 \cong G \) such that \( K_0 \triangle G_0 \), and \( G_0/K_0 \cong (H, \mathfrak{U}_0; \iota) \).

**3. Further mappings on \( \phi(K) \).** Let \( W \) be a group, and let \( G = (H, K; \phi) \). For \( \alpha \in \text{Hom}((H, W), \beta \in \text{Hom}(K, W)) \), define a mapping \( \gamma \) on \( G \) into \( W \) by \( \gamma(h, k) = \alpha(h)\beta(k) \). It is easy to verify that \( \gamma \in \text{Hom}(G, W) \) if and only if

\[
(1) \quad \beta(k)\alpha(h) = \alpha(\phi_k(h))\beta(k)
\]

for every \( h \in H \) and for every \( k \in K \). If such a pair of mappings \( \alpha \) and \( \beta \) obeys (1), we write \( \gamma = \alpha \land \beta \in \text{Hom}(G, W) \). Conversely, if \( \gamma \in \text{Hom}(G, W) \), if
$\alpha(h) = \gamma(h, e), \beta(k) = \gamma(e, k)$, then $\alpha \in \text{Hom}(H, W)$, $\beta \in \text{Hom}(K, W)$ and $\gamma = \alpha \wedge \beta$ where $\alpha$ and $\beta$ satisfy (1). Recall that $\mathfrak{N}(A; A/B)$ is a subgroup of $\mathfrak{N}(A)$. We have

**Theorem 1.** Let $G = (H, K; \phi)$, and suppose that $\alpha \in \text{Hom}(H, W)$ where $\ker \alpha$ is a characteristic subgroup of $H$, that $\beta \in \text{Hom}(K, W)$ where $\phi \wedge \gamma \in \text{Hom}(G, W)$, that $\sigma \in \mathfrak{N}(H)$ and that $\tau \in \mathfrak{N}(K; K/\ker \alpha)$. Then $\alpha \sigma \wedge \beta \tau \in \text{Hom}(G, W)$ if and only if $\sigma \in \mathfrak{N}(H; H/\ker \alpha) \div \phi(K)$.

**Proof.** Since $\alpha \wedge \beta \in \text{Hom}(G, W)$, $\beta(k) = \alpha(\phi_k(h))\beta(k)$ for all $h \in H$ and for all $k \in K$. For $\sigma \in \mathfrak{C}(H)$, $\tau \in \mathfrak{C}(K)$, $\sigma \wedge \beta \tau \in \text{Hom}(G, W)$ if and only if $\beta \tau(k) \alpha \sigma(h) = \alpha(\phi_k(h))\beta \tau(k)$. Since $\sigma$ and $\tau$ are automorphisms, in the first of these identities we can replace $k$ by $\tau(k)$ and $h$ by $\sigma(h)$. Then $\beta \tau(k) \alpha \sigma(h) = \alpha(\phi_{\tau(k)}(\sigma(h)))\beta \tau(k)$. Thus $\alpha \sigma \wedge \beta \tau \in \text{Hom}(G, W)$ if and only if $\alpha \phi_{\tau(k)}(\sigma(h)) = \alpha \sigma \phi_k(h)$ for every $h \in H$ and for every $k \in K$. Equivalently, $\phi_{\tau(k)}(\sigma(h)) \equiv \sigma \phi_k(h) \mod \ker \alpha$. By hypothesis, $\phi_{\tau(k)} = \phi_k$, and $\sigma$ has an inverse $\sigma^{-1}$. Then the condition that $\alpha \sigma \wedge \beta \tau \in \text{Hom}(G, W)$ reduces to $\phi_k \sigma(h) \equiv \sigma \phi_k(h) \mod \ker \alpha$. Since $\ker \alpha$ is characteristic, this latter congruence is the same as $\phi_k^{-1} \sigma \phi_k \equiv \phi_k \equiv \mod \ker \alpha$. Since $\ker \alpha$ is characteristic, $\mathfrak{N}(H; H/\ker \alpha)$ is $\sigma$-admissible for every $k \in K$. Corresponding to Theorem 1 is

**Theorem 2.** Let $G = (H, K; \phi)$ with ample $\phi$. For $\beta \in \text{Hom}(K, \phi(K))$, the following are equivalent: (a) $\theta \wedge \beta \in \text{Hom}(G, \phi(K))$; (b) $\beta_k \equiv \phi_k \mod (\mathfrak{N}_1(H \wedge \phi(K))$ for every $k \in K$; (c) for each $k \in K$ there exists $\zeta_k \in \text{Hom}(H, Z_1(H))$ with $\beta_k = \phi_k + \zeta_k$.

Suppose that $G = (H, K; \phi)$ with ample $\phi$. For $\alpha \in \text{Hom}(H, \phi(K))$, $\alpha \wedge \phi \in \text{Hom}(G, \phi(K))$ if and only if $\phi_k \alpha_k = \alpha \phi_k(\phi_k)$ for every $h \in H$ and for every $k \in K$, where we write $\alpha(h) = \alpha_k$. If $\alpha \wedge \phi \in \text{Hom}(G, \phi(K))$, then this mapping is a homomorphism onto $\phi(K)$. If $h \in \ker \alpha$, $\phi_k = \alpha \phi_k(\phi_k)$ so that $\phi_k \in \ker \alpha$, and $\ker \alpha$ is $\phi_k$-admissible for every $k \in K$. Corresponding to Theorem 2 is

**Theorem 3.** Let $G = (H, K; \phi)$. For $\alpha \in \text{Hom}(H, \phi(K))$ any two of the following imply the third: (a) $\alpha \wedge \phi \in \text{Hom}(G, \phi(K))$. (b) $\phi(K) \subset \mathfrak{N}(H; H/\ker \alpha)$. (c) $\alpha(H) \subset Z_1(\phi(K))$.

**Corollary.** (a) $G = (H, K; \phi)$ with ample $\phi$, $\phi(K) \subset \mathfrak{N}_1(H)$ if and only if $\mathfrak{N}(H) \subset Z_1(\phi(K))$. (b) $\mathfrak{N}(H) \subset Z_1(\phi(K))$ implies that $F \Delta H$.

We now examine modifications of $\gamma$. Let $G = (H, K; \phi)$ with ample $\phi$. For $\sigma \in \mathfrak{C}(H)$, $\tau \in \mathfrak{C}(K)$, $\gamma = \theta \sigma \wedge \phi \tau \in \text{Hom}(G, \phi(K))$ if and only if

$$\phi_{\tau(k)}(\sigma(h)) \equiv \sigma(\phi_k(h)) \mod Z_1(H)$$
for every \( h \in H \) and for every \( k \in K \). Suppose, now, that
\[
\gamma = \theta \sigma \land \phi \tau \in \text{Hom}(G, \phi(K)).
\]
Then \( \psi \) on \( \text{kern} \ \gamma \) into \( G \) given by \( \psi(h, k) = (\sigma(h), \tau(k)) \) for every \((h, k) \in \text{kern} \ \gamma \) is readily seen to be into \( B_1 \). Conversely, if \((\sigma(h), \tau(k)) \in B_1 \), then \((h, k) \in \text{kern} \ \gamma \). This shows that if \( \psi \) is extended to all of \( G \), then \( \psi(\text{kern} \ \gamma) \subset B_1 \), and that the complete inverse image in \( G \) of \( B_1 \) under \( \psi \) is \( \text{kern} \ \gamma \).

**Lemma 2.** Let \( G = (H, K; \phi) \) with \( \phi \) ample, and let \( \sigma \in \mathfrak{S}(H), \tau \in \mathfrak{S}(K) \) generate \( \psi \in \mathfrak{S}(G) \) by \( \psi(h, k) = (\sigma(h), \tau(k)) \). Then \( \theta \sigma \land \phi \tau \in \text{Hom}(G, \phi(K)) \).

**Proof.** Necessary and sufficient for \( \psi \) to be an endomorphism when \( \sigma \) and \( \tau \) are endomorphisms is that \( \psi(h\phi_k(x), ky) = \psi(h, k)\psi(x, y) \) for every \( h, x \in H \) and for every \( k, y \in K \). The left side is \( (\sigma(h)\sigma\phi_k(x), \tau(ky)) \), while the right is \( (\sigma(h)\phi\tau(k)\sigma(x), \tau(ky)) \). Hence \( \psi \) is in \( \mathfrak{S}(G) \) if and only if
\[
(3) \quad \sigma\phi_k = \phi\tau(k)\sigma
\]
for every \( k \in K \). But this equality surely implies the congruence \( (2) \).

**Lemma 3.** (a) If \( \sigma \) is an automorphism and if \( \mathfrak{A} \) is \( \tau \)-admissible, then \( \text{kern} \ (\theta \sigma \land \phi \tau) \cap B_6 = B_1 \cap B_6 \). (b) If \( \sigma \) is an automorphism then \( (h, k) \in \text{kern} \ \gamma \) implies that \( a(h) = a(\phi\tau(h)) \mod \Xi\sigma(H) \) and that \( \phi\tau(k) = \sigma(h) \mod \Xi\sigma(H) \).

It should be noted that the latter conclusion has the equivalent form
\[
[\sigma(h), \phi\tau(k)] = 1.
\]
If \( H \) is abelian, then \( \theta = \nu \). For \( \tau \in \mathfrak{A}(K) \) one can readily verify that \( \gamma = \nu \land \phi \tau \in \text{Hom}(G, \phi(K)) \). Here \( \gamma(h, k) = \phi\tau(k) \).

If \( \phi \) is ample, and if \( \sigma \in \mathfrak{A}(H) \), then \( \gamma = \theta \sigma \land \phi \in \text{Hom}(G, \phi(K)) \) if and only if \( \phi\kappa\sigma(h) = \sigma\phi\kappa(h) \mod \Xi\sigma(H) \), from \( (2) \). Since \( \Xi\sigma(H) \) is characteristic and since \( \Xi\kappa(H) \Delta \mathfrak{A}(H) \), this latter congruence is equivalent to \( (3) \). \( \sigma \in \Xi\kappa(H) \land \phi(K) \). (If, in Theorem 1, we take \( W = \phi(K) \), the same conclusion results.) Let \( B_\tau = \text{kern} \ \gamma \). Now suppose that \( \sigma, \tau \in \Xi\kappa(H) \land \phi(K) \). Let \( \mu \) on \( B_\tau \) onto \( B_\tau \) be defined by \( \mu(h, k) = (\tau^{-1}\sigma(h), k) \) for \( (h, k) \in B_\tau \). One can show that \( \mu \) will be an isomorphism on \( B_\tau \) onto \( B_\tau \) if \( \phi\kappa[\tau^{-1}\sigma(h)] = \tau^{-1}\phi\kappa(h) \) for every \( h \in H \) and for every \( k \in K \). But this is equivalent to \( \sigma \equiv \tau \mod \mathfrak{C}(\phi(K), \mathfrak{A}(H)) \). Now \( \sigma \equiv \tau \mod \Xi\kappa(H) \) if the above congruence modulo \( \mathfrak{C} \) holds; for, \( \phi(K) \subset \Xi\kappa(H) \) so that \( \Xi\kappa(K) \subset \Xi\kappa(H) \). Then \( \sigma(h) = \tau(h) \mod \Xi\kappa(H) \) for every \( h \in H \). From this it follows that \( \theta\sigma(h) = \theta\tau(h) \) for every \( h \in H \). Hence \( \gamma = \theta \sigma \land \phi = \theta \tau \land \phi = \gamma \). We have proved
Theorem 4. Let $G = (H, K; \phi)$ with $\phi$ ample. If $\sigma, \tau \in \mathfrak{H}(H)$ with $\sigma \equiv \tau \mod \mathcal{E}(\phi(K), \mathfrak{H}(H))$ and with $\sigma, \tau \in \mathfrak{I}(H) \circ \phi(K)$, then $\theta \sigma \wedge \phi = \theta \tau \wedge \phi$.

Note that if $\sigma \in \mathfrak{I}(H)$, then $\theta \sigma \wedge \phi = \gamma_1$.

By (2), $\gamma^{(e)} = \theta \wedge \phi \in \text{Hom} (G, \phi(K))$, where $\sigma \in \mathfrak{H}(K)$, if and only if $\phi_{\sigma(k)}(h) \equiv \phi_k(h) \mod Z_1(H)$ for every $h \in H$ and for every $k \in K$. That is, $\phi_{\kappa^{-1} \sigma(k)} \in \mathfrak{I}(H)$ is equivalent to $\gamma^{(e)} \in \text{Hom} (G, \phi(K))$. Observe that if $f$ is $\sigma$-admissible then $\sigma$ induces an automorphism on $\phi(K)$ by $\phi_k \rightarrow \phi_{\sigma(k)}$.

Let $K$ be abelian so that $\omega \in \mathfrak{H}(K)$. If $\phi$ is ample, $\mathfrak{H}(H)$ is consequently abelian so that $H$ is of class 2. For $\sigma \in \mathfrak{E}(H)$, (2) shows that $\sigma \wedge \omega \in \text{Hom}$ if and only if $\phi_{\kappa^{-1} \sigma(h)} \equiv \sigma \phi_k(h) \mod Z_1(H)$ for every $h \in H$ and for every $k \in K$. For any $W$, it is easily seen that $\alpha \in \text{Hom} (H, W)$, $\kappa \in \mathfrak{E}(K)$, if and only if $\kappa \wedge \beta \in \text{Hom} (G, \mathfrak{H}(W))$, where $G = (H, K; \phi)$ and $K$ is abelian, imply that $\alpha \wedge \beta \omega \in \text{Hom} (G, W)$ if and only if each $\phi_k \in \mathfrak{H}(H; H \wedge \kappa \omega)$.

Let $H$ be a group of class 2, and let $\mathfrak{K}$ be an abelian extension in $\mathfrak{H}(H)$ of $\mathfrak{H}(H)$. Consider $G = (H, \mathfrak{K}; \iota)$, the relative holomorph of $\mathfrak{K}$ over $H$. For a given integer $n$, $\theta \wedge \pi_n \in \text{Hom} (G, \mathfrak{K})$ if and only if, by (1), $\kappa^n \theta_h(x) = \theta_{\kappa(h)} \kappa^n(x)$ for every $x, h \in H$ and for every $\kappa \in \mathfrak{K}$. Upon expansion, this turns out to be the equivalent of $\kappa^n(h) \equiv \kappa(h) \mod Z_1(H)$, or equally, $\kappa^n \equiv \kappa \mod \mathfrak{I}(H)$ and $\kappa^{n-1} \in \mathfrak{I}(H)$. Since, however, $\mathfrak{K}$ is an abelian extension of the abelian group $\mathfrak{H}(H, \mathfrak{K}) = \mathfrak{H}(H; \mathfrak{H}(H))$. Hence

Lemma 5. Let $H$ be a group of class 2, and let $\mathfrak{K}$ be an abelian extension in $\mathfrak{H}(H)$ of $\mathfrak{H}(H)$. Then for each integer $n$, $\theta \wedge \pi_n \in \text{Hom} ((H, \mathfrak{K}; \iota), \mathfrak{K})$.

4. Some properties of the holomorph. The results of §3 apply to $\mathfrak{H}(H)$ since $\phi = \iota$ is ample. For $\alpha \in \text{Hom} (H, W)$, $\Delta \in \text{Hom} (\mathfrak{H}(H), W)$, we have $\alpha \wedge \Delta \in \text{Hom} (\mathfrak{H}(H), W)$ if and only if

$$\Delta(\kappa) \alpha(h) = \alpha(\kappa(h)) \Delta(\kappa)$$

for every $h \in H$ and for every $\kappa \in \mathfrak{H}(H)$. Take $W = \mathfrak{H}(H)$ so that $\Delta \in \mathfrak{E}(\mathfrak{H}(H))$. With the notation $\alpha(h) = \alpha_h$, if $\alpha \in \text{Hom} (H, \mathfrak{H}(H))$, $\Delta \in \mathfrak{E}(\mathfrak{H}(H))$, then $\alpha \wedge \Delta \in \text{Hom} (\mathfrak{H}(H), \mathfrak{H}(H))$ implies that $\Delta(\theta_u) \equiv \alpha_u \mod \mathcal{E}(\alpha(H), \mathfrak{H}(H))$ for every $u \in H$. By Theorem 2, $\theta \wedge \Delta \in \text{Hom} (\mathfrak{H}(H), \mathfrak{H}(H))$ if and only if $\Delta(\kappa) \equiv \kappa \mod \mathfrak{I}(H)$ for every $\kappa \in \mathfrak{H}(H)$. If $\Delta \in \mathfrak{E}(\mathfrak{H}(H))$, then

$$\Delta \theta \wedge \Delta \in \text{Hom} (\mathfrak{H}(H), \mathfrak{H}(H)),$$

and $B_1 \subseteq \ker (\Delta \theta \wedge \Delta)$. If $\Delta \in \mathfrak{H}(\mathfrak{H}(H)) = \mathfrak{H}^2(H)$, $\ker (\Delta \theta \wedge \Delta) = B_1$.

Lemma 6. Let $H$ be an abelian group. If $\delta \in \text{Hom} (H, \mathfrak{H}(H))$ such that there exists $\Delta \in \mathfrak{H}^2(H)$ with $\Delta \delta \wedge \Delta \in \text{Hom} (\mathfrak{H}(H), \mathfrak{H}(H))$, then $H / \ker \delta$ is abelian with exponent $\leq 2$.

Proof. Since $\Delta \delta \wedge \Delta \in \text{Hom} (\mathfrak{H}, \mathfrak{H})$, (1) shows that
\[ \Delta(\kappa)\Delta(\delta(h)) = \Delta(\delta(\kappa(h)))\Delta(\kappa) \]

for every \( h \in H \) and for every \( \kappa \in \mathfrak{A}(H) \). Hence \( \kappa \delta_h \kappa^{-1} \delta_{\kappa(h)^{-1}} = e \). Now \( H \) is abelian so that we may choose \( \kappa = \omega \). Then, for \( x \in H \), \( \omega \delta_h \omega \delta_h(x) = x \), whence

\[ \delta_{\kappa}(x) = x, \quad \delta_{\kappa^2} = e \quad \text{and} \quad h^2 \quad \text{in} \quad \text{ker} \ \delta. \]

In particular, the lemma holds if \( \delta \wedge \iota \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \).

**Corollary.** Let \( H \) be an abelian group in which \( x^2 = h \in H \) always has a solution. Then for \( \delta \in \text{Hom}(H, \mathfrak{A}(H)) \) there exists \( \Delta \in \mathfrak{A}^2(H) \) such that \( \Delta \delta \wedge \Delta \in \text{Hom}(\mathfrak{F}(H), \mathfrak{A}(H)) \) if and only if \( \delta = \nu \). If \( \delta = \nu \), any \( \Delta \in \mathfrak{A}^2(H) \) may be chosen.

Observe that if \( \delta \in \text{Hom}(H, \mathfrak{A}(H)) \) and if \( \Delta_1, \Delta_2 \in \mathfrak{A}(\mathfrak{A}(H)) \) where \( \text{ker} \ \Delta_1 = \text{ker} \ \Delta_2 \) then \( \Delta_1 \delta \wedge \Delta_2 \in \text{Hom}(\mathfrak{F}(H), \mathfrak{A}(H)) \) if and only if \( \Delta_2 \delta \wedge \Delta_2 \) is in this same Hom. In particular, if \( \Delta \in \mathfrak{A}^2(H) \), then \( \Delta \delta \wedge \Delta \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) if and only if \( \delta \wedge \iota \in \text{Hom}(\mathfrak{F}(H), \mathfrak{A}(H)) \); i.e., \( \kappa \delta_h \kappa^{-1} = \delta_{\kappa(h)} \) for every \( \kappa \in \mathfrak{A}(H) \). For instance, \( \theta \wedge \iota \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \). If \( \delta \in \text{Hom}(H, \mathfrak{A}(H)) \), if \( H/\text{ker} \ \delta \) is abelian and if \( \delta \wedge \iota \in \text{Hom}(\mathfrak{F}(H), \mathfrak{A}(H)) \), then \( \delta(H) \subset \mathcal{T}_1(H) \).

**Theorem 5.** If \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}(H), \mathfrak{A}(H)) \) for every \( h \in H \), then \( H \) is of class 3.

**Proof.** Here \( \theta_h \) means that element of \( \mathfrak{A}(H) \) which maps \( \kappa \in \mathfrak{A}(H) \) onto \( \theta_h \kappa \theta_h^{-1} \). By the remark after the italicized statement following (1), \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) if and only if \( \theta_h \theta_h^{-1} = \kappa \mod \mathcal{T}_1(H) \); equivalently, \( \theta_h \in \mathcal{T}(\mathfrak{A}(H)) \)

But \( \theta_h \theta_h^{-1} \kappa^{-1} = [h \kappa(h^{-1})] x [h \kappa(h^{-1})]^{-1} = x \mod \mathcal{Z}_1(H) \) so that \( h \kappa(h^{-1}) \in \mathcal{Z}_1(H) \). Thus every \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) if and only if \( \mathfrak{A}(H) = \mathcal{T}_2(H) \). Then \( \theta(z)(h) = h \mod \mathcal{Z}_2(H) \) for every \( x, h \in H \). It follows that \( \mathfrak{A}(H) \subset \mathcal{T}_2(H) \) so that \( H \) is of class 3.

We have also proved that \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) implies that \( h \in \mathcal{Z}_3(H) \). Theorem 2 shows that \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \), where \( \lambda \in \mathfrak{A}(H) \), if and only if \( \lambda \in \mathcal{T}_2(H) \). Since this commutator quotient is included in \( \mathcal{T}_2(H) \), \( \lambda \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) implies that \( \lambda \in \mathcal{T}_3(H) \). If \( H \) is of class 2, then \( \mathfrak{A}(H) = \mathcal{T}_2(H) \) so that \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) for every \( h \in H \). It is easy to show that if all \( \theta \wedge \theta_h \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \), then \( \mathcal{A} \ker \theta \wedge \theta_h \) is the set of all \( (x, \theta_x^{-1}) \) where \( x \in \mathcal{Z}_2(H) \). Hence if \( H \) is of class 2 this intersection of kernels reduces to \( B_1 \). Likewise, \( \theta \wedge \iota \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) as we saw above, and \( \ker (\theta \wedge \iota) = B_1 \). Suppose that \( \alpha \) is an isomorphism of \( H \) into \( \mathfrak{A}(H) \) for which \( \alpha \wedge \iota \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \). It is easy to show that \( \alpha = \theta \) so that \( H \) is centerless. In the general case, if \( \alpha \in \text{Hom}(H, \mathfrak{A}(H)) \) and if \( \alpha \wedge \iota \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \), then \( \alpha(\mathcal{T}_1(H)) \subset \mathcal{T}_1(H) \cap \mathfrak{A}(H; H/\ker \alpha) \). If \( H \) is abelian and if \( \alpha \wedge \nu \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \), then \( H/\ker \alpha \) is abelian with exponent \( \leq 2 \). For, given any group \( H, \alpha \wedge \nu \in \text{Hom}(\mathfrak{F}, \mathfrak{A}) \) if and only if, from (1), \( \kappa(h) = h \mod \ker \alpha \) for every \( h \in H \) and for every \( \kappa \in \mathfrak{A}(H) \). If \( H \) is abelian, take \( \kappa = \omega \). Even if \( H \) is not abelian, we can choose \( \kappa = \theta_z \) and allow \( x \) to run over all of \( H \). Then \( H' \subset \ker \alpha \) so that \( \alpha \wedge \nu \)}
$\in \text{Hom } (\mathfrak{H}, \mathfrak{H})$ implies that $H/\text{kern } \alpha$ is abelian.

It is also valid that if $\delta \in \text{Hom } (H, \mathfrak{H}(H))$ and if $\delta \wedge \iota$ and $\delta \wedge \nu$ are both in $\text{Hom } (\mathfrak{H}(H), \mathfrak{H}(H))$, then $\delta(H) \subset Z_1(\mathfrak{H}(H))$. (Cf. the italicized statement just before Theorem 5.) If $H$ has no outer automorphisms, (say, if it is complete [9]), then $\alpha \wedge \nu \in \text{Hom } (\mathfrak{H}(H), \mathfrak{H}(H))$ if and only if $H/\text{kern } \alpha$ is abelian. If $H$ is abelian, if $\alpha \in \text{Hom } (H, \mathfrak{H}(H))$, if $\Delta \in \mathfrak{H}(\mathfrak{H}(H))$, if $H/\text{kern } \alpha$ has exponent 2 and if $\alpha \wedge \Delta \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$, then $\Delta(\omega) \in \mathfrak{H}(\alpha(H), \mathfrak{H}(H))$. For, from (1) with $\kappa = \omega$, $\Delta(\omega) \alpha_h = \alpha_{h^{-1}} \Delta(\omega)$. Since $H/\text{kern } \alpha$ has exponent 2, $(\alpha_h)^2 = e$ so that $\alpha_h = \alpha_{h^{-1}}$. Hence $\Delta(\omega) \in \mathfrak{H}$.

Now $\alpha \wedge \theta_h \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$ if and only if

\begin{equation}
\lambda \kappa \lambda^{-1} \alpha_h = \alpha_{\kappa(h)} \lambda \kappa \lambda^{-1}
\end{equation}

for every $\kappa \in \mathfrak{H}(H)$. Equivalently, replacing $\kappa$ by $\lambda^{-1} \kappa$, we have

\begin{equation}
\kappa \alpha_h = \alpha^{-1} \kappa \alpha(h) \kappa.
\end{equation}

Replacing $\kappa$ by $\alpha^{-1}$, $[\lambda^{-1} \alpha_h - \lambda(h)] h^{-1} \in \text{kern } \alpha$. In (3) take $\kappa = \lambda$ and $h \in \text{kern } \alpha$. Then $\lambda(h) \in \text{kern } \alpha$ so that $\text{kern } \alpha$ is $\lambda$-admissible. Hence $\alpha_h^{-1} \lambda(h) = \lambda(h) \mod \text{kern } \alpha$. Thus, replacing $h$ by $h^{-1}$, we have $\alpha_h \lambda(h) = \lambda(h) \mod \text{kern } \alpha$. In (2), replace $\kappa$ by $\theta_h$ and reduce. Combining, we have

\begin{equation}
\alpha_h \lambda(h) = \lambda(h) \mod (Z_1(H) \cap \text{kern } \alpha)
\end{equation}

for every $h \in H$ and for every $\alpha$ and $\lambda$ such that $\alpha \wedge \theta_h \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$.

In (2), take $\kappa = \lambda$ to obtain $\lambda \kappa \alpha_h = \alpha \lambda \alpha_h \lambda$. In (2), replace $\kappa$ by $\alpha_h$. Then, by the preceding formula, $\lambda \alpha \lambda^{-1} \alpha_h = \alpha \lambda \alpha(h) \kappa$. On the left of (2). On the right of (2), $\alpha \alpha \lambda(h) \lambda \alpha \lambda^{-1} = \alpha \alpha(h) \lambda(h)$, by (3). Thus $\alpha \wedge \theta_h \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$ implies that

\begin{equation}
\lambda(h) \equiv \alpha_h(h) \lambda(h) \mod \text{kern } \alpha.
\end{equation}

**Lemma 7.** (a) If $\alpha \in \text{Hom } (H, \mathfrak{H}(H))$ and if $\alpha \wedge \iota \in \text{Hom } (\mathfrak{H}(H), \mathfrak{H}(H))$, then $\alpha_h(h) = h$ mod $(Z_1(H) \cap \text{kern } \alpha)$ for every $h \in H$. (b) If $h \in H$ has the property that every conjugate of $h$ lies in the centralizer of $h$ and if $u \in H$ with $\alpha \wedge \theta_u \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$ then $\alpha(h) \equiv h$ mod kern $\alpha$.

**Proof.** (a) Since $\alpha \wedge \iota \in \text{Hom } (\mathfrak{H}, \mathfrak{H})$, let $\lambda = \iota$ in (4). (b) In (5) take $\lambda = \theta_u$. Then $u h u^{-1} = \alpha(h) u h u^{-1}$ mod kern $\alpha$. But $(u h u^{-1}) h = h (u h u^{-1})$, whence the conclusion is immediate.

We can obtain some direct information about the first few members of the ascending central series of $\mathfrak{H}(H)$, the center consisting, for instance, of all $(h, \iota)$ where $h \in F_1(H)$. For $(h, \alpha) \in Z_2(\mathfrak{H}(H))$, $\eta(h) \equiv h$ mod $F_1(H)$ for every $\eta \in \mathfrak{H}(H)$. From this, one can show that $\mathfrak{H}(H)$ is of class 2 if and only if $\mathfrak{H}(H) = \mathfrak{H}(H; H/F_1(H))$. In general, $(h, \alpha) \in Z_2(\mathfrak{H}(H))$ if and only if (a) $\eta(h) \equiv h$ mod $F_1(H)$ for every $\eta \in \mathfrak{H}(H)$, (b) $\alpha \in Z_1(\mathfrak{H}(H))$ and (c) $\theta_h \alpha \in \mathfrak{H}(H)$.
By induction, \((h, \alpha) \in Z_n(\mathfrak{H}(H))\) implies that \(\alpha \in Z_{n-1}(\mathfrak{H}(H))\). One can readily show that if \(\mathfrak{H}(H)\) is of class 3 then \(\alpha(h') \equiv h' \mod F_1(H)\) for every \(\alpha \in \mathfrak{H}(H)\) and for every \((h) h' \in H'\).

5. **Inclusion products.** For \((h, k) \in G = (H, K; \phi)\), let \(\gamma_7(h, k) = (\theta_h, \phi_k) \in Q = (\mathfrak{H}(H), \phi(K); \theta)\). \(\gamma_7\) is a homomorphism with kernel \(B_1 \cap B_2 \cong Z_1(H) \oplus I\). Let \(\gamma_8(\theta_h, \phi_k) = \theta_h \phi_k\) (if \(\phi\) is ample). \(\gamma_8\) is likewise a homomorphism, and kernel \(\gamma_8(B_1)\) with \(\gamma_1 = \gamma_8 \gamma_7\). Consider a pair of groups \(H\) and \(K\) where \(H \triangle K\). Then one can always form the semi-direct product \((H, K; \theta)\) which we shall abbreviate by \([H \triangle K]\). We shall call this an *inclusion product of the first kind*. If \(\phi\) is ample, \(Q\) above is such a product. Another is \((H, H; \theta)\), the *central square* of \(H\). Denote this square by \([H \triangle H]\), an extension of \(H\) by \(H\) for which \(Z_1[H \triangle H] \cong B_1 \cap B_2 \cong [Z_1(H) \triangle Z_1(H)]\). For \(K\) of class 2 and for fixed integers \(m\) and \(n\), the mapping \(\delta\) given by \(\delta(h, k) = \theta_{m \phi_k}\) lies in \(\text{Hom}(\{H \triangle K\}, \mathfrak{H}(K))\).

For \(\alpha\) and \(\beta \in \mathfrak{H}(K)\), a routine argument shows that \(\alpha \wedge \beta \in \text{Hom}(\{H \triangle K\}, \mathfrak{H}(K))\) if and only if \(\alpha(k) \equiv \beta(k) \mod (Z_1(K) + \alpha(H))\) for every \(k \in K\). In particular, \(\alpha \wedge \beta \in \text{Hom}(\{H \triangle H\}, \mathfrak{H}(H))\) if and only if \(\alpha \equiv \beta \mod \mathfrak{H}_2(H)\). Hence if \(K\) is of class 2, \(\alpha \wedge \beta \in \text{Hom}(\{H \triangle K\}, \mathfrak{H}(H))\) for every \(\alpha\) and \(\beta \in \mathfrak{H}(K)\); and if \(H\) is of class 2, \(\alpha \wedge \beta \in \text{Hom}(\{H \triangle H\}, \mathfrak{H}(H))\) for every \(\alpha, \beta \in \mathfrak{H}(H)\). In any event, for \(\alpha, \beta \in \text{Hom}(K, \mathfrak{H}(H))\), \(\alpha \wedge \beta \in \text{Hom}(\{H \triangle K\}, \mathfrak{H}(K))\) if and only if \(\alpha(k) \equiv \beta(k) \mod (\mathfrak{H}(\alpha(H), \mathfrak{H}(\beta(H)))\).

If \(L\) is a subgroup of \(K\) and if \(H \triangle K\), then \(L \cap H \triangle L\). At once we have, by a simple induction, that \(Z_n[H \triangle K] = [Z_n(K) \cap H \triangle Z_n(K)]\).

If \(L \triangle H \triangle K\), \(L \triangle K\), then \([H \triangle K]\) maps onto \([(H/L) \triangle (K/L)]\) via the obvious homomorphism with a kernel which is isomorphic to the central square of \(L\). We may take \([L \triangle L]\) as a normal subgroup of \([H \triangle K]\). Let \(\mathfrak{M}(L)\) be the set of all \((x, x^{-1}) \in [H \triangle K]\) where \(x \in L\). \(\mathfrak{M}(L)\) is readily seen to be a normal subgroup of \([H \triangle K]\). In fact, \(\theta_{0, k_1}(x, x^{-1}) = (kxk^{-1}, kx^{-1}k^{-1})\) where \(x \in L, h \in H, k \in K\). Recall [9] that \([H, K]\) is the mutual commutator group of \(H\) and \(K\). If \(H \triangle K\), then \([H, K] \triangle H, K. \mathfrak{M}(H) \triangle B_1\). We have

**Lemma 8.** \(\mathfrak{M}[H, K] \triangle [H \triangle K]\).

**Proof.** Observe that \([(h^{-1}, h), (x, k)] = ([k, x], [h, k])\). On the right is a typical generator of \(\mathfrak{M}[H, K]\) where \(h \in H, k \in K\). On the left is an element of \([H \triangle K]\) (where \(x\) may be chosen at will in \(H\)).

A trivial argument shows that \([H \triangle K] \triangle [L \triangle M]\) if and only if \(H \triangle M, H \triangle K \triangle M, H \triangle L \triangle M\) and \([L, K] \subset H\).

Suppose that \(H \supset K\). An *inclusion product of the second kind* is given by \((H, K; \theta)\) and is denoted by \([H \supset K]\). \([H \supset H] = [H \triangle H]\). The \(\theta\) of a product of the first kind is always ample, but this is not generally true for a product of the second kind. If \(H\) is of class 2 and if \(m\) and \(n\) are integers, one can verify that the mapping \(\delta\) given by \(\delta(h, k) = \theta_{m \phi_k}\) lies in \(\text{Hom}(\{H \supset K\}, \mathfrak{H}(K))\) as before for inclusion products of the first kind. If \(\alpha \in \mathfrak{E}(H), \beta \in \text{Hom}(K, H)\)
where $K \subset H$, then \( \theta \alpha \wedge \theta \beta \in \text{Hom} \left( \left[ H \supset K \right], \mathfrak{Z}(H) \right) \) if and only if (cf. above) \( \alpha(k) \equiv \beta(k) \mod (Z_1(H) \triangleright \alpha(H)) \) for every \( k \in K \). If \( \alpha \in \mathfrak{N}(H) \), then the congruence reduces to \( \alpha(k) \equiv \beta(k) \mod Z_2(H) \). By direct verification, one can show that \( [H \supset K] \triangle [L \supset M] \) if and only if \( K \subset H \triangle L \) and \( K \triangle M \subset L \).

**Lemma 9.** If \( K \supset H \), then a necessary and sufficient condition that the mapping \( \gamma : (h, k) \to (k, h) \) be an isomorphism on \( [H \supset K] \) onto \( [K \supset H] \) is that \( K \subset Z_1(H) \). Then these inclusion products are both isomorphic to \( H \oplus K \).

By \( A(\div)^n B \) we shall mean \( (A(\div)^{n-1}B) \div B \) where the symbol \((\div)^n\) is already defined, and \( A(\div)^1 B = A \div B \). \( A(\div)^0 B \) shall mean \( A \).

**Theorem 6.** If \( H \supset K \), then \( Z_n[H \supset K] \) is the set of all \((h, k), h \in H, k \in K, \) where \( h \in \mathfrak{C}(K, H)(\div)^{n-1} K \) and where \( hk \in Z_n(H) \).

**Proof.** \((h, k) \in Z_1[H \supset K] \) if and only if \((h, k)(x, y) = (x, y)(h, k) \) for every \((x, y) \in [H \supset K]\). Equivalently, \( hkxk^{-1} = xyh^{-1} \) and \( ky = yk \). Taking \( x = e \), we see that \( h \in \mathfrak{C}(K, H) \) and that \( hkxk^{-1} = xh \), or \((hk)x = x(hk) \), so that \( hk \in Z_1(H) \). Conversely, if \( hk \in Z_1(H) \) and if \( h \in \mathfrak{C}(K, H) \), then \((hk)xk^{-1} = xk^{-1}(hk) = xhk^{-1}k = xh \), while \( xyh^{-1} = xhyy^{-1} = xh \). Moreover, \( h \in \mathfrak{C}(K, H) \) and \( hk \in Z_1(H) \) imply \( k \in Z_1(K) \) so that \( ky = yk \). This establishes the theorem for the case \( n = 1 \).

Now suppose that the theorem holds for the case \( n \). \((h, k) \in Z_{n+1} \) if and only if each \([\{(h, k), \ (x, y)\} \in Z_n \). But this commutator reduces to \( ([hk, xy] \cdot [y, k], [k, y]) \). The assumption \( h'k' \in Z_n(H) \) leads to the equivalence of \( h' \in \mathfrak{C}(K, H)(\div)^n K \) and of \( k' \in \mathfrak{C}(K, H)(\div)^n K \), as a short separate argument shows. Consequently, the induction assumption \((h, k) \in Z_{n+1} \) implies that \([hk, xy][y, k][k, y] = [hk, xy] \in Z_n(H) \) and that both \([hk, xy][y, k] \) and \([k, y] \in \mathfrak{C}(K, H)(\div)^n K \). Since \( xy \) ranges over all of \( H \), \( hk \in Z_{n+1}(H) \). Since \( y \) ranges over all of \( K \), \( k \in \mathfrak{C}(K, H)(\div)^n K \div K = \mathfrak{C}(K, H)(\div)^n K \). Hence \( h \in \mathfrak{C}(K, H)(\div)^n K \), and the theorem is established.

If \( L \triangle H, L \subset K, \ [H \supset K] \) maps homomorphically onto \( \left[ (H/L) \supset (K/L) \right] \) with kernel \( [L \supset L] \), so that the latter is a normal subgroup of \( [H \supset K] \). We can define \( \mathfrak{M}^*(L) \) as the set of all \((x, x^{-1}), x \in L, \) in the inclusion product of the second kind \( [H \supset K] \) where \( L \subset K \). Suppose, further, that \( L \triangle H \). Then \( \mathfrak{M}^*(L) \triangle [H \supset K] \).

There exists an onto mapping \( \lambda_1 \in \text{Hom} \left( \left[ H \Delta K \right], K \right) \) given by \( \lambda_1(h, k) = hk \). kern \( \lambda_1 = \mathfrak{M}(H) \). There exists \( \lambda_2 \in \text{Hom} \left( \left[ H \supset K \right], H \right) \), an onto mapping, given by \( \lambda_2(h, k) = hk \). kern \( \lambda_2 = \mathfrak{M}^*(K) \), so that \( \mathfrak{M}^*(K) \triangle [H \supset K] \) even if \( K \) is not normal in \( H \).

**Theorem 7.** To each \( \phi \in \text{Hom} \left( K, \mathfrak{A}(H) \right) \) there exists an extension \( W \) of \( H \) with a normal subgroup \( H_0 \) anti-isomorphic to \( H \) such that \( W/H \cong W/H_0 \cong (H, K; \phi) \).

**Proof.** Let \( W = \left[ H^* \Delta (H, K; \phi) \right] \). Let \( H_0 \) be the set of all \(((h, e), (h^{-1}, e)) \)
\(\in W\) where \(h \in H\). That is, \(H_0 = W(H^*) \triangle W\), where we recall that \(H^*\) is the set of all \((h, e)\). Since \(((h, e), (h^{-1}, e))((t, e), (t^{-1}, e)) = ((ht, e), (h^{-1}t^{-1}, e))\), \(H\) and \(H_0\) are anti-isomorphic. By the remarks above on \(\lambda_1\), \(W/W(H^*)\) is isomorphic to \((H, K; \phi)\). But \(W/H^*\) (written, “by abuse of language,” as \(W/H\)) is also isomorphic to \((H, K; \phi)\). In particular, each endomorphism of \(\mathcal{A}(H)\) leads to such an extension.

6. Extensions from \(\mathcal{A}(H)\) to \(\mathcal{A}(G)\). We saw above in §2 that each element of \(\phi(K)\) can be extended to an element of \(\mathcal{A}(G)\). Suppose that \(g = (a, u)\), \(a \in H, u \in K\). For \(x \in H\), \((x, e)^g = (\sigma_\phi u(x), e)\), so that those automorphisms of \(H\) which extend to inner automorphisms of \(G\) are precisely the elements of \(\mathcal{A}(H)\phi(K)\). In particular, each element of \(\mathcal{A}(H)\) extends to an element of \(\mathcal{A}(G)\). \(\theta_1 \in \mathcal{A}(H)\) extends to \(\theta_2 \in \mathcal{A}(G)\) for all \(g \in G\) such that \(g = (b, u)\), \(b \in H, u \in K\), with \(\phi_u = \theta_b^{-1}\). If, for instance, \(\phi(K) = \mathcal{A}(H)\), then to each pair \(a \in H, u \in K\), there exists \(b \in H\) such that \(\theta_{(b, u)}\) extends from \(\theta_a\). More generally, if \(\phi\) is ample, then \(\alpha \in \mathcal{A}(H)\) extends to some element of \(\mathcal{A}(G)\) if and only if \(\alpha \in \phi(K)\). Since \(\theta_{(a, u)}(x, e) = (\sigma_\phi u(x), e)\), \(\theta_{(a, u)}\) induces an element of \(\mathcal{A}(H)\) if and only if \(\phi_u \in \mathcal{A}(H)\). But \(\phi^{-1}(\mathcal{A}(H)) \Delta K\) since \(\phi\) ample implies \(\mathcal{A}(H)\Delta \phi(K)\), whence it is easy to see that the set of all elements of \(\mathcal{A}(G)\), each of which induces an element of \(\mathcal{A}(H)\), is a normal subgroup of \(\mathcal{A}(G)\).

\(\theta_{(a, u)}\) and \(\theta_{(b, v)}\) induce the same automorphism of \(H\) if and only if \(\phi_{uv^{-1}} = \theta_{a^{-1}b}\), so that \(\theta_{(a, e)}\) and \(\theta_{(b, e)}\) generate the same element of \(\mathcal{A}(H)\) if and only if \(a \equiv b \mod Z_1(H)\), while \(\theta_{(u, e)}\) and \(\theta_{(v, e)}\) generate the same element of \(\mathcal{A}(H)\) if and only if \(u \equiv v \mod I\). There exist mappings \(\delta_1 \in \text{Hom}(H, \mathcal{A}(G))\) and \(\delta_2 \in \text{Hom}(K, \mathcal{A}(G))\) given by \(\delta_1(a) = \theta_{(a, e)}\), \(\delta_2(u) = \theta_{(e, u)}\). \(\ker \delta_1 = Z_1(H)\cap F_1(H)\) and \(\ker \delta_2 = Z_1(K)\cap I\). For a subgroup \(U\) of \(G\), let \(\mathcal{A}(U, G)\) be that subgroup of \(\mathcal{A}(G)\) consisting of all \(\theta_{(a, u)}\), \(u \in U\).

**Theorem 8.** Let \(G = (H, K; \phi)\). (a) There exists \(\phi(1) \in \text{Hom}(\mathcal{A}(H^*, G), \mathcal{A}(\mathcal{A}(H^*, G)))\) with \(\ker \phi(1)\) consisting of all \(\theta_{(e, k)}\) where \(\theta_k \in \mathcal{A}(H; H/(Z_1(H)\cap F_1(H)))\). (b) There exists a homomorphism \(\psi_1\) on \(G\) onto the semi-direct product \(G_1 = (\mathcal{A}(H^*, G), \mathcal{A}(K^*, G); \phi(1))\), and \(\ker \psi_1 = (Z_1(H)\cap F_1(H)) \oplus (Z_1(K)\cap I)\).

**Proof.** (a) Let \(\phi(1)(\theta_{(e, k)}) = \alpha_k\), that automorphism of \(\mathcal{A}(H^*, G) \cong H/(Z_1(H)\cap F_1(H))\) which is induced by \(\phi_k - \alpha_k\) exists since both \(Z_1(H)\) and \(F_1(H)\) are \(\phi_k\)-admissible and \(\phi_k\)-consistent. If \(k \equiv k' \mod (Z_1(K)\cap I)\), then \(\phi_k = \phi_{k'}\), and \(\alpha_k = \alpha_{k'}\), whence \(\phi(1)\) is single-valued. It is immediate that \(\phi(1)\) is a homomorphism. \(\ker \phi(1)\) consists of all \(\theta_{(e, k)}\) for which \(\phi_k \equiv h \mod (Z_1(H)\cap F_1(H))\). (b) Define \(\psi_1\) by \(\psi_1(h, k) = (\delta_1(h), \delta_2(k)) \in G_1\), \(\psi_1\) is onto \(G_1\), and \(\ker \psi_1\) is the set of all \((h, k)\) where \(h \in \ker \delta_1\) and \(k \in \ker \delta_2\). If \(h \in \ker \delta_1, k \in \ker \delta_2, (i = 1, 2)\), then \((h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)\) so that \(\ker \psi_1 = \ker \delta_1 \oplus \ker \delta_2\). If \(g_1 = (h, k), g_2(x, y) \in G_1\), then \(\psi_1(g_1g_2) = \psi_1(h\phi_k(x), k\psi_2) = (\delta_1(h)\delta_1(\phi_k(x)), \delta_2(k)\delta_2(y))\). Since \(\phi(1)(\delta_2(x)) = \alpha_2(\theta_{(x, e)} = \theta_{(\delta_2, e)} = \delta_2(\phi_k(x)))\), it follows that \(\psi_1(g_1g_2) = \psi_1(g_1)\psi_1(g_2)\) so that \(\psi_1\) is a homomorphism.

**Corollary.** If \(\phi\) is ample, then \(\ker \psi_1 = Z_1(G),\) and \(G_1 \cong \mathcal{A}(G)\).
Proof. \( \phi \) ample implies that \( \mathfrak{Z}(H) \subseteq \phi(\mathfrak{Z}(K)) \) so that \( Z_1(H) \supseteq F_1(H) \). \((h, k) \in \ker \varphi \) if and only if \( h \in Z_1(H) \cap F_1(K) = F_1(H) \) and \( k \in Z_1(K) \cap \mathfrak{f} \). If \((h, k) \in \ker \varphi \), \( k \in \mathfrak{f} \) and \( \varphi_k = \psi \), while \( h \in F_1(H) \cap Z_1(H) \) implies that \( \theta_{h^{-1}} = \psi \). By our earlier determination, \((h, k) \in Z_1(G) \). Conversely, if \((h, k) \in Z_1(G) \), \( h \in F_1(H) = Z_1(H) \cap F_1(H) \), and \( k \in Z_1(K) \) with \( \theta_{h^{-1}} = \psi \). Since \( \phi \) is ample, \( F_1(H) \subseteq Z_1(H) \), whence \( \theta_{h^{-1}} = \psi \). Therefore \( \varphi_k = \psi \), and \( k \in \mathfrak{f} \), \( Z_1(K) \cap \mathfrak{f} \).

**Lemma 10.** If \( \phi \) is ample, then \( \phi^{(1)} \) is ample.

We shall use this lemma later in determining the ascending central series of semi-direct products with ample homomorphisms.

Suppose that \( Z_1(H) \subseteq F_1(H) \), and that \( \phi(Z_1(K)) \subseteq Z_1(H) \). (For instance, if \( \phi \) is ample, \( \phi(Z_1(K)) \subseteq Z_1(H) \).) Then, as in Theorem 8, we can show that there exists \( \phi^{(1)} \in \text{Hom}(\mathfrak{z}(K), \mathfrak{z}(H/F_1(H))) \) given by \( \phi_k^{(1)}(hF_1(H)) = \phi_k(h)F_1(H) \), and there exists a homomorphism \( \psi^{(1)} \) on \((H, K; \phi)\) onto \((H/F_1(H), J(K); \phi^{(1)})\) given by \( \psi^{(1)}(h, k) = (hF_1(H), kZ_1(K)) \), and \( \ker \psi^{(1)} \approx F_1(H) \oplus Z_1(K) \).

7. **Extensions from \( \mathfrak{Z}(K) \) to \( \mathfrak{Z}(G) \).**

**Lemma 11.** \( \mathfrak{Z}(K) \) is precisely that subset of \( \mathfrak{Z}(K) \), the elements of which can be extended to elements of each \( \mathfrak{Z}(H, K; \phi) \).

If \( \mathfrak{f} = K \), then \( \mathfrak{Z}(K) \) can be injected into \( \mathfrak{Z}(G) \). A related result is

**Theorem 9.** Let \( H \) be a group for which the ascending central series breaks off with \( Z_1(H) \). If \( \phi(K) = \mathfrak{Z}(H) \), then \( \mathfrak{Z}(K) \) can be injected into \( \mathfrak{Z}(H, K; \phi) \).

**Proof.** The mapping \( \gamma: \theta \to \theta^{(e, u)} \) is single valued if and only if \( Z_1(K) \subseteq \mathfrak{f} \). If \( \gamma \) is single valued, it is clearly in Hom \( (\mathfrak{Z}(K), \mathfrak{Z}(G)) \). \( \theta^{(e, u)} = \psi \) if and only if \( \psi = \psi \) where \( u \in Z_1(K) \), so that, if \( \gamma \) is single valued, it is an isomorphism. Now suppose that \( \phi(K) = \mathfrak{Z}(H) \). Then \( \mathfrak{Z}(H) \subseteq \phi(K) \) implies that each \( \varphi_k \) with \( k \in Z_1(K) \) is a normal automorphism. Since \( \phi(K) \subseteq \mathfrak{Z}(H) \), to each \( k \in Z_1(K) \) there exists \( h \in H \) with \( \varphi_k = \theta_h \). For \( x \in H \), \( \varphi_k(x) = h_x h^{-1} x_z \), \( z \in Z_1(H) \). Thus \( h \in Z_2(H) \). Since, however, \( Z_2(H) = Z_1(H) \), \( \varphi_k(x) = x \) so that \( Z_1(K) \subseteq \mathfrak{f} \), and \( \gamma \) is the required injection.

**Corollary.** If \( Z_1(K) \subseteq \mathfrak{f} \), then the image of \( \mathfrak{Z}(K) \) in \( \mathfrak{Z}(II, K; \phi) \) under the above injection \( \gamma \) is a normal subgroup of \( \mathfrak{Z}(H, K; \phi) \) if and only if \( \phi(K) \subseteq \mathfrak{Z}(H; II/(F_1(H) \cap Z_1(H))) \).

In particular, note that \( \mathfrak{Z}(K) \) can be injected into \( \mathfrak{Z}(H \Delta K) \) and if \( Z_1(K) \subseteq Z_1(H) \), also into \( \mathfrak{Z}(H \supseteq K) \). The image under injection is a normal subgroup of \( \mathfrak{Z}(H \Delta K) \) if and only if \( H \subseteq Z_2(K) \); of \( \mathfrak{Z}(H \supseteq K) \) if and only if \( K \subseteq Z_2(H) \) (where \( Z_1(K) \subseteq Z_1(H) \)).

8. **Pair extensions.** An automorphism \( \Gamma \) of \( G = (H, K; \phi) \) is called a pair extension of \( \alpha \) and of \( \beta \) over \( G \) if \( \Gamma \) induces an automorphism \( \alpha \) on \( H \) and an automorphism \( \beta \) on \( K \). Then \( \Gamma(h, k) = \Gamma(h, e) \Gamma(e, k) = (\alpha(h), e)(e, \beta(k)) \).
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= (α(h), β(k)). (Cf. Lemma 2.) Write Γ = Γα,β = (α, β). Let $G = G(G)$ be the set of all pair extensions over $G$.

**Lemma 12.** $G$ is a group under automorphism composition; and for $α ∈ G(H)$, $β ∈ H(K)$, $(α, β) ∈ G$ if and only if

$$αφ_k = φ_{β(k)}α$$

for every $k ∈ K$.

**Lemma 13.** (a) $(α, β) ∈ G$ if and only if $α ∈ G(ϕ(K), G(H))$. (b) $(α, β) ∈ G$ if and only if $β ∈ G(K; K/t)$. (c) If $(α, β) ∈ G$, then $α ∈ G(ψ(K), G(H))$ if and only if $β ∈ G(ϕ(K), G(H))$.

If $G = [H, K]$ then $G(H, K; ϕ)$ becomes $G(H, K; ϕ) = Ξ_1(H)$, while $G(K; K/t)$ reduces to $G(H; H/Z_1(H))$, a group which is also equal to $Ξ_1(H)$. Hence, by (c), if $(α, β) ∈ G(H, K; ϕ)$, $α$ is normal if and only if $β$ is normal, while $α$ and $β$ normal imply $(α, β) ∈ G(H, K; ϕ)$. It is easy to see that $(α, β) ∈ G(H, K; ϕ)$ if $α, β ∈ Ξ_1(H)$. Thus if $α ∈ Ξ_1(H)$, $(α, α) ∈ Ξ_1(H; K)$ so that $Ξ_1(H)$ nontrivial implies that $Ξ_1(H; K)$ is nontrivial. In particular, if $H$ is of class 2, if $(α, β) ∈ G(H; K)$ and if one of $α$ or $β$ is inner, then the other is normal.

**Lemma 14.** If $G = (H, K; ϕ)$, then $G(G) ∩ G(G)$ consists of all $θ_{(h, k)}$ such that $h ∈ F_1(H)$, so that $G(G) ∩ G(G)$ equals $F_1(H) ∘ K/Z_1(G)$. If $θ_{(h, k)} ∈ G(G)$, then $θ_{(h, k)} = θ_{(ϕ_h, ϕ_k)}$.

By Lemma 13a, each $(θ_h, ϕ) ∈ G(H, K; ϕ)$ if and only if $3(H) ⊂ G(ϕ(K), G(H))$, or equivalently $ϕ(K) ⊂ Ξ_1(H)$. If $(θ_h, ϕ) ∈ G$ under these conditions, then $(ϕ_h, ϕ) ∈ G$. Likewise $(θ_h, ϕ) ∈ G$ with $a ∈ F_1(H)$ implies $(ϕ_h, ϕ) ∈ G$. For then, $ϕ_hθ_a = ϕ_aθ_h$ whence $θ_a ∈ G(ϕ(K), G(H))$ so that Lemmas 13c, b are applicable. $θ_{(ϕ_h, ϕ)}$ induces inner automorphisms on both $H$ and $K$ if and only if $a ∈ F_1(H)$ and $k ∈ ϕ^{-1}(3(H))$.

**Theorem 10.** For a group $H$ of class 2 and for $ϕ ∈ HOM (K, G(H))$, let $n$ be the set of all $θ_{(h, k)}$ such that $θ_{(h, k)} ∈ G(H, K; ϕ)$. Then $n ∩ G(H, K; ϕ)$, and $n = Ξ(H) ∘ G(H; K/t)$.

**Proof.** Since $H$ is of class 2, $Ξ(H) ⊂ Ξ_1(H)$. Since $ϕ(K) ⊂ Ξ(H)$, the remarks before the theorem show that $(θ_h, ϕ) ∈ G$ implies that $(ϕ_h, ϕ) ∈ G$, so that, by Lemma 13b, $β ∈ G(K; K/t)$. Conversely, suppose that $β ∈ G(K; K/t)$. If $h ∈ H$, $ϕ_hθ_h = ϕ_hθ_h$ for every $k ∈ K$, since $ϕ(K) ⊂ Ξ_1(H)$. But $β(k) = k$ mod $t$ so that $ϕ_hθ_h = ϕ_hθ_h$. By (1) (of this section), $(θ_h, β) ∈ G$. If we define $n$ as above, we have proved that $n = Ξ(H) ⊂ Ξ(H; K/t)$. To show that $n ∩ G$, suppose that $(γ, δ) ∈ G$ where $γ ∈ Ξ(H)$, $δ ∈ G(K)$. Then, for $(θ_h, β) ∈ n$, $(θ_h, β)(γ, δ) = (γθ_hγ^{-1}, δβδ^{-1})$. Since $Ξ(H) ∩ G(K), γθ_hγ^{-1} ∈ Ξ(H)$. For $γ ∈ K$, $δβδ^{-1}(y) = δ [δ^{-1}(y)k]$ where $k ∈ t$. By (1), $γθ_hγ = φ_{θ_hγ}$. Hence $k ∈ t$ implies that $γ = φ_{θ_hγ}$ so that $δ(k) ∈ t$, whence $δβδ^{-1} ∈ G(K; K/t)$, and we have proved that $n ∩ G$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
If one could find a class 2 group $H$ without outer automorphisms (the existence of such is an open question) $\mathcal{G}(H, K; \phi)$ would be isomorphic to $\mathfrak{N}(H) \oplus \mathfrak{N}(K; K/\mathfrak{I})$. In particular, for such an $H$, $\mathcal{G}[H \Delta H] \cong \mathfrak{N}(H) \oplus \mathfrak{I}(H)$.

**Theorem 11.** If $G = (H, K; \phi)$, then the set of all pair extensions of the form $(\alpha, \iota)$ where $\alpha$ is a normal automorphism of $H$ which induces the identity on $H/F_1(H)$, and of all pair extensions of the form $(\iota, \beta)$ where $\beta$ is a normal automorphism of $K$, generates the group of pair extensions which are also normal automorphisms of $G$.

**Proof.** $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{I}$ if and only if $\alpha \phi_k = \phi_k \alpha$ for every $\gamma \in \mathfrak{I}$ and $(\alpha, \beta) = (hr, ks)$ where $r \in F_1(H), s \in \mathbb{Z}_1(K)$ and $\theta_r = \phi_k^{-1}$. Hence, if $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{I}$, $(\alpha, \beta)(h, e) = (hr, e)$ where $\theta_r = \iota$, (i.e, $r \in \mathbb{Z}_1(H)$). Then $(\alpha, \beta)(h, k) = (hr, ks)$ implies $s \in \mathfrak{I}$. Since $\alpha \phi_k = \phi_k \alpha$, where $\phi_k = \phi_k$, $\alpha \in \mathfrak{C}(\phi(K), \mathfrak{N}(H))$ so that, by Lemma 13a, $(\alpha, \iota) \in \mathcal{G}$. It follows that $(\iota, \beta) \in \mathcal{G}$. Now $(\alpha, \iota) \in \mathfrak{I} \cap \mathcal{G}$ if and only if $(\alpha, \iota) \in \mathcal{G}$ and $(\alpha(h) = hr$ where $r \in F_1(H)$ and $(\iota, \beta) \in \mathfrak{I} \cap \mathcal{G}$ if and only if $(\iota, \beta) \in \mathcal{G}$ and $\beta(k) = ks$ where $s \in \mathbb{Z}_1(K) \cap \mathfrak{I}$. If $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{I}$, we saw above that $r \in F_1(H) \cap \mathbb{Z}_1(H)$. Hence $(\alpha, \iota) \in \mathcal{G} \cap \mathfrak{I}$. At once, $(\iota, \beta) \in \mathfrak{I} \cap \mathcal{G}$ so that $(\alpha, \beta) \in \mathfrak{I} \cap \mathcal{G}$.

Let $K_*$ be that subgroup of a group $K$ which is generated by all $x^{-1}a(x)$ where $a$ ranges over all of $\mathfrak{N}(K)$ and $x$ ranges over all of $K$. $K' \subset K_*$, and if $K$ is abelian, $x^2 \in K_*$ for every $x \in K$. We might call $K_*$ the generalized derived subgroup of $K$.

**Theorem 12.** $\mathcal{G}(H, K; \phi) \cong \mathfrak{N}(H) \oplus \mathfrak{N}(K)$ (in the natural way) if and only if $\mathfrak{I} \cap K_*$ and $\phi(K) \subset \mathbb{Z}_1(\mathfrak{N}(H))$.

If $a \in \mathfrak{N}(H)$, there is one and only one pair extension of $a$ with $a$ as the first component of the pair over $\mathfrak{N}(H)$. Unless, therefore, $\Delta \in \mathfrak{N}(\mathfrak{N}(H))$, there is no pair extension of $\Delta \in \mathfrak{N}(H)$ with $\Delta$ as second component over $\mathfrak{N}(H)$. If $\Delta = \theta_a \in \mathfrak{N}(\mathfrak{N}(H))$, then each $(\alpha', \Delta) \in \mathfrak{N}(H, K; \phi)$ where $\alpha' = \alpha \mod \mathbb{Z}_1(\mathfrak{N}(H))$. Note that $(\alpha, \theta_a) = (\iota, a)$ on $\mathfrak{N}(H)$. Since each $(\alpha', \theta_a)$ above can be written $(\alpha', \theta_a)$, $\mathfrak{N}(\mathfrak{N}(H))$ consists of all $(\alpha, \theta_a) = (\iota, a)$.

$\mathfrak{N}(F_1(H), H) \subset \mathfrak{C}(\phi(K), \mathfrak{N}(H)$ since $\phi_k h(x) = h \phi_k(x) h^{-1}$ if $h \in F_1(H)$. If $\phi$ is ample there is a homomorphism $\gamma_9$ on $F_1(H) \oplus K$ onto $\phi(K)$ given by $\gamma_9(h, k) = \theta_h \phi_k$ for every $h \in F_1(H)$ and for every $k \in K$, where $\gamma_9$ restricted to the same domain of definition is $\gamma_9$. If $\ker \gamma_9 = B_9$, $\mathbb{Z}_1(H, K; \phi) \subset B_9 \subset B_1$ for ample $\phi$.

We should note carefully the following: by Lemma 14, $\theta_{(a, k)}$ is a pair extension, namely $(\theta_a \phi_k, \theta_k)$, if and only if $h \in F_1(H)$. But each product $\theta_a \phi_k$
can be extended, by §6, to the inner automorphism \( \theta(h,k) \) of \( (H, K; \phi) \). What \( \theta(h,k) \) does not do, in general, is to induce an inner automorphism on \( K \), unless \( h \in F_1(H) \), so that \( \theta(h,k) \) is always an extension of an automorphism of \( H \), even though it is only exceptionally a pair extension.

Since \( \mathcal{S}_1(H) = \mathcal{C}(\mathcal{H}(H), \mathcal{C}(H), \mathcal{Z}(H)) = \mathcal{C}(\mathcal{S}_1(H), \mathcal{C}(H), \mathcal{Z}(H)) \). Let \( \mathcal{B} \subset \mathcal{B} \subset \mathcal{C}(\mathcal{H}(H)) \) where \( \mathcal{B} \) is a group of automorphisms which we shall call an ample group of automorphisms; and form \( \mathcal{S}(H; \mathcal{B}) \), the relative holomorph [9] of \( \mathcal{B} \) over \( H \). Let \( F_1(H) \) be computed for the semi-direct product \( \mathcal{S}(H; \mathcal{B}) \). Note that \( F_1(H) \subset Z_1(H) \), so that \( \text{Hom}(\mathcal{B}, F_1(H)) \) is a group under homomorphism addition. There exists a \( \Theta \in \text{Hom}(\mathcal{A}(H; H/F_1(H)), \mathcal{A}(\text{Hom}(\mathcal{B}, F_1(H)))) \) given by \( \Theta: \gamma \rightarrow \Theta \gamma \) for every \( \gamma \in \mathcal{A}(H; H/F_1(H)) \) and for every \( \Gamma \in \text{Hom}(\mathcal{B}, F_1(H)) \). We can show that the group of normal automorphisms of \( \mathcal{S}(H; \mathcal{B}) \) has a faithful representation as a splitting extension of an abelian group by a group of normal automorphisms of \( H \). We have

**Theorem 13.** Let \( \mathcal{B} \) be an ample group of automorphisms of a group \( H \). If \( F_1(H) \) is computed for the relative holomorph of \( \mathcal{B} \) over \( H \), then the normal automorphism group of this relative holomorph is isomorphic to

\[
(\text{Hom}(\mathcal{B}, F_1(H)), \mathcal{A}(H; H/F_1(H)); \Theta)
\]

where \( \Theta \gamma \Gamma = \gamma \Gamma \) for every \( \gamma \in \mathcal{A}(H; H/F_1(H)) \) and for every

\[
\Gamma \in \text{Hom}(\mathcal{B}, F_1(H)).
\]

**Proof.** For \( \Psi \in \mathcal{S}_1(\mathcal{S}(H; \mathcal{B})) \) and for \( (x, \beta) \in \mathcal{S}(H; \mathcal{B}) \), \( \Psi(x, \beta) = (x, \beta) \). Now \( Z_1(\mathcal{S}) \) is the set of all \( (y^{-1}, \theta_y) \) where \( y \in F_1(H) \) and where \( \theta_y \in Z_1(\mathcal{B}) \). Since \( \mathcal{B} \subset \mathcal{C}(\mathcal{H}(H), F_1(H) \subset Z_1(H) \) so that \( Z_1(\mathcal{S}) \) is the set of all \( (f, \iota) \) where \( f \in F_1(H) \). Hence \( \Psi(x, \iota) = (xf(x), \iota) = (\gamma(x), \iota) \) where \( f(x) \in F_1(H) \), and \( \Psi(e, \beta) = (\Gamma(\beta), \beta) \) where \( \Gamma(\beta) \in F_1(H) \). It is easy to show that \( \gamma \in \mathcal{A}(H; H/F_1(H)) \) and that \( \Gamma \in \text{Hom}(\mathcal{B}, F_1(H)) \). It is clear that the semi-direct product \( \Phi = (\text{Hom}(\mathcal{B}, F_1(H)), \mathcal{A}(H; H/F_1(H)); \Theta) \) exists. Let \( \Omega(\Psi) = (\Gamma, \gamma) \in \Phi \). One can verify that \( \Omega \) is a homomorphism with trivial kernel. It remains to show that \( \Omega \) is onto. For \( \Gamma \in \text{Hom}(\mathcal{B}, F_1(H)) \) construct \( \Psi_1 \) by \( \Psi_1(x, \beta) = (x, \beta) \). A routine check which uses the fact that \( F_1(H) \subset Z_1(H) \) shows that \( \Psi_1 \) is an automorphism. Since \( (\beta^{-1}(x^{-1}), \beta^{-1}) \Psi_1(x, \beta) = (\Gamma(\beta), \iota) \in Z_1(\mathcal{S}), \) \( \Psi_1 \) is normal. Likewise, if \( \gamma \in \mathcal{A}(H; H/F_1(H)) \), construct \( \Psi_2 \) by \( \Psi_2(x, \beta) = (\gamma(x), \beta) \). Then \( \Psi_2(x_1, \beta_1(x_2), \beta_2) = (\gamma(x_1), \gamma_1(x_2), \beta_2) \). Since, however, \( \mathcal{B} \subset \mathcal{C}(\mathcal{H}(H), \mathcal{A}(H; H/F_1(H)); \Theta) = \mathcal{C}(\mathcal{H}(H), \mathcal{A}(H; H/F_1(H))) \), \( \mathcal{B} \subset \mathcal{C}(\mathcal{H}(H), \mathcal{A}(H; H/F_1(H))); \Theta) = \mathcal{C}(\mathcal{H}(H), \mathcal{A}(H; H/F_1(H)) \) is a homomorphism. That \( \Psi_2 \) is onto with a trivial kernel is immediate, and the fact \( (\beta^{-1}(x^{-1}), \beta^{-1}) \gamma(x), \beta = (\beta^{-1}(x^{-1} \gamma(x)), \iota) \) with \( x^{-1} \gamma(x) \in F_1(H) \) shows that \( \Psi_2 \) is normal. It is clear that \( \Omega(\Psi_1 \Psi_2) = (\Gamma, \gamma) \), so that \( \Omega \) is onto \( \Phi \), and the theorem is established.
Let \( \mathfrak{B}(H) = \mathfrak{A}(H; Z_1(H)) \cap \mathfrak{I}_1(H) \), the set of all normal automorphisms of \( H \) which reduce to the identity on the center, the abelian group of centrally normal automorphisms of \( H \). Let \( \mathfrak{B}(H) = \mathfrak{F}(H; \mathfrak{B}(H)) \), the centrally normal holomorph of \( H \).

**Corollary.** Let \( H \) be of class 2. Then the group of centrally normal automorphisms of the centrally normal holomorph of \( H \) splits into a direct sum of a group of automorphisms which extend the identity on \( H \), isomorphic to the group of homomorphisms of the group of centrally normal automorphisms of \( H \) into the center of \( H \), and of a group of inner automorphisms which are pair extensions over the centrally normal holomorph of \( H \), isomorphic to the group of centrally normal automorphisms of \( H \).

**Proof.** Since \( H \) is of class 2, \( \mathfrak{B}(H) \supseteq \mathfrak{Y}(H) \). If we identify \( \mathfrak{B}(H) \) with the \( \mathfrak{B} \) of the theorem (ignoring the upper bound given there for \( \mathfrak{B} \) and if we compute \( F_1(H) \) for \( \mathfrak{F}(H; \mathfrak{B}) \), we find that \( Z_1(\mathfrak{F}) \) is the set of all \( (f, \iota) \), \( f \in F_1(H) \). Since \( \mathfrak{B} \subseteq \mathfrak{I}(H; Z_1(H)) \), \( F_1(H) \supseteq Z_1(H) \) so that \( Z_1(\mathfrak{F}) = Z_1(\mathfrak{B}(H)) \) is the set of all \( (f, \iota) \), \( f \in Z_1(H) \). If \( \Psi \in \mathfrak{B}(\mathfrak{B}(H)) \), \( \Psi(x, \iota) = (\gamma(x), \iota) \), as in the proof of the theorem, whence \( x^{-1} \gamma(x) \in Z_1(H) \). Since \( \Psi \in \mathfrak{I}(\mathfrak{B}(H); Z_1(\mathfrak{B}(H))) \), one can show that \( \gamma \in \mathfrak{I}(H; Z_1(H)) \cap \mathfrak{I}_1(H) = \mathfrak{B}(H) \). Likewise \( \Psi(\eta, \iota) = (\Gamma(\beta), \beta) \) where \( \Gamma \in \text{Hom} \ (\mathfrak{B}, Z_1(H)) \). Let \( \Phi = \text{Hom} \ (\mathfrak{B}(H), Z_1(H)) \oplus \mathfrak{B}(H) \), and set \( \Omega(\Psi) = (\Gamma, \gamma) \in \Phi \). That \( \Omega \) is an isomorphism into is immediate. If \( (\Gamma, \gamma) \in \Phi \), let \( \Psi \) be defined by \( \Psi(x, \beta) = (x \Gamma(\beta), \beta) \). It is clear that \( \Psi \) is an automorphism of \( \mathfrak{B}(H) \) and that \( \Omega(\Psi) = (\Gamma, \gamma) \), so that \( \Omega \) is onto \( \Phi \). Observe that \( \gamma \) is the mapping of \( H \) induced by \( \Phi \). For \( \gamma \in \mathfrak{B}(H) \), define \( \Delta \) by \( \Delta(x, \beta) = (\gamma(x), \beta) \). Then \( \Omega(\Delta) = (\nu, \gamma) \) so that \( \Delta \) is that pair extension over \( \mathfrak{B}(H) \) which induces \( \gamma \) on \( H \) and \( \iota \) on \( \mathfrak{B} \) and which lies in \( \mathfrak{B}(\mathfrak{B}(H)) \). For

\[
\Gamma \in \text{Hom} \ (\mathfrak{B}(H), Z_1(H)),
\]

define \( \Psi \) by \( \Psi(x, \beta) = (x \Gamma(\beta), \beta) \), so that \( \Psi \) is a member of \( \mathfrak{B}(\mathfrak{B}(H)) \) which induces the identity on \( H \). Note that \( H \) of class 2 implies that \( \mathfrak{B}(H) \) is of class 2 since \( (x, \beta)(y, \alpha) = (y \alpha(x) \beta(y^{-1}), \beta) = (y \alpha, \beta) \) where \( z \in Z_1(H) \) for \( (x, \beta), (y, \alpha) \in \mathfrak{B}(H) \). Hence the inner automorphisms of \( \mathfrak{B}(H) \) lie in \( \mathfrak{B}(\mathfrak{B}(H)) \), and \( \Omega(\theta_{(\nu, \gamma)}) = (\Gamma, \gamma) \) where \( \Gamma(\beta) = y \beta(y^{-1}) \) and \( \gamma = \theta_{e, \gamma} \lambda \). It follows that \( \Omega(\theta_{(e, \gamma)}) = (\nu, \gamma) \). The group of pair extension automorphisms over the centrally normal holomorph of \( H \) which occurs in the corollary is now seen to be a group of inner automorphisms.

9. The ascending central series. Returning to the situation of Theorem 8, we recall the existence of \( \phi^{(1)} \in \hom (K_1, \mathfrak{K}(H_1)) \), where \( K_1 = K/(Z_1(K) \cap f) \) and \( H_1 = H/(Z_1(H) \cap F_1(H)) \), such that \( k^* = k(Z_1(K) \cap f) \) implies \( \phi^{(1)}(h(Z_1(H) \cap F_1(H))) = \phi_k(h)(Z_1(H) \cap F_1(H)) \) for every \( h \in H \) and for every \( k \in K \). Let \( f_{11} = \ker \phi^{(1)} \), and let \( f_{11}^* \) be all \( (e, k) \) where \( k \in f_{11} \). Define \( f_{11}^* \) by \( (\psi_{11}^{-1}(f_{11}^*)) \cap K^* \) where \( \psi_1 \) is the homomorphism on \( G = (H, K; \phi) \) onto \( G_1 = (H_1, K_1; \phi^{(1)}) \) which was described in Theorem 8. We follow the practice that if \( U \subseteq H \) (respec-
tively $K$) then $U^*$ is the set of all $(u, e)$ (respectively $(e, u)$). Then $t_1$ is defined as the set of all $k \in K$ such that $(e, k) \in t_1^*$. If $k \in t^* = \ker \phi$, $\phi^*_k(h) = h$ for every $h \in H$ so that $t \subseteq t_1$. Also $t_1 \triangle K$. Let $[F_2(H)]^* = H^* \cap \psi_1^{-1}[F_1(H_1)]^*$. "By abuse of language," we make such identifications as of $H_1$ with its isomorphic image $\psi_1(H_1, G)$. $h(Z_1(H) \cap F_1(H)) \subseteq F_1(H_1)$ if and only if

$$\phi_k(h) \equiv h \mod (Z_1(H) \cap F_1(H))$$

for every $h \in H$. Hence if $h \in F_1(H)$, then $h \in F_2(H)$ so that $F_1(H) \subseteq F_2(H)$. From $G_1$, we can, by the same process which yielded $G_1 = \psi_1(G)$, construct $G_2 = (H_2, K_2; \phi^{(2)})$ where there exists a homomorphism $\psi_2$ on $G_1$ such that $\psi_2(G_1) = G_2$. Then $\psi_2 = \psi_1 \psi_1^*$ carries $G$ onto $G_2$ homomorphically. Let $\ker \phi^{(2)} = t_2$, and let $t_2^* = K^* \cap \psi_2^{-1}(t_2^*)$ where $t_2^*$ is the set of all $(e, k) \in G_1$ with $k \in t_2$. Since $K_1^* \cap \psi_2^{-1}(t_2^*) = t_2^*$, $t_2^* = K^* \cap \psi_2^{-1}(t_2^*) = K^* \cap \psi_2^{-1}(t_2^*) = K^* \cap \psi_1^{-1}(t_2^*)$, so that $t_1 \subseteq t_2$. Continuing in this way, we can construct a sequence of groups $\{G_n\} = \{(H_n, K_n; \phi^{(n)})\}$ and an ascending chain of normal subgroups of $K: t_1 \subseteq t_2 \subseteq \cdots$. Likewise, since $[F_2(H_1)]^* = H_1^* \cap \psi_1^{-1}[F_1(H_2)]^* \supseteq [F_1(H_1)]^*$, $[F_2(H_2)]^* = H_2^* \cap \psi_2^{-1}[F_1(H_2)]^* \supseteq H_1^* \cap \psi_1^{-1}[F_1(H_1)]^* = [F_2(H_1)]^*$. Hence an ascending chain of subgroups of $H$ is constructed: $F_1(H) \subseteq F_2(H) \subseteq \cdots$.

Suppose that $R$ is a subgroup of $H$ which is $\phi_k$-admissible for every $k$ in a subgroup $S$ of $K$. Then the set $(h, k)$ of all $h \in R$ and all $k \in S$ is a subgroup $T$ of $G = (H, K; \phi)$, and $T \subseteq (R, S; \phi^*)$ where $\phi^*$ is $\phi$ restricted to $S$ with each $\phi_k, k \in S$, restricted to $R$. We shall write, “by abuse of language,” $T = (R, S; \phi) \subseteq G = (H, K; \phi)$.

**Theorem 14.** If $\phi$ is ample for $G = (H, K; \phi)$, then for $n = 1, 2, 3, \cdots$, $G/Z_n(G) \cong (H_n, K_n; \phi^{(n)})$ where $Z_n(G) = (F_n(H), Z_n(K) \cap \bigcap_{i+j=n-1} (t_i^{(\pm)} K_n; \phi)$ and where $H_n = H/F_n(H)$, $K_n = K/(Z_n(K) \cap \bigcap_{i+j=n-1} t_i^{(\pm)} K_n)$.

**Proof.** By the corollary to Theorem 8 and by Lemma 12, $G/Z_n(G) \cong (H_n, K_n; \phi^{(n)})$. For $n = 1$, $(h, k) \in Z_1(G)$ if and only if $h \in F_1(H)$, $k \in Z_1(K)$ and $\theta^{-1}_1 = \phi_k$. Since $\phi$ is ample, $F_1(H) \subseteq Z_1(H)$, whence $\theta^{-1}_n = \phi$. This places $k \in t^*$ so that $Z_1(G) = (F_1(H), Z_1(K) \cap t; \phi)$. Since $t = \bigcap_{i+j=0} t_i^{(\pm)} K_n, Z_1(G)$ has the required form. $G_1 = (H/(Z_1(H) \cap F_1(H)), K/(Z_1(K) \cap t); \phi^{(1)}) = (H/F_1(H), K/(Z_1(K) \cap t); \phi^{(1)}) \cong G/Z_1(G)$, so that the latter has the required form.

Now suppose that the theorem has been established through the index $n$. Then $(h, k) \in Z_{n+1}(G)$ if and only if (a) $h F_n(H) \subseteq F_1(H)$ and (b) $k Z_{n}(K) \cap L_{n-1} \subseteq Z_{n}(K) \cap t_{n}$ where $L_{n-1} = \bigcap_{i+j=n-1} t_i^{(\pm)} K_n$ and $t_{n} = \ker \phi^{(n)}$. $\phi^{(n)}$ is ample, by Lemma 10. $F_1(H) \Delta H$ since $F_1(H) \subseteq Z_1(H)$. If $F_1(H) \Delta H$, then $F_1(H/F_n(H)) = F_{n+1}(H)/F_n(H) \Delta H/F_n(H)$ so that $F_{n+1}(H) \Delta H$. Now (a) is equivalent to (a') $h \in F_{n+1}(H)$, and (b) is equivalent to (b') that both $k \in t_{n}$ and that $[k, y] \in Z_{n}(K) \cap L_{n-1}$ for every $y \in K$. Equivalently, $k \in Z_{n}(K) \cap K \cap (\bigcap_{i+j=n-1} (t_i^{(\pm)} i+1 K_n)) \cap t_{n} = Z_{n+1}(K) \cap L_{n}$, (as we see by replacing $j+1$ by...
Thus $Z_{n+1}$ has the required form. Finally, a short induction establishes the given isomorphisms for $H_n$ and $K_n$. One can also show that if $H$ is abelian, and if $F_n(H)$ is computed for $\mathfrak{S}(H)$, then $2^n$ is an exponent for $F_n(H)$.

We saw that $Z_n[H \triangle H] = [Z_n(H) \triangle Z_n(H)]$. If $Z_n(H, K; \phi) = (Z_n(H), Z_n(K); \phi)$ for $n = 1, 2, \ldots, m$, we say that the ascending central series of $(H, K; \phi)$ is regular through $m$. If this series is regular for every $m$, the series is called regular. By using Theorems 14 and 8a, we can prove

**Theorem 15.** If $\phi$ is ample for $G = (H, K; \phi)$, then the ascending central series of $G$ is regular (regular through $m$) if and only if

(a) $\phi(K) \subseteq \mathfrak{S}(H; Z_1(H)/Z_1(K))$, $j = 1, 2, 3, \ldots, (j = 1, 2, 3, \ldots, m)$,

(b) $\phi(Z_j(K)) \subseteq \mathfrak{S}(j-1(H))$, $j = 1, 2, 3, \ldots, (j = 1, 2, 3, \ldots, m)$.

**Corollary 1.** If $H$ is abelian, then $G = (H, K; \phi)$ has a regular ascending central series if and only if $G = H \oplus K$.

**Corollary 2.** If $\phi(K) = \mathfrak{S}(H)$ and if $\phi(Z_j(K)) \subseteq \mathfrak{S}(Z_j(H))$, $j = 1, 2, \ldots$, then $G = (H, K; \phi)$ has a regular ascending central series. (E.g., $G = [H \triangle H]$.)

Despite Theorems 6 and 14, it seems difficult to determine the ascending central series of $(H, K; \phi)$ for nonample $\phi$. However, we have

**Theorem 16.** $(h, k) \in Z_2(H, K; \phi)$ if and only if $(a') s = h\phi_k(x)\phi_{[k, y]}(\phi_y(h^{-1})x^{-1}) \in F_1(H)$,

(b') $[k, y] \in Z_1(K)$ and (c') $\phi_{[k, y]} = 1$, all for all $(x, y)$. Now assume that $(a')$, (b') and (c') hold. From $(a')$, with $x = e$, $h\phi_k(h^{-1}) \in F_1(H)$. Since $y^k$ ranges over all of $K$, $h\phi_y(h^{-1}) \in F_1(H)$; so that $(a_1)$ $\phi_y(h) = g(y)h$, $g(y) \in F_1(H)$. From $(a')$, with $y = e$, $\phi_k(x) \in Z_1(H) \cap F_1(H)$, and we have $(c'_1)$ $\phi_k(x) = h^{-1}hf_1(x)$. In $(a')$, let $x = \phi_{k-1}(h^{-1})$. Then $s$ reduces to $\phi_{[k, y]}(\phi_y(h^{-1})\phi_{k-1}(h)) \in F_1(H)$ so that $\phi_y(h^{-1})\phi_{k-1}(h) \in F_1(H)$. From $(c'_1)$, $\phi_k(h) = hf_1(h)$ whence $\phi_{k-1}(h) = h^{-1}hf_1(h)$. Hence $\phi_y(h^{-1})h^{-1}hf_1(h) = \phi_{k-1}(h)f_1(h)x^{-1} = f_1(h)x^{-1} = h^{-1}hf_1(h)f_1(x)$ so that $\phi_{k-1}(x) = h^{-1}hf_1(h)f_1(x)$.

We saw that $Z_n[H \triangle H] = [Z_n(H) \triangle Z_n(H)]$. If $Z_n(H, K; \phi) = (Z_n(H), Z_n(K); \phi)$ for $n = 1, 2, \ldots, m$, we say that the ascending central series of $(H, K; \phi)$ is regular through $m$. If this series is regular for every $m$, the series is called regular. By using Theorems 14 and 8a, we can prove

**Theorem 15.** If $\phi$ is ample for $G = (H, K; \phi)$, then the ascending central series of $G$ is regular (regular through $m$) if and only if

(a) $\phi(K) \subseteq \mathfrak{S}(H; Z_1(H)/Z_1(K))$, $j = 1, 2, 3, \ldots, (j = 1, 2, 3, \ldots, m)$,

(b) $\phi(Z_j(K)) \subseteq \mathfrak{S}(j-1(H))$, $j = 1, 2, 3, \ldots, (j = 1, 2, 3, \ldots, m)$.

**Corollary 1.** If $H$ is abelian, then $G = (H, K; \phi)$ has a regular ascending central series if and only if $G = H \oplus K$.

**Corollary 2.** If $\phi(K) = \mathfrak{S}(H)$ and if $\phi(Z_j(K)) \subseteq \mathfrak{S}(Z_j(H))$, $j = 1, 2, \ldots$, then $G = (H, K; \phi)$ has a regular ascending central series. (E.g., $G = [H \triangle H]$.)

Despite Theorems 6 and 14, it seems difficult to determine the ascending central series of $(H, K; \phi)$ for nonample $\phi$. However, we have

**Theorem 16.** $(h, k) \in Z_2(H, K; \phi)$ if and only if $(a') s = h\phi_k(x)\phi_{[k, y]}(\phi_y(h^{-1})x^{-1}) \in F_1(H)$,

(b') $[k, y] \in Z_1(K)$ and (c') $\phi_{[k, y]} = 1$, all for all $(x, y)$. Now assume that $(a')$, (b') and (c') hold. From $(a')$, with $x = e$, $h\phi_k(h^{-1}) \in F_1(H)$. Since $y^k$ ranges over all of $K$, $h\phi_y(h^{-1}) \in F_1(H)$; so that $(a_1)$ $\phi_y(h) = g(y)h$, $g(y) \in F_1(H)$. From $(a')$, with $y = e$, $\phi_k(x) \in Z_1(H) \cap F_1(H)$. But (c'), with $y = e$, reduces to $\phi_k(x) \in Z_1(H) \cap F_1(H)$, and we have $(c'_1)$ $\phi_k(x) = h^{-1}hf_1(x)$. In $(a')$, let $x = \phi_{k-1}(h^{-1})$. Then $s$ reduces to $\phi_{[k, y]}(\phi_y(h^{-1})\phi_{k-1}(h)) \in F_1(H)$ so that $\phi_y(h^{-1})\phi_{k-1}(h) \in F_1(H)$. From $(c'_1)$, $\phi_k(h) = hf_1(h)$ whence $\phi_{k-1}(h) = h^{-1}hf_1(h)$. Hence $\phi_y(h^{-1})h^{-1}hf_1(h) = \phi_{k-1}(h)f_1(h)x^{-1} = f_1(h)x^{-1} = h^{-1}hf_1(h)f_1(x)$ so that $\phi_{k-1}(x) = h^{-1}hf_1(h)f_1(x)$.

We saw that $Z_n[H \triangle H] = [Z_n(H) \triangle Z_n(H)]$. If $Z_n(H, K; \phi) = (Z_n(H), Z_n(K); \phi)$ for $n = 1, 2, \ldots, m$, we say that the ascending central series of $(H, K; \phi)$ is regular through $m$. If this series is regular for every $m$, the series is called regular. By using Theorems 14 and 8a, we can prove
Conversely, if (a), (b) and (c) hold, then (b') is implied by (b). \( s \) reduces, by (c), to \( f(t) = ft_{-1}ft \mod (Z_{i}(H) \cap F_{i}(H)) \), as we saw above. By (a), \( \phi_{y}(h^{-1})h \in F_{i}(H) \) so that (a') \( s \in F_{i}(H) \). As for (c'), \( \phi_{[k, y]}(t^s) = tf_{1}(t)f_{1}(\phi_{y}(t^{-1})) = t \), by (c), so that (c') \( \phi_{[k, y]}t_{s} = t \).

**Corollary 1.** \((H, K; \phi)\) is nilpotent of class 2 if and only if \( H \) and \( K \) are of class 2 and \( \psi(K) \subseteq \mathfrak{H}(H; H/(F_{i}(H) \cap Z_{i}(H))) \).

**Proof.** By the theorem, \( G = (H, K; \phi) = Z_{2}(G) \) if and only if (a'') \( h^{-1}\phi_{k}(h), \phi_{k}(h)h^{-1} \in F_{i}(H) \) for every \( (h, k) \), (b'') every \( k \in Z_{2}(K) \) and (c'') \( \phi_{k}(x) = h^{-1}xhf_{1}(x) \) where \( f_{1}(\phi_{y}(x)) = f_{1}(x) \in Z_{i}(H) \cap F_{i}(H) \) for every \( y, k \in K \). Suppose that \( G \) is of class 2. From (c''), with \( h = e, \phi_{k}(x) \equiv x \mod (Z_{i}(H) \cap F_{i}(H)) \), and the second conclusion follows. From (b''), \( K \) is of class 2. In (c''), take \( k = e \) so that all \( [x, h] \in Z_{i}(H) \), and \( H \) is of class 2. Conversely, if the conclusions hold, then \( \phi_{k}(x) = xf_{1} = fx \) where \( f \in F_{i}(H) \) so that (a'') holds. Since \( \phi_{k}(x) \equiv x \mod Z_{i}(H), \phi(K) \subseteq \mathfrak{H}(1) \) so that \( H' \subseteq F_{i}(H) \). From the second conclusion, there exists \( f(x) \in F_{i}(H) \cap Z_{i}(H) \) such that \( \phi_{k}(x) = xf(x) = h^{-1}xhf_{1}(x) \), since \( H' \subseteq Z_{i}(H) \). But \( x^{-1}xh \in F_{i}(H) \cap Z_{i}(H) \) since \( H' \subseteq F_{i}(H) \). Thus \( \phi_{k}(x) = h^{-1}xhf_{1}(x) \) where \( f_{1}(x) \in F_{i}(H) \cap Z_{i}(H) \). Now \( h^{-1}x^{-1}hf_{k}(x) = f_{1}(x) \) for every \( x \in H \). Replace \( x \) by \( \phi_{y}(x) \) and apply \( \phi_{y}^{-1} \) to both sides. Then \( \phi_{y}^{-1}(h^{-1})x^{-1}h\phi_{y}^{-1}(h\phi_{y}^{-1}(x)) = f_{1}(\phi_{y}(x)) \). Since \( \phi_{y} \in \mathfrak{H}(1) \), one can reduce the above to \( h^{-1}x^{-1}h\phi_{y}^{-1}(x) = f_{1}(\phi_{y}(x)) \). By what has already been proved, \( \phi_{y}(x) = xa \) and \( \phi_{k}(x) = xb \) where \( a, b \in F_{i}(H) \cap Z_{i}(H) \). Hence \( \phi_{y}^{-1}(x) = xh = \phi_{k}(x) \) so that \( f_{1}(x) = h^{-1}x^{-1}hf_{k}(x) = f_{1}(\phi_{y}(x)) \). This completes the proof.

We should note that \((H, K; \phi)\) of class 2 implies that \( H' \subseteq F_{i}(H) \) and that \( \phi(K) \) is abelian.

**Corollary 2.** Necessary and sufficient that \( G = (H, K; \phi) \) be nilpotent of class 2 is that \( H \) and \( K \) be of class 2 and that \( \psi(G) \) be direct (\( \phi^{(1)} = \nu \)).

**10. The derivative.** \[ [(a, b), (c, d)] = (a\phi_{b}(c)\phi_{b, a}[\phi_{d}(a^{-1})c^{-1}], [b, d]) = (\rho(a, b, c, d), [b, d]). \] \( \rho(a, b, c, d) \) may be called a skew commutator. Now \[ [(e, b), (e, d)] = (e, [b, d]), \] so that the generators of the derivative \( G' \) are all \( (\rho, e) \) and all \( (e, [b, d]) \). Hence every element of \( G' \) can be expressed as \( (s, t) \) with \( t \in K' \), and where \( s \) is a product of skew commutators, their inverses, and the transforms of these skew commutators and of their inverses under the automorphisms from \( \phi(K) \). A trivial verification shows that the inverse of a skew commutator is a skew commutator: \( \rho(a, b, c, d)^{-1} = \rho(a', b', c', d') \) where \( a' = \phi_{[b, a]}(c), b' = d^{[b, d]}, c' = \phi_{[b, a]}(a), \) and \( d' = b^{[b, d]} \). Let us denote by \( H^{s} \) the subgroup of \( H \) generated by the skew commutators. \( H' \subseteq H^{s} \Delta H \).

**Theorem 17.** If \( G = (H, K; \phi), \) then \( G' = (H^{s}, K'; \phi), \) and \( G/G' \cong (H/H^{s}) \oplus (K/K') \).

**Proof.** It is readily established that \( \phi_{k}(\rho(a, b, c, d)) = \rho(\phi_{k}(a), kb, c, d) \)

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\( \cdot \rho(\phi[k_b, d](c), d^{k_b}, e, k) \) so that \( \phi_k(H^s) \subseteq H^s \) for every \( k \in K \), and the first conclusion follows from the preliminary material. Since \( \phi_k(c)^{-1} = \rho(e, b, c, e) \), \( \phi(K) \subseteq \mathfrak{A}(H; H/H^s) \). Let \( X \) be the direct sum \( (H/H^s) \oplus (K/K') \). Define a map \( \xi \) on \( G \) into \( X \) by \( \xi(h, k) = (hH^s, kK') \). It is clear that \( \xi \) is onto \( X \). Since \( \phi(K) \subseteq \mathfrak{A}(H; H/H^s) \), \( \xi(h\phi_k(h_2), k_1k_2) = (h_1h_2H^s, k_1k_2K') = (h_1H^s, k_1K')(h_2H^s, k_2K') \), so that \( \xi \in \text{Hom}(G, X) \). Finally, \( (h, k) \in \ker \xi \) if and only if \( h \in H^s \) and \( k \in K' \); that is, if and only if \( (h, k) \in G' \).

Observe that for \([H \Delta H] \) we have \( H^s = H' \), so that \( D^{(i)}[H \Delta H] = D^{(i)}[H] \triangle D^{(i)}[H] \), \( j = 1, 2, \ldots \).

11. Normalizers and centralizers. Recall that \( \mathcal{B}(G) \) is the set of all \( (h, k) \) with \( \theta_k \phi = \iota \). Let \( B_{11} = B_{11}(G) \) and \( B_{12} = B_{12}(G) \) be the respective projections on \( H \) and on \( K \) of \( B_1 \) via the mappings \( (h, k) \to h \) and \( (h, k) \to k \). The sets \( B_{1i} \) are subgroups. We shall make such abbreviations as \( \mathcal{C}(H; G) = \mathcal{C}(H, G) \). Direct calculations establish that \( \mathcal{C}(H, G) = B_{11} \), that \( \mathcal{C}^{(2)}(H, G) \) is the set of all \( (h, k) \) with \( \phi_k \in \mathfrak{A}(H; B_{11}) \) and \( k \in \mathcal{C}(B_{12}, K) \), that \( \mathcal{C}(K, G) \subseteq F_1(H) \oplus Z_1(K) \), and that \( \mathcal{C}^{(2)}(K, G) \) is the set of all \( (h, k) \in G \) for which \( \phi_k(h) = h \) for every \( y \in Z_1(K) \) and \( h \in \mathcal{C}(F_1(H), H) \). We have, at once, the result of Jordan [9, p. 51] that \( \mathfrak{A}(H, \mathfrak{S}(H)) \) is the set of all \( (h, \theta_{\lambda^{-1}}) \in \mathfrak{S}(H) \). \( G = \mathfrak{C}(H, G) \) if and only if \( G \) reduces to \( H \oplus K \) with \( H \) abelian. \( G = \mathcal{C}^{(2)}(H, G) \) if and only if \( B_{11} \subseteq F_1(H) \) and \( B_{12} \subseteq Z_1(K) \). \( G = \mathfrak{C}(K, G) \) if and only if \( G \) reduces to \( H \oplus K \) with \( K \) abelian. \( G = \mathcal{C}^{(3)}(K, G) \) if and only if \( F_1(H) \subseteq Z_1(H) \) and \( Z_1(K) \subseteq \mathfrak{C} \). If \( H \) is abelian, then \( \mathfrak{C}(H, G) \cong H \oplus \mathfrak{C} \) so that \( \mathfrak{C} \) and \( \mathfrak{C} \) are each nilpotent of the same class or both are non-nilpotent. If \( H \) is abelian, \( \mathfrak{C}^{(3)}(H, G) = H \oplus Z_1(\mathfrak{C}) \), so that \( \mathfrak{C}^{(2)} = Z_1(\mathfrak{C}^{(1)}) \). If \( H \) is abelian, and if \( H \) is maximal with respect to the property \( H \Delta G \), then \( G \) is direct, or \( \phi \) is an isomorphism and \( G = (H, \phi(K); \iota) \), a relative holomorph. One can show that \( G = \mathcal{C}^{(4)}(H, G) \) if and only if \( \mathfrak{C}(H, G) \subseteq \mathfrak{C}(K, G) \), while \( G = \mathcal{C}^{(3)}(K, G) \) if and only if \( \mathfrak{C}(K, G) \subseteq Z_1(H) \oplus \mathfrak{C} = \mathfrak{C}(H, G) \cap (H \oplus \mathfrak{C}) \).

Let us define a sequence of sets \{ \( E_n(H) \) \} by \( E_1(H) = F_1(H) \), \( \ldots \), \( E_{n+1}(H) = \) all \( h \) such that \( a) h \in \mathfrak{N}(E_n(H), H) \) and \( b) h\phi_y(h^{-1}) \in E_n(H) \) for every \( y \in K \). An inductive proof shows that the \( E_n(H) \) form, under set inclusion, an ascending sequence of subgroups of \( H \), each of which is \( \phi_k \)-admissible for every \( k \in K \). \( E_n(H) \Delta E_{n+1}(H) = E_0(H) = (c). E_2(H) \) is the set of all \( h \in H \) such that \( h^{-1}\phi_y(h) \in Z_1(F_1(H)) \) for every \( y \in K \).

Lemma 15. \( \mathfrak{N}^{(i)}(K, G) = (E_j(H), K; \phi) \) so that \( a) G = \mathfrak{N}^{(i)}(K, G) \) if and only if \( H = E_j(H) \); \( b) G = \mathfrak{N}^{(2)}(K, G) \) if and only if \( \phi(K) \subseteq \mathfrak{A}(H; H \mod Z_1(\cdot F_1(H))) \); \( c) if \( (H, K; \phi) \) is of class \( c \), then there exists \( j \leq c \) with \( H = E_j(H) \); \( d) \phi \) is ample and if \( G = \mathfrak{N}^{(3)}(K, G) \), then \( H \) is of class 2.

Proof. (d). By \( b) \), \( h^{-1}\phi_y(h) \in Z_1(F_1(H)) \) from \( F_1(H) \). But each \( \theta_k \) is some \( \phi_y \), whence \( h^{-1}h^z \in F_1(H) \) for every \( x \) and \( h \in H \). Thus \( H' \subseteq F_1(H) \subseteq Z_1(H) \).

Note that \( \phi \) ample or \( \phi(K) \subseteq Z_1(H) \) implies that \( E_n(H) \Delta H \) for all natural
Under such normality, the subgroups $R$ of $H$ such that $E_{n+1} \supseteq R \supseteq E_n$ are characterized by the property $\phi(K) \subseteq \mathfrak{N}(H; R/E_n)$.

If $A, B \triangle H$ with $A \subseteq B$, then $\phi(A, B) = \phi(A, B)$. Let $\mathfrak{C}_1(A, B) = \mathfrak{C}(A, B)$. If $\mathfrak{C}_n(A, B)$ is defined as a subgroup of $B$, let $\mathfrak{C}_{n+1}(A, B) = \mathfrak{C}(A, B) \supseteq \mathfrak{C}_n(A, B)$. $\mathfrak{C}_n(A, B)$ is a subgroup of $B$ normal in $H$. If $B \triangle H$, $B \subseteq V \triangle H$, then $\mathfrak{C}_n(A, B) \subseteq \mathfrak{C}_n(A, V)$. In particular, $Z_n(A) = \mathfrak{C}_n(A, A) \subseteq \mathfrak{C}_n(A, B)$ for every $A, B \triangle H$, $A \subseteq B$. $\mathfrak{C}_n(A, B) = (e)( / )_B$. 

**Theorem 18.** If $G = (H, K; \phi)$ where each $E_n(H) \triangle H$, then for each $n$, $E_{n+1}(H)$ is the set of all $h \in H$ for which $h^{-1}\phi_y(h) \in \mathfrak{C}_n(F_1(H), E_n(H))$ for every $y \in K$.

**Proof.** We saw above that $E_2(H)$ is the set of all $h$ such that all $t = h^{-1}\phi_y(h) \in Z_1(F_1(H)) = \mathfrak{C}_1(F_1(H), E_1(H))$. Now suppose that the theorem is valid for the index $n$. Since $E_n(H) \triangle H$, $y \in K$, $h \in H$ and $x \in E_n(H)$ imply the existence of $z_1, z_2 \in \mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H))$ such that $\phi_y(x^{z_1}) = x^{z_2}$. Hence $\phi_y(x^{z_1}) = x^{z_2}$. $\mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H)) \triangle H$ so that $\phi_y = \phi_{y^{z_1}}$. That is, $t \in E_{n-1}(F_1(H), E_{n-1}(H))$.

**Lemma 16.** $\mathfrak{N}(K, G)/\mathfrak{C}(K, G) \cong \mathfrak{Z}(K)$.

The specific homomorphism on $\mathfrak{N}$ to $\mathfrak{Z}$ is $\gamma(f, k) = \theta_k$ for $f \in F_1(H)$. This quotient group is independent of $H$. Cf. [9, p. 47]. Note that the normalizer and centralizer of $K$ coincide if and only if $K$ is abelian.

Let $R$ be a $\phi$-admissible subgroup of $H$ for every $k \in K$. $W = (R, K; \phi)$ is a subgroup of $G = (H, K; \phi)$. $(a, b) \in \mathfrak{C}(W, G)$ where $a \in H$, $b \in K$, if and only if $a \phi_b(b) = r \phi_k(a)$ and $b \in Z_1(K)$ for every $r \in R$ and for every $k \in K$. Choosing $r = e$, we have $a = \phi_k(a)$, so that $a \in F_1(H)$. Then $a \phi_b(b) = r a$ so that on $R$, $\phi_b \theta_a = \theta_a$. Conversely, $a \in F_1(H), b \in Z_1(K)$ and $\phi_b \theta_a = \theta_a$ on $R$ imply $(a, b) \in \mathfrak{C}(W, G)$. Likewise, $(a, b) \in \mathfrak{C}(W, G)$ implies that $a \in \mathfrak{N}(R, H)$. $\mathfrak{C}(W, G)$ has the “least possible value” $Z_1(G)$ if and only if $R$ has the property (P) if $\alpha \in \mathfrak{N}(R)$ can be extended to $\phi \in \phi(K)$, where $b \in Z_1(K)$, and if $a$ can be extended to $\theta_{a^{-1}} \in \mathfrak{Z}(H)$, where $a \in F_1(H)$.

Let $\mathfrak{N}(K, G)$ be an abbreviation for $\mathfrak{C}(\mathfrak{N}(K, G), G)$.

**Theorem 19.** If $G = (H, K; \phi)$, and if $\mathfrak{C}(F_1(H), H) \subseteq F_1(H)$, then $\mathfrak{N}^2(K, G)/\mathfrak{C}(K, G) \cong (\mathfrak{N}(F_1(H), H)/\mathfrak{C}(F_1(H), H)) \oplus \mathfrak{Z}(K)$.

**Proof.** Let $R = F_1(H)$. Suppose that $\phi_b \theta_a = \theta_a$ on $F_1(H)$ where $b \in Z_1(K)$ and $a \in F_1(H)$. $\phi_b = \theta_a$ on $F_1(H)$. Hence $a \in Z_1(F_1(H))$. Conversely, if $a \in Z_1(F_1(H))$
and if \( b \in Z_1(K) \), then \( f \in F_1(H) \) and \( k \in K \) imply that \((a, b)(f, k) = (af, bk) = (fa, kb) = (f, k)(a, b)\) so that \((a, b) \in \mathfrak{N}(K, G)\). We have proved that \( \mathfrak{N}(K, G) = (Z_1(F_1(H), Z_1(K); \phi) = (Z_1(F_1(H)) \oplus Z_1(K)) \). Let us define a mapping \( \gamma \) on \( \mathfrak{N}^{(2)}(K, G) \) into \( (\mathfrak{N}(F_1(H), H) \oplus \mathfrak{N}(F_1(H), H)) \oplus \mathfrak{Z}(K) \) by \( \gamma(h, k) = (\theta^*_h, \theta_k) \) where \( \theta_h^* \) is \( \theta_h \) restricted to \( F_1(H) \). Here \( h \in E_2(H) \) and \( k \in K \). For \( h_1, h_2 \in E_2(H), k_1, k_2 \in K, h_1k_1(h_2) = h_1h_2z \), where \( z \in Z_1(F_1(H)) \). Now \( \theta_h^* = \iota \) on \( F_1(H) \) so that \( \theta_h^* h_2 = \theta_h \theta_h^* \), and \( \gamma \) is a homomorphism. Since \( \mathfrak{N}(F_1(H), H) \subset F_1(H) \) by hypothesis, it is clear that \( \text{kern } \gamma = (Z_1(F_1(H)), Z_1(K); \phi) = \mathfrak{N}(K, G) \). Hence \( \mathfrak{N}^{(2)} / \mathfrak{N} \cong T \oplus \mathfrak{Z}(K) \) where \( T \subset \mathfrak{N}(F_1(H), H) / \mathfrak{N}(F_1(H), H) \).

For a subgroup \( K \) of a group \( G \), let \( \mathfrak{N}_i(K, G) = \mathfrak{N}_i(K) \), the inner normal hull of \( K \) in \( G \), be the largest subgroup of \( K \) which is normal in \( G \). Let \( \mathfrak{N}_0(K, G) = \mathfrak{N}_0(K) \), the outer normal hull of \( K \) in \( G \), be the smallest normal subgroup of \( G \) in which \( K \) is included. \( \mathfrak{N}_i = \mathfrak{N}_i \) if and only if \( K \Delta G \), whence \( \mathfrak{N}_i = \mathfrak{N}_0 = K \). As generators, \( \mathfrak{N}_0(K) \) has all \((e, k)(x, w) = (x\phi_k(x^{-1}), k^v)\); that is, all \((x\phi_k(x), y)\).

In \( [H \wedge K] \) or \([H \supset K]\), the generators of \( \mathfrak{N}_0(K) \) are all \((h, k), k \) where \( h \in H, k \in K \). The inner normal hull of \( K \) in \( (H, K; \iota) \) is \( \mathfrak{N} \). If a group \( K \) can be represented faithfully as a group of automorphisms of a group \( H \), then one can minimize the inner normal hull by forming the relative holomorph \((H, K; \iota) \) of \( K \) over \( H \). Here \( \mathfrak{N}_i(K) = \mathfrak{N} = (e) \). We write, "by abuse of language," \( \mathfrak{C}(f, G) \) rather than \( \mathfrak{C}(f^*, G) \). \((x, y) \) is in the latter if and only if \( y \in \mathfrak{C}(f, K) \) as a short argument shows, and \( \mathfrak{C}(f, G) = (H, \mathfrak{C}(f, K); \phi) \). If, as in \([K \wedge K] \), \( \mathfrak{C} \subset Z_1(K) \), then \( \mathfrak{C}(f, G) = G \) so that \( f \subset Z_1(G) \).

Let \( K \) now be a group in which the ascending central series breaks off for some index \( n \) such that \( Z_n(K) \neq K \). Then \( K/Z_n(K) \) is a centerless group and is thus representable faithfully as a subgroup of its group of automorphisms. Form the product \( G = (K/Z_n(K), K; \phi) \) where \( \phi \) is the natural mapping on \( K \) onto \( K/Z_n(K) \). Let \( a' \), for \( a \in K \), be the coset \( aZ_n(K) \). For \( a, b, c \in K, (e', c)(a^x.b^y, c^z) = (a'b^y.c'^z.a'^{-1}b'^{-1}c'^{-1}a'^{-1}b'^{-1}c'^{-1}, c^z) = ([acd^{-1}(c^{-1})']^z, c^z) = ([a, c^z])^z, c^z) \) so that \( \mathfrak{N}_0(K) \) has for a complete set of generators all \((h, k'), k \). Since \( K \subset \mathfrak{N}_0(K) \), all \((h, k), e \) are included among the generators. Recall that groups \( K \) for which \( K = D(K) \) are called perfect [9]. If we take \( G \) as indicated, we have proved

**Theorem 20.** Let \( K \) be a group for which the ascending central series terminates with \( Z_n(K) \neq K \) and for which \( K/Z_n(K) \) is perfect. Then there exists a proper extension \( G \) of \( K \) such that the outer normal hull of \( K \) in \( G \) is \( G \).

Since any homomorphic image of a perfect group is perfect, we can have \( K \) rather than \( K/Z_n(K) \) perfect in the hypothesis.
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