ON SOME SIMPLE GROUPS DEFINED BY C. CHEVALLEY

BY

RIMHAK REE

Introduction. For any semi-simple Lie algebra $\mathfrak{g}$ over the field of complex numbers and an arbitrary field $K$, C. Chevalley [2] defined a group $\mathfrak{G}$ by a uniform method. The group $\mathfrak{G}$ turns out to be simple when $\mathfrak{g}$ is simple (except for a few exceptional cases), and yields some new classes of simple groups not contained in the theory of classical groups, if, in particular, $\mathfrak{g}$ is one of the exceptional simple Lie algebras.

In this paper first we consider the case where $\mathfrak{g}$ is one of the classical algebras $A_n, B_n, C_n, D_n$, and identify the group $\mathfrak{G}$ with classical groups, thus answering a question raised in the last section of [2]. Our results are as follows:

(a) if $\mathfrak{g}$ is of the type $A_n$, then $\mathfrak{G}$ is the special projective group $\text{PSL}(n+1, K)$;
(b) if $\mathfrak{g}$ is of the type $C_n$, then $\mathfrak{G}$ is the quotient group of the symplectic group $\text{Sp}(2n, K)$ over its center;
(c) if $\mathfrak{g}$ is of the type $D_n$, then $\mathfrak{G}$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=1}^{n} \xi_i \xi_{-i}$ ($3 \leq n$);
(d) if $\mathfrak{g}$ is of the type $B_n$ and if $K$ is not of characteristic 2 then $\mathfrak{G}$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=0}^{n} \xi_i \xi_{-i}$ ($2 \leq n$); if $\mathfrak{g}$ is of type $B_n$ and if $K$ is of characteristic 2 then $\mathfrak{G}$ is a subgroup of $\text{Sp}(2n, K)$; if, moreover, $K$ is perfect\(^{(1)}\) then $G=\text{Sp}(2n, K)$. The author has been unable to identify $\mathfrak{G}$ in case $K$ is not perfect and of characteristic 2.

Secondly, we consider the case where $\mathfrak{g}$ is the exceptional algebra $G_2$, and identify the group $\mathfrak{G}$ with the groups defined by L. E. Dickson [4; 5].

1. The group defined by Chevalley. Let $\mathfrak{g}$ be a semi-simple Lie algebra over the field of complex numbers and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Denote by $(X, Y)$ the Killing form of $\mathfrak{g}$. For a root\(^{(2)}\) $r$ of $\mathfrak{g}$, we set $H_r=2(r, r)^{-1}r$, which is called the co-weight (co-poids) attached to $r$. It is known that the additive group $\mathfrak{H}$ generated by all co-weights $H_r$ is a free abelian group of rank $n$, where $n$ is the dimension of $\mathfrak{h}$. Chevalley proves the existence of a system $\{X_r\}$ of root vectors satisfying the following conditions (1.1)–(1.2):

\begin{equation}
[X_r, X_{-r}] = H_r \quad \text{for all roots } r;
\end{equation}

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\(\quad\)\(^{(1)}\) A field $K$ of characteristic $p>0$ is called perfect if every element in $K$ is a $p$th power of an element in $K$.

\(\quad\)\(^{(2)}\) Roots of a semi-simple algebra over the field of complex numbers can be identified with elements in the Cartan subalgebra by which the roots are defined. See [7].
(1.2) if \( r, s \), and \( r+s \) are roots then \([X_r, X_s] = N_{r,s}X_{r+s}\), with \( N_{r,s} = \pm (p+1)\), where \( p \) is the greatest integer \( i \geq 0 \) such that \( s - ir \) is a root.

If a system \( \{X_r\} \) of root vectors satisfies (1.1) and (1.3) below then it also satisfies (1.2) (see [2, pp. 22–23]).

(1.3) \( N_{-r,-s} = N_{r,s} \) for any two roots \( r, s \) such that \( r+s \) is a root.

Let now \( \{\tilde{H}_1, \ldots, \tilde{H}_n\} \) be a basis of the group \( \mathfrak{X} \) and \( \{X_r\} \) a system of root vectors satisfying (1.1)–(1.2). Then the set \( \{\tilde{H}_1, \ldots, \tilde{H}_n\} \) together with \( \{X_r\} \) forms a basis \( \{\tilde{H}_i, X_r\} \) of \( \mathfrak{g} \). It is easily seen that the coefficients appearing in the multiplication table of the basis \( \{\tilde{H}_i, X_r\} \) are all integers. Chevalley proves that the coefficients of the matrix \( A_r(t) \) representing the automorphism

\( x_r(t) = \exp t(ad X_r) \),

where \( t \) is a complex variable, of \( \mathfrak{g} \) with respect to the basis \( \{\tilde{H}_i, X_r\} \) are all polynomials in \( t \) with integral coefficients. Therefore for any field \( K \) we can define a Lie algebra \( \mathfrak{g}_K \) over \( K \) by using the basis \( \{\tilde{H}_i, X_r\} \), and also automorphisms \( x_r^*(t) \), where \( t \in K \), of \( \mathfrak{g}_K \) by using the matrix \( A_r(t) \). Then the group considered by Chevalley is the group \( \mathfrak{G}(\mathfrak{g}; K) \) (denoted by the symbol \( \mathfrak{G}' \) in [2]) generated by all \( x_r^*(t) \), where \( r \) runs over all roots of \( \mathfrak{g} \) while \( t \) takes on all elements in \( K \). This group was proved to be simple except for a few exceptional cases.

Since every \( \tilde{H}_i \) is a linear combination of the co-weights \( H_r \) with integral coefficients, it follows that an element in \( \mathfrak{G}(\mathfrak{g}; K) \) is uniquely determined by its effect on \( H_r^* \) and \( X_r^* \), where \( H_r^* \) and \( X_r^* \) are elements in \( \mathfrak{g}_K \) corresponding to \( H_r \) and \( X_r \) in \( \mathfrak{g} \), respectively. In particular an element in \( \mathfrak{G} \) is the identity if and only if it leaves every \( H_r^* \) and every \( X_r^* \) invariant.

2. The classical algebras. For the classical algebras \( A_n, B_n, C_n, D_n \) the co-weights \( H_r \) and the system \( \{X_r\} \) of root vectors satisfying (1.1) and (1.3) can be computed easily (cf. [7]).

The algebra \( A_n \) is the algebra of all \((n+1) \times (n+1)\) matrices of trace zero. Denote by \( E_{ij} \) the \((n+1) \times (n+1)\) matrix whose \((i, j)\)-entry is 1 and whose other entries are all zero. Then

\[
\{H_r\} = \{E_{ii} - E_{jj} \mid i \neq j\}, \quad \{X_r\} = \{E_{ij} \mid i \neq j\},
\]

and if \( H_r = E_{ii} - E_{jj} \) then \( X_r = E_{ij}, X_{-r} = E_{ji} \). The mapping \( \phi \) which maps a matrix \( T \) to \(-T'\), where \( T' \) is the transpose of \( T \), is an automorphism of \( A_n \).

We have \( \phi(E_{ij}) = -E_{ji} \), and hence it is easily seen that the system \( \{X_r\} \) satisfies the condition (1.3).

The algebra \( D_n \) is the algebra of all \( 2n \times 2n \) matrices \( X \) which satisfy

\[
f_D(X\xi, \eta) + f_D(\xi, X\eta) = 0\text{ identically, where}
\]

\[
f_D(\xi, \eta) = \xi_{-n}\eta + \cdots + \xi_{-1}\eta_1 + \xi_1\eta_{-1} + \cdots + \xi_n\eta_{-n}.
\]
Denote by \( E_{ij} \) the \( 2n \times 2n \) matrix such that \( E_{ij} \xi = \xi' \), where \( \xi' = \xi_j \), \( \xi' = 0 \) for \( k \neq i \), and set

\[
F_{ij} = E_{ij} - E_{ji}, \quad H_{ij} = F_{i,-i} + F_{j,-j}.
\]

Then we have

\[
\{ H_r \} = \{ H_{ij} \mid i < j, i \neq -j \}, \quad \{ X_r \} = \{ F_{ij} \mid i < j, i \neq -j \},
\]

where if \( H_r = H_{ij} \) then \( X_r = F_{ij}, X_r = F_{-j,-i} \). The mapping \( \phi(T) = -T' \) is also an automorphism of \( D_n \), and hence the system \( \{ X_r \} \) satisfies (1.3).

The algebra \( B_n \) may be regarded as the algebra of all \( (2n+1) \times (2n+1) \) matrices \( X \) satisfying \( f_B(X \xi, \eta) + f_B(\xi, X \eta) = 0 \) identically, where

\[
f_B(\xi, \eta) = 2\xi_0\eta_0 + \sum_{i=1}^{n} (\xi_{-i}\eta_i + \xi_i\eta_{-i}).
\]

It is readily seen that a matrix \( X = (x_{ij}) \) belongs to \( B_n \) if and only if

\[
x_{-ij} + x_{-ji} = 0, \quad x_{-ij} + 2x_{0i} = 0, \quad x_{00} = 0
\]

for all \( i \neq 0 \) and \( j \neq 0 \). Define the matrices \( E_{ij} \) as in the case of algebra \( D_n \), and set

\[
F_{ij} = E_{ij} - E_{ji}, \quad F_{0i} = 2E_{-i0} - E_{0i}, \quad F_{0j} = -F_{j0},
\]

\[
H_{ij} = F_{i,-i} + F_{j,-j}, \quad H_{0i} = 2F_{i,-i}, \quad H_{0j} = 2F_{j,-j}
\]

for \( i \neq 0, j \neq 0 \). Then we see easily that

\[
\{ H_r \} = \{ H_{ij} \mid i < j, i \neq -j \}
\]

is the set of all co-weights (with respect to a certain Cartan subalgebra) and that

\[
\{ X_r \} = \{ F_{ij} \mid i < j, i \neq -j \}
\]

is a system of root vectors satisfying (1.1), where, \( X_r = F_{ij} \) and \( X_r = F_{-j,-i} \) if \( H_r = H_{ij} \). This system also satisfies (1.3). This is seen as follows: Let \( P = 2E_{00} + \sum_{i=1}^{n} (E_{ii} + E_{-i,-i}) \). Then the mapping \( \phi(X) = -P^{-1}X'P \) is an automorphism of \( B_n \) such that \( \phi(F_{ij}) = -F_{-j,-i} \), that is, \( \phi(X_r) = -X_r \) for all roots \( r \), and hence (1.3) follows immediately.

The algebra \( C_n \) is the algebra of all \( 2n \times 2n \) matrices \( X \) which satisfy \( f_C(X \xi, \eta) + f_C(\xi, X \eta) = 0 \) identically, where

\[
f_C(\xi, \eta) = \xi_{-n}\eta_n + \cdots + \xi_{-1}\eta_1 - \xi_1\eta_{-1} - \cdots - \xi_n\eta_{-n}.
\]

Define \( E_{ij} \) as in the case of the algebra \( D_n \) and set

\[
F_{ij} = \begin{cases} 
E_{ij} + \text{sign} (ij) E_{-ji}, & \text{if } i \neq j, \\
E_{-ii} & \text{if } i = j,
\end{cases}
\]
where sign \( k \) denotes +1 if \( k > 0 \), and −1 if \( k < 0 \), and
\[
H_{ij} = \begin{cases} 
F_{i,-i} + F_{j,-j}, & \text{if } i \neq j, \\
F_{i,-i} & \text{if } i = j.
\end{cases}
\]

Then we have
\[
\{H_r\} = \{H_{ij} \mid i \leq j, i + j \neq 0\},
\]
(2.6)
\[
\{X_r\} = \{F_{ij} \mid i \leq j, i + j \neq 0\},
\]
and if \( H_r = H_{ij} \) then \( X_r = F_{ij}, X_{-r} = F_{-j,-i} \). The mapping \( \phi: X \rightarrow X' \) is an automorphism of \( C_n \) such that \( \phi(X_r) = -X_{-r} \) for all roots. Thus (1.3) is also satisfied by the system \( \{X_r\} \).

3. The group \( \mathfrak{g}(g, K) \). For any two matrices \( X, Y \) with complex coefficients we have the formula
\[
\exp(\text{ad } X) \cdot Y = (\exp X) Y (\exp (-X)),
\]
(3.1)
where \( (\text{ad } X) \cdot Y = XY - YX \). This can be verified easily if we use the identity
\[
\frac{(\text{ad } X)^k}{k!} \cdot Y = \sum_{i+j=k} \frac{X^i}{i!} Y \frac{(-X)^j}{j!}.
\]

For the matrix algebras \( g = A_n, C_n, D_n \), every root vector \( X_r = F_{ij} \) (we set \( F_{ij} = E_{ij}, i \neq j \), for the algebra \( A_n \)) given in the preceding section satisfies \( X^2_r = 0 \), and hence
\[
\exp(tX_r) = I + tF_{ij},
\]
(3.2)
where \( I \) is the unit matrix. For the algebra \( g = B_n \), every root vector \( X_r = F_{ij} \) such that \( i \neq 0, j \neq 0 \) satisfies \( X^2_r = 0 \), and hence we have (3.2) for this \( X_r \) also. For the root vector \( X_r = F_{i0} \), of the algebra \( B_n \) we have \( X^2_r = -2E_{-i0}, X^4_r = 0 \). Therefore
\[
\exp(tX_r) = I + t(2E_{-i0} - E_{00}) - t^4E_{-i0},
\]
(3.3)
for \( i = \pm 1, \pm 2, \ldots, \pm n \). From (3.2) and (3.3) we see that \( \exp(tX_r) \) can always be represented as a matrix whose entries are polynomials in \( t \) with integral coefficients. Now, the matrix \( A_r(t) \) representing the automorphism \( x_r(t) \) of \( g \) with respect to the basis \( \{\tilde{H}_i, X_r\} \) can be obtained from the formula
\[
x_r(t) \cdot Y = (\exp(tX_r)) Y (\exp(-tX_r)), \quad Y \in \mathfrak{g},
\]
(3.4)
which follows from (3.1), and the expression for \( \exp(tX_r) \) given in (3.2)–(3.3).

Let \( K \) be an arbitrary field and \( g \) one of the algebras \( A_n, B_n, C_n, D_n \). In the preceding section all co-weights \( H_r \) and root vectors \( X_r \) of \( g \) are represented by matrices with integral coefficients, and consequently, every element in the basis \( \{\tilde{H}_i, X_r\} \) is also represented as a matrix with integral coefficients.
Therefore we see that $\mathfrak{g}_K$ is represented in this way as a matrix algebra over $K$. For $t \in K$, define the matrix $\exp(tX^*)$, by the right hand side of (3.3) if $X_r$ is one of the root vectors $F_{i_0}$, $F_{o_j}$ (in case $g = B_n$), and by that of (3.2) if $X_r$ is not, and consider the automorphism $x_r^*(t)$ defined by $A_r(t)$. Then (3.4) yields

\begin{equation}
(3.5) \quad x_r^*(t) \cdot Y^* = (\exp(tX^*))Y^*(\exp(-tX^*))
\end{equation}

for any $Y^*$ in $\mathfrak{g}_K$. Denote by $\mathcal{G}(\mathfrak{g}, K)$ the multiplicative group of matrices generated by all $\exp(tX^*)$, where $r$ runs over all roots of $\mathfrak{g}$ while $t$ runs over all elements in $K$. For any $S \in \mathcal{G}$ define the mapping $S^\prime$ of $\mathfrak{g}_K$ into itself by $S^\prime \cdot Y^* = SY^*S^{-1}$, where $Y^* \in \mathfrak{g}_K$. Then (3.5) shows that $S^\prime$ is a product of certain automorphisms of $\mathfrak{g}_K$ of the form $x_r^*(t)$, and hence belongs to the group $\mathcal{G}(\mathfrak{g}, K)$ defined in §1. The mapping $\sigma: S \mapsto S^\prime$ is clearly a homomorphism of $\mathcal{G}$ onto $\mathcal{G}$. Denote by $\mathcal{Z}$ the kernel of the homomorphism $\sigma$. By the remark given at the end of §1, we see that an element $S$ in $\mathcal{G}$ belongs to $\mathcal{Z}$ if and only if $S$ commutes with all $X_i^*$ and $H_i^*$. If $S \in \mathcal{Z}$ commutes with all $X_i^*$ then $S$ belongs to the center of $\mathcal{G}$ for $\mathcal{G}$ is generated by elements of the form $\exp(tX^*)$. Conversely, suppose that $S$ belongs to the center of $\mathcal{G}$. Then $S$ commutes with all $\exp(tX^*)$. If $\mathfrak{g}$ is one of the algebras $A_n$, $C_n$, $D_n$, then $\exp(tX^*) = I + tX^*$. Hence $S$ commutes with all $X_i^*$. Since $[X_i^*, X_r^*] = H_i^*$, we see that $S$ also commutes with all $H_i^*$. Therefore $S$ belongs to $\mathcal{Z}$. Thus we have proved that $\mathcal{Z}$ coincides with the center of $\mathcal{G}(\mathfrak{g}; K)$ if $\mathfrak{g}$ is one of the algebras $A_n$, $C_n$, $D_n$. Consider now the case $\mathfrak{g} = B_n$. If $S$ belongs to the center of $\mathcal{G}(B_n, K)$ then $S$ commutes with $\exp X_i^*$, and hence with $(\exp X_i^* - I)^2 = (X_i^*)^2$. Therefore, if $K$ is of characteristic $\neq 2$, then $S$ commutes with all $X_i^*$ and hence belongs to $\mathcal{Z}$, as before. If $K$ is of characteristic 2, then from (3.3) we see that $S$ commutes with all $E_{0i} + E_{-ii}$, $i = \pm 1, \pm 2, \cdots, \pm n$. Then it follows easily that $S$ is of the form $aI$, with $a \in K$. Therefore $S$ commutes with all $X_i^*$ and $H_i^*$, and consequently it belongs to $\mathcal{Z}$. Thus we have proved

\begin{equation}
(3.6) \quad \mathcal{G}(\mathfrak{g}, K) \cong \mathcal{G}(\mathfrak{g}, K)/\mathcal{Z},
\end{equation}

where $\mathcal{Z}$ is the center of $\mathcal{G}(\mathfrak{g}, K)$.

4. **The group $\mathcal{G}(A_n, K)$**. It is well known that the matrices of the form $I + tE_{ij}$, where $t \in K$, $i \neq j$, generate the special linear group $\text{SL}(n + 1, K)$. Therefore $\mathcal{G}(A_n, K) = \text{SL}(n + 1, K)$. By (3.6) it follows that $\mathcal{G}(A_n, K)$ is isomorphic to the special projective group $\text{PSL}(n + 1, K)$.

5. **The group $\mathcal{G}(C_n, K)$**. We show that the group $\mathcal{G}(C_n, K)$ coincides with the symplectic group $\text{Sp}(2n, K)$ (in its matrix representation). It is known [6] that the group $\text{Sp}(2n, K)$ is generated by the matrices

\begin{equation}
I + tE_{-ii}, I + t(E_{-ij} - E_{-ji}),
\end{equation}

and

\begin{equation}
S_i = I - E_{ii} - E_{-i,-i} + E_{-ii} - E_{i,-i},
\end{equation}
where \( t \in K, i, j = 1, 2, \cdots, n \), except when \( n = 1 \) and \( K \) is of order 2 or 3. In view of (2.5)–(2.6), the first two sets of matrices belong to \( \mathfrak{O}(C_n, K) \). In order to obtain the equality \( \mathfrak{O} = \text{Sp}(2n, K) \), it thus suffices to show that the matrices \( S_i \) also belong to \( \mathfrak{O} \). As is mentioned in the preceding section, however, the group \( \text{SL}(2, K) \) is generated by matrices of the form \( I + tE_{ij} \), where \( t \in K, i \neq j \). Therefore \( S_i \) can be written as a product of matrices of the forms \( I + tE_{-i,-i} \) and \( I + tE_{-i,-i} \), which belong to \( \mathfrak{O} \) by definition. Except when \( n = 1 \) and \( K \) is of order 2 or 3, therefore we obtain \( \mathfrak{O} = \text{Sp}(2n, K) \). Hence, by (3.6), \( \mathfrak{O}(C_n, K) \) is isomorphic to the quotient of the symplectic group \( \text{Sp}(2n, K) \) over its center. It should be noted that \( \text{Sp}(2n, K) \) is the group of all \( 2n \times 2n \) matrices \( X \) which satisfy
\[
f_c(X\xi, X\eta) = f_c(\xi, \eta) \text{ identically.}
\]

6. The group \( \mathfrak{O}(D_n, K) \). First we shall show that the group \( \mathfrak{O}(D_n, K) \) contains the commutator subgroup \( \mathfrak{O}' \) of the orthogonal group \( \mathfrak{O} \) consisting of all \( 2n \times 2n \) matrices which leave the quadratic form \( \sum_{i=1}^{n} \xi_i \xi_i \) invariant. By (2.2), the group \( \mathfrak{O} \) is generated by the matrices
\[
W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i}),
\]
where \( t \in K \) and \( i, j = \pm 1, \cdots, \pm n; i \neq \pm j \). Denote by \( P_{ij}, Q_i \) and \( R_{i,t} \) the matrices corresponding to the permutations \( (\xi_i, \xi_j)(\xi_{-i}, \xi_{-j}) \) and \( (\xi_{-i}, \xi_{-j}) \) and the transformations \( \xi_i = t\xi_i, \xi_j = t^{-1}\xi_j, \xi_{-i} = t\xi_{-i}, \xi_{-j} = t^{-1}\xi_{-j} \), respectively. It is known [3] that the group \( \mathfrak{O} \) is generated by the matrices \( W_{i,j,t}, P_{ij}, Q_i, \) and \( R_{i,t} \). Therefore, \( \mathfrak{O}' \leq \mathfrak{O} \) is proved if we can show that for any distinct \( |i|, |j|, |k| \) the matrices \( (P_{ik}P_{ij})^2, Q_iQ_j, R_{i,t}R_{j,t}^{-1}, \) and \( R_{i,t}^2 \) are in \( \mathfrak{O} \). By elementary computations we have
\[
(P_{ik}P_{ij})^2 = W_{j,-k,1}W_{j,k,1}W_{j,-k,1}W_{i,j,1}W_{i,j,1}W_{k,i,1}W_{k,i,1},
\]
\[
Q_iQ_j = W_{j,-i,1}W_{i,j,1}W_{j,-i,1}W_{i,j,1}W_{i,j,1}W_{i,j,1},
\]
\[
R_{i,t}R_{j,t}^{-1} = W_{i,-j,1}W_{j,i,1}W_{i,-j,1}W_{j,i,1},
\]
(6.1)
\[
R_{i,t}R_{j,t} = W_{-i,-j,1}W_{i,j,1}W_{-i,-j,1}W_{i,j,1}W_{i,j,1},
\]
where \( s = t^{-1} - 1 \). Thus the matrices \( (P_{ik}P_{ij})^2, Q_iQ_j, R_{i,t}R_{j,t}^{-1} \) are shown to be in \( \mathfrak{O} \). Since \( R_{i,t} \) and \( R_{j,t} \) commute, we have \( R_{i,t}^2 = (R_{i,t}R_{j,t})(R_{i,t}R_{j,t}) \). Hence \( R_{i,t}^2 \) is also in \( \mathfrak{O} \). Therefore \( \mathfrak{O}' \leq \mathfrak{O} \) is proved.

Now we shall show \( \mathfrak{O} \leq \mathfrak{O}' \), assuming that \( 3 \leq n \). For any \( W_{i,j,t} \) take \( k \) such that \( |k| \neq |i|, |j| \). Then we have
\[
W_{i,j,t} = W_{-k,j,-1}W_{i,k,1}W_{-k,j,-1}W_{i,k,1},
\]
which shows that \( \mathfrak{O} \leq \mathfrak{O}' \). Therefore we have \( \mathfrak{O} = \mathfrak{O}' \) provided that \( 3 \leq n \), and hence \( \mathfrak{O}(D_n, K) \) is the commutator group of projective orthogonal group defined by the quadratic form \( \sum_{i=1}^{n} \xi_i \xi_i \).
7. The group \( \mathfrak{S}(B_n, K) \). First we consider the case where \( K \) is not of characteristic 2. We shall show that the group \( \mathfrak{S}(B_n, K) \) is isomorphic to the commutator group \( \mathfrak{S}' \) of the orthogonal group \( \mathfrak{S} \) defined by the quadratic form \( \sum_{i=0}^{n} \xi_i \xi_{-i} \), provided that \( 2 \leq n \). By (2.4), the group \( \mathfrak{S} \) is generated by the matrices

\[
W_{i,j,t} = I + t(E_{i,j} - E_{j,i}), \\
V_{i,t} = I + t(2E_{i0} - E_{0i}) - t^2E_{-i,i},
\]

where \( t \in K, i, j = \pm 1, \cdots, \pm n; i \neq \pm j \). Define the matrices \( P_{ij}, Q_{i}, R_{i,t} \) as in \( \S 6 \) for \( i, j \neq 0 \). It is known [4] that the group \( \mathfrak{S} \) is generated by the matrices \( W_{i,j,t}, V_{i,t}, P_{ij}, Q_{i}, R_{i,t} \). The formula (6.1) shows \( \mathfrak{S}' \leq \mathfrak{S} \).

We shall show \( \mathfrak{S} \leq \mathfrak{S}' \) under the assumption that \( 2 \leq n \). By an elementary computation we have

\[
W_{i,j,2t} = V_{i,t}^{-1}W_{i,j,t}^{-1},
\]

which shows that every \( W_{i,j,t} \) is in \( \mathfrak{S}' \), since \( K \) is not of characteristic 2. Also we have

\[
V_{i,t} = W_{j,t} \phi(W_{j,t,1}^{-1}W_{j,-t}^{-1}W_{-j,t,1}^{-1}),
\]

Hence we see that \( V_{i,t} \) is also in \( \mathfrak{S}' \). Thus \( \mathfrak{S} \leq \mathfrak{S}' \), and hence \( \mathfrak{S} = \mathfrak{S}' \) is proved.

Therefore if \( K \) is not of characteristic 2 then \( \mathfrak{S}(B_n, K) \) is the commutator group of the projective orthogonal group defined by the quadratic form \( \sum_{i=0}^{n} \xi_i \xi_{-i} \).

Consider now the case where \( K \) is of characteristic 2. Denote by \( \mathcal{M} \) a vector space over \( K \) spanned by the indeterminates \( \xi_i, i=0, \pm 1, \cdots, \pm n \), and by \( \mathcal{M}' \) the subspace of \( \mathcal{M} \) spanned by \( \xi_i, i=\pm 1, \cdots, \pm n \). The matrix \( W_{i,j,t} \) is the matrix of the linear transformation of \( \mathcal{M} \):

\[
(7.1) \quad \xi_i' = \xi_i + t\xi_{-j}, \quad \xi_j' = \xi_j + t\xi_{-i}, \quad \xi_k' = \xi_k \quad (k \neq i, j),
\]

while \( V_{i,t} = I + tE_{0i} + t^2E_{-i,i} \) is the matrix of the linear transformation:

\[
(7.2) \quad \xi_0' = \xi_0 + t\xi_i, \quad \xi_{-i} = \xi_{-i} + t^2\xi_i, \quad \xi_i' = \xi_i \quad (k \neq 0, -i).
\]

Therefore the subspace \( \mathcal{M}' \) reduces the group \( \mathfrak{S} \). Moreover, (7.1) and (7.2) shows that the linear transformations in \( \mathfrak{S} \) leave the bilinear form \( f_c(\xi, \eta) = \sum_{i=1}^{n} (\xi_i \eta_{-i} - \xi_{-i} \eta_i) \) invariant. It was noted in \( \S 3 \) that the center of \( \mathfrak{S} \) consists of elements of the form \( aI \) with \( a \in K \). Since \( aI \) leaves \( f_c(\xi, \eta) \) invariant, we have \( a^2=1 \) and hence \( a = 1 \). Then \( \mathfrak{S} = \mathfrak{S} \) and hence \( \mathfrak{S} \) is simple. (We exclude the exceptional cases. See \([2, p. 65]\)). Therefore the representation of \( \mathfrak{S} \) induced in the space \( \mathcal{M}' \) is faithful. Thus we may regard \( \mathfrak{S} \) as a multiplicative group of \( 2n \times 2n \) matrices generated by \( W_{i,j,t} = I + t(E_{i,j} - E_{j,i}) \) and \( V_{i,t} = I + t^2E_{-i,i} \), where \( t \in K \) and \( i, j = \pm 1, \pm 2, \cdots, \pm n \). We have mentioned in \( \S 5 \) that the symplectic group \( \text{Sp}(2n, K) \) is generated by \( W_{i,j,t} \) and \( I + tE_{-i,i} \). Thus the group \( \mathfrak{S}(B_n, K) \), in the case when \( K \) is of characteristic 2,
is a subgroup of $\text{Sp}(2n, K)$. If, however, the field $K$ is perfect then $\Theta = \text{Sp}(2n, K)$.

8. The algebra $G_2$ as a subalgebra of $B_3$. E. Cartan [1, p. 146] gave a representation of $G_2$ as a subalgebra of the algebra $B_3$, which we shall derive here in a form convenient for our purposes. The algebra $B_3$ is defined as the algebra of all $7 \times 7$ matrices $X$ satisfying $f_B(X \xi, \eta) + f_B(\xi, X \eta) = 0$ identically, where

$$f_B(\xi, \eta) = 2\xi_0 \eta_0 + \sum_{i=1}^{3} (\xi_i \eta_i + \xi_i \eta_{-i}).$$

Denote by $E_{ij}$ the $7 \times 7$ matrix such that $\xi' = E_{ij} \xi$ where $\xi'_i = \xi_i$, $\xi'_k = 0$ for $k \neq i$, and set

$$F_{ij} = E_{-ij} - E_{-ji}, \quad F_{i0} = 2E_{-i0} - E_{0i}, \quad F_{0i} = -F_{i0},$$

$$H_{ij} = F_{i,-i} + F_{j,-j}, \quad H_{i0} = 2F_{i,-i}, \quad H_{0i} = 2F_{i,-i}$$

for $i, j = 1, 2, 3$. Then $\{H_r\} = \{H_{ij} | i < j, i \neq -j\}$ is the set of all co-weights of $B_3$ with respect to the Cartan subalgebra $\mathfrak{h}$ spanned by all $H_r$, and $\{X_r\} = \{F_{ij} | i < j, i \neq -j\}$ is a system of root vectors such that if $X_r = F_{ij}$ then $H_r = H_{ij}$ and $X_{-r} = F_{-j,-i}$. It was shown in [5] that there is an automorphism $\phi$ of $B_3$ such that $\phi(X_r) = -X_{-r}$ for all roots $r$ of $B_3$. Cartan showed that the six elements

$$U_{ij} = F_{i,-j}, \quad (0 < i, 0 < j, i \neq j),$$

together with the six elements

$$U_{0i} = F_{i0} + F_{i'i''}, \quad U_{i0} = F_{0,-i} + F_{-i'i''},$$

where $i = 1, 2, 3$ and where $(i' i'')$ is a even permutation of $1, 2, 3$, form a system of root vectors of a subalgebra $\mathfrak{g}$ of $B_3$ isomorphic to $G_2$. The Cartan subalgebra of $\mathfrak{g}$ is given by $\mathfrak{g} \cap \mathfrak{h}$. We have

$$[U_{ij}, U_{jl}] = H_{i,-j},$$

$$[U_{0i}, U_{i0}] = H_{i0} + H_{i'i''},$$

where $i \neq 0, j \neq 0$, since $[F_{i0}, F_{-i'i''}] = 0$, $[F_{0,-i}, F_{i'i''}] = 0$. We have also the identities

$$[H_{i,-j}, U_{ij}] = 2U_{ij}, \quad [H_{i0} + H_{i'i''}, U_{0i}] = 2U_{0i}.$$

Hence we see that the elements on the right hand side of (8.1) are co-weights of the algebra $\mathfrak{g}$. Moreover, it is easily seen that the automorphism $\phi$ of $B_3$ induces an automorphism of the subalgebra $\mathfrak{g}$ such that $\phi(U_{ij}) = -U_{ij}$, $\phi(U_{0i}) = -U_{i0}$. Therefore the system $\{U_{ij}, U_{0i}, U_{i0}\}$ of root vectors can be used to define the group $\Theta(G_2, K)$.
its center of the multiplicative group \( \mathfrak{S}(G_2, K) \) generated by the matrices

\[
\begin{align*}
\exp(tU_{ij}) &= I + tU_{ij}, \\
\exp(tU_{0i}) &= I + tU_{0i} - t^2E_{-ii}, \\
\exp(tU_{i0}) &= I + tU_{i0} - t^2E_{-ii},
\end{align*}
\]

(9.1)

where \( t \in K \) and \( i, j = 1, 2, 3 \).

Dickson [4, §9; 5], guided by the above mentioned representation of the algebra \( G_2 \) given by Cartan, considered the multiplicative group \( \mathfrak{S} \) consisting of all \( 7 \times 7 \) matrices \( X \) with entries in \( K \) which satisfy the conditions (9.2)–(9.4):

(9.2) \( \det X = 1 \);
(9.3) \( X \) leaves the quadratic form

\[
q(\xi) = \xi_0^2 + \xi_1\xi_{-1} + \xi_2\xi_{-2} + \xi_3\xi_{-3}
\]

invariant, that is, \( q(X\xi) = q(\xi) \) identically;
(9.4) \( X \) leaves invariant the system of equations

\[
\xi_0\eta_i - \xi_i\eta_0 + \xi_{-\nu}\eta_{-\nu'} - \xi_{-\nu'}\eta_{-\nu} = 0
\]

where \( (i\ i'\ i'') \) are cyclic even permutations of 1, 2, 3, or of \(-1, -2, -3\), when \( X \) operates cogrediently upon the two sets of variables \( (\xi_i), (\eta_i) \), \(-3 \leq i \leq 3\).

Dickson proved, assuming that \( K \) has more than two elements, that the group \( \mathfrak{S} \) is simple and generated by the matrices in (9.1) and the matrices \( R_i, R_j^{-1}, i \neq j, i \neq 0, j \neq 0, t \in K \), where \( R_i, t \) is the matrix of the transformation

\[
\begin{align*}
\xi_i' &= t\xi_i, \\
\xi_{-i}' &= t^{-1}\xi_{-i}, \\
\xi_k' &= \xi_k \quad (k \neq i, -i).
\end{align*}
\]

In view of the formulas (6.1), the matrix \( R_i, tR_j^{-1} \) can be represented as a product of matrices in (9.1), and hence it follows that the group \( \mathfrak{S} \) is generated by the matrices in (9.1). Therefore we have \( \mathfrak{S} = \mathfrak{S}(G_2, K) \). Since \( \mathfrak{S} \) is simple, we have also \( \mathfrak{S}(G_2, K) = \mathfrak{S}(G_2, K) = \mathfrak{S} \).

**References**


**University of British Columbia,**
**Vancouver, B. C.**