# CLOSED COVERINGS IN ČECH HOMOLOGY THEORY

BY

E. E. FLOYD

1. **Introduction.** We treat here such topics as homology local connectedness, regular convergence, and the Vietoris mapping theorem(1). In general, we are interested in those topics of topology which are based on the technique of chain-realizations. We present an alternative technique, based on one central theorem (Theorem 2.3). One justification for our technique is that it allows the use of compact coefficient groups as well as fields. Moreover, we hope that with further study it will provide a certain amount of unification of the topics treated.

The paper is divided into two parts. The first part consists of §§2, 3, and 4. In §2 we state the basic theorem and some of its corollaries. In §§3 and 4 we give a few applications of our theorem to the topics already noted. In the second part, which is not dependent on the first part, we prove the basic theorem. In §§5 and 6 we develop in detail a Kelley-Pitcher theory of finite closed coverings of compact spaces. This theory was partially developed in the well-known Kelley-Pitcher paper on exact sequences [3, pp. 703–706]. In §7 we use this development to prove the basic theorem. We would be interested in knowing whether or not the basic theorem can be proved by more elementary means.

2. **The basic theorems.** In this paper, a space will always be a Hausdorff space. Suppose that $\alpha=(A^1, \cdots, A^r)$ is an ordered, finite covering of a space $X$. In $r$-space, let $\tilde{A}^i$ be the point whose $j$th coordinate is $\delta^i_j$. Then the nerve of $\alpha$ will be the collection of all simplices $(\tilde{A}^{i_0}, \cdots, \tilde{A}^{i_q})$ with $A^{i_0}\cap \cdots \cap A^{i_q}\neq\emptyset$. If $X$ is compact, $H_n(X)$ will denote the $n$-dimensional Čech homology group with coefficients in a fixed group $\mathfrak{G}$, which may be either a field or a compact abelian group; $H_0(X)$ will denote the reduced 0-dimensional group. For a covering $\alpha$, $H_n(\alpha)$ denotes the homology group of the nerve of $\alpha$.

If $u=(U^1, \cdots, U^r)$ is an open covering of $X$, there is the projection homomorphism $\pi_u: H_n(X)\rightarrow H_n(u)$ which assigns to each element of $H_n(X)$ its $u$-coordinate.

By a *closed covering* $\alpha$ of a compact space $X$, we will always mean an ordered, finite covering $\alpha=(A^1, \cdots, A^r)$ by closed sets in $X$ such that every point of $X$ is in the interior of some $A^i$. Given such an $\alpha$, there exists an open covering $u=(U^1, \cdots, U^r)$ of $X$ such that $U^i\supset A^i$ and $U^{i_0}\cap \cdots \cap U^{i_q}\neq\emptyset$

Presented to the Society, April 20, 1956; received by the editors December 11, 1955.

(1) For bibliographical notes concerning homology local connectedness, see Wilder [6]; for a bibliography on regular convergence, see White [4]; for a modern treatment of the Vietoris mapping theorem, see Begle [1].

if and only if $A^{i_0}\cap \cdots \cap A^{i_q}\neq\emptyset$ (this is a special case of (6.1)). Then $u$ and $\alpha$ have the same nerve and $H_n(\alpha)=H_n(u)$. Define a projection homomorphism $\pi_\alpha:H_n(X)\to H_n(\alpha)$ by $\pi_\alpha=\pi_u$. It may be checked that $\pi_\alpha$ is independent of $u$.

If $\alpha$ and $\beta$ are finite collections of closed subsets of $X$, then $\beta$ *refines* $\alpha$, or $\beta>\alpha$, if and only if given an element $B$ of $\beta$ there exists an element $A$ of $\alpha$ with $B\subset A$. A projection $\pi_{\beta\alpha}$ assigns to each $B$ in $\beta$ such an $A$. The induced homomorphism of $H_n(\beta)$ into $H_n(\alpha)$ will also be denoted by $\pi_{\beta\alpha}$ (or occasionally by just $\pi$). If $\alpha$ and $\beta$ are closed coverings of $X$ with $\beta>\alpha$, it may be checked that $\pi_{\beta\alpha}\pi_\beta=\pi_\alpha$.

If $A$ and $B$ are closed subsets of $X$ with $A\subset B$, we denote by $I_{AB}:H_n(A)\to H_n(B)$, or occasionally by $I$, the injection homomorphism which is induced by inclusion. We denote by $H_n(A;B)$ the image of $I_{AB}$.

(2.1) Definition. If $\alpha$ and $\beta$ are finite collections of closed subsets of $X$ and $n$ is a non-negative integer, we write $\beta^n>\alpha$ ($\beta$ *n-refines* $\alpha$) if and only if given an element $B$ of $\beta$ there exists an element $A$ of $\alpha$ with $B\subset A$ and $H_j(B;A)=0$ for all $j\leqq n$.

We write $\beta^n\gg\alpha$ ($\beta$ *strongly n-refines* $\alpha$) if and only if $\beta>\alpha$ and there exists a projection $\pi_{\beta\alpha}:\beta\to\alpha$ such that $H_j(B^{i_0}\cap \cdots \cap B^{i_q};\pi B^{i_0}\cap \cdots \cap \pi B^{i_q})=0$ for all $B^{i_0},\cdots,B^{i_q}$ in $\beta$ and all $j\leqq n$.

(2.2) *If $\gamma^n>\beta$ and $\beta$ star-refines $\alpha$, then $\gamma^n\gg\alpha$.*

**Proof.** Since $\gamma^n>\beta$, there exists a projection $\pi':\gamma\to\beta$ such that $H_j(C;\pi'C)=0$ for each $C$ in $\gamma$ and each $j\leqq n$. Since $\beta$ star-refines $\alpha$, there is a projection $\pi'':\beta\to\alpha$ such that if $B$ is in $\beta$ then every element of $\beta$ which intersects $B$ is in $\pi''B$. Let $\pi=\pi''\pi'$. Suppose $C^{i_0}\cap \cdots \cap C^{i_q}\neq\emptyset$. Then

$$C^{i_0}\cap \cdots \cap C^{i_q}\subset C^{i_0}\subset \pi'C^{i_0}\subset \pi C^{i_0}\cap \cdots \cap \pi C^{i_q}.$$

Then $H_j(C^{i_0}\cap \cdots \cap C^{i_q};\ \pi C^{i_0}\cap \cdots \cap \pi C^{i_q})=0$ since $H_j(C^{i_0};\ \pi'C^{i_0})=0$, $j\leqq n$.

The following is the basic theorem of the paper; its proof is deferred to §7.

(2.3) Theorem. *Suppose that $A_{-1}\supset A_0\supset \cdots \supset A_n$ is a sequence of closed subsets of a compact space $X$, and that $\alpha_{-1}\ll^n\alpha_0\ll^n \cdots \ll^n\alpha_n$, where $\alpha_i$ is a closed covering of $A_i$. Then*

(i) *the kernel of $\pi_{\alpha_n}:H_j(A_n)\to H_j(\alpha_n)$ is contained in the kernel of the injection $I:H_j(A_n)\to H_j(A_{-1})$ for all $j\leqq n$, and*

(ii) *the image of $\pi_{\alpha_n,\alpha_0}:H_j(\alpha_n)\to H_j(\alpha_0)$ is contained in the image of the projection $\pi_{\alpha_0}:H_j(A_0)\to H_j(\alpha_0)$ for all $j\leqq n+1$.*

We now study the implications of the theorem in case $A_i=X$, all $i$.

(2.4) Definition. Let $\alpha$, $\beta$ be closed coverings of the compact space $X$. We say that $\alpha$, $\beta$ *determine* $H_n(X)$ if and only if $\alpha<\beta$ and $\pi_\alpha:H_n(X)\to H_n(\alpha)$ maps $H_n(X)$ isomorphically onto the image of the projection $\pi_{\beta\alpha}:H_n(\beta)\to H_n(\alpha)$. We say that $\alpha$, $\beta$ *partially determine* $H_n(X)$ if and only if image

$\pi_\alpha = \text{image } \pi_{\beta\alpha}$ (this latter is equivalent to the classical notion that $\beta$ is a normal refinement [7, p. 140] of $\alpha$).

(2.5) *If $\alpha$, $\beta$ (partially) determine $H_n(X)$ and $\gamma > \beta$, then $\alpha$, $\gamma$ (partially) determine $H_n(X)$.*

**Proof.** It is always true that image $\pi_\alpha \subset$ image $\pi_{\gamma\alpha}$. But image $\pi_{\gamma\alpha} \subset$ image $\pi_{\beta\alpha} =$ image $\pi_\alpha$, so image $\pi_\alpha =$ image $\pi_{\gamma\alpha}$.

(2.6) *If $\beta$, $\gamma$ partially determine $H_n(X)$ and if $\beta > \alpha$ then $\alpha$, $\gamma$ partially determine $H_n(X)$.*

**Proof.** We have image $\pi_{\gamma\alpha} = \pi_{\beta\alpha}$ (image $\pi_{\gamma\beta}$) $= \pi_{\beta\alpha}$ (image $\pi_\beta$) $=$ image $\pi_\alpha$.

The following consequence of (2.3) is used as the basis for the rest of the paper.

(2.7) THEOREM. *If $X$ is a compact space and if $\alpha_{-1} \ll^n \alpha_0 \ll^n \cdots \ll^n \alpha_{2n}$, where $\alpha_i$ is a closed covering of $X$, then $\alpha_n$, $\alpha_{2n}$ determine $H_j(X)$, for all $j \leqq n$, and partially determine $H_{n+1}(X)$.*

**Proof.** In (2.3 i), let $A_i = X$. Hence according to (2.3 i) the kernel of $\pi_{\alpha_n}$ is contained in the kernel of the identity map of $H_j(X)$, $j \leqq n$, and hence $\pi_{\alpha_n}$ is an isomorphism into for $j \leqq n$. In (2.3 ii), set $A_i = X$ and consider the coverings $\alpha_n, \cdots, \alpha_{2n}$. In (2.3 ii), the image of $\pi_{\alpha_{2n},\alpha_n}: H_j(\alpha_{2n}) \to H_j(\alpha_n)$ is contained in the image of $\pi_{\alpha_n}$ for $j \leqq n+1$. Since the opposite inclusion always holds, the two images are equal. Hence $\alpha_n$, $\alpha_{2n}$ partially determine $H_j(X)$ for $j \leqq n+1$. The theorem follows.

3. **Locally connected space; the Vietoris mapping theorem.** We consider here a few properties of locally connected spaces; these are known for the case when the coefficient group is a field [7, Chap. 6]. On the basis of these, we give a new proof of the Vietoris mapping theorem as given by Begle [1].

(3.1) DEFINITION. A compact space $X$ is said to be $lc^n$, $n$ a non-negative integer, if and only if given $x \in X$ and a closed neighborhood $U$ of $x$, there exists a closed neighborhood $V$ of $x$ with $H_j(V; U) = 0$, all $j \leqq n$.

(3.2) *If $X$ is an $lc^n$, compact space, then given a closed covering $\alpha$ of $X$, there exists a closed covering $\beta$ of $X$ with $\beta^n \gg \alpha$.*

**Proof.** Suppose that $\gamma$ is a closed covering which star-refines $\alpha$. For each $x \in X$, there is an element $C$ of $\gamma$ which contains $x$ in its interior. Hence there is a closed neighborhood $V_x$ of $x$ with $H_j(V_x; C) = 0$ for $j \leqq n$. Let $\beta$ be a finite collection of the $V_x$ whose interiors cover $X$. Then $\beta^n > \gamma$ and $\gamma$ star-refines $\alpha$. By (2.2), $\beta^n \gg \alpha$.

The following is equivalent to a classical theorem [7, p. 180] when the coefficient group is a field.

(3.3) THEOREM. *Suppose that $X$ is a compact $lc^n$ space. For each sufficiently small covering $\alpha$ of $X$, there exists a closed covering $\beta > \alpha$ such that $\alpha$, $\beta$ determine $H_j(X)$, all $j \leqq n$, and partially determine $H_{n+1}(X)$.*

**Proof.** Let $\alpha_{-1}$ be a closed covering of $X$. By (3.2), there are closed coverings $\alpha_0, \cdots, \alpha_{n-1}$ with $\alpha_{-1} \ll^n \cdots \ll^n \alpha_{n-1}$. By (3.2), for all sufficiently small $\alpha_n$ we have $\alpha_n{}^n \gg \alpha_{n-1}$. Given such an $\alpha_n$, we get a sequence $\alpha_{-1} \ll^n \cdots \ll^n \alpha_{2n}$. By (2.7), $\alpha_n$, $\alpha_{2n}$ determine $H_j(X)$, $j \leqq n$, and partially determine $H_{n+1}(X)$.

(3.4) Definition. We shall say that the coefficient group $G$ is *elementary* if it is either a field or an elementary compact group(²). In case $G$ is elementary, we say that $H_n(X)$, or a subgroup thereof, is elementary if it is a finite dimensional vector space when $G$ is a field, or an elementary compact group when $G$ is an elementary compact group.

Condition (b) of the following theorem is similar to property $(P, Q)_n$ of Wilder [7, p. 193].

(3.5) Theorem. *If the coefficient group is elementary, then the following are equivalent for a compact space $X$:*

(a) *$X$ is $lc^n$;*

(b) *if $A$ and $B$ are closed subsets of $X$ with $A$ in the interior of $B$, then $H_j(A; B)$ is elementary for $j \leqq n$;*

(c) *if $A$ is a closed subset of $X$ and $U$ is a closed neighborhood of $A$, then there is a closed neighborhood $V$ of $A$ with $H_j(V; U) = H_j(A; U)$ for all $j \leqq n$.*

**Proof.** To show that (a) implies (b), suppose (a) holds and that $A$ is contained in the interior of $B$ (written $A \subset\subset B$). Let $A_{-1} = B$. We may find a sequence $B = A_{-1}, A_0, \cdots, A_n = A$ such that $A_{i+1} \subset\subset A_i$. Let $\alpha_{-1}$ be a closed covering of $A_{-1}$. There exists a closed covering $\beta$ of $A_{-1}$ which star-refines $\alpha_{-1}$. Since $A_0$ is in the interior of $A_{-1}$ and $X$ is $lc^n$, there is a closed covering $\alpha_0$ of $A_0$ with $\alpha_0{}^n > \beta$. By (2.2), $\alpha_0{}^n \gg \alpha_{-1}$. Similarly there exist closed coverings $\alpha_i$ of $A_i$ with $\alpha_{-1} \ll^n \alpha_0 \ll^n \cdots \ll^n \alpha_n$. According to (2.3 i), the kernel $K$ of $\pi_{\alpha_n}: H_j(A) \to H_j(\alpha_n)$ is contained in the kernel $K'$ of $I_{AB}: H_j(A) \to H_j(B)$ for $j \leqq n$. Now $H_j(\alpha_n)$ is elementary, together with its subgroups and factor groups. Hence $H_j(A)/K \approx$ image $\pi_{\alpha_n}$ is elementary. Then

$$H_j(A)/K' \approx (H_j(A)/K)/(K'/K)$$

is elementary, being a factor group of an elementary group. But $H_j(A; B) \approx H_j(A)/K'$; hence (a) implies (b).

Suppose now that (b) holds and that $A \subset\subset U$. For each closed $V$ with $A \subset\subset V \subset U$ let $K(V) = \bigcap H_j(V'; V)$ where the intersection is taken over all $V'$ with $A \subset\subset V' \subset\subset V$. Since each $H_j(V'; V)$ is elementary, there is a $V'$ with $K(V) = H_j(V'; V)$. We note that $I_{VU}$ maps $K(V)$ onto $K(U)$. For if $x \in K(U)$, then $x \in H_j(V'; U)$ and there is a $y \in H_j(V')$ with $I_{V'U}(y) = x$. Then $I_{VU}(I_{V'V}(y)) = x$, and $I_{V'V}(y) \in K(V)$. Hence the $K(V)$, together with the $I_{V'V}$, constitute an inverse mapping system of elementary groups, and $I_{VU}$ maps $K(V)$ onto $K(U)$. Hence, given $x \in K(U)$, there is a function assigning to each $V$ an $x(V) \in K(V)$ with $x(U) = x$ and $I_{V'V}(x(V')) = x(V)$. Hence there

(²) An elementary compact group is defined to the direct sum of a finite number of groups, each of which is the reals mod 1 or a finite cyclic group.

is a $y \in H_j(\bigcap V) = H_j(A)$ with $I_{AV}(y) = x(V)$. In particular, $I_{AU}(y) = x$. Then $H_j(A; U) = \bigcap H_j(V; U)$. But each $H_j(V; U)$ is elementary, so that for some $V$, $H_j(A; U) = H_j(V; U)$. It follows that (b) implies (c).

It is easy to see that (c) implies (a), by taking for $A$ an arbitrary one-point set.

(3.6) Vietoris mapping theorem. *Suppose that $f$ is a continuous map of a compact space $X$ onto a compact space $Y$ such that $H_j(f^{-1}(y)) = 0$ for all $j \leqq n$ and $y \in Y$, where the coefficient group is elementary. Then $f_*: H_j(X) \to H_j(Y)$ is an isomorphism onto for $j \leqq n$, and is onto for $j = n+1$.*

**Proof.** *Case* I: $X$ is $lc^{n+1}$. Let $\beta_{-1}$ be an arbitrary closed covering of $Y$. Suppose that $\gamma$ is a star-refinement of $\beta_{-1}$; then $f^{-1}(\gamma)$ is a star-refinement of $f^{-1}(\beta_{-1})$. Suppose $y \in Y$. Then $y$ is interior to some element $C$ of $\gamma$, and $f^{-1}(y)$ is interior to $f^{-1}(C)$. Since $X$ is $lc^n$ and $H_j(f^{-1}(y)) = 0$ for $j \leqq n$, there exists by (3.5) a $V$ with $f^{-1}(y) \subset\subset V \subset f^{-1}(C)$ and $H_j(V; f^{-1}(C)) = 0$, $j \leqq n$. There exists a closed neighborhood $B_y$ of $y$ with $f^{-1}(B_y) \subset V$. Let $\beta_0$ be a finite subcollection of $\{B_y\}$ whose interiors cover $Y$. Then $f^{-1}(\beta_0)^n > f^{-1}(\gamma)$ and $f^{-1}(\gamma)$ star-refines $f^{-1}(\beta_{-1})$; by (2.2), $f^{-1}(\beta_0)^n \gg f^{-1}(\beta_{-1})$. In a similar manner we obtain a sequence $\beta_{-1} < \beta_0 < \cdots < \beta_{2n}$ of closed coverings of $Y$ such that if $\alpha_i = f^{-1}(\beta_i)$ then $\alpha_{-1} \ll^n \cdots \ll^n \alpha_{2n}$. Then by (2.7) $\alpha_n$, $\alpha_{2n}$ determine $H_j(X)$, $j \leqq n$, and partially determine $H_{n+1}(X)$. Since $\beta_n$ can be made arbitrarily small, *for every sufficiently small covering $\beta$ of $Y$ there exists a refinement $\beta'$ of $\beta$ such that the coverings $\alpha = f^{-1}(\beta)$, $\alpha' = f^{-1}(\beta')$ determine $H_j(X)$, $j \leqq n$, and partially determine $H_{n+1}(X)$.*

Consider the diagram

$$\begin{tikzcd}
H_j(X) \arrow[dr, "\pi_\alpha"'] \arrow[drr, "\pi_{\alpha'}"] \arrow[ddd, "f_*"'] & & \\
& H_j(\alpha) \arrow[d, equal] & H_j(\alpha') \arrow[d, equal] \\
& H_j(\beta) & H_j(\beta') \arrow[l, "\pi_{\beta'\beta}"'] \\
H_j(Y) \arrow[ur, "\pi_\beta"] \arrow[urr, "\pi_{\beta'}"'] & &
\end{tikzcd}$$

where $\alpha$, $\beta$, $\alpha'$, $\beta'$ are as above. There is commutativity: $\pi_\beta f_* = \pi_\alpha$, $\pi_{\beta'} f_* = \pi_{\alpha'}$, etc.

Note first that $f_*$ is an isomorphism into for $j \leqq n$. For $\pi_\alpha = \pi_\beta f_*$ and $\pi_\alpha$ is an isomorphism into; hence $f_*$ is an isomorphism into.

By (3.3), $H_j(X)$ is elementary for $j \leqq n+1$; hence $K = \text{image } f_*$ is elementary. Moreover, image $\pi_\beta f_* = \text{image } \pi_\beta$ for $j \leqq n+1$. For if $y \in H_j(Y)$ then

$$\pi_\beta(y) = \pi_{\beta'\beta}(\pi_{\beta'}(y)) = \pi_{\alpha'\alpha}(\pi_{\beta'}(y)).$$

Since $\alpha$, $\alpha'$ partially determine $H_j(X)$, there is an $x \in H_j(X)$ with $\pi_\alpha(x) = \pi_\beta(y)$. Then $\pi_\beta f_*(x) = \pi_\beta(y)$. *Hence*

$$\pi_\beta(K) = \pi_\beta(H_j(Y)) \qquad \textit{for all sufficiently small } \beta.$$

Suppose now that $y \in H_j(Y)$. $K-y$ denotes the set $\{k-y : k \in K\}$. For each $\beta$ there is a $k \in K$ with $\pi_\beta(k) = \pi_\beta(y)$, so that $k-y \in L_\beta = (\text{kernel } \pi_\beta) \cap (K-y)$. Now $K-y$ is a translate of $K$, which is elementary. The $L_\beta$ are nonempty and decrease with $\beta$. Hence $\bigcap_\beta L_\beta \neq \emptyset$. If $z \in \bigcap_\beta L_\beta$, then $\pi_\beta(z) = 0$ for all $\beta$, so that $z = 0$. Hence $0 \in K-y$ and $y \in K$. Hence $f_*$ maps $H_j(X)$ onto $H_j(Y)$, $j \leqq n+1$.

*Case* II: $X$ a compact space. We may consider $X$ as embedded in an $lc^{n+1}$ compact space $X'$ (for example, $X'$ a product of intervals). The map $f$ generates a decomposition of $X$ whose elements are the sets $f^{-1}(y)$, $y \in Y$. Extend this decomposition to a decomposition of $X'$ by admitting as elements the one-point sets of $X'-X$. This upper semi-continuous decomposition of $X'$ has a decomposition space $Y'$ and a decomposition map $F: X' \to Y'$. We may identify $Y$ with $F(X)$ and $f$ with $F|X$. We have the diagram

$$\begin{array}{ccccccccc}
H_{j+1}(X') & \to & H_{j+1}(X', X) & \to & H_j(X) & \to & H_j(X') & \to & H_j(X', X) \\
\downarrow F_{*_1} & & \downarrow F_*' & & \downarrow f_* & & \downarrow F_{*_2} & & \downarrow F_*' \\
H_{j+1}(Y') & \to & H_{j+1}(Y', Y) & \to & H_j(Y) & \to & H_j(Y') & \to & H_j(Y', Y)
\end{array}$$

where the downward homomorphisms are induced by $F$. Now both the $F_*'$ are isomorphisms onto, since $F$ maps $X'-X$ homeomorphically onto $Y'-Y$. For $j \leqq n$, $F_{*2}$ is an isomorphism onto by Case I, and $F_{*1}$ is onto. By the five-lemma of Eilenberg-Steenrod [2, p. 16], $f_*$ is an isomorphism onto. For $j = n+1$, $F_{*2}$ is onto; by the five-lemma, $f_*$ is then onto. The theorem follows.

4. **Regular convergence.** In this section, we prove some theorems concerning regular convergence. For the known facts where the coefficient group is a field, see [5].

(4.1) Definition. A sequence $(A_i)$ of closed subsets of a compact space $X$ is said to converge *n-regularly* to the subset $A$ of $X$ if and only if $(A_i)$ converges to $A$, and given $x \in A$ and a closed neighborhood $U$ of $x$ there exist a closed neighborhood $V$ of $x$ and a positive integer $N$ such that $H_j(V \cap A_i; U \cap A_i) = 0$ for $j \leqq n$ and $i \geqq N$.

If $\beta = (B^1, \cdots, B^r)$ is a covering of $X$ and $A \subset X$ then $\beta \cap A$ denotes the covering $(B^1 \cap A, \cdots, B^r \cap A)$ of $A$.

(4.2) *If the sequence $(A_i)$ converges n-regularly to $A$, then given a closed covering $\alpha$ of $X$ there exist a closed covering $\beta$ of $X$ and a positive integer $N$ such that $\alpha \cap A_i \ll^n \beta \cap A_i$ for all $i \geqq N$.*

**Proof.** Let $\gamma$ be a closed covering of $X$ such that $\gamma$ star-refines $\alpha$. For each

$x \in X$ there is a $C$ in $\gamma$ with $x$ in its interior. Hence there is a closed neighborhood $B_x$ of $x$ and a positive integer $N_x$ such that $H_j(B_x \cap A_i;\ C \cap A_i) = 0$ for $j \leqq n$ and $i \geqq N_x$. Let $\beta$ be a finite subcollection of $B_x$ whose interiors cover $X$. Then $\beta \cap A_i \overset{n}{>} \gamma \cap A_i$ for $i$ sufficiently large. Since $\gamma \cap A_i$ star-refines $\alpha \cap A_i$, then $\beta \cap A_i \overset{n}{\gg} \alpha \cap A_i$ for $i$ sufficiently large.

(4.3) *If $(A_i)$ converges $n$-regularly to $A$, then for each sufficiently small closed covering $\alpha$ of $X$, there exists a closed covering $\beta > \alpha$ and a positive integer $N$ such that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_j(A_i)$ for $j \leqq n$ and $i \geqq N$ and partially determine $H_{n+1}(A_i)$ for $i \geqq N$.*

**Proof.** The proof follows easily from (4.2) and (2.7).

(4.4) Definition. Suppose that $A$ is a closed subset of the compact space $X$, and that $\alpha$ is a closed covering of $X$. We say that $\alpha$ is in *general position relative to $A$* if and only if whenever $A^{i_0}, \cdots, A^{i_q}$ are elements of $\alpha$ with $A^{i_0} \cap \cdots \cap A^{i_q} \cap A \neq \emptyset$ then $\operatorname{int} A^{i_0} \cap \cdots \cap \operatorname{int} A^{i_q} \cap A \neq \emptyset$.

(4.5) *If $A$ is a closed subset of the compact space $X$ and $\alpha$ is a closed covering of $X$, there is a refinement $\beta$ of $\alpha$ in general position relative to $A$.*

**Proof.** Suppose $\alpha = (A^1, \cdots, A^r)$. For each $i_0, \cdots, i_q$, select a point $x_{i_0, \cdots, i_q}$ of $\operatorname{int} A^{i_0} \cap \cdots \cap \operatorname{int} A^{i_q} \cap A$, if such a point exists. Let $\beta = (B^1, \cdots, B^r)$ be a closed covering of $X$ such that $B^i$ is in the interior of $A^i$ and such that $B^i$ contains (in its interior) all the $x_{i_0, \cdots, i_q}$ which belong to $\operatorname{int} A^i$. Then $\beta$ is the desired covering.

It will be noted that if $(A^i)$ converges to $A$, and $\alpha$ is general position relative to $A$, then the nerves of $\alpha \cap A$ and $\alpha \cap A_i$ coincide for $i$ sufficiently large.

(4.6) Theorem. *Suppose that the sequence $(A_i)$ of closed subsets of the compact space $X$ converges to $A$, and that for each sufficiently small closed covering $\alpha$ of $X$ there exists a closed covering $\beta$ of $X$ and a positive integer $N$ such that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$ for $i \geqq N$. Then for each $\alpha$, there is a $\beta$ such that $\alpha \cap A$, $\beta \cap A$ determine $H_n(A)$.*

**Proof.** We consider closed coverings $\alpha < \beta < \gamma$ which are in general position relative to $A$. We use the diagram

$$\begin{tikzcd}
H_n(A) \arrow[r, "\pi_3"] \arrow[dr, "\pi_2"] \arrow[ddr, "\pi_1"'] & H_n(\gamma \cap A) & = \arrow[d, "\pi_2''"] & H_n(\gamma \cap A_i) & H_n(A_i) \arrow[l, "\pi_3'"'] \arrow[dl, "\pi_2'"'] \arrow[ddl, "\pi_1'"] \\
 & H_n(\beta \cap A) & = \arrow[d, "\pi_1''"] & H_n(\beta \cap A_i) & \\
 & H_n(\alpha \cap A) & = & H_n(\alpha \cap A_i) &
\end{tikzcd}$$

where all the $\pi$'s are projections and where $i$ is always large enough so that the indicated equalities hold.

Suppose that $x \in H_n(A)$, $x \neq 0$, and $\pi_1(x) = 0$. Suppose $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$, $i \geqq N$, that $\pi_2(x) \neq 0$, and that $\beta \cap A_i$, $\gamma \cap A_i$ determine $H_n(A_i)$, $i \geqq N$. Since $\pi_3(x) = \pi_2'' \pi_2(x)$, there is a $y \in H_n(A_i)$ with $\pi_2'(y) = \pi_2(x)$. Then $y \neq 0$ since $\pi_2'(y) \neq 0$. Then also $\pi_1'(y) \neq 0$, since $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$. But $\pi_1'(y) = \pi_1(x)$. This is a contradiction. Hence $\pi_1$ is an isomorphism into.

Suppose that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$, $i \geqq N$. We show that image $\pi_1 =$ image $\pi_1''$; we know that image $\pi_1 \subset$ image $\pi_1''$. Let $y \in H_n(\beta \cap A)$. There is an $x_i \in H_n(A_i)$ with $\pi_1'(x) = \pi_1''(y)$. For each closed covering $\gamma$ of $X$, $\gamma$ in general position relative to $A$, with $\gamma > \beta$, $\pi_1'' \pi_2''(\pi_3'(x_i)) = \pi_1'(x_i) = \pi_1''(y)$. Hence for each such $\gamma$ there is a $z_\gamma$ $(= \pi_3'(x_i))$ in $H_n(\gamma \cap A)$ with $\pi_1'' \pi_2''(z_\gamma) = \pi_1''(y)$. Hence there is an $x \in H_n(A)$ whose $(\alpha \cap A)$-coordinate is $\pi_1'(y)$. Then $\pi_1(x) = \pi_1''(y)$. Hence *if $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$, $i \geqq N$, and $\alpha$, $\beta$ are in general position relative to $A$ then $\alpha \cap A$, $\beta \cap A$ determine $H_n(A)$.*

(4.7) THEOREM. *Suppose that the sequence $(A_i)$ of closed subsets of the compact space $X$ converges n-regularly to the subset $A$. Then $H_n(A) \approx H_n(A_i)$ for $i$ sufficiently large.*

**Proof.** There exist, by (4.5) and (4.3), closed coverings $\alpha, \beta$ of $X$, in general position relative to $A$, such that $\alpha \cap A_i$, $\beta \cap A_i$ determine $H_n(A_i)$. By (4.6), $\alpha \cap A$, $\beta \cap A$ determine $H_n(A)$. Hence $H_n(A_i)$ and $H_n(A)$, for $i$ sufficiently large, are isomorphic to the image of the projection $\pi: H_n(\beta \cap A) \to H_n(\alpha \cap A)$, and are isomorphic to each other.

5. **The Kelley-Pitcher theory for simplicial pairs.** In §§5 and 6 we continue an investigation of Kelley and Pitcher [3, pp. 703–706] seeking relationships between the groups of a space, the groups of the nerve of a covering of the space, and the groups of intersections of elements of the covering. The treatment is self-contained; the portion of this section through (5.5) is due to Kelley-Pitcher.

By a *simplicial pair* we mean a pair $(X, \alpha)$ consisting of a finite simplicial complex $X$ and a covering $\alpha = (A^1, \cdots, A^r)$ by subcomplexes. $X_\alpha$ will denote the nerve of $\alpha$. However, $C_p(\alpha)$ and $H_p(\alpha)$ will indicate the chain group and homology group of the nerve $X_\alpha$. If a simplex of $X_\alpha$ is of the form $T_q = (\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ then define $\bigcap T_q = A^{i_0} \cap \cdots \cap A^{i_q}$. Consider $X_\alpha$ as having a $(-1)$-dimensional simplex $T_{-1}$ with $\bigcap T_{-1} = X$.

If $S_p$ is a $p$-simplex of $X$ let $\Lambda S_p$ denote the subcomplex of $X_\alpha$ consisting of all $T_q$ with $S_p \subset \bigcap T_q$. If $A^{i_0}, \cdots, A^{i_q}$ are all the elements of $\alpha$ which contain $S_p$, $p \geqq 0$, then $\Lambda S_p$ consists of all faces of the simplex $(\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$. If $S_{-1}$ is the $(-1)$-dimensional simplex of $X$, then $\Lambda S_{-1} = X_\alpha$. The boundary operator will be denoted by $\partial$.

*Denote by $C_{p,q}(\alpha)$ the set of all linear forms $\sum g S_p \cdot T_q$ where $\mathfrak{g} \in \mathfrak{G}$ and $S_p$, $T_q$ are oriented simplices of $X$, $X_\alpha$ respectively* with $S_p \subset \bigcap T_q$. Agree that $\mathfrak{g} S_p \cdot (-T_q) = \mathfrak{g}(-S_p) \cdot T_q = (-\mathfrak{g}) S_p \cdot T_q$ and that forms are to be added as usual. If $A_p = \sum g_i S_p^i$ is a $p$-chain of $\cap T_q$, define $A_p \cdot T_q = \sum g_i S_p^i \cdot T_q$. If $B_q = \sum g_i T_q^i$ is a $q$-chain of $\wedge S_p$, define $S_p \cdot B_q = \sum g_i S_p \cdot T_q^i$.

The group $C_{p,-1}(\alpha)$ will be identified with the chain group $C_p(X)$ under the identification $A_p \cdot T_{-1} \leftrightarrow A_p$. The group $C_{-1,q}(\alpha)$ will be identified with the chain group $C_q(\alpha)$ under the identification $S_{-1} \cdot B_q \leftrightarrow B_q$.

*A homomorphism* $\Delta: C_{p,q} \to C_{p-1,q}$ *is defined by* $\Delta(A_p \cdot S_q) = (\partial A_p) \cdot S_q$ *and linearity. A homomorphism* $D: C_{p,q} \to C_{p,q-1}$ *is defined by* $D(S_p \cdot B_q) = S_p \cdot (\partial B_q)$ *and linearity.*

(5.1) *If* $p \geqq 0$ *the sequence*

$$\cdots \to C_{p,q+1} \xrightarrow{D} C_{p,q} \xrightarrow{D} C_{p,q-1} \to \cdots$$

*is exact. If* $p = -1$, *the sequence coincides with*

$$\cdots \to C_{q+1}(\alpha) \xrightarrow{\partial} C_q(\alpha) \xrightarrow{\partial} C_{q-1}(\alpha) \to \cdots.$$

**Proof.** It is clear that $DD = 0$ since $\partial\partial = 0$. Suppose $p \geqq 0$ and that $D(\sum S_p^i \cdot B_q^i) = \sum S_p^i \cdot (\partial B_q^i) = 0$, where the $S_p^i$ form a basis for $C_p(X)$. Then $\partial B_q^i = 0$. Since $B_q^i$ is on the simplex $\wedge S_p^i$, then $B_q^i = \partial C_q^i$ for some $C_q^i$ in $\wedge S_p^i$. Then $D(\sum S_p^i \cdot C_q^i) = \sum S_p^i \cdot B_q^i$, and the sequence is exact for $p \geqq 0$. The statement for $p = -1$ can be easily seen.

(5.2) *The sequence*

$$\cdots \to C_{p+1,q} \xrightarrow{\Delta} C_{p,q} \xrightarrow{\Delta} \cdots$$

*has* $\Delta\Delta = 0$. *The homology group* $H_{p,q}(\alpha)$ *of the sequence corresponding to the indices* $p$, $q$ *is isomorphic to the direct sum* $\sum H_p(\cap T_q)$, *where the summation is extended over all* $q$-*simplices* $T_q = (\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ *with* $i_0 < \cdots < i_q$.

**Proof.** It is clear that $\Delta\Delta = 0$. If $q = -1$, $C_{p,-1} = C_p(X)$ and $\Delta = \partial$ on $C_p(X)$. Hence the homology group, if $q = -1$, is $H_p(X) = H_{p,-1}(\alpha)$. Suppose $q \geqq 0$. Elements $z$ of $C_{p,q}$ are uniquely represented as $z = \sum A_p^i \cdot T_q^i$, where the $T_q^i$ are the oriented $q$-simplices of the form $(\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ with $i_0 < \cdots < i_q$ and where $A_p^i \in C_p(\cap T_q^i)$. This representation sets up an isomorphism of $C_{p,q}$ with $\sum C_p(\cap S_q)$ and the statement follows easily.

*Define* $E_{p,q}$ *to be the kernel of* $D: C_{p,q} \to C_{p,q-1}$. For $p \geqq 0$, it follows from (5.1) that $E_{p,q}$ is the image of $D: C_{p,q+1} \to C_{p,q}$. We have the sequence

$$\cdots \to E_{p+1,q} \xrightarrow{\Delta} E_{p,q} \xrightarrow{\Delta} \cdots.$$

*Define* $K_{p,q}$ *to be the homology group of this sequence corresponding to indices* $p$, $q$.

(5.3) *We have* $K_{p,-1} = H_p(X)$.

**Proof.** $K_{p,-1}$ is the homology group of

$$\cdots \to E_{p+1,-1} \xrightarrow{\Delta} E_{p,-1} \xrightarrow{\Delta} E_{p-1,-1} \to \cdots.$$

Now $D: C_{p,-1} \to C_{p,-2} = 0$ so that $E_{p,-1} = C_{p,-1} = C_p(X)$. Moreover, $\Delta = \partial$ on $C_p(X)$. The result follows.

(5.4) *We have* $K_{-1,q} = H_q(\alpha)$.

**Proof.** We have $C_{-1,q} = C_q(\alpha)$. Also $D: C_{-1,q} \to C_{-1,q-1}$ coincides with $\partial: C_q(\alpha) \to C_{q-1}(\alpha)$. Hence $E_{-1,q}$ is the cycle group $Z_q(\alpha)$. Now $E_{0,q}$, is the image of $D: C_{0,q+1} \to C_{0,q}$. Hence the image of $\Delta: E_{0,q} \to E_{-1,q}$ is also the image of $\Delta D: C_{0,q+1} \to C_{-1,q}$. But $\Delta D(\sum A_0^i \cdot T_{q+1}^i) = \sum (\partial A_0^i) \cdot (\partial T_{q+1}^i)$ where $\partial A_0^i$ is of the form $\mathfrak{g}_i S_{-1}$ with $\mathfrak{g}_i$ the coefficient sum of $A_0^i$. It follows that the image of $\Delta: E_{0,q} \to E_{-1,q}$ is the bounding cycle group $B_q(\alpha)$. The result follows.

For each $p \geqq 0$, the sequence

$$0 \to E_{p,q} \xrightarrow{l'} C_{p,q} \xrightarrow{D} E_{p,q-1} \to 0$$

is exact, where $l'$ is inclusion. If $p = -1$ the sequence is exact except that $D$ may not be onto. Taking homology groups and using a well-known method of constructing exact sequences [3, p. 687] we get the exact sequence

$$\cdots \to K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q} \to \cdots$$

$\to K_{0,q-1} \to K_{-1,q} \to H_{-1,q}$, where $l$ is induced by inclusion $l'$, $m$ by $D$, and $\eta^{(p)}$ by the boundary operation. Since $H_{-1,q} = 0$, we get

(5.5) *The sequence*

$$\cdots \to K_{p,q} \xrightarrow{l} H_{p,q} \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q} \to \cdots \to K_{0,q-1} \to K_{-1,p} \to 0$$

*is exact.*

The map $\eta^{(p)}: K_{p,q-1} \to K_{p-1,q}$, $p \geqq 0$, is of particular interest. Suppose $z$ represents an element $x$ of $K_{p,q-1}$. Then $z = D(z')$ where $z' \in C_{p,q}$. Then $\Delta z'$ represents $\eta^{(p)}(x)$.

$$\text{In } H_p(X) = K_{p,-1} \xrightarrow{\eta^{(p)}} K_{p-1,0} \xrightarrow{\eta^{(p-1)}} K_{p-1,2} \to \cdots \to K_{0,p-1} \xrightarrow{\eta^{(0)}} K_{-1,p}$$
$$= H_p(\alpha), \text{ define } \eta = \eta^{(0)} \cdots \eta^{(p-1)}\eta^{(p)}.$$

(5.6) THEOREM. *Suppose* $(X, \alpha)$ *is a simplicial pair such that if* $S$ *is a simplex of* $X$ *there is a vertex* $v$ *of* $S$ *such that if* $v \in A^i$ *then* $S \subset A^i$. *Any function* $f$ *assigning to each vertex* $v$ *of* $X$ *a vertex* $\tilde{A}^i$ *of* $X_\alpha$ *with* $v \in A^i$ *is a simplicial map of* $X$ *into* $X_\alpha$. *The homomorphism* $f_*: H_p(X) \to H_p(\alpha)$ *is given by* $f_* = \epsilon\eta$, *where* $\epsilon = (-1)^{p(p+1)/2}$.

**Proof.** For a vertex $v$ of $X$, $\Lambda v$ denotes the simplex of $X_\alpha$ whose vertices are those $\tilde{A}^i$ with $v \in A^i$. Suppose $v \sim v'$ if and only if $\Lambda v = \Lambda v'$. This is an equivalence relation. Suppose the elements of each equivalence class are simply ordered by an ordering $>$. Suppose also that $v > v'$ whenever $\Lambda v$ contains $\Lambda v'$ as a proper face. We have then a partial ordering $>$ on the vertices of $X$

such that if $v>v'$ then $\Lambda v\supset\Lambda v'$, if $\Lambda v=\Lambda v'$ either $v>v'$ or $v'>v$, if $\Lambda v$ contains $\Lambda v'$ properly then $v>v'$.

Suppose $S$ is a simplex of $X$. There is a vertex $v$ such that if $v'$ is any other vertex of $S$ then $\Lambda v\subset\Lambda v'$. It follows that there is a vertex $v_0$ of $S$ (with $\Lambda v_0=\Lambda v$) such that $v_0<v'$ for all vertices $v'$ of $S$. It then follows that the vertices of $S$ may be ordered uniquely as $v_0<v_1<\cdots<v_p$. In this proof, all oriented simplices $S=(v_0\cdots v_p)$ will have $v_0<\cdots<v_p$. If $v_0\in A^i$ then $S\subset A^i$. Also if $v_i\in A^j$ then $v_p\in A^j$.

Consider $f$ as in the statement of the theorem. If $S=(v_0\cdots v_p)$ then $f(v_j)$ is a vertex $\tilde{A}^{i_j}$ with $v_j\in A^{i_0}$. Then $v_p\in A^{i_0}\cap\cdots\cap A^{i_p}$ and $f$ is simplicial.

A vertex of $X_\alpha$ will be denoted by a symbol of the form $w_j$. Define a homomorphism $\mathfrak{J}:C_{p,q}\to C_{p,q+1}$, $p\geqq 0$, by

$$g(v_0\cdots v_p)\cdot(w_0\cdots w_q)\to g(v_0\cdots v_p)\cdot(f(v_0)w_0\cdots w_q)$$

and linearity. If $S=(v_0\cdots v_p)$ then $(w_0\cdots w_q)\subset\Lambda S$ and $f(v_0)\epsilon\Lambda S$. Since $\Lambda S$ is a simplex, the definition has meaning.

It may be verified that $D\mathfrak{J}+\mathfrak{J}D=$identity. Hence if $D(z)=0$ then $D(T(z))=z$. Hence (*) *if $z$ represents an element $x$ of $K_{p,q}$, $p\geqq 0$, then $\Delta T(z)$ represents $\eta^{(p)}(x)$ in $K_{p-1,q+1}$.*

Let $z=\sum\mathfrak{g}_i(v_{0i}\cdots v_{pi})$ represent an element $x$ of $K_{p,-1}=H_p(X)$. By (*), $\eta^{(p)}(x)$ in $K_{p-1,0}$ is represented by

$$\sum\mathfrak{g}_i\partial(v_{0i}\cdots v_{pi})\cdot f(v_{0i})$$
$$=\sum\mathfrak{g}_i(v_{1i}\cdots v_{pi})\cdot f(v_{0i})+\sum_{j>0}(-1)^j g_i(v_{0i}\cdots\hat{v}_{ji}\cdots v_{pi})\cdot f(v_{0i}).$$

By (*) again, $\eta^{(p-1)}\eta^{(p)}(x)$ in $K_{p-1,2}$ is represented by

$$\sum\mathfrak{g}_i\partial(v_{1i}\cdots v_{pi})\cdot(f(v_{1i})f(v_{0i}))$$
$$=\sum\mathfrak{g}_i(v_{2i}\cdots v_{pi})\cdot(f(v_{1i})f(v_{0i}))$$
$$+\sum_{j>1}(-1)^{j-1}\mathfrak{g}_i(v_{1i}\cdots\hat{v}_{ji}\cdots v_{pi})\qquad(f(v_{1i})f(v_{0i})).$$

Continuing, the element $\eta^{(1)}\cdots\eta^{(p)}(x)$ in $K_{0,p-1}$ is represented by

$$\sum\mathfrak{g}_i\partial(v_{p-1i}v_{pi})(f(v_{p-1i})\cdots f(v_{0i}))$$
$$=\sum\mathfrak{g}_i v_{pi}\cdot(f(v_{p-1i})\cdots f(v_{0i}))-\sum\mathfrak{g}_i v_{p-1i}\cdot(f(v_{p-1i})\cdots f(v_{0i})).$$

Hence $\eta(x)$ is represented by

$$\sum\mathfrak{g}_i S_{-1}\cdot(f(v_{pi})\cdots f(v_{0i}))=(-1)^{p(p+1)/2}f(z).$$

The theorem follows for $p\geqq 0$. For $p=-1$ the theorem is trivial.

If $(X,\alpha)$ and $(Y,\beta)$ are simplicial pairs then *a map $F:(X,\alpha)\to(Y,\beta)$ is a pair $F=(f,g)$ consisting of a simplicial map $f:X\to Y$ and a function $g:\alpha\to\beta$ which assigns to each $A^i$ in $\alpha$ an element $gA^i=B^j$ of $\beta$ with $fA^i\subset gA^i$.* The corresponding simplicial map from $X_\alpha$ to $Y_\beta$ will also be denoted by $g$. Suppose $S_p \subset \bigcap T_q$ where $T_q$ is a simplex of $X_\alpha$. Then $f(S_p) \subset \bigcap gT_q$. If $A$ is a chain of $X$ then $fA$ will denote its image under $f$. *There is the function*

$$\sum A^i \cdot T_q^i \to \sum fA^i \cdot gT_q^i$$

*of* $C_{p,q}(\alpha)$ *into* $C_{p,q}(\beta)$, *which we denote by* $F_c$. It may be seen that $DF_c = F_cD$. Hence $F_c$ maps $E_{p,q}(\alpha)$ into $E_{p,q}(\beta)$. Moreover, since $\Delta F_c = F_c\Delta$, *there is induced a map* $F_k: K_{p,q}(\alpha) \to K_{p,q}(\beta)$. *Similarly* $F_c$ *induces a map* $F_h: H_{p,q}(\alpha) \to H_{p,q}(\beta)$. The following may be seen.

(5.7) *There is commutativity in each rectangle of*

$$\begin{array}{ccccccc} \cdots \to & K_{p,q}(\alpha) & \to & H_{p,q}(\alpha) & \to & K_{p,q-1}(\alpha) & \to \cdots \\ & \downarrow F_k & & \downarrow F_h & & \downarrow F_k & \\ \cdots \to & K_{p,q}(\beta) & \to & H_{p,q}(\beta) & \to & K_{p,q-1}(\beta) & \to \cdots . \end{array}$$

*Moreover,* $F_k: K_{p,-1}(\alpha) \to K_{p,-1}(\beta)$ *coincides with* $f_*: H_p(X) \to H_p(Y)$ *and* $F_k: K_{-1,q}(\alpha) \to K_{-1,q}(\beta)$ *with* $g_*: H_q(\alpha) \to H_q(\beta)$.

(5.8) *Suppose that* $F = (f, g)$ *and* $F' = (f', g)$ *are two maps of* $(X, \alpha)$ *into* $(Y, \beta)$ *such that given a simplex* $S$ *of* $A^i \in \alpha$ *then there is a simplex* $S'$ *of* $gA^i \in \beta$ *such that* $f(S) \cup f'(S) \subset S'$. *Then* $F_h = F_h'$ *and* $F_k = F_k'$.

**Proof.** For a simplex $S$ of $X$ suppose that $S'$ is the smallest simplex of $Y$ containing $f(S) \cup f'(S)$. If $S \in \bigcap T_q$ then $S' \in \bigcap gT_q$. Define $\mathfrak{J}: C_{p,q}(\alpha) \to C_{p+1,q}(\beta)$, $p \geqq 0$, by

$$\mathfrak{g}(v_0 \cdots v_p) \cdot T_q \to \left[ \sum (-1)^i \mathfrak{g}(f(v_0) \cdots f(v_i) f'(v_i) \cdots f'(v_p)) \right] \cdot gT_q$$

and linearity. Then $\Delta\mathfrak{J} + \mathfrak{J}\Delta = F_c' - F_c$ for $p \geqq 1$, $\Delta\mathfrak{J} = F_c' - F_c$ for $p = 0$. Moreover, $D\mathfrak{J} = \mathfrak{J}D$. Hence $\mathfrak{J}: E_{p,q}(\alpha) \to E_{p+1,q}(\beta)$, $p \geqq 0$. It follows that $F_h = F_h'$ and $F_k = F_k'$ for $p \geqq 0$.

The case $p = -1$ remains. On $C_{-1,p}$, $F_c = F_c'$. Hence the conclusion follows for $p = -1$.

The following theorem may be easily proved.

(5.9) *If*

$$(X, \alpha) \xrightarrow{F} (Y, \beta) \xrightarrow{F'} (Z, \gamma)$$

*then* $(F'F)_h = F_h' F_h$ *and* $(F'F)_k = F_k' F_k$.

6. **The Kelley-Pitcher theory of compact pairs.** In this section we extend the Kelley-Pitcher theory to pairs $(X, \alpha)$, where $X$ is a compact space and $\alpha = (A^1, \cdots, A^r)$ a collection of closed subsets of $X$ which covers $X$. We begin the section by summarizing the results.

For such a compact pair $(X, \alpha)$, *define* $H_{p,q}(\alpha) = \sum H_p(\bigcap T_q)$, *where the summation is over all q-simplices* $T_q = (\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ *of* $X_\alpha$ with $i_0 < \cdots < i_q$. If $T_q$ is a simplex as above and if $A_p \in H_p(\bigcap T_q)$ then denote by $A_p \cdot T_q$ the element of $H_{p,q}$ whose $T_q$-coordinate is $A_p$ and all of whose other coordinates are 0. Agree also that $A_p\cdot(-T_q)=(-A_p)\cdot T_q$. Then $H_{p,q}$ is the set of linear forms $\sum A_p^i\cdot T_q^i$ where $A_p^i\in H_p(\bigcap T_q^i)$.

*A map $F$ of a pair $(X, \alpha)$ into a pair $(Y, \beta)$ will be a pair $F=(f, g)$ where $f:X\to Y$ is continuous and where $g:\alpha\to\beta$ assigns to each $A^i$ in $\alpha$ an element $gA^i=B^j$ in $\beta$ such that $fA^i\subset gA^i$.* The simplicial map of $X_\alpha$ into $Y_\beta$ will also be denoted by $g$. If $T_q$ is an unoriented simplex of $X_\alpha$ then $f(\bigcap T_q)\subset\bigcap gT_q$. The map $f$ of $\bigcap T_q$ into $\bigcap gT_q$ is denoted by $f|\bigcap T_q$. There is the induced homomorphisms $(f|\bigcap T_q)_*:H_p(\bigcap T_q)\to H_p(\bigcap gT_q)$. *A homomorphism $F_h:H_{p,q}(\alpha)\to H_{p,q}(\beta)$ is defined by $A_p\cdot T_q\to[(f|\bigcap T_q)_*A_p]\cdot gT_q$ and linearity.*

If

$$(X, \alpha)\xrightarrow{F}(Y, \beta)\xrightarrow{F'}(Z, \gamma)$$

where $F=(f, g)$ and $F'=(f', g')$ then the *composition* $F'F:(X, \alpha)\to(Z, \gamma)$ is defined to be $(f'f, g'g)$.

We define for each $(X, \alpha)$ a group $K_{p,q}(\alpha)$; a map $F:(X, \alpha)\to(Y, \beta)$ will induce a homomorphism $F_k:K_{p,q}(\alpha)\to K_{p,q}(\beta)$. There will also be homomorphisms

$$K_{p,q}\xrightarrow{l}H_{p,q}\xrightarrow{m}K_{p,q-1}\xrightarrow{\eta^{(p)}}K_{p-1,q}.$$

The following theorems constitute the theory for compact pairs, and are proved later in the section.

THEOREM I. *For each $(X, \alpha)$ and each $q$ the sequence*

$$\cdots\to K_{p,q}\xrightarrow{l}H_{p,q}\xrightarrow{m}K_{p,q-1}\xrightarrow{\eta^{(p)}}K_{p-1,q}\xrightarrow{l}\cdots\to K_{0,q-1}\xrightarrow{\eta^{(0)}}K_{-1,q}\to 0$$

*is exact.*

THEOREM II. *For each $(X, \alpha)$ we have $K_{p,-1}=H_p(X)$ and $K_{-1,q}=H_p(\alpha)$.*

THEOREM III. *The homomorphism $\eta=\eta^{(0)}\cdots\eta^{(p)}$ of $H_p(X)$ into $H_p(\alpha)$ given by*

$$H_p(X) = K_{p,-1}\xrightarrow{\eta^{(p)}}K_{p-1,0}\to\cdots\to K_{0,p-1}\xrightarrow{\eta^{(0)}}K_{-1,p} = H_p(\alpha)$$

*is such that $\eta=(-1)^{p(p+1)/2}\pi_\alpha$, where $\pi_\alpha$ is the projection homomorphism of $H_p(X)$ into $H_p(\alpha)$.*

THEOREM IV. *If*

$$(X, \alpha)\xrightarrow{F}(Y, \beta)\xrightarrow{F'}(Z, \gamma)$$

*then $(F'F)_h=F_h'F_h$ and $(F'F)_k=F_k'F_k$.*

THEOREM V. *If $F=(f,g)$ maps $(X,\alpha)$ into $(Y,\beta)$ then $F_k:K_{p,-1}(\alpha)\to K_{p,-1}(\beta)$*

*coincides with* $f_*: H_p(X) \to H_p(Y)$; *also* $F_k: K_{-1,q}(\alpha) \to K_{-1,q}(\beta)$ *coincides with* $g_*: H_q(\alpha) \to H_q(\beta)$.

THEOREM VI. *If* $F: (X, \alpha) \to (Y, \beta)$, *there is commutativity in each rectangle of*

$$\begin{array}{ccccccccc} \cdots \to & K_{p,q}(\alpha) & \xrightarrow{l} & H_{p,q}(\alpha) & \xrightarrow{m} & K_{p,q-1}(\alpha) & \xrightarrow{\eta^{(p)}} & \cdots \\ & \downarrow F_k & & \downarrow F_h & & \downarrow F_k & & \\ \cdots \to & K_{p,q}(\beta) & \xrightarrow{l} & H_{p,q}(\beta) & \xrightarrow{m} & K_{p,q-1}(\beta) & \longrightarrow & \cdots \end{array}$$

We now begin the proofs.

(6.1) LEMMA. *Suppose that* $X$ *is a compact Hausdorff space, and that* $\alpha = (A^1, \cdots, A^r)$ *is a closed covering of* $X$. *If* $\beta = (B^1, \cdots, B^s)$ *is a collection of closed subsets of* $X$, *there is a collection* $\gamma = (C^1, \cdots, C^s)$ *of closed subsets of* $X$ *with* $C^i$ *containing* $B^i$ *in its interior*, $i = 1, \cdots, s$, *and with a set of the type* $C^{i_0} \cap \cdots \cap C^{i_p}$ *intersecting some* $A^j$ *if and only if* $B^{i_0} \cap \cdots \cap B^{i_p}$ *intersects* $A^j$.

**Proof.** It is sufficient to prove that if $1 \leqq k \leqq s$ then there is a $\gamma = (C^1, \cdots, C^s)$ with $C^i = B^i$ for $i \neq k$, $C^k$ containing $B^k$ in its interior, and $C^{i_0} \cap \cdots \cap C^{i_p} \cap A^j \neq \emptyset$ if and only if $B^{i_0} \cap \cdots \cap B^{i_p} \cap A^j \neq \emptyset$. Suppose that $C^k$ contains $B^k$ in its interior and that $C^k$ intersects any set of the type $B^{i_0} \cap \cdots \cap B^{i_p} \cap A^j$ if and only if $B^k$ intersects $B^{i_0} \cap \cdots \cap B^{i_p} \cap A^j$. Let $C^i = B^i$ for $i \neq k$. Then $C^{i_0} \cap \cdots \cap C^{i_p} \cap A^j \neq \emptyset$ if and only if $B^{i_0} \cap \cdots \cap B^{i_p} \cap A^j \neq \emptyset$. The assertion follows.

(6.2) DEFINITION. Suppose that $(X, \alpha)$ is a compact pair. Then a finite open covering $u$ of $X$ is a *special* covering of $(X, \alpha)$ if and only if

(i) whenever $U^{i_0}, \cdots, U^{i_p}$ are elements of $u$ with $U^{i_0} \cap \cdots \cap U^{i_p}$ intersecting each of $A^{j_0}, \cdots, A^{j_q}$ then $U^{i_0} \cap \cdots \cap U^{i_p}$ intersects $A^{j_0} \cap \cdots \cap A^{j_q}$;

(ii) whenever $U^{i_0}, \cdots, U^{i_p}$ are elements of $u$ with $U^{i_0} \cap \cdots \cap U^{i_p} \neq \emptyset$, there is an element $U^{i_k}$ such that if $U^{i_k} \cap A^j \neq \emptyset$ then $U^{i_0} \cap \cdots \cap U^{i_p} \cap A^j \neq \emptyset$. The set of all such special coverings will be denoted by $U(\alpha)$.

(6.3) THEOREM. *Suppose that* $(X, \alpha)$ *is a compact pair. If* $v$ *is an open covering of* $X$, *there is a special open covering* $u$ *of* $(X, \alpha)$ *which refines* $v$.

**Proof.** Let $Y^m$ consist of all $x \in X$ with $x \in A^i$ for at least $m$ values of $i$. Then $Y^m$ is closed, $Y^{m+1} \subset Y^m$ and $Y^1 = X$. We prove the following by induction for $r \geqq m \geqq 1$.

$L_m$: there exists a covering $\beta_m = (B^1, \cdots, B^s)$ of $Y^m$ by closed subsets of $Y^m$ such that $\beta_m$ refines $v$ and such that $\beta_m$ satisfies (i) and (ii) of the definition of a special covering with the $U^i$'s replaced by $B^i$'s.

$L_r$ may be seen to be true. Suppose $L_m$ is true and that $\beta_m = (B^1, \cdots, B^s)$ is the desired covering of $Y^m$. By Lemma 6.1, there is a collection $\gamma = (C^1, \cdots, C^s)$ of closed subsets of $Y^{m-1}$ with $C^i$ containing $B^i$ in its interior relative to $Y^{m-1}$ and with $C^{i_0}\cap \cdots \cap C^{i_p}$ intersecting $A^j$ if and only if $B^{i_0}\cap \cdots \cap B^{i_p}$ intersects $A^j$. It may be seen that $\gamma$ satisfies (i) and (ii) of Definition 6.2, since $\beta_m$ does. We may also suppose $\gamma$ sufficiently small so that $\gamma$ refines $v$.

Now suppose $x\in Y^{m-1}-Y^m$. Then $x$ is contained in exactly $m-1$ $A^i$'s. There is also a neighborhood of $x$ relative to $Y^{m-1}$ every point of which is contained in exactly those $A^i$'s which contain $x$. Hence we may expand $\gamma$ to a covering

$$\beta_{m-1} = (C^1, \cdots, C^s, C^{s+1}, \cdots, C^{s+t})$$

of $Y^{m-1}$ by closed sets, where all points of $C^{s+i}$ are contained in exactly the same $A^i$'s, and where $\beta_{m-1}$ refines $v$. Suppose $C^{i_0}\cap \cdots \cap C^{i_p}$ intersects each of $A^{j_0}, \cdots, A^{j_q}$. If $i_0, \cdots, i_p \leqq s$, then $C^{i_0}\cap \cdots \cap C^{i_p}$ intersects $A^{j_0}\cap \cdots \cap A^{j_q}$ since $\gamma$ satisfied (i). If some $i_k > s$ then $A^{j_0}, \cdots, A^{j_q}$ must each be one of the $A^i$'s which contains $C^{i_k}$. Then

$$A^{j_0} \cap \cdots \cap A^{j_q} \supset C^{i_k} \supset C^{i_0} \cdots C^{i_p}.$$

Hence $\beta_{m-1}$ satisfies (i).

Suppose now that $C^{i_0}\cap \cdots \cap C^{i_p} \neq \emptyset$. If $i_0, \cdots, i_p \leqq s$, there is a $C^{i_k}$ such that if $C^{i_k}\cap A^j \neq \emptyset$ then $C^{i_0}\cap \cdots \cap C^{i_p} \cap A^j \neq \emptyset$, since $\gamma$ satisfies (ii). Suppose $i_k > s$. If $C^{i_k}\cap A^j\neq\emptyset$ then $A^j \supset C^{i_k}$. Then $C^{i_0}\cap \cdots \cap C^{i_p}\cap A^j = C^{i_0}\cap \cdots \cap C^{i_p} \neq \emptyset$. Hence $\beta_{m-1}$ satisfies (ii). The induction is complete.

Now $L_1$ is true, with $\beta_1 = (B^1, \cdots, B^s)$. Let $u = (U^1, \cdots, U^s)$ be such that $u$ refines $v$, $U^i \supset B^i$, and such that $U^{i_0}\cap \cdots \cap U^{i_p}$ intersects $A^j$ if and only if $B^{i_0}\cap \cdots \cap B^{i_p}$ intersects $A^j$. Then $u$ is a special covering, and the theorem follows.

Suppose that $(X, \alpha)$ is a compact pair. For each special covering $u \in U(\alpha)$ there is the nerve $X_u$. There is also a covering $\alpha_u = (A_u^1, \cdots, A_u^r)$ of $X_u$, where $A_u^i$ consists of all simplexes $\tau^p = (\tilde{U}^0 \cdots \tilde{U}^p)$ of $X_u$ with $U^0\cap \cdots \cap U^p \cap A^i \neq \emptyset$. Furthermore, from (i) of Definition 6.2, $A_u^{i_0}\cap \cdots \cap A_u^{i_q}$ consists of those simplexes $\tau^p$ of $X_u$ with $U^0\cap \cdots \cap U^p$ intersecting $A^{i_0}\cap \cdots \cap A^{i_q}$. In particular, the nerve $(X_u)_{\alpha_u}$ of $\alpha_u$ coincides with the nerve $X_\alpha$. If $T_q = (\tilde{A}^{i_0} \cdots \tilde{A}^{i_q})$ is a simplex of $X_\alpha$ then $T_q^u = (\tilde{A}_u^{i_0} \cdots \tilde{A}_u^{i_q})$ is a simplex of $(X_u)_{\alpha_u}$ and $T_q = T_q^u$. Moreover, $\bigcap T_q^u = A_u^{i_0}\cap \cdots \cap A_u^{i_q}$ consists of all those simplices $\tau^p$ of $X_u$ with $U^0\cap \cdots \cap U^p$ intersecting $\bigcap T_q$.

If $u, v \in U(\alpha)$ and $v$ refines $u$, let $\pi_{vu}$ denote a projection of $X_v$ into $X_u$. It may be checked that $\pi_{vu}$ maps $\bigcap T_q^v$ into $\bigcap T_q^u$; this map is denoted by $\pi_{vu} | \bigcap T_q^v$. It follows that $(\pi_{vu}, id)$, $id$ the identity function, maps the simplicial pair $(X_v, \alpha_v)$ into $(X_u, \alpha_u)$. Define $\tilde{F}^{vu} = (\pi_{vu}, id)$.

(6.4) *For a simplex $T_q$ of $X_\alpha$, the limit group of the inverse system* $[H_q(\bigcap T_q^u), (\pi_{vu} | \bigcap T_q^v)_*]$, *indexed by* $U(\alpha)$, *is* $H_p(\bigcap T_q)$.

**Proof.** We have that $\bigcap T_q^u$ consists of all simplices of $X_u$ whose intersection intersects $\bigcap T_q$. Also, $\pi_{vu} | \bigcap T_q^v$ is a projection of $\bigcap T_q^v$ into $\bigcap T_q^u$. Since $U(\alpha)$ is cofinal in the set of all open coverings of $X$, the limit group may be identified with $H_p(\bigcap T_q)$.

(6.5) *If $v$, $u \in U(\alpha)$ and $v$ refines $u$, then $(\tilde{F}^{vu})_h = \tilde{F}_h^{vu}$ and $(\tilde{F}^{vu})_k = \tilde{F}_k^{vu}$ are independent of the projection $\pi_{vu}$. If also $w$ refines $v$ then $\tilde{F}_h^{vu}\tilde{F}_h^{wv} = \tilde{F}_h^{wu}$ and $\tilde{F}_k^{vu}\tilde{F}_k^{wv} = \tilde{F}_k^{wu}$.*

**Proof.** If $\pi_{vu}$ and $\pi'_{vu}$ are projections of $X_v$ into $X_u$, then the maps $(\pi_{vu}, id)$ and $(\pi'_{vu}, id)$ satisfy the hypothesis of (5.8). Hence $F_h^{vu}$ and $F_k^{vu}$ are independent of the projection.

(6.6) *The limit group of the inverse system $[H_{p,q}(\alpha_u), \tilde{F}_h^{vu}]$, indexed by $U(\alpha)$, is isomorphic with $H_{p,q}(\alpha)$. Henceforth we identify the two.*

**Proof.** Consider an element $A_p \cdot T_q$ of $H_{p,q}(\alpha)$, where $A_p \in H_p(\bigcap T_q)$. According to (6.4), $A_p = (A_p(u) : u \in U(\alpha))$, where $A_p(u)$ is the coordinate of $A_p$ in $H_p(\bigcap T_q^u)$, and where, if $v$ refines $u$, then $(\pi_{vu} | \bigcap T_q^v)_* A_p(v) = A_p(u)$. For each $v$, $A_p(v) \cdot T_q^v$ is an element of $H_{p,q}(\alpha_v)$ and if $v$ refines $u$ then $\tilde{F}_h^{vu}$ maps $A_p(v) \cdot T_q^v$ into $A_p(u) \cdot T_q^u$. The correspondence

$$A_p \cdot T_q \to (A_p(u) \cdot T_q^u : u \in U(\alpha))$$

together with linearity, yields an isomorphism of $H_{p,q}(\alpha)$ onto the limit group.

(6.7) Definition. For a pair $(X, \alpha)$, $K_{p,q}(\alpha)$ is defined to be the limit group of the inverse system $[K_{p,q}(\alpha_u), \tilde{F}_k^{vu}]$, indexed by $U(\alpha)$. The maps

$$K_{p,q}(\alpha) \xrightarrow{l} H_{p,q}(\alpha) \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q}(\alpha)$$

are defined as limits of

$$K_{p,q}(\alpha_u) \xrightarrow{l} H_{p,q}(\alpha_u) \xrightarrow{m} K_{p,q-1} \xrightarrow{\eta^{(p)}} K_{p-1,q}(\alpha_u).$$

Theorem I then follows from (5.5), together with the fact that the groups of $\alpha_u$ are either compact or finite-dimensional vector spaces.

Theorem II. *We have $K_{p,-1}(\alpha) = H_p(X)$, $K_{-1,q}(\alpha) = H_q(\alpha)$.*

**Proof.** Consider the inverse system $[K_{p,-1}(\alpha_u), \tilde{F}_k^{vu}]$. According to (5.3), $K_{p,-1}(\alpha_u) = H_p(u)$. According to (5.7), $F_k : K_{p,-1}(\alpha_v) \to K_{p,-1}(\alpha_u)$ coincides with $\pi_{vu} : H_p(v) \to H_p(u)$. Hence the inverse system coincides with $[H_p(u), \pi_{vu}]$ and $K_{p,-1} = H_p(X)$. Consider now the inverse system $[K_{-1,q}(\alpha_u), \tilde{F}_k^{vu}]$. Now $K_{-1,q}(\alpha_u) = H_q((X_u)\alpha_u) = H_q(u)$ and $\tilde{F}_k^{vu}$ is the identity by (5.7). Hence $K_{-1,q}$ may be identified with $H_q(\alpha)$.

Theorem III. *The homomorphism $\eta : H_p(X) \to H_p(\alpha)$ is such that $\eta = (-1)^{p(p+1)/2}\pi_\alpha$.*

**Proof.** Let $W$ be an open covering $(W^0, \cdots, W^r)$ of $X$ where $W^i \supset A^i$ and where $W^{i_0} \cap \cdots \cap W^{i_q} \neq \emptyset$ if and only if $A^{i_0} \cap \cdots \cap A^{i_q} \neq \emptyset$. Then $X_w = X_\alpha$ and $\pi_w = \pi_u$. Consider any special covering $u$ of $X$ such that if $U$ is an element of $u$ and $U\cap A^i\neq\emptyset$ then $U\subset W^i$. Since $u$ is a special covering, if $U^0\cap\cdots\cap U^p\neq\emptyset$, where the $U^i$'s are elements of $u$, there is an element $U^i$ such that if $U^i\cap A^j\neq\emptyset$ then $U^0\cap\cdots\cap U^p\cap A_j\neq\emptyset$. It follows that $(X_u, \alpha_u)$ satisfies the hypothesis of (5.6). Let $f$ be a function assigning to each vertex $U$ of $X_u$ an element $A_u^i$ of $\alpha_u$ with $U\cap A^i\neq\emptyset$ (that is, $U$ is a vertex of $A_u^i$). According to (5.6), $f:X_u\to(X_u)\alpha_u=X_\alpha$ is simplicial and $f_*:H_p(X_u)\to H_p((X_u)\alpha_u)$ is such that $f_*=(-1)^{p(p+1)/2}\eta_u$ where $\eta_u$ is the composition

$$H_p(X_u) = K_{p,-1}(\alpha_u) \xrightarrow{\eta^{(p)}} \cdots \xrightarrow{\eta^{(0)}} K_{-1,p}(\alpha_u) = H_p((X_u)\alpha_u).$$

In the diagram

$$X_u \xrightarrow{f} (X_u)\alpha_u = X_\alpha = X_w$$

note that $u$ and $w$ are chosen so that $f:X_u\to X_w$ is a projection $\pi_{uw}$ of $X_u$ into $X_w$. So $\pi_{uw*}=f_*$ and

$$\pi_{uw*} = (-1)^{p(p+1)/2}\eta_u, \qquad \pi_w = (-1)^{p(p+1)/2}\eta.$$

Since $X_\alpha=X_w$ and $\pi_\alpha=\pi_w$, the theorem follows.

(6.8) Definition. Suppose $F=(f, g)$ maps the compact pair $(X, \alpha)$ into the compact pair $(Y, \beta)$ where $\beta=(B^1, \cdots, B^s)$. Denote by $P(F)$ the set of all pairs $(u, v)$, where $u\in U(\alpha)$ and $v\in U(\beta)$ and where if $U$ is an element of $u$ then there is an element $V$ of $v$ with $f(U)\subset V$. Let $f_{uv}:X_u\to X_v$ assign to each vertex $U$ of $X_u$ a vertex $V$ of $Y_v$ with $f(U)\subset V$. Then $f_{uv}$ is simplicial.

There are the coverings $\alpha_u=(A_u^1, \cdots, A_u^r)$ of $X$ and $\beta_v=(B_v^1, \cdots, B_v^s)$ of $Y_v$. Moreover $(X_u)\alpha_u=X_\alpha$ and $(Y_v)\beta_v=Y_\beta$. It may be seen that $F^{uv}=(f_{uv}, g)$ maps the pair $(X_u, \alpha_u)$ into the pair $(Y_v, \beta_v)$. Moreover, by (5.8), $F_h^{uv}:H_{p,q}(\alpha_u)\to H_{p,q}(\beta_v)$ and $F_k^{uv}$ are independent of the particular $f_{uv}$ chosen. Also, if $u'\in U(\alpha)$ and $u'$ refines $u$ we have $\tilde F^{u'u}=(\pi_{u'u}, id)$ mapping $(X_{u'}, \alpha_{u'})$ into $(X_u, \alpha_u)$. If $v$ refines $v'\in U(\beta)$ there is $F^{vv'}=(\pi_{vv'}, id)$ mapping $(Y_v, \beta_v)$ into $(Y_{v'}, \beta_{v'})$. Then $\tilde F^{vv'}F^{uv}\tilde F^{u'u}=F^{u'v'}$. Hence

$$(*)\qquad \tilde F_h^{vv'}F_h^{uv}\tilde F_h^{u'v}=F_h^{u'v'}, \qquad \tilde F_k^{vv'}F_k^{uv}\tilde F_k^{u'u}=F_k^{u'v'}.$$

Consider the inverse systems $[K_{p,q}(\alpha_u), \tilde F_k^{u'u}]$, indexed by $U(\alpha)$, and $[K_{p,q}(\beta_v), \tilde F_k^{vv'}]$, indexed by $U(\beta)$. For $(u, v)\in P(F)$, there is the map $F_k^{vu}:K_{p,q}(\alpha_u)\to K_{p,q}(\beta_v)$ such that (*) holds. *There is induced, in the limit, a map* $F_k:K_{p,q}(\alpha)\to K_{p,q}(\beta)$.

(6.9) *The limit of the map* $F_h^{uv}:H_{p,q}(\alpha_u)\to H_{p,q}(\beta_v)$, $(u, v)\in P(F)$, *is the map* $F_h$ *defined earlier.*

**Proof.** It is sufficient to check the equality of the two maps on elements $A_p\cdot T_q$, $T_q$ an oriented $q$-simplex of $\alpha$ and $A_p\in H_p(\cap T_q)$. Now $F_h$ maps $A_p\cdot T_q$ into $[(f|\cap T_q)_*A_p]\cdot gT_q$. Also, according to (6.6), $A_p\cdot T_q=(A_p(u)\cdot T_q^u: u\in U(\alpha))$. Now $F_h^{vu}$ maps $A_p(u)\cdot T_q^u$ into $[(f_{uv}|\cap T_q^u)_*A_p(u)]\cdot gT_q^u$. This is the coordinate in $H_{p,q}(\beta_v)$ of $[(f|\cap T_q)_*A_p]\cdot gT_q$.

We leave the proofs of Theorems IV, V, and VI to the reader.

**7. Proof of the basic theorem.**

THEOREM. *Suppose that $A_{-1} \supset A_0 \supset \cdots \supset A_n$ is a sequence of closed subsets of a compact space $X$, and that $\alpha_{-1} \ll^n \cdots \ll^n \alpha_n$, where $\alpha_i$ is a closed covering of $A^i$. Then*

(i) *the kernel of $\pi_{\alpha_n}: H_j(A_n) \to H_j(\alpha_n)$ is contained in the kernel of the injection $I: H_j(A_n) \to H_j(A_{-1})$ for $j \leqq n$;*

(ii) *the image of $\pi_{\alpha_n,\alpha_0}: H_j(\alpha_n) \to H_j(\alpha_0)$ is contained in the image of $\pi_{\alpha_0}$: $H_j(A_0) \to H_j(\alpha_0)$ for $j \leqq n+1$.*

**Proof.** For each $i$ there is a projection $\pi = \pi_{\alpha_i,\alpha_{i-1}}$ of $\alpha_i$ into $\alpha_{i-1}$ such that $H_j(A^{i_0} \cap \cdots \cap A^{i_q}; \pi A^{i_0} \cap \cdots \cap \pi A^{i_q}) = 0$ for all $j \leqq n$. Let $I_i^{\prime}: A_i \to A_{i-1}$ denote the inclusion map. Then $F^i = (I_i^{\prime}, \pi)$ maps $(A_i, \alpha_i)$ into $(A_{i-1}, \alpha_{i-1})$ and $F_h^i: K_{p,q}(\alpha_i) \to K_{p,q}(\alpha_{i-1})$ is trivial for $0 \leqq p \leqq n$.

Consider the diagram

$$\begin{array}{ccccccccc}
\cdots \to H_{p,q}(\alpha_i) & \xrightarrow{m} & K_{p,q-1}(\alpha_i) & \xrightarrow{\eta^{(p)}} & K_{p-1,q}(\alpha_i) & \xrightarrow{l} & H_{p-1,q}(\alpha_i) \to \cdots \\
\downarrow F_h^i & & \downarrow F_k^i & & \downarrow F_k^i & & \downarrow F_h^i \\
\cdots \to H_{p,q}(\alpha_{i-1}) & \xrightarrow{m} & K_{p,q-1}(\alpha_{i-1}) & \xrightarrow{\eta^{(p)}} & K_{p-1,q}(\alpha_{i-1}) & \xrightarrow{l} & H_{p-1,q}(\alpha_{i-1}) \to \cdots
\end{array}$$

Now (1) *if $x \in K_{p,q-1}(\alpha_i)$, $p \leqq n$, and $\eta^{(p)}(x) = 0$, then $F_k^i(x) = 0$.* For if $\eta^{(p)}(x) = 0$ then $x = m(y)$ and $F_k^i(x) = F_k^i m(y) = mF_h^i(y) = 0$. Moreover, (2) *if $x \in K_{p-1,q}(\alpha_i)$, $p \leqq n+1$, then $F_k^i(x) \in K_{p-1,q}(\alpha_{i-1})$ is of the form $\eta^{(p)}(y)$ for some $y \in K_{p,q-1}(\alpha_{i-1})$.*

To prove (i), suppose $\pi_{\alpha_n}(x) = 0$, $x \in H_j(A_n) = K_{j,-1}(\alpha_n)$. Then, by Theorem III, $\eta(x) = \eta^{(0)} \cdots \eta^{(j)}(x) = 0$ in $K_{-1,j}(\alpha_n) = H_j(\alpha_n)$. By (1), $F_k^n \eta^{(1)} \cdots \eta^{(j)}(x) = \eta^{(1)} \cdots \eta^{(j)} F_k^n(x) = 0$ in $K_{0,j-1}(\alpha_{n-1})$. By (1) again, $F_k^{n-1} \eta^{(2)} \cdots \eta^{(j)} F_k^n(x) = \eta^{(2)} \cdots \eta^{(j)} F_k^{n-1} F_k^n(x) = 0$ in $K_{1,j-2}(\alpha_{n-2})$. Continuing, $F_k^{n-j} \cdots F_k^{n-1} F_k^n(x) = 0$ in $K_{j,-1}(\alpha_{n-j-1})$. But $F_k^{n-j} \cdots F_k^n(x)$ is, by Theorem V, the injection $I$ of $H_j(A_n)$ into $H_j(A_{n-j-1})$. Hence (i) follows.

To prove (2), suppose $x \in H_j(\alpha_n) = K_{-1,j}(\alpha_n)$. The map

$$K_{0,j-1} \xrightarrow{\eta^{(0)}} K_{-1,j}$$

is onto, by Theorem I, so $x = \eta^{(0)}(x_0)$, $x_0 \in K_{0,j-1}(\alpha_n)$. By (2), $F_k^n(x_0) \in K_{0,j-1}(\alpha_{n-1})$ is of the form $\eta^{(1)}(x_1)$ for some $x_1 \in K_{1,j-2}(\alpha_{n-1})$. Continuing by use of (2), $F_k^{n-j+2}(x_{j-1}) \in K_{j-1,0}(\alpha_{n-j+1})$ is of the form $\eta^{(j)}(x_j)$ for some $x \in K_{j,-1}(\alpha_{n-j+1})$. Then

$$\begin{aligned}
\eta^{(0)} \cdots \eta^{(j)}(x_j) &= \eta^{(0)} \cdots \eta^{(j-1)} F_k^{n-j+2}(x_{j-1}) \\
&= F_k^{n-j+2} \eta^{(0)} \cdots \eta^{(j-2)} F_k^{n-j+3}(x_{j-2}) \\
&= F_k^{n-j+2} \cdots F_k^n(x).
\end{aligned}$$

By Theorems IV and V, $F_k^{n-j+2} \cdots F_k^n$ is the projection $\pi$ of $H_j(\alpha_n)$ into $H_j(\alpha_{n-j+1})$. Hence image $\pi_{\alpha_n, \alpha_{n-j+1}} \subset$ image $\pi_{\alpha_{n-j+1}}$. For $j \leqq n+1$, it follows readily that image $\pi_{\alpha_n, \alpha_0} \subset$ image $\pi_{\alpha_0}$.

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University of Virginia,
Charlottesville, Va.