ON THE REPRESENTATION OF $\alpha$-COMPLETE BOOLEAN ALGEBRAS

BY

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Let $\alpha$ be an infinite cardinal. A Boolean algebra $A$ is $\alpha$-complete if every subset of $A$ with power (cardinality) at most $\alpha$ possesses a least upper bound in $A$. An ideal $I$ in a Boolean algebra is $\alpha$-complete in case the least upper bound (if it exists) of every subset of $I$ with power at most $\alpha$ belongs to $I$. A Boolean algebra that is $\aleph_\alpha$-complete is also called $\sigma$-complete. A field of sets $B$ is a Boolean algebra where the operations $\cup$, $\cap$, and $\neg$ are respectively the operations of set-union, set-intersection, and complementation with respect to the unit element of $B$. A field of sets $B$ is $\alpha$-complete if the union of any subset of $B$ with power at most $\alpha$ belongs to $B$.

By a theorem of Stone [7], every Boolean algebra is isomorphic to a field of sets. On the other hand, not every $\alpha$-complete Boolean algebra is isomorphic to some $\alpha$-complete field of sets; a necessary and sufficient condition for such a representation is that every principal ideal of the algebra be contained in an $\alpha$-complete maximal ideal (cf. [5]). In 1947, Loomis [3] proved that every $\sigma$-complete Boolean algebra $A$ is isomorphic to a $\sigma$-complete field of sets $B$ modulo a $\sigma$-complete ideal of $B$. The question was raised as to whether this result holds for all infinite cardinals $\alpha$. In 1948, Sikorski [5] showed that the Boolean algebra $L$ of Lebesgue measurable subsets of the unit interval modulo the sets of measure zero is $2^{\aleph_\alpha}$-complete but not isomorphic to any $2^{\aleph_\alpha}$-complete field of sets modulo a $2^{\aleph_\alpha}$-complete ideal.

It is the object of this note to give a necessary and sufficient condition for $\alpha$-complete Boolean algebras $A$ to be $\alpha$-representable, i.e., to be isomorphic to an $\alpha$-complete field of sets $B$ modulo an $\alpha$-complete ideal of $B$ (2). It turned out that to each $\alpha$-complete Boolean algebra $A$ there is associated an ideal $R_\alpha(A)$ which plays the role of a radical with respect to $\alpha$-representation, i.e., a homomorphic image of $A$ is $\alpha$-representable if, and only if, the kernel of the homomorphism includes $R_\alpha(A)$ (cf. Theorem 3). Our characterization, presented in Theorem 2, may be regarded as a generalization of the theorem of Loomis since every $\sigma$-complete Boolean algebra satisfies our condition with

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(2) This result was first announced in the abstract [2]. It represents the first known characterization of $\alpha$-representable Boolean algebras and gives a complete solution to Problem 80 in [1, p. 168].
α = ℵ₀ (cf. Theorem 4). Recently, other characterizations of α-representable Boolean algebras have been found (Scott and Tarski [4]) where the proofs given are metamathematical in nature. We shall give a purely algebraic and direct proof (Theorem 6) of the equivalence of the characterization given here and the one given in [4]. This equivalence can be established by analyzing certain properties which individual elements of a Boolean algebra must possess; in so doing, we have proved a theorem (Theorem 5) which, aside from its use in the problem of α-representation, is of some interest in itself.

Let A be a Boolean algebra. We shall denote by \( \sum_{i \in I} a_i \) (\( \prod_{i \in I} a_i \)) the least upper bound (greatest lower bound) in A (if it exists) of the set \( \{a_i; i \in I\} \). If I is of power at most \( \alpha \), then an element of the form \( \sum_{i \in I} a_i \) (\( \prod_{i \in I} a_i \)) is called an \( \alpha \)-sum (\( \alpha \)-product). A system of elements \( a_{i,j} \) indexed by the sets I and J (i.e., \( a_{i,j} \) is an element of the Boolean algebra for \( i \in I \) and \( j \in J \)) is called an \( \alpha \)-system in case the sets I and J have powers at most \( \alpha \).

As usual, the complement of an element \( a \) of A shall be denoted by \( a \). For typographical reasons, we shall denote the complement of a group of letters by enclosing the group of letters in square brackets followed by a bar, e.g., \( [\sum_{i \in I} a_i]^- \). Whenever there is no possibility of confusion, we simply let \( a_i = [a_i]^- \) and \( a_{i,j} = [a_{i,j}]^- \). 0 and 1 shall denote respectively the zero and unit elements of A. For arbitrary sets A, \( P(A) \) denotes the set of all functions \( f \) with domain A and such that \( f(x) \in x \) for each \( x \in A \). We let \( J^f \) denote the set of all functions \( f \) with domain I and range included in \( J \). If \( f \) is a function and \( X \) is a set, \( f^*(X) \) is the image of \( X \) under \( f \). We assume that ordinals have been defined in such a way that every ordinal coincides with the set of smaller ordinals. A cardinal can be understood as an ordinal which has larger power than every smaller ordinal.

**Definition.** If A is a Boolean algebra and \( \alpha \) an infinite cardinal, then \( R_\alpha(A) \) shall denote the set of all elements \( x \in A \) for which there exists an \( \alpha \)-system of elements \( a_{i,j} \in A \) indexed by the sets I and J such that

\[
(i) \quad \prod_{j \in J} a_{i,j} = 0 \text{ for each } i \in I,
\]

and

\[
(ii) \quad \text{for each function } f, f \in J^f, \text{ the set of elements } \{a_{i,f(i)}; i \in I\} \text{ either contains } x \text{ or else contains some complementary pair of elements } b \text{ and } \bar{b}.
\]

We see readily from the definition that \( 0 \in R_\alpha(A) \).

**Theorem 1.** If A is an \( \alpha \)-complete Boolean algebra, then \( R_\alpha(A) \) is an \( \alpha \)-complete ideal of A and \( A/R_\alpha(A) \) is α-representable.

**Proof.** We shall first prove that there exists a homomorphism \( f \) of A onto an \( \alpha \)-complete field of sets B modulo an \( \alpha \)-complete ideal \( N \) of B, then prove that this homomorphism \( f \) preserves \( \alpha \)-sums, and, finally, that the kernel of this homomorphism is \( R_\alpha(A) \).
For each $x \in A$, we let $x^* = \{ x, \bar{x} \}$ and $A^* = \{ x^*; x \in A \}$. We define a function $g$ on the elements of $A$ to subsets of $P(A^*)$ such that

$$g(x) = \{ h; h \in P(A^*) \text{ and } h(x^*) = x \}.$$

It is clear that for every $x \in A$, $g(x) \cap g(\bar{x}) = \emptyset$ and $g(x) \cup g(\bar{x}) = P(A^*)$. We let $B$ be the $\alpha$-complete field of sets generated by the elements of $g^*(A)$ in the (complete) field of all subsets of $P(A^*)$. Furthermore, let $M = \{ \bigcap_{i \in I} g(x_i); I \text{ has at most power } \alpha, x_i \in A \text{ for each } i \in I, \text{ and } \prod_{i \in I} x_i = 0 \}$ and $N$ be the $\alpha$-complete ideal generated by $M$ in $B$. We now wish to show that $B/N$ is a homomorphic image of $A$ by the following mapping $f$:

$$f(x) = g(x)/N.$$

It is clear that $f^*(A)$ generates $B/N$. Now $f(\bar{x}) = g(\bar{x})/N = [P(A^*) \setminus g(x)]/N = [g(x)]^-/N$ and hence $f$ preserves complementation. Let $a = \sum_{i \in I} a_i$ be an $\alpha$-sum of elements of $A$. In order to show $f(a) = \sum_{i \in I} f(a_i)$ it is sufficient to prove that the symmetric difference of $g(a)$ and $\bigcup_{i \in I} g(a_i)$ is an element of $N$. Since $I$ has power at most $\alpha$ and $a \cdot \prod_{i \in I} a_i = 0$, we obtain $g(a) \cap \bigcap_{i \in I} g(a_i) \in N$. But

$$g(a) \cap \bigcap_{i \in I} g(a_i) = g(a) \cap \bigcap_{i \in I} [g(a_i)]^- = g(a) \cap \left[ \bigcup_{i \in I} g(a_i) \right],$$

hence

$$g(a) \cap \left[ \bigcup_{i \in I} g(a_i) \right] \in N. \tag{1}$$

On the other hand, $\bar{a} \cdot a_i = 0$ for each $i \in I$, thus $g(\bar{a}) \cap g(a_i) = [g(a)]^- \cap g(a_i) \in N$ for each $i \in I$. Since $N$ is $\alpha$-complete, $\bigcup_{i \in I} ([g(a)]^- \cap g(a_i)) \in N$, and

$$[g(a)]^- \cap \bigcup_{i \in I} g(a_i) \in N. \tag{2}$$

It follows from (1) and (2) that the symmetric difference of $g(a)$ and $\bigcup_{i \in I} g(a_i)$ belongs to $N$. Thus $f$ preserves $\alpha$-sums and $f^*(A)$ is an $\alpha$-complete subalgebra of $B/N$. Hence $f^*(A) = B/N$ and $f$ maps $A$ homomorphically onto $B/N$ preserving all $\alpha$-sums of elements of $A$.

It remains to prove that the kernel of $f$ is the set $R_\alpha(A)$. If $f(x) = 0$, then $g(x) \in N$. We see that the condition $g(x) \in N$ is equivalent to the following: there exists an $\alpha$-system of elements $a_{i,j}$ indexed by the sets $I$ and $J$ such that

(i) $\prod_{j \in J} a_{i,j} = 0$ for each $i \in I,$

(ii) $\bigcap_{j \in J} g(a_{i,j}) \in M$ for each $i \in I,$

(9) The idea of using the elements of $P(A^*)$ as points in the representation was discussed in [3].
and

\[ g(x) \subseteq \bigcup_{i \in I} \bigcap_{j \in J} g(a_{i,j}). \]

By the set-theoretical distributive law,

\[ \bigcup_{i \in I} \bigcap_{j \in J} g(a_{i,j}) = \bigcap_{h \in J^*} \bigcup_{i \in I} g(a_{i,h(i)}). \]

Hence (iii) together with (3) imply

\[ g(x) \subseteq \bigcup_{i \in I} g(a_{i,h(i)}) \quad \text{for each } h \in J^*. \]

If the set \( \{a_{i,h(i)}; i \in I\} \) does not contain a complementary pair of elements, then any two different elements \( a_{i,h(i)} \) and \( a_{j,h(j)} \), with \( i \neq j \), belong to different elements of \( A^* \). If, in addition, the set \( \{a_{i,h(i)}; i \in I\} \) does not contain \( x \), then clearly there exists a function \( k \in P(A^*) \) such that \( k(x^*) = x \) and \( k(a_{i,h(i)^*}) = a_{i,h(i)} \) for each \( i \in I \), i.e.,

\[ k \in g(x) \quad \text{and} \quad k \notin g(a_{i,h(i)}) \quad \text{for each } i \in I. \]

(5) is a contradiction to (4). Hence \( x \in R_\alpha(A) \). On the other hand, let \( x \in R_\alpha(A) \) and let \( a_{i,j} \) be the associated \( \alpha \)-system of elements indexed by the sets \( I \) and \( J \). Clearly conditions (i), (ii), and (4) are satisfied by the elements \( a_{i,j} \). By (3) and (i) we see that \( \bigcap_{h \in J^*} \bigcup_{i \in I} g(a_{i,h(i)}) \subseteq N \), which, together with (4) imply (iii). Thus \( g(x) \subseteq N \) and \( f(x) = 0 \). The theorem has been proved(4).

**Theorem 2.** Let \( A \) be an \( \alpha \)-complete Boolean algebra. Then \( A \) is \( \alpha \)-representable if, and only if, \( R_\alpha(A) = \{0\} \).

**Proof.** Obviously, if \( R_\alpha(A) = \{0\} \), then by Theorem 1 \( A \) is \( \alpha \)-representable. Let \( f \) be a homomorphism of an \( \alpha \)-complete field of sets \( B \) onto \( A \) and such that \( f \) preserves all \( \alpha \)-sums of \( B \). Let \( x \in R_\alpha(A) \) and let \( a_{i,j} \) be the associated \( \alpha \)-system of elements. We choose an inverse \( f^{-1} \) to the function \( f \) satisfying the condition: \( f^{-1}(\tilde{y}) = [f^{-1}(y)]^* \) for each \( y \in A \). It is evident that such an inverse can always be chosen. Since \( \prod_{j \in J} a_{i,j} = 0 \) for each \( i \in I \) and since \( f \) preserves all \( \alpha \)-sums (and hence all \( \alpha \)-products), we obtain

\[ \bigcap_{j \in J} f^{-1}(a_{i,j}) \subseteq B \quad \text{and} \quad f \left( \bigcap_{j \in J} f^{-1}(a_{i,j}) \right) = 0 \quad \text{for every } i \in I. \]

From (1), it follows that \( \bigcup_{i \in I} \bigcap_{j \in J} f^{-1}(a_{i,j}) \subseteq B \), \( f(\bigcup_{i \in I} \bigcap_{j \in J} f^{-1}(a_{i,j})) = 0 \), and, by an application of the set-theoretical distributive law,

\[ f \left( \bigcap_{h \in J^*} \bigcup_{i \in I} f^{-1}(a_{i,h(i)}) \right) = 0. \]

(4) The fact that the ideal \( R_\alpha(A) \) is \( \alpha \)-complete can be proved without resorting to the homomorphism \( f \) and without even the assumption of the \( \alpha \)-completeness of \( A \).
By our choice of the inverse function $f^{-1}$, we see that

$$\text{for every } h \in J, \text{ either } f^{-1}(x) \subseteq \bigcup_{i \in I} f^{-1}(a_{i,h(i)}),$$

or else $\bigcup_{i \in I} f^{-1}(a_{i,h(i)}) = 1.$

Clearly (3) leads to the condition

$$f^{-1}(x) \subseteq \bigcup_{i \in I} f^{-1}(a_{i,h(i)}) \text{ for each } h \in J.$$  

Applying now the function $f$ to both sides of the inclusion of (4) and by the use of (2), we obtain the desired conclusion $x=0$. The theorem has been proved.

The condition $R_\alpha(A) = \{0\}$, as has been proved in Theorem 2, is both necessary and sufficient for $A$ to be $\alpha$-representable. Earlier, Smith [6] gave a sufficient condition for $A$ to be $\alpha$-representable and which he has shown to be not necessary. Furthermore, he pointed out (in [6]) that all those $\alpha$-complete Boolean algebras in which the so-called $\alpha$-distributive law holds satisfy his sufficient condition and, consequently, are $\alpha$-representable. We see quite easily from our definition of $R_\alpha(A)$ that if $R_\alpha(A) \neq \{0\}$ then clearly $A$ will not satisfy the $\alpha$-distributive law. One can also give a simple and direct argument that the condition $R_\alpha(A) = \{0\}$ is implied by his sufficient condition; however, we point out here that our condition was obtained without the knowledge of the results to be found in [6] and that the two approaches are entirely different.

The next theorem studies more closely the role that the ideals $R_\alpha(A)$ play in the problem of $\alpha$-representation.

**Theorem 3.** Let $A$ be an $\alpha$-complete Boolean algebra and let $N$ be an $\alpha$-complete ideal of $A$. Then $A/N$ is $\alpha$-representable if, and only if, $R_\alpha(A) \subseteq N$.

**Proof.** Assume that $A/N$ is $\alpha$-representable, i.e., $R_\alpha(A/N) = \{0/N\}$. Let $x \in R_\alpha(A)$ and $a_{i,j}$ be the associated $\alpha$-system of elements. It is evident that the elements $a_{i,j}/N$ of $A/N$ satisfy

$$(i) \quad \prod_{j \in J} [a_{i,j}/N] = 0/N \text{ for each } i \in I,$$

and

$$(ii) \quad \text{for every } h \in J, \text{ the set of elements } \{a_{i,h(i)}/N; i \in I\} \text{ contains either } x/N \text{ or a complementary pair.}$$

Thus, it follows from (i) and (ii) that $x/N \in R_\alpha(A/N)$ and $x \in N$.

On the other hand, assume that $R_\alpha(A) \subseteq N$. We shall prove that $R_\alpha(A/N) = \{0/N\}$. Let $x/N \in R_\alpha(A/N)$ and let $(a/N)_{i,j}$ be the associated $\alpha$-system of elements. Let us now pick representatives $a_{i,j}$ out of the cosets $(a/N)_{i,j}$ such
that if \((a/N)_{i,j} = x/N\), then \(a_{i,j} = x\), and such that if \((a/N)_{i,j} = \neg((a/N)_{i',j'})\), then \(a_{i,j} = \neg a_{i',j'}\). From this choice of representatives, it follows that

\[
\prod_{j \in J} a_{i,j} \in N \text{ for every } i \in I
\]

and

2. for each \(h \in J'\), the set of elements \(\{a_{i,h(i)}; i \in I\}\) either contains \(x\) or else contains a complementary pair.

Let now \(y = x \cdot \prod_{i \in I} \sum_{j \in J} a_{i,j}\) and let us pick a \(j' \in J\) and set \(J' = J \cup \{j'\}\). We define an \(\alpha\)-system of elements \(b_{i,j}\) indexed by the sets \(I\) and \(J'\) as follows:

(i) \(b_{i,j} = a_{i,j}\) if \(j \neq j'\) and \(a_{i,j} \neq x\),
(ii) \(b_{i,j} = y\) if \(j \neq j'\) and \(a_{i,j} = x\),

and

(iii) \(b_{i,j'} = y\).

It follows from (1), (2), and the definition of \(b_{i,j}\) that

\[
\prod_{j \in J'} b_{i,j} = 0 \text{ for each } i \in I,
\]

and

4. for every \(h \in J''\), the set of elements \(\{b_{i,h(i)}; i \in I\}\) either contains the element \(y\) or else contains a complementary pair.

Conditions (3) and (4) show that the element \(y \in R_\alpha(A)\) and hence, by our hypothesis, \(y \in N\). However, \(x = x \cdot y + y = x \cdot \prod_{i \in I} \sum_{j \in J} a_{i,j} + y\) and whence, by (2), \(x \in N\) and \(x/N = 0/N\). The proof is now complete. (It actually follows from the proof of Theorem 3 that \(R_\alpha(A/N) = R_\alpha(A)/N\) for any \(\alpha\)-complete ideal \(N\) of \(A\).)

Due to Theorem 3 we may now justly regard the ideal \(R_\alpha(A)\) as the \(\alpha\)-radical of an \(\alpha\)-complete Boolean algebra with respect to \(\alpha\)-representation. \(R_\alpha(A)\) is unique in the sense that any \(\alpha\)-complete ideal \(N'\) of \(A\) satisfying Theorem 3 with \(R_\alpha(A)\) replaced by \(N'\) must be identical with \(R_\alpha(A)\), i.e., \(R_\alpha(A) = N'\). Furthermore, we see that if \(\alpha\) and \(\beta\) are infinite cardinals and \(\beta \leq \alpha\), then \(R_\beta(A) \subseteq R_\alpha(A)\). It follows then for each \(\alpha\)-complete Boolean algebra \(A\) either \(A\) is \(\alpha\)-representable or else there exists a least \(\beta \leq \alpha\) for which \(R_\beta(A)\) does not vanish. The problem is open whether for all cardinals \(\alpha\) and \(\beta\) with \(\aleph_0 < \beta \leq \alpha\) there exists an \(\alpha\)-complete Boolean algebra \(A\) for which \(\beta\) is the least cardinal such that \(R_\beta(A)\) does not vanish. We shall see from Theorem 4 that if \(\beta = \aleph_0\), then \(R_\beta(A) = \{0\}\).

From the results in [5] and Theorem 2, the algebra \(L\) of Lebesgue measurable sets modulo the sets of measure zero is such that \(R_\gamma(L) \neq \{0\}\), where for the discussion in this paragraph we let \(\gamma = 2^{\aleph_0}\). Since the algebra \(L\) is
known to be of the power of the continuum, complete, and homogeneous\(^{(6)}\), we see immediately that \(R_\gamma(L)\) is a principal ideal of \(L\) and, what is more interesting, \(R_\gamma(L)\) simply coincides with \(L\). Another interesting example is the Boolean algebra \(B\) of the Borel sets modulo the sets of first category in a separable complete metric space \(S\). This Boolean algebra is also known as the algebra of regular open sets of \(S^{(6)}\). It is known that \(B\) is complete and is of the power of the continuum. Hence \(R_\gamma(B)\) is again a principal ideal. It is not difficult to see that for any regular open set \(x\) there exists a sequence of sets \(\{x_{i_0, i_1, \ldots, i_n}\}\) where each \(i_j\) is either 0 or 1,

\[x = x_0 + x_1,\]

and

\[x_{i_0, \ldots, i_n} = x_{i_0, \ldots, i_n, 0} + x_{i_0, \ldots, i_n, 1}\]

for each \(n\),

and such that for every choice of the index \(i\)

\[\prod_{n} x_{i_0, i_1, \ldots, i_n} = 0^{(7)}\]

From the above and Theorem 3.1 in [5] we see that again \(R_\gamma(B) = B\). Thus in the above two instances, not only are the algebras themselves not \(2^{\aleph_0}\)-representable, but any nontrivial \(2^{\aleph_0}\)-complete homomorphic image is also not \(2^{\aleph_0}\)-representable.

It should also be mentioned that Theorem 3 may be obtained in a metamathematical fashion by using Theorem 2 and the fact that the class of all \(\alpha\)-complete Boolean algebras which are \(\alpha\)-representable forms an equational class of algebras. As a matter of fact, Scott and Tarski have shown that the characterization given in Theorem 2 can be transformed in a natural way to yield a set of characterizing equations for the class of all \(\alpha\)-representable Boolean algebras\(^{(8)}\).

The connection between the result of Loomis \([3]\) concerning \(\sigma\)-complete Boolean algebras and Theorem 2 will be made clear by the following theorem.

**Theorem 4.** For any Boolean algebra \(A\), \(R_{\aleph_0}(A) = \{0\}\).

**Proof.** Let \(x \in R_{\aleph_0}(A)\) and let \(a_{i,j}\) be the associated \(\aleph_0\)-system of elements indexed by the sets \(I\) and \(J\) where we may assume \(I = J = \{\text{the set of all natural numbers}\}\). Suppose that \(x \neq 0\), thus \(\bar{x} \neq 1\). Hence \(1 \neq \bar{x} + \prod_{j \in J} a_{0,j} \) and \(1 \neq \prod_{j \in J} (\bar{x} + a_{0,j})\). We can now pick a \(j_0\) such that \(\bar{x} + a_{0,j_0} \neq 1\). If we proceed

\(^{(6)}\) For some details on the algebra \(L\), cf. [1, pp. 168–169 and p. 184].

\(^{(7)}\) For some details on the algebra \(B\), cf. [1, pp. 176–179].

\(^{(8)}\) Cf. footnote 5.

\(^{(8)}\) This result may be found in [4, Theorem 1].
in this fashion, we will pick an infinite sequence of elements $a_{0,i_0}$, $a_{1,i_1}$, $a_{2,i_2}$, \ldots such that

$$\bar{x} + a_{0,i_0} + a_{1,i_1} + \cdots + a_{i,i_i} \neq 1 \text{ for each } i \in I.$$ 

This clearly means that the function $h$ defined by the condition $h(i) = j_i$ for each $i \in I$ will yield a set of elements $\{a_{i,h(i)}; i \in I\}$ which will not contain $x$ and will not contain a complementary pair. Hence $x = 0$ and the theorem is proved.

For our subsequent discussion we introduce the following notion. An ideal $N$ of an $\alpha$-complete Boolean algebra $A$ preserves the $\alpha$-system of elements $a_{i,j}$ (of $A$) indexed by the sets $I$ and $J$ if, and only if,

$$(*) \quad \text{for each } i \in I, \sum_{j \in J} a_{i,j} \notin N \text{ if, and only if, } \bar{a}_{i,j} \in N \text{ for some } j \in J.$$

We see that if, in particular, $N$ is a maximal ideal, then condition $(*)$ can be replaced by the condition

$$(** \quad \text{for each } i \in I, \sum_{j \in J} a_{i,j} \in N \text{ if, and only if, } a_{i,j} \in N \text{ for every } j \in J.$$

In general, we see that condition $(*)$ implies the corresponding notion defined in [4] and which in turn implies condition $(**)$; however, for maximal ideals $N$ all three notions are equivalent. The following lemma will require no proof.

**Lemma.** An ideal $N$ preserves the $\alpha$-system of elements $a_{i,j}$ indexed by $I$ and $J$ if, and only if, $N$ preserves the $\alpha$-system of elements $b_{i,j}$ indexed by the sets $I$ and $J \cup \{j'\}$ ($j' \in J$) where $b_{i,j} = a_{i,j} \text{ if } j \neq j'$ and $b_{i,j'} = [\sum_{j \in J} a_{i,j}]^{-1}$ for each $i \in I$.

It follows from the lemma that if an ideal $N$ preserves all $\alpha$-systems of elements $a_{i,j}$ where \(1 = \sum_{j \in J} a_{i,j}\) for each $i \in I$, then $N$ preserves all $\alpha$-systems of elements.

**Theorem 5.** For any element $x$ of an $\alpha$-complete Boolean algebra $A$ the following four conditions are equivalent.

(i) $x \in R_\alpha(A)$.

(ii) For any $\alpha$-system of elements of $A$, there exists a proper ideal $N$ containing $\bar{x}$ and preserving the $\alpha$-system of elements and which is $\beta$-complete for every cardinal $\beta$ such that $\alpha^\beta \text{ has at most the power } \alpha$.

(iii) For any $\alpha$-system of elements of $A$, there exists a proper ideal $N$ containing $\bar{x}$ and preserving the $\alpha$-system of elements.

(iv) For any $\alpha$-system of elements of $A$, there exists a maximal ideal $M$ containing $\bar{x}$ and preserving the $\alpha$-system of elements.

**Proof.** The equivalence of (iii) and (iv) follows from the fact that if $N$ preserves an $\alpha$-system of elements, then any of its maximal extensions $M$ will
also preserve the same $\alpha$-system of elements. The implication (ii) to (iii) is obvious. We shall now show the implication of (iii) to (i) by contradiction. Suppose $x \in R_\alpha(A)$ and let $a_{i,j}$ be the associated $\alpha$-system of elements. Consider now a proper ideal $N$ which preserves the $\alpha$-system of elements $\tilde{a}_{i,j}$ and which contains $\tilde{x}$. Since $1 = \sum_{j \in J} \tilde{a}_{i,j}$ for each $i \in I$, it follows that

$$\text{(1)} \quad \text{for each } i \in I, a_{i,j} \in N \text{ for some } j \in J.$$ 

By (1), we define a function $h$, $h \in J'$, such that

$$\text{(2)} \quad a_{i,h(i)} \in N \text{ for each } i \in I.$$ 

Using (2) and the fact that $N$ is a proper ideal containing $\tilde{x}$, we see that the set of elements $\{a_{i,h(i)}; i \in I\}$ cannot contain the element $x$ nor a complementary pair. Hence we have a contradiction and $x \in R_\alpha(A)$.

Next we prove (ii) from (i). Let $a_{i,j}$ be an $\alpha$-system of elements indexed by $I$ and $J$ and such that

$$\text{(3)} \quad 1 = \sum_{j \in J} a_{i,j} \text{ for each } i \in I.$$ 

We may assume without loss of generality that the sets $I$ and $J$ have precisely the power $\alpha$. It follows from the lemma that it is sufficient if we can prove the existence of an ideal $N$ preserving $\alpha$-systems of the above special form. Notice that (3) leads to

$$\text{(4)} \quad 0 = \prod_{j \in J} \tilde{a}_{i,j} \text{ for each } i \in I.$$ 

Let $\beta$ be any cardinal such that $\alpha^\beta$ has at most the power $\alpha$. We let

$$I_{\beta} = \{\tilde{a}_{i,j}; i \in I, j \in J\}^\beta,$$

and

$$I' = I \cup \bigcup (I_{\beta}; \alpha^\beta \text{ has the power at most } \alpha).$$

Since the set $\{a_{i,j}; i \in I, j \in J\}$ has the power $\alpha$, it is clear that each set $I_{\beta}$ has the power at most $\alpha$ and the set $I'$ also has power at most $\alpha$. We now define (in any manner we wish) an $\alpha$-system of elements $b_{i,j}$ indexed by the sets $I' \cup \{i'\} (i' \in I')$ and $J$ and satisfying the following conditions:

$$\text{(5)} \quad b_{i,j} = \tilde{a}_{i,j} \text{ for } i \in I \text{ and } j \in J.$$ 

$$\text{(6)} \quad b_{i,j} = 0 \text{ for all } i \in J.$$ 

$$\text{(7)} \quad \{\tilde{x}; y \in f^*(\beta)\} \cup \{x\} \cup \left\{\sum_{\rho \in \beta} f(\rho) + \tilde{x}\right\} = \{b_{f,j}; j \in J\} \text{ for each } f \in I_{\beta}.$$ 

It follows readily from (4)-(7) that

$$\text{(8)} \quad \prod_{j \in J} b_{i,j} = 0 \text{ for each } i \in I' \cup \{i'\}.$$ 

Since $x \in R_\alpha(A)$ and $b_{i,j}$ is an $\alpha$-system of elements satisfying (8), there exists a function $h \in J'' \cup \{i'\}$ such that
the set of elements $K = \{ b_{i,h(i)} ; i \in I \cup \{ i' \} \}$ does not contain $x$ and does not contain a complementary pair.

From (6) and (9), we see that

(10) \[ 0 \in K \text{ and } 1 \notin K. \]

Let $L = \{ b_{i,h(i)} ; i \in I \} = \{ \bar{a}_{i,h(i)} ; i \in I \} \subseteq K$. For any $\beta$ such that $\alpha^\beta$ has power at most $\alpha$ and for any subset $L'$ of $L$ with power $\beta$ we can find a function $f$, $f \in I_{\beta}$, such that $L' = f^*(\beta)$. From (7) and (9) we see that

(11) \[ \sum_{\rho \in \beta} f(\rho) + \bar{x} = b_{f,h(f)} \in K. \]

(10) and (11) clearly imply that the least upper bound of any subset $L'$ of $L \cup \{ \bar{x} \}$ of power $\beta$ is different from 1. We now simply let

\[ N = \left\{ y; \text{for some } L' \subseteq L \cup \{ \bar{x} \}, L' \text{ has power } \beta, \text{ and } \alpha^\beta \text{ has power } \alpha, \text{ and } \right\}. \]

Obviously, $N$ is a proper ideal containing $\bar{x}$ and preserving the $\alpha$-system of elements $a_{i,j}$. Suppose $\alpha^\beta$ has power $\alpha$. For each $\xi \in \beta$, let $\gamma_\xi \in N$ and such that for some $\beta_\xi$, $\alpha^{\beta_\xi}$ has power $\alpha$ and

(13) \[ y_\xi \leq \sum_{\rho \in \beta_\xi} y_{\xi,\rho} \]

where

(14) \[ y_{\xi,\rho} \in L \cup \{ \bar{x} \} \text{ for each } \rho \in \beta_\xi. \]

Let $C$ be the cardinal sum of all the sets $\beta_\xi$ as $\xi \in \beta$. By the set-theoretical law on exponents, we see that $\alpha^C$ is set-theoretically equivalent to $P( \{ \alpha^{\beta_\xi} ; \xi \in \beta \} )$. Since $\alpha^{\beta_\xi}$ has the power $\alpha$ for each $\xi \in \beta$, we see that $P( \{ \alpha^{\beta_\xi} ; \xi \in \beta \} )$ is simply set-theoretically equivalent to $\alpha^\beta$ which again has the power $\alpha$. Hence $\alpha^C$ has power $\alpha$. From this and (14) it follows that there exists a subset $L'$ of $L \cup \{ \bar{x} \}$ with power $\beta'$ where $\alpha^{\beta'}$ has power at most $\alpha$ such that

(15) \[ \sum_{\xi \in \beta} \sum_{\rho \in \beta_\xi} y_{\xi,\rho} = \sum_{x \in L'} x. \]

Hence by (13) and (15), $\sum_{\xi \in \beta} y_\xi \leq \sum_{x \in L'} x$ and, by (12), $\sum_{\xi \in \beta} y_\xi \in N$. Thus we see that $N$ is $\beta$-complete for every $\beta$ such that $\alpha^\beta$ has the power $\alpha$. The theorem is proved.

Using Theorem 5 we can now present several characterizations of $\alpha$-representable Boolean algebras in the following(9):

\[ (*) \text{ The equivalence of 6(i) with 6(iv) is precisely a condition given in [4, Theor. 2]. This is easily seen from our remarks concerning the equivalence of conditions (\*) and (\*\*) in case } N \text{ is maximal.} \]
Theorem 6. For any \( \alpha \)-complete Boolean algebra \( A \) the following four conditions are equivalent:

(i) \( A \) is \( \alpha \)-representable.

(ii) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements and which is \( \beta \)-complete for every \( \beta \) such that \( \alpha^\beta \) has power at most \( \alpha \).

(iii) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements.

(iv) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a maximal ideal \( M \) containing \( x \) and preserving the \( \alpha \)-system of elements.

Proof. By Theorem 2 and Theorem 5.

In conclusion we point out the significance of condition 6(ii) in the following special application: If a Boolean algebra \( A \) is continuously-representable (i.e. \( 2^{2^{\aleph_0}} \)-representable), then for any \( x \neq 1 \) and any continuum-system of elements there exists a denumerably-complete proper ideal \( N \) containing \( x \) and preserving the continuum-system of elements.

References

2. C. C. Chang, A necessary and sufficient condition for an \( \alpha \)-complete Boolean algebra to be an \( \alpha \)-homomorphic image of an \( \alpha \)-complete field of sets, Bull. Amer. Math. Soc. Abstract 61-4-579.

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