ON THE REPRESENTATION OF $\alpha$-COMPLETE
BOOLEAN ALGEBRAS\(^{(1)}\)

BY

C. C. CHANG

Let $\alpha$ be an infinite cardinal. A Boolean algebra $A$ is $\alpha$-complete if every subset of $A$ with power (cardinality) at most $\alpha$ possesses a least upper bound in $A$. An ideal $I$ in a Boolean algebra is $\alpha$-complete in case the least upper bound (if it exists) of every subset of $I$ with power at most $\alpha$ belongs to $I$. A Boolean algebra that is $\aleph_\alpha$-complete is also called $\sigma$-complete. A field of sets $B$ is a Boolean algebra where the operations $\cup$, $\cap$, and $\complement$ are respectively the operations of set-union, set-intersection, and complementation with respect to the unit element of $B$. A field of sets $B$ is $\alpha$-complete if the union of any subset of $B$ with power at most $\alpha$ belongs to $B$.

By a theorem of Stone [7], every Boolean algebra is isomorphic to a field of sets. On the other hand, not every $\alpha$-complete Boolean algebra is isomorphic to some $\alpha$-complete field of sets; a necessary and sufficient condition for such a representation is that every principal ideal of the algebra be contained in an $\alpha$-complete maximal ideal (cf. [5]). In 1947, Loomis [3] proved that every $\sigma$-complete Boolean algebra $A$ is isomorphic to a $\sigma$-complete field of sets $B$ modulo a $\sigma$-complete ideal of $B$. The question was raised as to whether this result holds for all infinite cardinals $\alpha$. In 1948, Sikorski [5] showed that the Boolean algebra $L$ of Lebesgue measurable subsets of the unit interval modulo the sets of measure zero is $2^{\aleph_\alpha}$-complete but not isomorphic to any $2^{\aleph_\alpha}$-complete field of sets modulo a $2^{\aleph_\alpha}$-complete ideal.

It is the object of this note to give a necessary and sufficient condition for $\alpha$-complete Boolean algebras $A$ to be $\alpha$-representable, i.e., to be isomorphic to an $\alpha$-complete field of sets $B$ modulo an $\alpha$-complete ideal of $B$\(^{(2)}\). It turned out that to each $\alpha$-complete Boolean algebra $A$ there is associated an ideal $R_\alpha(A)$ which plays the role of a radical with respect to $\alpha$-representation, i.e., a homomorphic image of $A$ is $\alpha$-representable if, and only if, the kernel of the homomorphism includes $R_\alpha(A)$ (cf. Theorem 3). Our characterization, presented in Theorem 2, may be regarded as a generalization of the theorem of Loomis since every $\sigma$-complete Boolean algebra satisfies our condition with

---

\(^{(1)}\) The preparation of this paper was supported in part by the United States Navy under Contract No. NONR 401(20)-NR 043-167 monitored by the Office of Naval Research; reproduction in whole or in part is permitted for any purpose of the United States government.

\(^{(2)}\) This result was first announced in the abstract [2]. It represents the first known characterization of $\alpha$-representable Boolean algebras and gives a complete solution to Problem 80 in [1, p. 168].
\( \alpha = \aleph_0 \) (cf. Theorem 4). Recently, other characterizations of \( \alpha \)-representable Boolean algebras have been found (Scott and Tarski [4]) where the proofs given are metamathematical in nature. We shall give a purely algebraic and direct proof (Theorem 6) of the equivalence of the characterization given here and the one given in [4]. This equivalence can be established by analyzing certain properties which individual elements of a Boolean algebra must possess; in so doing, we have proved a theorem (Theorem 5) which, aside from its use in the problem of \( \alpha \)-representation, is of some interest in itself.

Let \( A \) be a Boolean algebra. We shall denote by \( \sum_{i \in I} a_i \) (\( \prod_{i \in I} a_i \)) the least upper bound (greatest lower bound) in \( A \) (if it exists) of the set \( \{a_i; i \in I\} \). If \( I \) is of power at most \( \alpha \), then an element of the form \( \sum_{i \in I} a_i \) (\( \prod_{i \in I} a_i \)) is called an \( \alpha \)-sum (\( \alpha \)-product). A system of elements \( a_{i,j} \) indexed by the sets \( I \) and \( J \) (i.e., \( a_{i,j} \) is an element of the Boolean algebra for \( i \in I \) and \( j \in J \)) is called an \( \alpha \)-system in case the sets \( I \) and \( J \) have powers at most \( \alpha \).

As usual, the complement of an element \( a \) of \( A \) shall be denoted by \( \bar{a} \). For typographical reasons, we shall denote the complement of a group of letters by enclosing the group of letters in square brackets followed by a bar, e.g., \([\sum_{i \in I} a_i]^-\). Whenever there is no possibility of confusion, we simply let \( \bar{a}_i = [a_i]^- \) and \( \bar{a}_{i,j} = [a_{i,j}]^- \). \( 0 \) and \( 1 \) shall denote respectively the zero and unit elements of \( A \). For arbitrary sets \( A \), \( P(A) \) denotes the set of all functions \( f \) with domain \( A \) and such that \( f(x) \in x \) for each \( x \in A \). We let \( J' \) denote the set of all functions \( f \) with domain \( I \) and range included in \( J \). If \( f \) is a function and \( X \) is a set, \( f^*(X) \) is the image of \( X \) under \( f \). We assume that ordinals have been defined in such a way that every ordinal coincides with the set of smaller ordinals. A cardinal can be understood as an ordinal which has larger power than every smaller ordinal.

**Definition.** If \( A \) is a Boolean algebra and \( \alpha \) an infinite cardinal, then \( R_\alpha(A) \) shall denote the set of all elements \( x \in A \) for which there exists an \( \alpha \)-system of elements \( a_{i,j} \in A \) indexed by the sets \( I \) and \( J \) such that

(i) \[ \prod_{j \in J} a_{i,j} = 0 \text{ for each } i \in I, \]

and

(ii) for each function \( f, f \in J' \), the set of elements \( \{a_{i,f(i)}; i \in I\} \) either contains \( x \) or else contains some complementary pair of elements \( b \) and \( \bar{b} \).

We see readily from the definition that \( 0 \in R_\alpha(A) \).

**Theorem 1.** If \( A \) is an \( \alpha \)-complete Boolean algebra, then \( R_\alpha(A) \) is an \( \alpha \)-complete ideal of \( A \) and \( A/R_\alpha(A) \) is \( \alpha \)-representable.

**Proof.** We shall first prove that there exists a homomorphism \( f \) of \( A \) onto an \( \alpha \)-complete field of sets \( B \) modulo an \( \alpha \)-complete ideal \( N \) of \( B \), then prove that this homomorphism \( f \) preserves \( \alpha \)-sums, and, finally, that the kernel of this homomorphism is \( R_\alpha(A) \).
For each \( x \in A \), we let \( x^* = \{x, \bar{x}\} \) and \( A^* = \{x^*; x \in A\} \). We define a function \( g \) on the elements of \( A \) to subsets of \( P(A^*) \) such that
\[
g(x) = \{ h; h \in P(A^*) \text{ and } h(x^*) = x \}.
\]
It is clear that for every \( x \in A \), \( g(x) \cap g(\bar{x}) = \emptyset \) and \( g(x) \cup g(\bar{x}) = P(A^*) \). We let \( B \) be the \( \alpha \)-complete field of sets generated by the elements of \( g^*(A) \) in the (complete) field of all subsets of \( P(A^*) \). Furthermore, let \( M = \{ \bigcap_{i \in I} g(x_i); I \) has at most power \( \alpha \), \( x_i \in A \) for each \( i \in I \), and \( \prod_{i \in I} x_i = 0 \} \) and \( N \) be the \( \alpha \)-complete ideal generated by \( M \) in \( B \). We now wish to show that \( B/N \) is a homomorphic image of \( A \) by the following mapping \( f \):
\[
f(x) = g(x)/N.
\]
It is clear that \( f^*(A) \) generates \( B/N \). Now \( f(\bar{x}) = g(\bar{x})/N = [P(A^*) \sim g(x)]/N = [g(x)]^\sim/N \) and hence \( f \) preserves complementation. Let \( a = \sum_{i \in I} a_i \) be an \( \alpha \)-sum of elements of \( A \). In order to show \( f(a) = \sum_{i \in I} f(a_i) \) it is sufficient to prove that the symmetric difference of \( g(a) \) and \( \bigcup_{i \in I} g(a_i) \) is an element of \( N \). Since \( I \) has power at most \( \alpha \) and \( a \cdot \prod_{i \in I} \bar{a}_i = 0 \), we obtain \( g(a) \cap \bigcap_{i \in I} g(\bar{a}_i) \in N \). But
\[
g(a) \cap \bigcap_{i \in I} g(\bar{a}_i) = g(a) \cap \bigcap_{i \in I} [g(a_i)]^\sim = g(a) \cap \left[ \bigcup_{i \in I} g(a_i) \right]^\sim,
\]
hence
\[(1) \quad g(a) \cap \left[ \bigcup_{i \in I} g(a_i) \right]^\sim \in N.
\]
On the other hand, \( \bar{a} \cdot a = 0 \) for each \( i \in I \), thus \( g(\bar{a}) \cap g(a_i) = [g(a)]^\sim \cap g(a_i) \in N \) for each \( i \in I \). Since \( N \) is \( \alpha \)-complete, \( \bigcup_{i \in I} ([g(a)]^\sim \cap g(a_i)) \in N \), and
\[(2) \quad [g(a)]^\sim \cap \bigcup_{i \in I} g(a_i) \in N.
\]
It follows from (1) and (2) that the symmetric difference of \( g(a) \) and \( \bigcup_{i \in I} g(a_i) \) belongs to \( N \). Thus \( f \) preserves \( \alpha \)-sums and \( f^*(A) \) is an \( \alpha \)-complete subalgebra of \( B/N \). Hence \( f^*(A) = B/N \) and \( f \) maps \( A \) homomorphically onto \( B/N \) preserving all \( \alpha \)-sums of elements of \( A \).

It remains to prove that the kernel of \( f \) is the set \( R_\alpha(A) \). If \( f(x) = 0 \), then \( g(x) \in N \). We see that the condition \( g(x) \in N \) is equivalent to the following: there exists an \( \alpha \)-system of elements \( a_{i,j} \) indexed by the sets \( I \) and \( J \) such that
\[
\begin{align*}
(i) \quad \prod_{j \in J} a_{i,j} &= 0 \text{ for each } i \in I, \\
(ii) \quad \bigcap_{j \in J} g(a_{i,j}) &\in M \text{ for each } i \in I,
\end{align*}
\]

(9) The idea of using the elements of \( P(A^*) \) as points in the representation was discussed in [3].
and

\( g(x) \subseteq \bigcup_{i \in I} \bigcap_{j \in J} g(a_{i,j}). \)

By the set-theoretical distributive law,

\( \bigcup_{i \in I} \bigcap_{j \in J} g(a_{i,j}) = \bigcap_{h \in J'} \bigcup_{i \in I} g(a_{i,h(i)}). \)

Hence (iii) together with (3) imply

\( g(x) \subseteq \bigcup_{i \in I} g(a_{i,h(i)}) \) for each \( h \in J'. \)

If the set \( \{a_{i,h(i)}; i \in I\} \) does not contain a complementary pair of elements, then any two different elements \( a_{i,h(i)} \) and \( a_{j,h(j)} \), with \( i \neq j \), belong to different elements of \( A^* \). If, in addition, the set \( \{a_{i,h(i)}; i \in I\} \) does not contain \( x \), then clearly there exists a function \( k \in P(A^*) \) such that \( k(x^*) = x \) and \( k(a_{i,h(i)^*}) = a_{i,h(i)} \) for each \( i \in I \), i.e.,

\( k \in g(x) \) and \( k \notin g(a_{i,h(i)}) \) for each \( i \in I \).

(5) is a contradiction to (4). Hence \( x \in R_\alpha(A) \). On the other hand, let \( x \in R_\alpha(A) \) and let \( a_{i,j} \) be the associated \( \alpha \)-system of elements indexed by the sets \( I \) and \( J \). Clearly conditions (i), (ii), and (4) are satisfied by the elements \( a_{i,j} \). By (3) and (i) we see that \( \bigcap_{h \in J'} \bigcup_{i \in I} g(a_{i,h(i)}) \subseteq N \), which, together with (4) imply (iii). Thus \( g(x) \subseteq N \) and \( f(x) = 0 \). The theorem has been proved(4).

**Theorem 2.** Let \( A \) be an \( \alpha \)-complete Boolean algebra. Then \( A \) is \( \alpha \)-representable if, and only if, \( R_\alpha(A) = \{0\} \).

**Proof.** Obviously, if \( R_\alpha(A) = \{0\} \), then by Theorem 1 \( A \) is \( \alpha \)-representable. Let \( f \) be a homomorphism of an \( \alpha \)-complete field of sets \( B \) onto \( A \) and such that \( f \) preserves all \( \alpha \)-sums of \( B \). Let \( x \in R_\alpha(A) \) and let \( a_{i,j} \) be the associated \( \alpha \)-system of elements. We choose an inverse \( f^{-1} \) to the function \( f \) satisfying the condition: that \( f^{-1}(y) = [f^{-1}(y)]^{-} \) for each \( y \in A \). It is evident that such an inverse can always be chosen. Since \( \prod_{i \in I} a_{i,j} = 0 \) for each \( i \in I \) and since \( f \) preserves all \( \alpha \)-sums (and hence all \( \alpha \)-products), we obtain

\[ \bigcap_{j \in J} f^{-1}(a_{i,j}) \subseteq B \quad \text{and} \quad f \left( \bigcap_{j \in J} f^{-1}(a_{i,j}) \right) = 0 \quad \text{for every} \quad i \in I. \]

From (1), it follows that \( \bigcup_{i \in I} f^{-1}(a_{i,j}) \subseteq B \), \( f(\bigcup_{i \in I} \bigcap_{j \in J} f^{-1}(a_{i,j})) = 0 \), and, by an application of the set-theoretical distributive law,

\[ f \left( \bigcap_{h \in J'} \bigcup_{i \in I} f^{-1}(a_{i,h(i)}) \right) = 0. \]

(4) The fact that the ideal \( R_\alpha(A) \) is \( \alpha \)-complete can be proved without resorting to the homomorphism \( f \) and without even the assumption of the \( \alpha \)-completeness of \( A \).
By our choice of the inverse function $f^{-1}$, we see that for every $h \in J^I$, either $f^{-1}(x) \subseteq \bigcup_{i \in I} f^{-1}(a_{i,h(i)})$, or else $\bigcup_{i \in I} f^{-1}(a_{i,h(i)}) = 1$. Clearly (3) leads to the condition:

$$f^{-1}(x) \subseteq \bigcup_{i \in I} f^{-1}(a_{i,h(i)}) \text{ for each } h \in J^I.$$  

Applying now the function $f$ to both sides of the inclusion of (4) and by the use of (2), we obtain the desired conclusion $x = 0$. The theorem has been proved.

The condition $R_\alpha(A) = \{0\}$, as has been proved in Theorem 2, is both necessary and sufficient for $A$ to be $\alpha$-representable. Earlier, Smith [6] gave a sufficient condition for $A$ to be $\alpha$-representable and which he has shown to be not necessary. Furthermore, he pointed out (in [6]) that all those $\alpha$-complete Boolean algebras in which the so-called $\alpha$-distributive law holds satisfy his sufficient condition and, consequently, are $\alpha$-representable. We see quite easily from our definition of $R_\alpha(A)$ that if $R_\alpha(A) \neq \{0\}$ then clearly $A$ will not satisfy the $\alpha$-distributive law. One can also give a simple and direct argument that the condition $R_\alpha(A) = \{0\}$ is implied by his sufficient condition; however, we point out here that our condition was obtained without the knowledge of the results to be found in [6] and that the two approaches are entirely different.

The next theorem studies more closely the role that the ideals $R_\alpha(A)$ play in the problem of $\alpha$-representation.

**THEOREM 3.** Let $A$ be an $\alpha$-complete Boolean algebra and let $N$ be an $\alpha$-complete ideal of $A$. Then $A/N$ is $\alpha$-representable if, and only if, $R_\alpha(A) \subseteq N$.

**Proof.** Assume that $A/N$ is $\alpha$-representable, i.e., $R_\alpha(A/N) = \{0/N\}$. Let $x \in R_\alpha(A)$ and $a_{i,j}$ be the associated $\alpha$-system of elements. It is evident that the elements $a_{i,j}/N$ of $A/N$ satisfy:

(i) $$\prod_{j \in J} [a_{i,j}/N] = 0/N \text{ for each } i \in I,$$

and

(ii) for every $h \in J^I$, the set of elements $\{a_{i,h(i)}/N; i \in I\}$ contains either $x/N$ or a complementary pair.

Thus, it follows from (i) and (ii) that $x/N \in R_\alpha(A/N)$ and $x \in N$.

On the other hand, assume that $R_\alpha(A) \subseteq N$. We shall prove that $R_\alpha(A/N) = \{0/N\}$. Let $x/N \in R_\alpha(A/N)$ and let $(a/N)_{i,j}$ be the associated $\alpha$-system of elements. Let us now pick representatives $a_{i,j}$ out of the cosets $(a/N)_{i,j}$ such...
that if \((a/N)_{i,j} = x/N\), then \(a_{i,j} = x\), and such that if \((a/N)_{i,j} = [(a/N)_{i',j'}]^\prime\), then \(a_{i,j} = a_{i',j'}^\prime\). From this choice of representatives, it follows that

\[(1) \prod_{j \in J} a_{i,j} \in N \text{ for every } i \in I\]

and

\[(2) \text{ for each } h \in J', \text{ the set of elements } \{a_{i,h(h)}; i \in I\} \text{ either contains } x \text{ or else contains a complementary pair.}\]

Let now \(y = x \cdot \prod_{i \in I} \sum_{j \in J} a_{i,j} \) and let us pick a \(j' \in J\) and set \(J' = J \cup \{j'\}\). We define an \(\alpha\)-system of elements \(b_{i,j}\) indexed by the sets \(I\) and \(J'\) as follows:

\[(i) b_{i,j} = a_{i,j} \text{ if } j \neq j' \text{ and } a_{i,j} \neq x,\]

\[(ii) b_{i,j} = y \text{ if } j \neq j' \text{ and } a_{i,j} = x,\]

and

\[(iii) b_{i,j'} = y.\]

It follows from (1), (2), and the definition of \(b_{i,j}\) that

\[(3) \prod_{j \in J'} b_{i,j} = 0 \text{ for each } i \in I,\]

and

\[(4) \text{ for every } h \in J'', \text{ the set of elements } \{b_{i,h(h)}; i \in I\} \text{ either contains the element } y \text{ or else contains a complementary pair.}\]

Conditions (3) and (4) show that the element \(y \in \mathcal{R}_a(A)\) and hence, by our hypothesis, \(y \in N\). However, \(x = x \cdot \bar{y} + y = x \cdot \sum_{i \in I} \prod_{j \in J} a_{i,j} + y\) and whence, by (2), \(x \in N\) and \(x/N = 0/N\). The proof is now complete. (It actually follows from the proof of Theorem 3 that \(\mathcal{R}_a(A/N) = \mathcal{R}_a(A)/N\) for any \(\alpha\)-complete ideal \(N\) of \(A\).)

Due to Theorem 3 we may now justly regard the ideal \(\mathcal{R}_a(A)\) as the \(\alpha\)-radical of an \(\alpha\)-complete Boolean algebra with respect to \(\alpha\)-representation. \(\mathcal{R}_a(A)\) is unique in the sense that any \(\alpha\)-complete ideal \(N'\) of \(A\) satisfying Theorem 3 with \(\mathcal{R}_a(A)\) replaced by \(N'\) must be identical with \(\mathcal{R}_a(A)\), i.e., \(\mathcal{R}_a(A) = N'\). Furthermore, we see that if \(\alpha\) and \(\beta\) are infinite cardinals and \(\beta \leq \alpha\), then \(\mathcal{R}_\beta(A) \subseteq \mathcal{R}_\alpha(A)\). It follows then for each \(\alpha\)-complete Boolean algebra \(A\) either \(A\) is \(\alpha\)-representable or else there exists a least \(\beta \leq \alpha\) for which \(\mathcal{R}_\beta(A)\) does not vanish. The problem is open whether for all cardinals \(\alpha\) and \(\beta\) with \(\aleph_0 < \beta \leq \alpha\) there exists an \(\alpha\)-complete Boolean algebra \(A\) for which \(\beta\) is the least cardinal such that \(\mathcal{R}_\beta(A)\) does not vanish. We shall see from Theorem 4 that if \(\beta = \aleph_0\), then \(\mathcal{R}_\beta(A) = \{0\}\).

From the results in [5] and Theorem 2, the algebra \(L\) of Lebesgue measurable sets modulo the sets of measure zero is such that \(R_\gamma(L) \neq \{0\}\), where for the discussion in this paragraph we let \(\gamma = 2^{\aleph_0}\). Since the algebra \(L\) is
known to be of the power of the continuum, complete, and homogeneous\(^{(6)}\), we see immediately that \(R_\gamma(L)\) is a principal ideal of \(L\) and, what is more interesting, \(R_\gamma(L)\) simply coincides with \(L\). Another interesting example is the Boolean algebra \(B\) of the Borel sets modulo the sets of first category in a separable complete metric space \(S\). This Boolean algebra is also known as the algebra of regular open sets of \(S\)\(^{(6)}\). It is known that \(B\) is complete and is of the power of the continuum. Hence \(R_\gamma(B)\) is again a principal ideal. It is not difficult to see that for any regular open set \(x\) there exists a sequence of sets \(\{x_{i_0,i_1,\ldots,i_n}\}\) where each \(i_j\) is either 0 or 1,

\[ x = x_0 + x_1, \]

and

\[ x_{i_0,\ldots,i_n} = x_{i_0,\ldots,i_n,0} + x_{i_0,\ldots,i_n,1} \]

for each \(n\), and such that for every choice of the index \(i\)

\[ \prod_{n} x_{i_0,i_1,\ldots,i_n} = 0 \]

From the above and Theorem 3.1 in [5] we see that again \(R_\gamma(B) = B\). Thus in the above two instances, not only are the algebras themselves not \(2^{\aleph_0}\)-representable, but any nontrivial \(2^{\aleph_0}\)-complete homomorphic image is also not \(2^{\aleph_0}\)-representable.

It should also be mentioned that Theorem 3 may be obtained in a metamathematical fashion by using Theorem 2 and the fact that the class of all \(\alpha\)-complete Boolean algebras which are \(\alpha\)-representable forms an equational class of algebras. As a matter of fact, Scott and Tarski have shown that the characterization given in Theorem 2 can be transformed in a natural way to yield a set of characterizing equations for the class of all \(\alpha\)-representable Boolean algebras\(^{(8)}\).

The connection between the result of Loomis [3] concerning \(\sigma\)-complete Boolean algebras and Theorem 2 will be made clear by the following theorem.

**Theorem 4.** For any Boolean algebra \(A\), \(R_{\aleph_0}(A) = \{0\}\).

**Proof.** Let \(x \in R_{\aleph_0}(A)\) and let \(a_{i,j}\) be the associated \(\aleph_0\)-system of elements indexed by the sets \(I\) and \(J\) where we may assume \(I = J = \text{the set of all natural numbers}\). Suppose that \(x \neq 0\), thus \(\bar{x} \neq 1\). Hence \(1 \neq \bar{x} + \prod_{i \in J} a_{0,i}\) and \(1 \neq \prod_{i \in J} (\bar{x} + a_{0,i})\). We can now pick a \(j_0\) such that \(\bar{x} + a_{0,j_0} \neq 1\). If we proceed

\(^{(5)}\) For some details on the algebra \(L\), cf. [1, pp. 168-169 and p. 184].

\(^{(6)}\) For some details on the algebra \(B\), cf. [1, pp. 176-179].

\(^{(7)}\) Cf. footnote 5.

\(^{(8)}\) This result may be found in [4, Theorem 1].
in this fashion, we will pick an infinite sequence of elements \( a_{0, i_0}, a_{1, i_1}, a_{2, i_2}, \cdots \) such that
\[
x + a_{0, i_0} + a_{1, i_1} + \cdots + a_{i, i_i} \neq 1 \text{ for each } i \in I.
\]
This clearly means that the function \( h \) defined by the condition \( h(i) = j_i \) for each \( i \in I \) will yield a set of elements \( \{ a_{i, h(i)} ; i \in I \} \) which will not contain \( x \) and will not contain a complementary pair. Hence \( x = 0 \) and the theorem is proved.

For our subsequent discussion we introduce the following notion. An ideal \( N \) of an \( \alpha \)-complete Boolean algebra \( A \) preserves the \( \alpha \)-system of elements \( a_{i, j} \) (of \( A \)) indexed by the sets \( I \) and \( J \) if, and only if,
\[
(*) \quad \text{for each } i \in I, \sum_{j \in J} a_{i, j} \in N \text{ if, and only if, } a_{i, J} \in N \text{ for some } j \in J.
\]
We see that if, in particular, \( N \) is a maximal ideal, then condition \( (*) \) can be replaced by the condition
\[
(**) \quad \text{for each } i \in I, \sum_{j \in J} a_{i, j} \in N \text{ if, and only if, } a_{i, j} \in N \text{ for every } j \in J.
\]
In general, we see that condition \( (*) \) implies the corresponding notion defined in \([4]\) and which in turn implies condition \( (**) \); however, for maximal ideals \( N \) all three notions are equivalent. The following lemma will require no proof.

**Lemma.** An ideal \( N \) preserves the \( \alpha \)-system of elements \( a_{i, j} \) indexed by \( I \) and \( J \) if, and only if, \( N \) preserves the \( \alpha \)-system of elements \( b_{i, j} \) indexed by the sets \( I \) and \( J \cup \{ j' \} (j' \in J) \) where \( b_{i, j} = a_{i, j} \) if \( j \neq j' \) and \( b_{i, j'} = \{ \sum_{j \in J} a_{i, j} \}^{-1} \) for each \( i \in I \).

It follows from the lemma that if an ideal \( N \) preserves all \( \alpha \)-systems of elements \( a_{i, j} \) where \( 1 = \sum_{j \in J} a_{i, j} \) for each \( i \in I \), then \( N \) preserves all \( \alpha \)-systems of elements.

**Theorem 5.** For any element \( x \) of an \( \alpha \)-complete Boolean algebra \( A \) the following four conditions are equivalent.

(i) \( x \in R^\alpha(A) \).
(ii) For any \( \alpha \)-system of elements of \( A \), there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements and which is \( \beta \)-complete for every cardinal \( \beta \) such that \( \alpha^\beta \) has at most the power \( \alpha \).
(iii) For any \( \alpha \)-system of elements of \( A \), there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements.
(iv) For any \( \alpha \)-system of elements of \( A \), there exists a maximal ideal \( M \) containing \( x \) and preserving the \( \alpha \)-system of elements.

**Proof.** The equivalence of (iii) and (iv) follows from the fact that if \( N \) preserves an \( \alpha \)-system of elements, then any of its maximal extensions \( M \) will
also preserve the same \( \alpha \)-system of elements. The implication (ii) to (iii) is obvious. We shall now show the implication of (iii) to (i) by contradiction. Suppose \( x \in R_\alpha(A) \) and let \( a_{i,j} \) be the associated \( \alpha \)-system of elements. Consider now a proper ideal \( N \) which preserves the \( \alpha \)-system of elements \( \bar{a}_{i,j} \) and which contains \( \bar{x} \). Since \( 1 = \sum_{i \in J} \bar{a}_{i,j} \) for each \( i \in I \), it follows that

\[
(1) \quad \text{for each } i \in I, \ a_{i,j} \in N \text{ for some } j \in J.
\]

By (1), we define a function \( h, h \in J_1 \), such that

\[
(2) \quad a_{i,h(i)} \in N \text{ for each } i \in I.
\]

Using (2) and the fact that \( N \) is a proper ideal containing \( \bar{x} \), we see that the set of elements \( \{ a_{i,h(i)} ; i \in I \} \) cannot contain the element \( x \) nor a complementary pair. Hence we have a contradiction and \( x \notin R_\alpha(A) \).

Next we prove (ii) from (i). Let \( a_{i,j} \) be an \( \alpha \)-system of elements indexed by \( I \) and \( J \) and such that

\[
(3) \quad 1 = \sum_{j \in J} a_{i,j} \text{ for each } i \in I.
\]

We may assume without loss of generality that the sets \( I \) and \( J \) have precisely the power \( \alpha \). It follows from the lemma that it is sufficient if we can prove the existence of an ideal \( N \) preserving \( \alpha \)-systems of the above special form. Notice that (3) leads to

\[
(4) \quad 0 = \prod_{j \in J} \bar{a}_{i,j} \text{ for each } i \in I.
\]

Let \( \beta \) be any cardinal such that \( \alpha^\beta \) has at most the power \( \alpha \). We let

\[
I_\beta = \{ \bar{a}_{i,j} ; i \in I, j \in J \}^\beta,
\]

and

\[
I' = I \cup \bigcup (I_\beta ; \alpha^\beta \text{ has the power at most } \alpha).
\]

Since the set \( \{ a_{i,j} ; i \in I, j \in J \} \) has the power \( \alpha \), it is clear that each set \( I_\beta \) has the power at most \( \alpha \) and the set \( I' \) also has power at most \( \alpha \). We now define (in any manner we wish) an \( \alpha \)-system of elements \( b_{i,j} \) indexed by the sets \( I' \cup \{ i' \} (i' \in I') \) and \( J \) and satisfying the following conditions:

\[
(5) \quad b_{i,j} = \bar{a}_{i,j} \text{ for } i \in I \text{ and } j \in J.
\]

\[
(6) \quad b_{i'} = 0 \text{ for all } i' \in J.
\]

\[
(7) \quad \{ y ; y \in f^*(\beta) \} \cup \{ x \} \cup \left\{ \sum_{j \in J} f(\rho) + \bar{x} \right\} = \{ b_{j,i} ; j \in J \} \text{ for each } f \in I_\beta.
\]

It follows readily from (4)–(7) that

\[
(8) \quad \prod_{j \in J} b_{i,j} = 0 \text{ for each } i \in I' \cup \{ i' \}.
\]

Since \( x \in R_\alpha(A) \) and \( b_{i,j} \) is an \( \alpha \)-system of elements satisfying (8), there exists a function \( h \in J'' \cup \{ i' \} \) such that
the set of elements $K = \{ b_{i,h(t)} ; i \in I' \cup \{ i' \} \}$ does not contain $x$ and does not contain a complementary pair.

From (6) and (9), we see that

(10) $0 \in K$ and $1 \notin K$.

Let $L = \{ b_{i,h(t)} ; i \in I \} = \{ a_{i,h(t)} ; i \in I \} \subseteq K$. For any $\beta$ such that $\alpha^{\beta}$ has power at most $\alpha$ and for any subset $L'$ of $L$ with power $\beta$ we can find a function $f$, $f \in I_{\beta}$, such that $L' = f^{*}(\beta)$. From (7) and (9) we see that

(11) $\sum_{\rho \in \beta} f(\rho) + x = b_{f,h(t)} \in K$.

(10) and (11) clearly imply that the least upper bound of any subset $L'$ of $L \cup \{ x \}$ of power $\beta$ is different from 1. We now simply let

$$N = \{ y; \text{for some } L' \subseteq L \cup \{ x \}, L' \text{ has power } \beta, \text{ and } \alpha^{\beta} \text{ has power } \alpha, \text{ and}$$

(12) $y \leq \sum_{x \in L'} x \}$.  

Obviously, $N$ is a proper ideal containing $x$ and preserving the $\alpha$-system of elements $a_{i,j}$. Suppose $\alpha^{\beta}$ has power $\alpha$. For each $\xi \in \beta$, let $y_{\xi} \in N$ and such that for some $\beta_{\xi}$, $\alpha^{\beta_{\xi}}$ has power $\alpha$ and

(13) $y_{\xi} \leq \sum_{\rho \in \beta_{\xi}} y_{\xi,\rho}$

where

(14) $y_{\xi,\rho} \in L \cup \{ x \} \text{ for each } \rho \in \beta_{\xi}$.

Let $C$ be the cardinal sum of all the sets $\beta_{\xi}$ as $\xi \in \beta$. By the set-theoretical law on exponents, we see that $\alpha^{\beta}$ is set-theoretically equivalent to $P(\{ \alpha^{\beta_{\xi}} ; \xi \in \beta \})$. Since $\alpha^{\beta_{\xi}}$ has the power $\alpha$ for each $\xi \in \beta$, we see that $P(\{ \alpha^{\beta_{\xi}} ; \xi \in \beta \})$ is simply set-theoretically equivalent to $\alpha^{\beta}$ which again has the power $\alpha$. Hence $\alpha^{\beta}$ has power $\alpha$. From this and (14) it follows that there exists a subset $L'$ of $L \cup \{ x \}$ with power $\beta'$ where $\alpha^{\beta'}$ has power at most $\alpha$ such that

(15) $\sum_{\xi \in \beta} \sum_{\rho \in \beta_{\xi}} y_{\xi,\rho} = \sum_{x \in L'} x$.

Hence by (13) and (15), $\sum_{\xi \in \beta} y_{\xi} \leq \sum_{x \in L'} x$ and, by (12), $\sum_{\xi \in \beta} y_{\xi} \in N$. Thus we see that $N$ is $\beta$-complete for every $\beta$ such that $\alpha^{\beta}$ has the power $\alpha$. The theorem is proved.

Using Theorem 5 we can now present several characterizations of $\alpha$-representable Boolean algebras in the following:

(*) The equivalence of 6(i) with 6(iv) is precisely a condition given in [4, Theo 2]. This is easily seen from our remarks concerning the equivalence of conditions (*) and (***) in case $N$ is maximal.
Theorem 6. For any \( \alpha \)-complete Boolean algebra \( A \) the following four conditions are equivalent:

(i) \( A \) is \( \alpha \)-representable.

(ii) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements and which is \( \beta \)-complete for every \( \beta \) such that \( \alpha^{\beta} \) has power at most \( \alpha \).

(iii) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a proper ideal \( N \) containing \( x \) and preserving the \( \alpha \)-system of elements.

(iv) For any \( x \neq 1 \) and any \( \alpha \)-system of elements, there exists a maximal ideal \( M \) containing \( x \) and preserving the \( \alpha \)-system of elements.

Proof. By Theorem 2 and Theorem 5.

In conclusion we point out the significance of condition 6(ii) in the following special application: If a Boolean algebra \( A \) is continuously-representable (i.e. \( 2^{K_0} \)-representable), then for any \( x \neq 1 \) and any continuum-system of elements there exists a denumerably-complete proper ideal \( N \) containing \( x \) and preserving the continuum-system of elements.

References

2. C. C. Chang, A necessary and sufficient condition for an \( \alpha \)-complete Boolean algebra to be an \( \alpha \)-homomorphic image of an \( \alpha \)-complete field of sets, Bull. Amer. Math. Soc. Abstract 61-4-579.