ON THE FUNDAMENTAL THEOREMS OF THE CALCULUS

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1. **Introduction.** As is well known the fundamental theorem of the differential calculus states the following: *If f is continuous in the closed interval $[a, b]$ and if $f'=0$ in the interior of $[a, b]$ then f is constant.* A principal application of this theorem occurs in the theory of primitive functions, for it implies that the difference of any two primitives of a given function is constant and so establishes the connection between the elementary indefinite and definite integrals. This is known as the fundamental theorem of the integral calculus. In one form this theorem states: *If f has a continuous derivative in the closed interval $[a, b]$ then $\int_a^b f'(x)dx=f(b)-f(a)$.*

On the one hand the concept of definite integrals can be extended to various classes of real functions for example by use of the Lebesgue and Perron integration and on the other hand the definition of primitive functions can be changed from the classical definition at will. One is interested in such generalizations of primitive functions for which the above connection between primitives and definite integrals remains valid. This is the main reason for trying to generalize the fundamental theorem of the differential calculus. The best known among such generalizations is Dini's theorem [3, p. 204] which follows from an interesting result of Zygmund [3, p. 203] and a recent result in this direction is due to Aumann [1, p. 222].

The object of the present paper is to prove a generalization of the fundamental theorem of the differential calculus which includes all earlier results as special cases. The hypothesis of the new result is a combination of one condition given by Zygmund and another one given by Aumann.

2. **Results.** In order to state the theorems in a simple form we shall say that a property holds *nearly everywhere* if it holds everywhere except possibly on a countable set of points. As usual the word countable means finite or enumerable. It is also convenient to use the word interval to denote any open, closed or half-open half-closed interval of nonzero length. The right upper derivative will be denoted by D^+f.

Occasionally we shall use the notations

$$\Lambda^-(x) = \limsup_{\xi\to x-0} f(\xi) \quad \text{and} \quad \Lambda^+(x) = \limsup_{\xi\to x+0} f(\xi).$$

As is well known $\Lambda^-(x)\leqq f(x)$ is equivalent to the left upper semicontinuity of f at x.

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THEOREM 1. *Let the real valued f be defined in the finite closed interval $[a, b]$ and*

1° *let $\Lambda^-(x) \leqq f(x) \leqq \Lambda^+(x)$ for every $x \in [a, b]$,*

2° *let $D^+f \geqq 0$ almost everywhere in $[a, b]$,*

3° *let $D^+f > -\infty$ nearly everywhere in $[a, b]$.*

Then $f(a) \leqq f(b)$.

Condition 1° occurs in Zygmund's result and it implies the left upper semicontinuity of f at the point x. Conditions 2° and 3° first occur in Titchmarsh [4, p. 372]. Aumann assumes the left upper and right lower semicontinuity of f in addition to 2° and 3°. Titchmarsh considers only continuous functions and Bourbaki [2] assumes besides continuity the existence of a finite right derivative nearly everywhere in the interval. Titchmarsh uses the theory of Lebesgue measure, Dini's proof is based on the least upper bound property of the reals. Aumann, Zygmund and Bourbaki work along the ideas of Dini but Aumann freely uses the theory of the exterior measure.

The present method is based on the explicit use of the compactness of the finite closed interval. It can be used also to prove various other results of the above type among which we mention the following simple ones: *If f is absolutely continuous in $[a, b]$ and if $D^+f \geqq 0$ almost everywhere then f is increasing in $[a, b]$*. Of course it is well known that an absolutely continuous function is differentiable almost everywhere but the point is that the above theorem can be proved without any reference to this deeper result. Another simple application of the present method will be given in the proof of a simple monotony criterion: *If ϕ is upper semicontinuous in the closed interval $[a, b]$ and for every x; $a \leqq x < b$ there is a y; $x < y < b$ such that $\phi(x) < \phi(y)$ then ϕ is increasing.*

In the second part of this paper we shall use our method to prove the following extension of the above theorem:

THEOREM 2. *Let the real valued f be defined in the finite closed interval $[\alpha, \beta]$ and*

1° *let $\Lambda^-(x) \leqq f(x) \leqq \Lambda^+(x)$ for every $x \in [\alpha, \beta]$,*

2° *let $\mu(t)$ $(-\infty \leqq t \leqq 0)$ denote the exterior measure of the set*

$$S(t) = [x \mid \alpha \leqq x < \beta \quad \text{and} \quad D^+f(x) < t],$$

and suppose that $\mu(t) \to 0$ as $t \to -\infty$,

3° *let $D^+f > -\infty$ nearly everywhere in $[\alpha, \beta]$.*

Then

$$f(\beta) - f(\alpha) \geqq \int_{-\infty}^{0} t d\mu(t).$$

The proof given in §4 is completely elementary and self contained. We shall use only the simplest properties of open sets on the real line, the compactness of a finite closed interval and the concept of the exterior measure.

The integral can be interpreted as the Riemann integral of the increasing function $\mu(t)$. Since $d\mu(t)$ is non-negative the integral converges to a finite negative value or is $-\infty$. If the integral is finite then the lower variation of f over $[\alpha, \beta]$ is finite.

For continuous functions there is a sharper lower estimate for $f(\beta)-f(\alpha)$ than the one given in Theorem 2. For we can prove the following:

THEOREM 3. *Let the real valued f be continuous in the finite closed interval $[a, b]$ and let $D^+f > -\infty$ nearly everywhere in $[a, b]$. Denote by $\mu(t)$ $(-\infty \leqq t \leqq +\infty)$ the measure of the set*

$$S(t) = [x \mid a \leqq x < b \text{ and } D^+f(x) < t].$$

Then

$$f(b) - f(a) \geqq \int_{-\infty}^{+\infty} t d\mu(t)$$

whenever the integral $\int_{-\infty}^{0} t d\mu(t)$ converges.

It is well known that D^+f is a measurable function whenever f is measurable [3, p. 113] We need this in the proof of the theorem.

3. **Proof of Theorem 1.** In order to keep the main idea of the proof from getting lost among the details of the proof we first give an outline of the proof: We assume that f satisfies the hypotheses of Theorem 1 and $-\infty < a < b + \infty$. The object is to prove that $f(a) \leqq f(b)$.

Let $\epsilon > 0$ be given and let ϕ be defined by $\phi(x) = f(x) + \epsilon(x-a)$ for $a \leqq x \leqq b$. We are going to determine a sequence of points

$$a = \xi_1 < \xi_2 < \cdots < \xi_r < \cdots < \xi_s < \xi_{s+1} = b$$

in such a way that $\phi(\xi_r) \leqq \phi(\xi_{r+1}) + \epsilon_r$ and $\sum \epsilon_r < \epsilon$. Then $\phi(a) \leqq \phi(b) + \epsilon$, that is to say $f(a) \leqq f(b) + \epsilon(b-a+1)$ and since $\epsilon > 0$ is arbitrary $f(a) \leqq f(b)$. To this end we construct a suitable covering of the finite closed interval $[a, b]$ by open intervals I_x and apply the Heine-Borel theorem to select a finite subcovering of the compact interval $[a, b]$.

The covering of $[a, b]$ by the collection I_x is assured if we define an open interval I_x containing the point x for every $x \in [a, b]$. The length of I_x will be determined by the value of D^+f at x and the modulus of upper semicontinuity of f at x.

We recall a simple lemma [3, p. 261] from the theory of arbitrary functions of a real variable:

LEMMA 1. *If ϕ is any real valued function defined on an interval I then*

$$\limsup_{\xi \to x-0} \phi(\xi) \leqq \limsup_{\xi \to x+0} \phi(\xi)$$

nearly everywhere in I.

This lemma has interesting implications concerning the continuity of ϕ which unfortunately can not be found in the textbook literature. Of these only two are relevant here. One relates to the condition given by Aumann: If ϕ is left upper and right lower semicontinuous nearly everywhere in I then ϕ is continuous nearly everywhere in I. This corollary implies immediately the well known result about the continuity of increasing functions. The other consequence of the lemma is needed in the present proof:

LEMMA 2. *If ϕ is left upper semicontinuous nearly everywhere in I then ϕ is upper semicontinuous nearly everywhere in I.*

Now suppose that f satisfies the hypothesis of the theorem. Let $\epsilon>0$ be fixed. We define ϕ by $\phi(x)=f(x)+\epsilon(x-a)$ for $a\leqq x\leqq b$. We can find a $\delta=\delta(\epsilon)>0$ such that $\phi(\xi)<\phi(b)+\epsilon$ whenever $b-\delta<\xi\leqq b$. We assume that $\delta<b-a$.

A simple consequence of the left upper semicontinuity of ϕ will be used: If $\phi(x)<\phi(y)+\eta$ for some x and y with $a<x<y<b$ then $\phi(\xi)<\phi(y)+\eta$ for every $\xi\leqq x$ which is sufficiently close to x. A similar statement holds at those points x at which ϕ is right upper semicontinuous: If ϕ is right upper semicontinuous at x and $\phi(x)<\phi(y)+\eta$ with $a<x<y<b$ then $\phi(\xi)<\phi(y)+\eta$ for every $\xi\geqq x$ which is sufficiently close to x. Consequently we have the following:

LEMMA 3. *If $\phi(x)<\phi(y)+\eta$ for some x and y with $a<x<y<b$ and ϕ is right upper semicontinuous at x then there is an open interval I_x such that $x\in I_x$, $y\notin I_x$ and $\phi(\xi)<\phi(y)+\eta$ for every $\xi\in I_x$.*

Now we shall determine a collection $\{I_x\}$ of open intervals. For every $a\leqq x<b$ exactly one of the following must hold:

CASE 1. *The point x is such that $-\infty<D^+f(x)<\epsilon$.*

CASE 2. *f is upper semicontinuous at x and $D^+f(x)\geqq\epsilon$.*

CASE 3. *f is not upper semicontinuous at x or $D^+f(x)=-\infty$.*

For the point b we define I_b to be the interval $b-\delta<\xi<b+\delta$. For the point x; $a\leqq x<b$ the open intervals I_x are determined in the following way:

CASE 1. Let S_k denote the set of those points $x<b$ for which $\epsilon-k\leqq D^+\phi(x)<\epsilon-(k-1)$. Since $D^+\phi$ is finite ϕ is right upper semicontinuous at every $x\in S_k$ and so ϕ is upper semicontinuous at every such point. By the hypothesis of the theorem S_k is a set of measure zero for every $k=1, 2, \cdots$. Let S_k be covered by a system $\{I_{kl}\}$ of open intervals whose total length is so small that $k\sum|I_{kl}|<\epsilon 2^{-k}$.

If $x\in S_k$ then $x\in I_{kl}$ for some index l and $\epsilon-k\leqq D^+\phi(x)$. Hence using the fact that I_{kl} is open we can find a y; $x<y<b$ such that $y\in I_{kl}$ and $k(x-y)<\phi(y)-\phi(x)$. By the upper semicontinuity of ϕ at x we can determine an open interval I_x such that $x\in I_x$, $y\notin I_x$ and $\phi(\xi)<\phi(y)+k(y-x)$ for every $\xi\in I_x$. We may choose I_x so that its length is at most δ.

CASE 2. Now we consider the set of those points x; $a\leqq x<b$ at which ϕ

is upper semicontinuous and $D^+\phi \geqq \epsilon$. To every such point x there is a y; $x<y<b$ such that $\phi(x)<\phi(y)$. Since ϕ is upper semicontinuous at x we may apply the above lemma and determine an open interval I_x containing x not y and such that $\phi(\xi)<\phi(y)$ for every $\xi \in I_x$. We may assume that the length of I_x is at most δ.

CASE 3. The set of those points x; $a \leqq x<b$ at which ϕ is not upper semicontinuous is countable because ϕ is left upper semicontinuous everywhere in I. By the hypothesis of the theorem the set of those points x at which $D^+f(x) = -\infty$ is also a countable set. Therefore the set of points x; $a \leqq x<b$ which come under Case 3 is a countable set. Let $x_1, \cdots, x_n, \cdots$ be an arbitrary ordering of the set. By the definition of the limit superior of ϕ on the one hand there is a point y; $x_n<y<b$ such that $\Lambda^+(x_n)<\phi(y)+\epsilon 2^{-(n+1)}$ and on the other hand $\phi(\xi)<\Lambda^+(x_n)+\epsilon 2^{-(n+1)}$ for every $\xi>x_n$ whenever ξ is sufficiently close to x_n. The above inequality holds also for $\xi=x_n$; in fact, by hypothesis 1° we have $\phi(x_n) \leqq \Lambda^+(x_n)$. Hence $\phi(\xi)<\phi(y)+\epsilon 2^{-n}$ for every $\xi \geqq x_n$ whenever ξ is close to x_n. Therefore by the left upper semicontinuity of ϕ at x_n the inequality remains valid for every $\xi \leqq x_n$ which is sufficiently close to x_n. Thus to every x_n there is a y; $x_n<y<b$ and an open interval I_{x_n} such that $x_n \in I_{x_n}$, $y \notin I_{x_n}$ and $\phi(\xi)<\phi(y)+\epsilon 2^{-n}$ for every $\xi \in I_{x_n}$. Of course we may assume that the length of I_{x_n} is at most δ.

The collection of all open intervals I_x; $a \leqq x \leqq b$ covers the interval $[a, b]$. By the compactness of the finite closed interval $[a, b]$ we can select a finite subcollection of the system $\{I_x\}$ which also covers $[a, b]$. Let the intervals of the subcollection be denoted by $I(1), \cdots, I(m)$.

Now we construct a sequence of points

$$a = \xi_1 < \xi_2 < \cdots < \xi_r < \cdots < \xi_s < \xi_{s+1} = b$$

as follows: Let $\xi_1=a$. $\xi_1 \notin I_b$ because $\delta<b-a$. Since ξ_1 is covered by at least one of the intervals $I(1), \cdots, I(m)$ there is a point y to the right of this interval such that at least one of the following inequalities hold:

$$\phi(\xi_1) < \phi(y),$$
$$\phi(\xi_1) < \phi(y) + k_1(y - \xi_1),$$
$$\phi(\xi_1) < \phi(y) + \epsilon 2^{-n_1}.$$

We put $\xi_2=y$. Let us now assume that the points $\xi_1<\xi_2< \cdots <\xi_r<b$ are already determined. Then the point ξ_r is covered by at least one of the intervals $I(1), \cdots, I(m)$. Either ξ_r is covered by I_b in which case we define $r=s$ and $\xi_{s+1}=b$, or else there is a y right of the interval covering ξ_r such that at least one of the following inequalities hold for $y=\xi_{r+1}$:

$$\phi(\xi_r) \leqq \phi(\xi_{r+1}),$$
$$\phi(\xi_r) \leqq \phi(\xi_{r+1}) + k_r(\xi_{r+1} - \xi_r),$$
$$\phi(\xi_r) \leqq \phi(\xi_{r+1}) + \epsilon 2^{-n_r}.$$

Here k_r and n_r denote the particular values of k and n which correspond to the interval I_x covering ξ_r. After at most m steps the procedure stops because ξ_r does not belong to any of the intervals which were constructed earlier and we obtain the desired sequence such that at least one of the inequalities given above holds for each $r<s$.

We have

$$\phi(\xi_1) < \phi(\xi_s) + \sum k_r(\xi_{r+1} - \xi_r) + \epsilon \sum 2^{-n_r}$$

where the summations are extended over some, not necessarily all indices r. The n_r's are distinct positive integers so that $\sum 2^{-n_r}<1$. Moreover according to the construction in Case 1 ξ_r and ξ_{r+1} belong to the same interval $I_{k_r l}$ and so

$$\sum k_r(\xi_{r+1} - \xi_r) \leqq \sum_{k=1}^{\infty} \sum_{k_r=k} k_r(\xi_{r+1} - \xi_r) \leqq \sum_{k=1}^{\infty} k \sum_{l=1}^{\infty} |I_{kl}| \leqq \sum_{k=1}^{\infty} \epsilon 2^{-k}.$$

Consequently $\phi(\xi_1)<\phi(\xi_s)+2\epsilon$. Since $\xi_1=a$ and $\xi_s\in I_b$ we have, by the definitions of I_b and δ,$\phi(\xi_s)<\phi(b)+\epsilon$ and so we proved that $\phi(a)<\phi(b)+3\epsilon$ where $\epsilon>0$ is arbitrary. Thus $f(a)\leqq f(b)$ and the theorem is proved.

4. **Proof of Theorem 2.** First let us recall that given a set S on the real line and $\epsilon>0$ there is a countable set of *disjoint open intervals* $\{I_n\}$ such that the union of these intervals covers S and $\sum|I_n|<\mu+\epsilon$ where μ denotes the exterior measure of S. For let a system of open intervals with total length less than $\mu+\epsilon$ be given such that it covers S. The union of all open intervals of this covering is an open set and hence it is the union of countably many disjoint open intervals I_n. We may also assume that the disjoint open covering $\{I_n\}$ of S does not contain adjacent open intervals. If a covering of a set S by disjoint open intervals has this additional property we say that $\{I_n\}$ is a *normal covering* of the set S.

Another simple remark concerns the covering of a set and any one of its subsets: Let S be a set on the real line and let $\{I_n\}$ be a covering of S by open intervals. If $S^*\subseteq S$ and if μ^* is the exterior measure of S^* then for any $\epsilon^*>0$ there is a covering $\{I_m^*\}$ of S^* such that $\sum|I_m^*|<\mu^*+\epsilon^*$ and *each* I_m^* *is a subinterval* of some I_n. In fact S^* can be covered by a system of open intervals J_n^* whose total length is less than $\mu^*+\epsilon^*$. Hence S is covered by the intersection of the two open sets $\bigcup_n J_n^*$ and $\bigcup_n I_n$ because $S^*\subseteq S$. The intersection is also open and so it is the union of countably many disjoint open intervals I_m^* with $\sum|I_m^*|\leqq\sum|J_n^*|<\mu^*+\epsilon^*$.

If $\{I_m^*\}$ and $\{I_n\}$ are systems of open intervals such that every I_m^* is a subinterval of some I_n then we say that $\{I_m^*\}$ is a *refinement* of $\{I_n\}$. If the intervals of the system $\{I_n\}$ are disjoint then $\{I_m^*\}$ is a refinement of $\{I_n\}$ if and only if $\bigcup I_m^*\subseteq\bigcup I_n$.

Using these remarks we can easily prove the following

LEMMA 4. *Let $S_0\supseteq S_1\supseteq\cdots\supseteq S_r\supseteq\cdots$ be a sequence of sets on an interval and let μ_r denote the exterior measure of S_r. Then given any sequence of errors*

$\epsilon_r > 0$ there is a sequence of normal coverings $\{I_{nr}\}$ such that $\sum_n |I_{nr}| < \mu_r + \epsilon_r$ and $\{I_{n,r+1}\}$ is a refinement of $\{I_{nr}\}$ for every r.

A similar statement holds for the interior measure of an increasing sequence of sets on the real line which is a direct consequence of the foregoing lemma. However the use of this statement in the proofs can be easily avoided.

Now comes the proof of the theorem: Let $t_0 = 0$ and let $\{t_r\}$ be any strictly increasing sequence or real numbers $t_0 = 0 < t_1 < \cdots < t_r < \cdots$ such that $t_{r+1} - t_r > \epsilon > 0$ for every r. For every value of $r = 0, 1, \cdots$ we consider the set

$$S_r = S(-t_r) = [x \mid \alpha \leqq x < \beta \text{ and } D^+f(x) < -t_r].$$

If no confusion can arise we shall use the simpler notation S_r. Clearly we have $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_r \supseteq \cdots$. According to the definition of $\mu(t)$ given in the text of the theorem the exterior measure of S_r is $\mu(-t_r) = \mu_r$. We choose $\epsilon(t) = \epsilon(t^3 + t^{-3})^{-1}$ where $\epsilon > 0$ and apply the above lemma to the sequence $\{S_r\}$.

By the lemma we can find a sequence of normal coverings $\{I_{n,r}\}$ of the sets S_r such that each covering system is a refinement of its predecessor in the sequence of coverings. For every $r = 0, 1, \cdots$ let $\{J_{m,r}\}$ denote a countable set of open intervals which together cover the set $\bigcup_n I_{n,r} - \bigcup_n I_{n,(r+1)}$ and satisfy

$$\sum_m |J_{m,r}| < \mu(-t_r) - \mu(-t_{r+1}) + \epsilon(-t_r) \tag{1}$$

The set $S = \bigcap_r \bigcup_n I_{n,r}$ has measure zero because $\mu(S) \leqq \mu(\bigcup_n I_{n,r}) < \mu_r + \epsilon_r$ and by hypothesis $\mu_r \to 0$ as $r \to \infty$. Therefore the set of those points $x \in S$ for which $-t_{r+1} \leqq D^+f(x) < -t_r$ form a set of measure zero. We can enlarge the covering system $\{J_{m,r}\}$ such that the enlarged system covers these points and its total length still satisfies (1). Denote the new covering system again by $\{J_{m,r}\}$. Then we have

If $\alpha \leqq x < \beta$ and $-\infty < D^+f(x) < 0$ then $x \in J_{m,r}$ for some indices m, r.

Our next object is to cover $[\alpha, \beta]$ by a suitable system of open intervals. We shall construct for each x; $\alpha \leqq x \leqq \beta$ an open interval I_x which contains x and so $[\alpha, \beta]$ will be certainly covered by the union of the intervals I_x. We distinguish three different cases:

CASE 1. *The point x is such that $-\infty < D^+f(x) < 0$.*

CASE 2. *The point x is such that f is upper semicontinuous at x and $D^+f(x) \geqq 0$.*

CASE 3. *The point x is such that f is not upper semicontinuous at x or $D^+f(x) = -\infty$.*

The point β does not belong to any of the above groups. Since f is left upper semicontinuous at β there is a $\delta > 0$ such that $f(\xi) < f(\beta) + \epsilon$ for every ξ; $\beta - \delta < \xi < \beta$. We assume that $\delta < \beta - \alpha$. We define I_β to be the interval $\beta - \delta < \xi < \beta + \delta$.

CASE 1. If $-\infty < D^+f(x) < 0$ then there is a unique index $r \geqq 0$ such that $x \in J_{m,r}$ for some index m and so $-t_{r+1} \leqq D^+f(x)$. Consequently we can find a y; $x < y < \beta$ such that $y \in J_{m,r}$ and $f(x) < f(y) + (t_{r+1} + \epsilon) \cdot (y - x)$. Hence f being upper semicontinuous at x there is an open interval $I_x \in J_{m,r}$ such that $x \in I_x$, $y \notin I_x$ and

$$f(\xi) < f(y) + (t_{r+1} + \epsilon)(y - \xi) \text{ for every } \xi \in I_x.$$

We may choose I_x so small that its length is less than δ.

CASE 2. If x comes under this case then $\alpha \leqq x < \beta$ and $-\epsilon < D^+f(x)$. Hence there is a y such that $x < y < \beta$ and $f(x) < f(y) + \epsilon(y - x)$. By the upper semicontinuity of f at x we can select an open interval I_x such that $x \in I_x$, $y \notin I_x$ and

$$f(\xi) < f(y) + \epsilon(y - \xi) \text{ for every } \xi \in I_x.$$

We assume again that the length of I_x is less than δ.

CASE 3. There are at most enumerably many points which come under this case. For f is left upper semicontinuous everywhere and so it is upper semicontinuous nearly everywhere in $[\alpha, \beta]$. Let $x_1, x_2, \cdots, x_n, \cdots$ be any ordering of this enumerable set. By the hypothesis of the theorem $f(x_n) \leqq \Lambda^+(x_n)$ for every $n = 1, 2, \cdots$. Hence there is a point y; $x_n < y < \beta$ such that $f(x_n) \leqq \Lambda^+(x_n) < f(y) + \epsilon 2^{-(n+1)}$. Moreover $f(\xi) < \Lambda^+(x_n) + \epsilon 2^{-(n+1)}$ for every $\xi > x_n$ whenever ξ is sufficiently close to x_n. Therefore $f(\xi) < f(y) + \epsilon 2^{-n}$ whenever $\xi = x_n$ or $\xi > x_n$ but is close to x_n. By the left upper semicontinuity of f at x_n the inequality remains valid for every $\xi < x_n$ which is sufficiently close to x_n. Thus to every x_n there is a y; $x_n < y < \beta$ and an open interval I_{x_n} such that $x_n \in I_{x_n}$, $y \notin I_{x_n}$ and $f(\xi) < f(y) + \epsilon 2^{-n}$ for every $\xi \in I_{x_n}$. We choose I_{x_n} so that its length is less than δ.

The system of all open intervals I_x: $\alpha \leqq x \leqq \beta$ covers $[\alpha, \beta]$. Hence by the Heine-Borel theorem we can select a finite subcovering of $[\alpha, \beta]$. Let $I(1), \cdots, I(m)$ denote the intervals of this finite covering. Now we select a sequence

$$\alpha = \xi_1 < \xi_2 < \cdots < \xi_r < \cdots < \xi_s < \xi_{s+1} = \beta$$

as follows: We put $\xi_1 = \alpha$. Then $\xi_1 \notin I_\beta$. We consider the point y such that

$$f(\xi_1) < f(y) + (t(1) + \epsilon)(y - \xi_1),$$
$$f(\xi_1) < f(y) + \epsilon(y - \xi_1),$$

or

$$f(\xi_1) < f(y) + \epsilon 2^{-n_1}$$

according as the interval $I(k_1)$ which covers ξ_1 comes under Case 1, 2 or 3. Here $t(1)$ denotes the value of t_{p+1} which corresponds to the interval $I(k_1)$ covering $\xi_1 = a$. Similarly n_1 is the exponent which corresponds to the point $x = x_{n_1}$ if x comes under Case 3. Let us assume that the points $\xi_1 < \cdots < \xi_r < \beta$

are already determined in a suitable way. Then ξ_r belongs to at least one of the intervals $I(1), \cdots, I(m)$. If ξ_r is covered by I_β then we define $r=s$ and $\xi_{s+1}=\beta$. If $\xi_r \notin I_\beta$ then there is a y right of the interval $I(k_r)$ covering ξ_r such that

$$f(\xi_r) < f(y) + (t(r) + \epsilon)(y - \xi_r),$$
$$f(\xi_r) < f(y) + \epsilon(y - \xi_r),$$

or

$$f(\xi_r) < f(y) + \epsilon 2^{-n_r}$$

according as $I(k_r)$ belongs to Case 1, 2 or 3. The number $t(r)$ denotes that particular value of t_{p+1} which corresponds to the interval $I(k_r)$ covering ξ_r. We put $y=\xi_{r+1}$.

After a finite number of steps this process stops because the number of intervals $I(k)$ is finite and the point ξ_r does not belong to any of the intervals $I(k_1), \cdots, I(k_r)$. According to the construction for each index $r<s$ at least one of the following inequalities holds:

$$f(\xi_r) < f(\xi_{r+1}) + (t(r) + \epsilon)(\xi_{r+1} - \xi_r),$$
$$f(\xi_r) < f(\xi_{r+1}) + \epsilon(\xi_{r+1} - \xi_r),$$
$$f(\xi_r) < f(\xi_{r+1}) + \epsilon 2^{-n_r}.$$

Choosing the right inequality for each index r and adding these inequalities we obtain by $\xi_1=\alpha$

$$f(\alpha) < f(\xi_s) + \sum (t(r) + \epsilon)(\xi_{r+1} - \xi_r) + \epsilon \sum (\xi_{r+1} - \xi_r) + \sum \epsilon 2^{-n_r}$$

where the positive integers n_r are all distinct. Hence

$$f(\alpha) < f(\xi_s) + \sum t(r)(\xi_{r+1} - \xi_r) + 2\epsilon \sum (\xi_{r+1} - \xi_r) + \epsilon.$$

Since the sequence ξ_r is increasing the sum $\sum(\xi_{r+1}-\xi_r)$ does not exceed $\beta-\alpha$. Moreover according to the construction under Case 1 ξ_r and ξ_{r+1} belong to the same interval $J_{m,p}$ whenever ξ_r comes under Case 1. Therefore by (1)

$$\begin{aligned}\sum t(r)(\xi_{r+1} - \xi_r) &< \sum_p t_p \sum_m |J_{m,p}| \\ &< \sum_p t_p[\mu(-t_p) - \mu(-t_{p+1})] + \sum_p t_p\epsilon(-t_p).\end{aligned}$$

Hence by the definition of $\epsilon(t)$

$$f(\alpha) < f(\xi_s) - \int_{-\infty}^{0} t d\mu(t) + c\epsilon$$

where c depends on the sequence $\{t_p\}$ but is independent of ϵ. We have $\xi_j \in I_\beta$

and so by the definitions of I_β and $\delta>0$, $f(\xi_s)<f(\beta)+\epsilon$. Consequently we proved that

$$f(\alpha) < f(\beta) - \int_{-\infty}^{0} t d\mu(t) + C\epsilon$$

where $\epsilon>0$ is arbitrary and C is a constant which is independent of ϵ. Thus the theorem is proved.

5. Additional results and applications. Theorem 1 in its present form is a montony criterion. In fact if the hypotheses 1°, 2° and 3° are satisfied for an interval $[a, b]$ then they are satisfied as well for any subinterval $[\alpha, \beta]$ of $[a, b]$ and so $f(\alpha)\leqq f(\beta)$. If we are interested only in the inequality $f(a)\leqq f(b)$ then it is possible to generalize Theorem 1 as follows:

THEOREM 4. *Let f be defined in the interval $[a, b]$ and*

1°. *let $\Lambda^-(x)\leqq f(x)\leqq \Lambda^+f(x)$ for every $x\in[a, b]$,*

2°. *let $\operatorname*{lub}_{x<y} \dfrac{f(y)-f(x)}{y-x}\geqq 0$ for almost all $x\in[a, b]$,*

3°. *let $D^+f(x)>-\infty$ nearly everywhere in $[a, b]$.*

Then $f(a)\leqq f(b)$.

The proof of this result is identical with the proof of Theorem 1. For condition 2° of Theorem 1 was used only to show the existence of a y; $x<y<b$ such that $\phi(x)<\phi(y)$. If f satisfies the present condition 2° the same conclusion holds for $\phi(x)=f(x)+\epsilon(x-a)$.

The third condition in Theorems 1 and 2 is necessary in the following sense: Let any nonenumerable subset S of $[a, b]$ be given. Then there exist real functions f satisfying conditions 1° and 2° and such that $D^+f(x)>-\infty$ for every $x\notin S$ but nevertheless $f(b)-f(a)<\int_{-\infty}^{0} t d\mu(t)$. For as is well known there exist singular decreasing functions such that $f'=0$ on the complement of S and $f(a)-f(b)>c$ where c is any positive value given in advance.

Theorem 1 has a few applications to Perron integration. For let any real valued function f in $[a, b]$ be given. We say that ϕ *is a minor function of f if $\phi(a)=f(a)$ and*

1°. $\Lambda_-(x)\geqq\phi(x)\geqq\Lambda^+(x)$ *everywhere in* $[a, b]$,

2°. $D^+\phi(x)\leqq f(x)$ *almost everywhere in* $[a, b]$,

3°. $D^+\phi(x)<+\infty$ *nearly everywhere in* $[a, b]$.

Similarly ψ *is a major function of f if $\psi(a)=f(a)$ and*

1°. $\Lambda^-(x)\leqq\psi(x)\leqq\Lambda+(x)$ *everywhere in* $[a, b]$,

2°. $f(x)\leqq D+\psi(x)$ *almost everywhere in* $[a, b]$,

3°. $-\infty<D+\psi(x)$ *nearly everywhere in* $[a, b]$.

The difference $\psi-\phi$ satisfies conditions 1°, 2° and 3° of Theorem 1 for any

subinterval $[\alpha, \beta]$ of $[a, b]$. Consequently $\psi(\beta)-\phi(\beta)\geqq\psi(\alpha)-\phi(\alpha)$ and also $\psi(\alpha)-\phi(\alpha)\geqq\psi(a)-\phi(a)=0$ so that

$$0 \leqq \psi(\alpha) - \phi(\alpha) \leqq \psi(\beta) - \phi(\beta) \tag{5}$$

for every subinterval $[\alpha, \beta]$ of $[a, b]$.

The rest follows a simple pattern of reasoning: For every f which has both minor and major functions we can define its lower and upper Perron integral over the range $[a, b]$ as

$$(\mathcal{P}_l)\int_a^b f = \operatorname{lub} \phi(b) \quad \text{and} \quad (\mathcal{P}^u)\int_a^b f = \operatorname{glb} \psi(b)$$

where the supremum and the infimum are understood over all minor and major functions of f respectively. The existence of these values follow from the inequality $\psi(b)-\phi(b)\geqq 0$ which was established by using Theorem 1. The function f is called Perron integrable over $[a, b]$ if $(\mathcal{P}_l)\int_a^b f=(\mathcal{P}^u)\int_a^b f=(\mathcal{P})\int_a^b f$. According to (5) if f is Perron integrable over $[a, b]$ then it is Perron integrable over every subinterval $[\alpha, \beta]$ of $[a, b]$.

The function Pf defined by the integral $(\mathcal{P})\int_a^x f$ for every $x\in[a, b]$ is the Perron indefinite integral or Perron primitive of f in $[a, b]$. In the usual definition of the Perron integral and the Perron primitive it is either required that the upper and lower functions be continuous in $[a, b]$ or else one uses stronger differentiability conditions than the present ones. (See [1, 239] and [3, 191].) The continuity of Pf under the present conditions follows immediately from hypothesis 1° for major and minor functions.

As usual we obtain an immediate necessary and sufficient condition for Perron integrability from the definition of the integral: *A real valued function f is Perron integrable over $[a, b]$ if and only if there exist minor and major functions ϕ and ψ such that $\psi(b)-\phi(b)<\epsilon$ where $\epsilon>0$ is given in advance.*

Using this definition of the Perron integral one can give several extensions of Theorem 3 to noncontinuous functions. For instance we have: *Let f be such that $\Lambda^-(x)\leqq f(x)\leqq\Lambda_+(x)$ everywhere and $D_+f(x)>-\infty$ nearly everywhere* in $[a, b]$. *If g is Perron integrable over $[a, b]$ and $D_+f(x)\geqq g(x)$ almost everywhere then $f(b)-f(a)\geqq(\mathcal{P})\int_a^b g$.* In fact ψ with $\psi(x)=f(x)-f(a)+g(a)$ is a major function of g and so the inequality follows immediately.

Stronger consequences can be derived by assuming that f is continuous everywhere and $D_+f(x)>-\infty$, $D^+f(x)<+\infty$ nearly everywhere in $[a, b]$. For instance (a) *if there is a Perron integrable function g such that $D^+f(x)\leqq g(x)\leqq D_+f(x)$ almost everywhere then $f(b)-f(a)=\int_a^b g$.* Also (b) *if f' exists almost everywhere then f' is Perron integrable and $f(b)-f(a)=(\mathcal{P})\int_a^b f'$.* (See [3, p. 205].) The proofs of these statements are immediate.

The following proof of Theorem 3 is based on the measurability of the Dini derivatives of a measurable function [3, p. 113]: Let us suppose that f satisfies the condition $\Lambda^-(x)\leqq f(x)\leqq\Lambda^+(x)$ everywhere in $[a, b]$ and let

$D^+f(x) > -\infty$ nearly everywhere. The same conditions hold good also for the function g defined by $g(x) = f(x) - f(a) - T(x-a)$. Moreover since $D^+g(x) = D^+f(x) - T$ the measure $\mu_g(t)$ of the set $S(t)$ where $D^+g(x) < t$ is the same as the measure $\mu_f(t+T)$ of the set $S_g(t+T)$ where $D^+f(x) < t+T$. By Theorem 2 we obtain $g(b) - g(a) \geqq \int_{-\infty}^{0} t d\mu_g$ and so $f(b) - f(a) \geqq T(b-a) + \int_{-\infty}^{0} t d\mu_f(t+T)$. Theorem 3 follows from the additivity of the measure.

In §2 we promised to give a simple proof for the following lemma:

If f is absolutely continuous and if $D^+f(x) \geqq 0$ almost everywhere in $[a, b]$ then $f(a) \leqq f(b)$. Hence f is increasing in $[a, b]$.

Indeed let $\epsilon > 0$ and let ϕ be defined by $\phi(x) = f(x) + \epsilon(x-a)$. We denote by S the set of those points $x \in [a, b]$ where $D^+\phi(x) < \epsilon$. By the hypothesis S is a set of measure zero, and so it can be covered by a system of open intervals $\{I(n)\}$ such that the variation of ϕ over $\bigcup I(n)$ is less than ϵ.

For every $x \notin \bigcup I(n)$ let I_x be determined as follows: $D^+\phi(x) \geqq \epsilon$ and so there is a y; $x < y < b$ such that $\phi(x) < \phi(y)$. Hence $\phi(\xi) < \phi(y)$ for every $\xi \in I_x$ provided $x \in I_x$, $y \in I_x$ and the length of I_x is sufficiently small. If $x \in I(n)$ for some index n then let I_x be an open subinterval of $I(n)$ containing x and let y be the right end point of I_x. We may assume that the length of each interval I_x is less than $\delta = \delta(\epsilon) > 0$ given in advance.

By the Heine-Borel theorem we can select a finite subsystem of the system of open intervals I_x $(a \leqq x \leqq b)$ which covers $[a, b]$. Using induction on $r < s$ we determine as usual a sequence

$$a = \xi_1 < \xi_2 < \cdots < \xi_r < \cdots < \xi_s < \xi_{s+1} = b$$

such that $\xi_s \in I_b$ and

$$\phi(\xi_r) < \phi(\xi_{r+1}) \quad \text{or} \quad \phi(\xi_r) < \phi(\xi_{r+1}) + \left| \phi(\xi_{r+1}) - \phi(\xi_r) \right|$$

for every $r < s$ according as ξ_r is covered by the interval for which $\phi(\xi_r) < \phi(y)$ or is covered only by one of the intervals $I(n)$.

Therefore summing over r we obtain

$$\phi(\xi_1) < \phi(\xi_s) + \sum \left| \phi(\xi_{r+1}) - \phi(\xi_r) \right|$$

where the summation is over such r's that ξ_r and ξ_{r+1} belong to the same $I(n)$. Since the variation of f over $\bigcup I(n)$ is less than ϵ we have $\phi(\xi_1) < \phi(\xi_s) + \epsilon$. Moreover $\xi_s \in I_b$ where the length of I_b is less than $\delta(\epsilon)$. So if $\delta(\epsilon)$ is chosen to be sufficiently small then it follows that $\phi(\xi_1) < \phi(\xi_{s+1}) + 2\epsilon$, i.e. $\phi(a) < \phi(b) + 2\epsilon$. Hence $f(a) \leqq f(b)$.

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