FRAGMENTS OF MANY-VALUED STATEMENT CALCULI

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1. Introduction. In 1930 (see [1]), Łukasiewicz and Tarski discussed the axiomatization of many-valued logics of a more general sort than originally axiomatized by Post in 1921 (see [2]). We propose to develop these ideas a bit further.

In particular, let us consider statement calculi based on a set, \( \mathcal{S} \), of truth-values. We make the following assumptions about \( \mathcal{S} \). If \( x \) is in \( \mathcal{S} \), then \( 0 \leq x \leq 1 \). \( \mathcal{S} \) is nonempty. \( \mathcal{S} \) is closed under application of the functions \( c \) and \( n \), where
\[
(1.1) \quad c(x, y) = \min (1, 1 - x + y),
\]
\[
(1.2) \quad n(x) = 1 - x.
\]
Obviously \( \mathcal{S} \) must contain 0 and 1, and may perhaps contain only these. If \( \mathcal{S} \) contains \( M \) members, then \( \mathcal{S} \) must consist of the rational numbers
\[
0, \frac{1}{M-1}, \frac{2}{M-1}, \ldots, \frac{M-2}{M-1}, 1,
\]
as one can conclude by an analysis like that given by McNaughton (see [3]).

A similar analysis shows that if \( \mathcal{S} \) has an infinite number of members, then these must be everywhere dense in the interval \([0, 1]\). Possibilities are that \( \mathcal{S} \) might consist of all rationals in this interval, or of all reals in this interval. Many other possibilities exist, such as that of choosing an irrational \( \theta \) and letting \( \mathcal{S} \) consist of all reals of the form \( a + b\theta \) with \( 0 \leq a + b\theta \leq 1 \); here one may set such requirements as that \( a \) and \( b \) should be integers, or that \( a \) and \( b \) should be rationals.

The members of \( \mathcal{S} \) are commonly called truth-values; we shall usually refer to them just as values.

It is common to separate \( \mathcal{S} \) into designated and undesignated values. This is done as follows. One chooses a real number \( S \) with \( 0 \leq S \leq 1 \). All members of \( \mathcal{S} \) less than \( S \) are undesignated and all members greater than \( S \) are designated. If \( S \) is itself a member of \( \mathcal{S} \), one must also specify whether \( S \) is designated or undesignated; however we set the requirement that 0 shall always be undesignated and 1 shall always be designated. If there are \( M \) values of which \( S \) are designated, then \( 1 \leq S < M \), and the designated values are just
\[
\frac{M-S}{M-1}, \frac{M-S+1}{M-1}, \ldots, \frac{M-2}{M-1}, 1.
\]

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An obvious linear transformation will identify this case with that considered by Rosser and Turquette (see [4]).

In many of the later sections, we leave 3 and S quite general. In other sections, we impose special conditions, such as that 3 have \( M \) members or that \( S = 1 \).

In various sections, we shall make use of selected ones of the statement functions, \( C, N, J, T, \) and \( D \). With each of these, we associate truth-value functions \( c, n, j, t, \) and \( d \), as follows:

\[
\begin{align*}
(1.1) & \quad c(x, y) = \min(1, 1 - x + y), \\
(1.2) & \quad n(x) = 1 - x, \\
(1.3) & \quad j(x) = \begin{cases} 1 & \text{if } x = 1, \\
0 & \text{if } x \neq 1, \end{cases} \\
(1.4) & \quad \ell(x) = \frac{M - 2}{M - 1}, \\
(1.5) & \quad d(x) = \begin{cases} \text{an undesignated value if } x \text{ is designated}, \\
\text{a designated value if } x \text{ is undesignated}. \end{cases}
\end{align*}
\]

In the above, \( x \) and \( y \) are restricted to lie in 3.

Since 3 contains both 0 and 1, it is closed under application of \( j \). We will use \( T \) only when 3 has \( M \) members, in which case 3 is closed under application of \( \ell \). Clearly (1.5) does not define \( d \) uniquely, but only puts certain restrictions on it. For our uses, it suffices that \( d \) be some specified one of the functions satisfying (1.5). Clearly 3 is closed under application of \( d \). Not uncommonly, \( d \) is made precise as follows

\[
d(x) = \begin{cases} 0 & \text{if } x \text{ is designated}, \\
1 & \text{if } x \text{ is undesignated}. \end{cases}
\]

However, we do not need to be so specific.

Of these functions, we shall take \( C, N, \) and \( T \) as undefined if we use them at all. If 3 has \( M \) members, then \( J \) and \( D \) can be defined in terms of \( C \) and \( N \) (see [3] or [4], for example) and we shall consider them as so defined. Note that the definition depends on \( M \), and in the case of \( D \) on \( S \) also. If 3 has an infinite number of members, then \( J \) and \( D \) are taken as undefined if they are used.

In various places, we will make use of certain of the functions \( A, K, B, L, I, \& \), and \( E \) defined as follows:

\[
\begin{align*}
(1.6) & \quad A_{PQ} \text{ for } C C P Q Q, \\
(1.7) & \quad K_{PQ} \text{ for } N A N P N Q, \\
(1.8) & \quad B_{PQ} \text{ for } C N P Q, \\
(1.9) & \quad L_{PQ} \text{ for } N C P N Q, \\
(1.10) & \quad I_{PQ} \text{ for } A D P Q,
\end{align*}
\]
(1.11) \&PQ for \textit{DADPDQ},
(1.12) \textit{EPQ} for \textit{LCPQCQP}.

Then each of these has an associated truth-value function as follows:

(1.13) \( a(x, y) = \max (x, y) \),
(1.14) \( k(x, y) = \min (x, y) \),
(1.15) \( b(x, y) = \min (1, x+y) \),
(1.16) \( l(x, y) = \max (0, x+y-1) \),
(1.17) \( a(x, y) \) is designated if either of \( x \) or \( y \) is designated,
(1.18) \( a(x, y) \) is undesignated if both of \( x \) and \( y \) are undesignated,
(1.19) \( i(x, y) \) is designated if \( x \) is undesignated or \( y \) is designated,
(1.20) \( i(x, y) \) is undesignated if \( x \) is designated and \( y \) is undesignated,
(1.21) \&(x, y) \) is designated if both of \( x \) and \( y \) are designated,
(1.22) \&(x, y) \) is undesignated if either of \( x \) or \( y \) is undesignated,
(1.23) \( e(x, y) = \min (1-x+y, 1-y+x) \).

In particular, \( e(x, y) = 1 \) if and only if \( x = y \).

Note that \( A \) and \( B \) are analogous to the inclusive “or” function of the usual two-valued statement calculus. Each will be seen to have some, but not all, of the familiar properties of the two-valued “or.” Similarly \( K, L, \) and \& are analogous to the two-valued “and,” \( C \) and \( I \) are analogous to the two-valued “implies,” \( N \) and \( D \) are analogous to the two-valued “not,” and \( E \) serves as an equivalence relation.

If \( \alpha \) is a non-negative integer, we allow ourselves to indicate \( \alpha \) repetitions of a block of symbols by enclosing the block in parentheses and adjoining the exponent \( \alpha \). Thus we may write

\[
(CP)^0Q \quad \text{for} \quad Q,
\]
\[
(CP)^1Q \quad \text{for} \quad CPQ,
\]
\[
(CP)^2Q \quad \text{for} \quad CPCPQ,
\]
\[
(CP)^\alpha Q \quad \text{for} \quad CPCPCPQ
\]

etc.

Then with \( (CP)^\alpha Q \), \( (BP)^\alpha Q \), and \( (LP)^\alpha Q \) there are associated truth-value functions as follows:

(1.24) \( c_\alpha(x, y) = \min (1, y+\alpha(1-x)) \) \( (0 \leq \alpha) \),
(1.25) \( b_\alpha(x, y) = \min (1, y+\alpha x) \) \( (0 \leq \alpha) \),
(1.26) \( l_\alpha(x, y) = \max (0, y-\alpha(1-x)) \) \( (0 \leq \alpha) \).

We define

(1.27) \( B_\alpha P \) for \( (BP)^\alpha NCPP \) \( (0 \leq \alpha) \),
(1.28) \( L_\alpha P \) for \( (LP)^\alpha CPP \) \( (0 \leq \alpha) \),
(1.29) \( V_\alpha P \) for \( EPNB_\alpha^{-1}P \) \( (1 \leq \alpha) \),
(1.30) \( W_\alpha PQ \) for \( EPB_\alpha Q \) \( (0 \leq \alpha) \).

These have associated truth-value functions as follows:

(1.31) \( b_\alpha(x) = \min (1, \alpha x) \) \( (0 \leq \alpha) \),
(1.32) \( l_\alpha(x) = \max (1, 1 - \alpha(1 - x)) \quad (0 \leq \alpha) \),
(1.33) \( v_\alpha(x) = 1 \) if and only if \( \alpha x = 1 \) \( (1 \leq \alpha) \),
(1.34) \( w_\alpha(x, y) = 1 \) if and only if \( x = \min (1, \alpha y) \) \( (0 \leq \alpha) \).

As in [4], we define a chain symbol \( \Gamma \) by the following recursion:

(1.35) If \( \beta < \alpha \), then \( \Gamma_{t=\alpha}^{\beta} P_t Q \) denotes \( Q \).
(1.36) If \( \beta \geq \alpha \), then \( \Gamma_{t=\alpha}^{\beta} P_t Q \) denotes \( CP_{\beta} \Gamma_{t=\alpha}^{\beta-1} P_t Q \).

The associated truth function is:

\[
\gamma_{t=\alpha}(x_i, y) = \begin{cases} 
  y & (\alpha \geq \beta + 1), \\
  \min \left( 1, \beta + 1 - \alpha + y - \sum_{i=\alpha}^{\beta} x_i \right) & (\alpha \leq \beta).
\end{cases}
\]

We note that if all the \( P_i \)'s are identical with \( P \), then

(1.37) \( \Gamma_{t=\alpha}^{\beta} P_t Q \) is \( (CP)^{\beta+1-\alpha} Q \) \( (\alpha \leq \beta + 1) \).

We also define a generalized summation by the following recursion:

(1.39) If \( \beta = \alpha \), then \( \sum_{t=\alpha}^{\beta} P_t \) denotes \( P_\alpha \).
(1.40) If \( \beta > \alpha \), then \( \sum_{t=\alpha}^{\beta} P_t \) denotes \( AP_{\beta} \sum_{t=\alpha}^{\beta-1} P_t \).

The associated truth-value function is

(1.41) \( \max (x_\alpha, x_{\alpha+1}, \ldots, x_\beta) \).

A perennial problem is to make some choice of \( \mathfrak{F}, \mathfrak{S} \), and of a set of undefined functions, and then to ask for a set of axioms and rules from which one can derive exactly those statement formulas (in the sense of [4, pp. 13-14]) whose corresponding truth-value functions take only designated values. We shall present some fragments of a general theory, and then enlarge these to give complete solutions in a number of special cases.

In [1], it is stated that Łukasiewicz conjectured that if \( \mathfrak{F} \) has an infinite number of members and \( \mathfrak{S} = 1 \), then the following rule and set of axiom schemes give a solution when \( C \) and \( N \) are chosen as the undefined functions.

Rule C. If \( P \) and \( CPQ \), then \( Q \).

Axiom schemes:

\[
\begin{align*}
CPCQP. \\
CCPQCCQRCPR. \\
CAPQAQP. \\
ACPQCCQ. \\
CCNPNCQCPQ.
\end{align*}
\]

In §13 we shall prove this conjecture. Incidentally, we note that C. A. Meredith and C. C. Chang have recently shown how to derive the fourth of these axioms from the rest.

In [8], on p. 240, M. Wajsberg announced that he had a proof of Łukasiewicz's conjecture. However, apparently Wajsberg's proof was never pub-
lished, since in [9], on p. 51, Tarski refers to Wajsberg's proof but cites only [8].

When $s = 1$ and 3 has an infinite number of members, then the set of formulas based on C and N which take designated values exclusively is independent of the further details of the composition of 3. This enables us to assume that 3 consists of the rationals in the interval $[0, 1]$ when dealing with axiomatization in the case when $s = 1$ and 3 has an infinite number of members and C and N are the undefined functions.

If $s < 1$, the situation is not so simple. For instance, take $s = 1/2$, take 3 to consist of the rationals and take 1/2 as not designated; then $C(CP)\neg QCPQ$ can take an undesignated value, namely 1/2. Alternatively, take $s = 1/2$ again but take 3 to consist of all reals of the form $a + b\theta$ with $0 \leq a + b\theta \leq 1$, where $\theta$ is a fixed irrational and $a$ and $b$ are integers. As 1/2 is not a member of 3, we need not specify if it is designated or not. In any case, whatever we decide about making 1/2 designated, we conclude that $C(CP)\neg QCPQ$ must take only designated values, since its minimum possible value is 1/2 and it cannot take that value since (with this specification of 3) the only rational value that $C(CP)\neg QCPQ$ can assume is 1.

We do not make an effort to furnish an axiomatization for any of the cases where $s < 1$ and 3 has an infinite number of members. We mention that if $s < 1$, then Rule C is not acceptable. Possible alternatives are:

Rule JC. If $JP$ and $JCPQ$, then $JQ$.

Rule I. If $P$ and $IPQ$, then $Q$.

When one considers the case where 3 has a finite number of members, the situation changes a bit. Even when $s = 1$ the set of statement formulas which take designated values exclusively depends on the number of members of 3. In particular, if $s = 1$ then $C(CP)\neg Q(CP)\neg Q$ takes only designated values if and only if 3 has $\alpha$ or fewer members.

In [4] have been given systems of axioms for each case in which 3 has a finite number of members. These axiom systems are very general with regard to which statement functions are taken as undefined. If C and N, or C and N and T, are taken as undefined, one can get systems of axioms with fewer axioms than are used in [4]. This we do, and the results are summarized here-with.

If $C$, $N$, and $T$ are taken as undefined and 3 has $M$ members, then:

(a) Rule C and six axiom schemes suffice if $s = 1$ (see §5),
(b) Rule I and eight axiom schemes suffice if $s < 1$ (see §8).

If $C$ and $N$ are taken as undefined and 3 has $M$ members, then:

(a) Rule C and five axiom schemes suffice if $s = 1$ (see §14). However, in §6 we give an alternative treatment involving Rule C and seven axiom schemes because the alternative development seems of interest and is fairly short. Also the alternative development is far simpler than the development depending on five axiom schemes, and avoids the excessive metamathematical difficulties of the more sophisticated development.
(b) If \( n \) is the number of axiom schemes required when \( S = 1 \), then Rule I and \( 3 + n \) axiom schemes suffice if \( S < 1 \) (see §9).

2. A fragment of the \( C \)-calculus. In this section, we derive a number of consequences of the following rule and axiom schemes.

Rule C. If \( P \) and \( CPQ \), then \( Q \).

- \( A1. CPCQP \).
- \( A2. CCPQCCQRCPR \).
- \( A3. CAPQAQP \).

We introduce the usual yields sign, \( \vdash \), (see [4, p. 34]) and let its signification depend on the current set of axioms and rules. Thus throughout this section, we shall use \( \vdash \) as depending on Rule C and axiom schemes \( A1, A2, A3 \). As we change axioms or rules, we shall make the corresponding change in the signification of \( \vdash \) without comment.

We introduce the special notation

\[
P_1, \ldots, P_n \vdash Q \equiv R
\]

to denote that both of

\[
P_1, \ldots, P_n \vdash CQR,
\]
\[
P_1, \ldots, P_n \vdash CRQ
\]

are valid. Obviously we have \( P_1, \ldots, P_n \vdash Q \equiv R \) if and only if we have \( P_1, \ldots, P_n \vdash R \equiv Q \); also, from \( A2 \) and Rule C, we infer that if \( P_1, \ldots, P_n \vdash Q \equiv R \) and \( P_1, \ldots, P_n \vdash R \equiv S \), then \( P_1, \ldots, P_n \vdash Q \equiv S \). We shall use these properties without comment.

Another principle which we shall usually use without comment is

\[
(2.1) \ CPQ, \ CQR \vdash CPR,
\]

which follows from \( A2 \) and Rule C.

By interchanging \( P \) and \( Q \) in \( A3 \), we infer:

\[
(2.2) \vdash APQ \equiv AQP.
\]

Since \( A2 \) gives \( CPQ \vdash CCQRCPR \) and \( CQP \vdash CCPRCQR \), we infer:

\[
(2.3) \ \text{If} \ S_1, \ldots, S_n \vdash P \equiv Q, \ \text{then} \ S_1, \ldots, S_n \vdash CPR \equiv CQR.
\]

By taking \( Q \) to be \( CQP \) in \( A1 \), we infer

\[
(2.4) \vdash CPAQP,
\]

whence we get

\[
(2.5) \vdash CPAPQ
\]

by \( A3 \). Consequently, we infer \( \vdash CQAQR \), from which by \( A2 \) we get

\[
\vdash CCAQRCPRCQCPR.
\]

However, by putting \( CQR \) for \( Q \) in \( A2 \), we get

\[
\vdash CCPCQRCACQRCPR.
\]
By these two formulas we get

\[ (2.6) \vdash CCPCQRCQCPR. \]

Interchanging \( P \) and \( Q \) gives

\[ (2.7) \vdash CPCQR \equiv CQPR. \]

By applying (2.6) to A2, we get

\[ (2.8) \vdash CCQRCCPQCPR. \]

Using this, we may reason as in our derivation of (2.3) to infer:

\[ (2.9) \text{If } S_1, \ldots, S_n \vdash P \equiv Q, \text{ then } S_1, \ldots, S_n \vdash CRP \equiv CRQ. \]

In (2.6) take \( R \) to be \( P \) and use A1. This gives \( \vdash CQCPR \). By taking \( Q \) to be any proved result, we get

\[ (2.10) \vdash CPP, \]

\[ (2.11) \vdash P \equiv P. \]

By use of (2.3), (2.9), and (2.11), we can prove the standard type of equivalence and substitution theorems to the effect that if \( S_1, \ldots, S_n \vdash P \equiv Q \), then under the hypotheses \( S_1, \ldots, S_n \) one can replace occurrences of \( P \) by \( Q \) at will in statement formulas built up by use of \( C \) alone. We shall make such substitutions without comment. In particular, because of (2.2), we now have full commutativity of \( A \), and will use it freely.

**Theorem 2.1.** Let \( Q_1, \ldots, Q_q \) denote an ordered set of statements among which each of \( P_1, \ldots, P_p \) occurs at least once. Then

\[ (2.12) \vdash CT_{i=1}^{p} P_i RT_{i=1}^{q} Q_i R. \]

We can use the proof given for Lemma 3.1.4 on p. 35 of [4].

Similarly, by using the proof given for Lemma 3.1.3 on p. 35 of [4], we can infer:

\[ (2.13) \vdash CCPQCT_{i=1}^{p} R_i PT_{i=1}^{q} S_i Q. \]

Suppose we have \( \Gamma_{i=1}^{q} S_i CPQ \). Then we get \( CPT_{i=1}^{p} S_i Q \) by Theorem 2.1, and then \( CT_{i=1}^{p} R_i PT_{i=1}^{q} R_i PT_{i=1}^{q} S_i Q \) by (2.13). Consequently

\[ (2.14) \Gamma_{i=1}^{q} R_i P, \Gamma_{i=1}^{q} S_i CPQ \vdash \Gamma_{i=1}^{q} R_i TT_{i=1}^{q} S_i Q. \]

By means of this, we can prove a generalized version of the familiar Deduction Theorem.

**Theorem 2.2.** If \( R_1, \ldots, R_n, P \vdash Q \), then there is a non-negative integer \( \alpha \) such that \( R_1, \ldots, R_n \vdash (CP)^{\alpha} Q \).

As we have \( \vdash CQCPR \) by A1 and \( P \vdash CQPQQ \) by (2.5), we infer

\[ (2.15) P \vdash Q \equiv CPQ. \]

Taking \( P \) to be \( CPQ \) and using (2.10) gives

\[ (2.16) Q \equiv AQP. \]

By A2,

\[ \vdash CCCQRCPRCAQPRAQR. \]

Combining this with A2 itself gives

\[ (2.17) \vdash CQPQCAPRAQR. \]
Commutativity of A gives
\[(2.18) \vdash CCPQCARPARQ.\]
These give
\[
\vdash CCPRCAPQARQ
\vdash CCQSCARQARS.
\]
From these last two by (2.14)
\[
\vdash CCPRCAPQCCQSARS.
\]
Then by (2.7)
\[(2.19) \vdash CCPRCCQSCAPQARS,
\]
from which by (2.16)
\[(2.20) CPR, CQR \vdash CAPQR.\]

**Theorem 2.3.** If \(P_1, \ldots, P_p, R \vdash T\) and \(Q_1, \ldots, Q_q, S \vdash T\), then \(P_1, \ldots, P_p, Q_1, \ldots, Q_q, ARS \vdash T\).

**Proof.** From \(P_1, \ldots, P_p, R \vdash T\) and \(Q_1, \ldots, Q_q, S \vdash T\) we get
(a) \(P_1, \ldots, P_p \vdash (CR)^\alpha T\),
(b) \(Q_1, \ldots, Q_q \vdash (CS)^\beta T\)
by Theorem 2.2. For any \(W\), we write \(ARS \vdash W\) as shorthand for \(P_1, \ldots, P_p, Q_1, \ldots, Q_q, ARS \vdash W\). We now prove by induction on \(\gamma\) the following lemma:

If \(\gamma\) is a positive integer, and \(\gamma \leq \alpha + \beta\), and \(U_1, \ldots, U_{\alpha+\beta-\gamma}\) are formulas each of which is either \(R\) or \(S\), then
(c) \(\varepsilon, ARS \vdash \Gamma_{i=1}^{\alpha+\beta-\gamma} U_i T\).

First let \(\gamma = 1\).

*Case 1.* There are fewer than \(\alpha\) \(R\)'s among \(U_1, \ldots, U_{\alpha+\beta-1}\). Then there are at least \(\beta\) \(S\)'s. So by Theorem 2.1
\[
\vdash (CS)^\beta T \Gamma_{i=1}^{\alpha+\beta-\gamma-1} U_i T.
\]
Then (c) holds by (b).

*Case 2.* There are at least \(\alpha\) \(R\)'s among \(U_1, \ldots, U_{\alpha+\beta-1}\). Then we can get (c) from (a) by similar reasoning.

Now assume the lemma for \(\gamma\). Using this and (1.36) we get both of
\[
\varepsilon, ARS \vdash CRM_{i=1}^{\alpha+\beta-\gamma-1} U_i T,
\]
\[
\varepsilon, ARS \vdash CST_{i=1}^{\alpha+\beta-\gamma-1} U_i T.
\]
From these by (2.20) we get
\[
\varepsilon, ARS \vdash \Gamma_{i=1}^{\alpha+\beta-\gamma-1} U_i T,
\]
so that the induction is established.

Finally, we conclude our theorem by taking \(\gamma = \alpha + \beta\) in the lemma.
We can prove the associative law for $A$, namely
\[(2.21) \vdash APAQR = AAPQR,\]
by the methods used to prove Formel (14) and Formel (15) of §11 of Chapter 1 of [5].

We close with some miscellaneous results that will be needed later.

By (2.7)
\[\vdash CCPQCRQ = CRAPQ.\]

Consequently, commutativity of $A$ gives
\[(2.22) \vdash CCPQCRQ = CCQPCR.\]
By (2.5)
\[(2.23) \vdash CCPQCCPQRR.\]
Taking $R$ to be $Q$ gives
\[\vdash CCPQCAPQQ.\]
Also, by (2.5) and A2
\[\vdash CCAPQQCPQ.\]

Thus we have shown
\[(2.24) \vdash CPQ = CAPQQ.\]
By A2
\[\vdash CCCPQRCCRPAPQ.\]

Using commutativity of $A$ followed by (2.7) gives
\[(2.25) \vdash CCCPQRCCQPCCRQP.\]
By (2.8) and (2.7), \[\vdash CCRSCPCCPRS.\] So by (2.14)
\[CCCPRSQ \vdash CCRSCPQ.\]

Thence we infer
\[(2.26) CCQSCPR, CSQ \vdash CCRSCPQ\]
by putting $Q$, $S$, and $CPR$ respectively for $P$, $Q$, and $R$ in (2.25).

3. A fragment of the $C$—$N$-calculus. In this section, we add one more axiom scheme, namely

\[A4. CCNPNCQCPQ,\]

to the three considered in the preceding section, and derive a number of consequences involving $N$.

By A1, \[\vdash CNNPCCNNQNNP.\] By two uses of A4, we get successively \[\vdash CNNPCCNPQ\] and \[\vdash CNNPQNPQ.\] Then (2.7) gives \[\vdash CQCNNPP.\] Taking $Q$ to be any proved result gives
\[(3.1) \vdash CNNPP.\]
From this, by A2, we get \[\vdash CCPQCNPNQ.\] Using A4 gives
\[(3.2) \vdash CCPQCNQCPNQ.\]
Interchanging $P$ and $Q$ gives
\[(3.3) \vdash CPNQ \equiv CQNP.\]
Putting $NP$ for $Q$ in (3.3) and using (2.10) we get $\vdash CPNNP$, so that by (3.1)
\[(3.4) \vdash P \equiv NNP.\]
Using $\vdash Q \equiv NNQ$ with (2.9) gives $\vdash CPQ \equiv CPNNQ$. Putting $NQ$ for $Q$ in
(3.3) gives $\vdash CPNNQ \equiv CNQNP$. Thus
\[(3.5) \vdash CPQ \equiv CNQNP.\]
Consequently
\[(3.6) \text{ If } R_1, \ldots, R_n \vdash P \equiv Q, \text{ then } R_1, \ldots, R_n \vdash NP \equiv NQ.\]
This enables us to extend the equivalence and substitution theorems to formulas involving $N$ as well as $C$. We can thus get many results by familiar transformations involving (3.4) and (3.5). We list a number of such, leaving the details to the reader. The first three are
\[(3.7) \vdash APQ \equiv NKNPNQ.\]
\[(3.8) \vdash BPQ \equiv NLNPNQ.\]
\[(3.9) \vdash LPQ \equiv NBNPNQ.\]
By applying (3.6) to (3.3), we get
\[(3.10) \vdash LPQ \equiv LQP,\]
whence we get
\[(3.11) \vdash BPQ \equiv BQP.\]
From the corresponding results for $A$ come
\[(3.12) \vdash KPO \equiv KBPQ,\]
\[(3.13) \vdash CKKPP,\]
\[(3.14) \vdash CKPQ,\]
\[(3.15) \vdash Q \equiv QQ,\]
\[(3.16) \vdash CPQCKPRKQR,\]
\[(3.17) \vdash CPQCKRQPDKRQ,\]
\[(3.18) \vdash CPQCCQSCCKPQKRS,\]
\[(3.19) CPQ, CPR \vdash CPKQR,\]
\[(3.20) \vdash KPKQR \equiv KKPQR,\]
\[(3.21) \vdash CQP \equiv CQKPQ.\]
Since $\vdash CPCQP$ by A1, we can use (3.21) to infer
\[(3.22) \vdash CPCQKPQ.\]
By (2.8)
\[(3.23) \vdash CPQCBRPBRQ,\]
whence we get
\[(3.24) \vdash CPQCBPRBQR,\]
\[(3.25) \vdash CPQRCCQSCBPQBRS,\]
\[(3.26) \vdash CPQCLRPLRQ,\]
\[(3.27) \vdash CPQCLRPLQR,\]
\[(3.28) \vdash CPQRCCQSCPQLRS.\]
Putting $NP$ and $NQ$ for $P$ and $Q$ in (2.7) gives
\[\vdash BPBQR \equiv BQBPQ.\]
Interchanging $Q$ and $R$ gives

$$\vdash BPBRQ = BRBPQ.$$ 

Then commutativity of $B$ gives

(3.29) $$\vdash BPBQR = BBPQR,$$
whence we get

(3.30) $$\vdash LPLQR = LLPQR.$$

Putting $NQ$ for $Q$ in A1 gives

(3.31) $$\vdash CPBQP,$$
whence we get

(3.32) $$\vdash CPBPQ,$$
(3.33) $$\vdash CLQPP,$$
(3.34) $$\vdash CLPQP.$$

By (3.5)

$$\vdash CPCQR = CPCNRNQ.$$ 

Then by (2.7)

$$\vdash CPCQR = CNRCPNQ.$$ 

Finally by (3.11)

(3.35) $$\vdash CPCQR = CLPQR.$$ 

Applying this to $\vdash CLPQLPQ$, we get

(3.36) $$\vdash CPCQLPQ.$$ 

By this and (3.33)

(3.37) $$P \vdash Q = LPQ.$$

Theorem 3.1. $R_1, \cdots, R_n \vdash P = Q$ if and only if $R_1, \cdots, R_n \vdash EPQ$.

Proof. By (3.36)

$$CPQ, CQP \vdash EPQ,$$

and by (3.34) and (3.33)

$$EPQ \vdash P = Q.$$ 

By use of this theorem, we can get

(3.38) $$\vdash EPP,$$
(3.39) $$EPO, EQR \vdash EPR$$
directly, and

(3.40) $$EPQ \vdash ENPNQ,$$
(3.41) $$EPQ, ERS \vdash ECPRCQ$$
by appealing respectively to (3.6), and to both of (2.3) and (2.9). By (3.10), we have

(3.42) $$\vdash CEPQEQP.$$
With both $A$ and $B$ serving as disjunctions and both $K$ and $L$ serving as
conjunctions, one can write a number of possible distributive laws. Some are not valid, and of the valid ones we have been able to prove only two from axiom schemes A1–A4. We now give the proofs.

By (2.5) and the commutativity of $L$

$$\vdash CLQPALPQLPR.$$  

Then by (3.35)

$$\vdash CQCPALPQLPR.$$  

Similarly

$$\vdash CRCPALPQLPR.$$  

Then by (2.20)

$$\vdash CAQRPCALPQLPR.$$  

Finally by (3.35) and the commutativity of $L$

(a) $$\vdash CLPAQRALPQLPR.$$  

By (2.5) and (3.26)

$$\vdash CLPQI.PAQR.$$  

By (2.4) and (3.26)

$$\vdash CLPRLPAQR.$$  

Then by (2.20)

$$\vdash CALPQLPRLPAQR.$$  

From this and (a) we get

(3.43) $$\vdash LPAQR=ALPQLPR.$$  

By replacing $P$, $Q$, and $R$ by $NP$, $NQ$, and $NR$, we get

(3.44) $$\vdash BPKQR=KBPQBPR.$$  

By (2.15)

(3.45) $$NP\vdash Q=BPQ.$$  

By (2.10) and A1, $$\vdash CPCRR.$$  

So by (3.36)

$$\vdash CCCRRPEPCRR.$$  

But by A1, $$\vdash CPCCRRP,$$ so that

(3.46) $$\vdash CPECRR.$$  

Then by Theorem 3.1,

(3.47) $$P\vdash P=CRR,$$

whence, by the transitivity of $=,$

(3.48) $$P, Q\vdash P=Q.$$  

By (3.47), (3.6), and (3.4)

(3.49) $$NP\vdash P=NCRR,$$
whence
(3.50) \( NP, NQ \vdash P \equiv Q \).

We close with some miscellaneous results that will be needed later. Negating all variables of (2.22) and applying (3.5) gives
(3.51) \( \vdash CCQPCQR \equiv CCPQCPR \).

We raise the question if this can be proved from A1, A2, and A3 alone. By (3.32) and A2
\( \vdash CCBPQQCPQ \).

This is
(3.52) \( \vdash CANPQCPQ \).

By A1, \( \vdash CNSCNRNS \), so that by (3.5)
(3.53) \( \vdash CNSCSR \).

A simple application of (3.4) gives
(3.54) \( \vdash BLPQR \equiv CCPNQR \).

If we put \( NQ \) and \( NS \) for \( Q \) and \( S \) in (2.26) and use (3.5), we get
\[ CQS, CCSQCPR \vdash CCRNSCPQ. \]

Another use of (3.5) gives
(3.55) \( CQS, CCSQCPR \vdash CLPQLRS. \)

By (3.32), \( P \vdash BPR \), so that by (3.36) \( P, Q \vdash LBPRQ \). However, by (3.52)
\[ ANPQ, P \vdash Q. \]

So
(3.56) \( ANPQ, P \vdash LBPRQ. \)

**Theorem 3.2.**
(3.57) \( ANPQ \vdash LBPRQ \equiv BPLQR. \)

**Proof.** By (3.37)
\[ Q \vdash BPR \equiv BPLQR, \]
\[ Q \vdash LQBPR \equiv BPR. \]

So by the commutativity of \( L \)
(a) \( Q \vdash LBPRQ \equiv BPLQR. \)

By (3.45)
\[ NP \vdash LQR \equiv BPLQR, \]
\[ NP \vdash LBPRQ \equiv LRQ. \]

So by the commutativity of \( L \)
(b) \( NP \vdash LBPRQ \equiv BPLQR. \)

Our theorem follows from (a) and (b) by Theorem 2.3.

4. **Special results for use in the finite-valued case.** We continue with the same four axiom schemes as in the preceding section.
By (2.10) and (3.4)
(4.1) \( \neg NB\phi P \).

Then by (3.50)
(4.2) \( \neg B\phi P \equiv B\phi Q \),
and by (3.45) and the commutativity of \( B \)
(4.3) \( \neg P \equiv BPB\phi Q \).

Taking \( Q \) to be \( P \) in this gives
(4.4) \( \neg P \equiv B_1 P \).

**Theorem 4.1.** If \( \alpha \) and \( \beta \) are non-negative integers, then
(4.5) \( \neg B_{\alpha+1} P \equiv BPB_{\alpha} P \),
(4.6) \( \neg B_{\alpha+1} P \equiv CBNB_{\alpha} PP \),
(4.7) \( \neg B_{\alpha+\beta} P \equiv BB_{\alpha} PB_{\beta} P \).

**Proof.** We infer (4.5) by (1.27), and then deduce (4.6) by the commutativity of \( B \). To prove (4.7), we use induction on \( \alpha \). When \( \alpha = 0 \), use (4.3) and the commutativity of \( B \). For the induction step, use (4.5) and the associativity of \( B \).

**Theorem 4.2.** If \( \alpha \) and \( \beta \) are non-negative integers, then
(4.8) \( \neg B_{\alpha\beta} P \equiv B_{\alpha} B_{\beta} P \).

**Proof by induction on \( \alpha \).** When \( \alpha = 0 \), use (4.2). For the induction step, use (4.7) and (4.5).

**Theorem 4.3.** If \( \alpha \) and \( \beta \) are non-negative integers, then
(4.9) \( \neg C B_{\alpha} PB_{\alpha+\beta} P \).

**Proof.** By (3.32)
(4.10) \( \neg C B_{\alpha} PBB_{\alpha} PB_{\beta} P \).

Now use (4.7).

**Theorem 4.4.** If \( \alpha \) and \( \beta \) are positive integers, then
(4.11) \( CB_{\alpha} PNB_{\beta} P \ |
\neg C B_{\alpha-1} PNB_{\beta+1} P \).

**Proof.** Assume
(a) \( CB_{\alpha} PNB_{\beta} P \).
By A2
(4.12) \( CCB_{\beta} PBCB_{\alpha} PP \).

So by (4.6)
(b) \( CB_{\beta+1} PBCB_{\alpha} PP \).
By (a) and (3.3)
(c) \( CB_{\beta} PNB_{\alpha} P \).
By (4.4) and (4.9)
(d) \( CPB_{\beta} P \).

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By (4.9) and (3.5)
(e) \( CNB_\alpha PNB_{\alpha-1}P. \)

By (c), (d), and (e)

\[ CPNB_{\alpha-1}P. \]

By (2.24), this gives

\[ CAPNB_{\alpha-1}PNB_{\alpha-1}P. \]

By the commutativity of A

\[ CCCNB_{\alpha-1}PPPBNB_{\alpha-1}P. \]

Then by (4.6)

\[ CCB_{\alpha}PPNB_{\alpha-1}P, \]

whence by (b)

\[ CB_{\beta+1}PNB_{\alpha-1}P. \]

By (3.3), we conclude our theorem.

**Theorem 4.5.** If \( \alpha \) and \( \beta \) are positive integers, and \( \gamma \) is a non-negative integer, and \( \gamma \leq \alpha \), then

\( (4.11) \ CB_{\alpha}PNB_{\beta}P \vdash CB_{\alpha-\gamma}PNB_{\beta+\gamma}P. \)

Proof by induction on \( \gamma \), using Theorem 4.4 for the induction step.

**Theorem 4.6.** If \( \alpha \) and \( \beta \) are positive integers, then

\( (4.12) \ CPNB_{\alpha+\beta-1}P \vdash CB_{\alpha}PNB_{\beta}P, \)
\( (4.13) \ CB_{\alpha}PNB_{\beta-\gamma}P \vdash CPNB_{\alpha+\beta-1}P. \)

**Proof.** First assume \( CPNB_{\alpha+\beta-1}P. \) By (3.3) and (4.4), \( CB_{\alpha+\beta-1}PNB_{1}P \), so that by (4.11) \( CB_{\alpha+\beta-1-\gamma}PNB_{1+\gamma}P \). Then we get (4.12) by taking \( \gamma = \beta - 1 \). To get (4.13), we take \( \gamma = \alpha - 1 \) in (4.11) and use (4.4).

**Theorem 4.7.** If \( \alpha \) is a positive integer, then

\( (4.14) \ V_{\alpha}P \vdash B_{\alpha}P. \)

**Proof.** This follows by (3.33) and (4.6).

**Theorem 4.8.** If \( \alpha \) and \( \beta \) are positive integers, then

\( (4.15) \ V_{\alpha+\beta}P \vdash EB_{\alpha}PNB_{\beta}P, \)
\( (4.16) \ EB_{\alpha}PNB_{\beta}P \vdash V_{\alpha+\beta}P. \)

**Proof.** First assume \( V_{\alpha+\beta}P. \) Then by (3.34) and (3.33), we get \( CPNB_{\alpha+\beta-1}P \) and \( CNB_{\alpha+\beta-\gamma}PP \). From the first, we get \( CB_{\alpha}PNB_{\beta}P \) by (4.12), and from the second we get \( BB_{\beta}PB_{\alpha}P \) by (4.6) and (4.7). Then we get \( EB_{\alpha}PNB_{\beta}P \) by (3.36). To get (4.16), we merely reverse the steps just given.
Theorem 4.9. If $\alpha$ is a positive integer, and $\gamma$ is a non-negative integer, and $\gamma \leq \alpha$, then

\begin{align*}
(4.17) & \quad V_{\alpha}P \vdash EB_{\gamma}PNB_{\alpha-\gamma}P, \\
(4.18) & \quad V_{\alpha}P \vdash EB_{\alpha-\gamma}PNB_{\gamma}P.
\end{align*}

Proof. We note that if (4.17) can be proved for all $\gamma$ with $0 \leq \gamma \leq \alpha$, then (4.18) follows by replacing $\gamma$ by $\alpha - \gamma$. If $0 < \gamma < \alpha$, then both (4.17) and (4.18) follow from (4.15). To handle the remaining cases, we note first that by (4.14) and (3.46)

(a) $V_{\alpha}P \vdash EB_{\alpha}PCPP$.

By (3.4), this gives

$$V_{\alpha}P \vdash EB_{\alpha}PNB_{\alpha}P,$$

which gives (4.17) for the case $\gamma = \alpha$. By applying (3.40) and (3.42) to (a), we get

$$V_{\alpha}P \vdash EB_{\alpha}PNB_{\alpha}P,$$

which gives (4.17) for the case $\gamma = 0$.

Theorem 4.10. If $\alpha$ is a positive integer, and $\gamma$ is a non-negative integer, and $\gamma \leq \alpha$, then

\begin{align*}
(4.19) & \quad V_{\alpha}R, W_{\gamma}PR \vdash W_{\alpha-\gamma}NPR.
\end{align*}

Proof. By (3.40)

$$W_{\gamma}PR \vdash ENPNB_{\gamma}R$$

and by (4.18)

$$V_{\alpha}R \vdash EB_{\alpha-\gamma}RNB_{\gamma}R.$$

Combining these by (3.39) and (3.42) gives (4.19).

Theorem 4.11. If $\alpha$ is a positive integer, and $\beta$ and $\gamma$ are non-negative integers, and $\beta \leq \alpha$, then

\begin{align*}
(4.20) & \quad V_{\alpha}P \vdash W_{\alpha-\beta+\gamma}CB_{\beta}PB_{\gamma}PP.
\end{align*}

Proof. By (4.17) and Theorem 3.1

$$V_{\alpha}P \vdash B_{\beta}P \equiv NB_{\alpha-\beta}P.$$ 

Then by (2.3) and Theorem 3.1

$$V_{\alpha}P \vdash ECB_{\beta}PB_{\gamma}PB_{\alpha-\beta}PB_{\gamma}P.$$

Finally we use (4.7) and (1.30).

Theorem 4.12. If $\alpha$ is a positive integer, and $\beta$ and $\gamma$ are non-negative integers, and $\beta \leq \alpha$, and $\eta = \min (\alpha, \alpha - \beta + \gamma)$, then

\begin{align*}
(4.21) & \quad V_{\alpha}R, W_{\beta}PR, W_{\gamma}QR \vdash W_{\eta}CPQR.
\end{align*}
Proof. Assume $V_\alpha R$, $W_\beta PR$, and $W_\gamma QR$. As we have

$$W_\beta PR \vdash P \equiv B_\beta R,$$

$$W_\gamma QR \vdash Q \equiv B_\gamma R,$$

by Theorem 3.1, we infer

(a) $W_{\alpha-\beta+\gamma}CPQR$

by (4.20), and we infer

(b) $B_\gamma R$

by (4.14). If $\alpha-\beta+\gamma \leq \alpha$, then (a) gives the desired result. So assume $\alpha < \alpha-\beta+\gamma$. Then by (b) and (4.9) we get $B_{\alpha-\beta+\gamma}R$, while by (a) and (3.33) we get $CB_{\alpha-\beta+\gamma}RCPQ$, so that we can conclude

(c) $CPQ$.

Then we conclude

(d) $W_\alpha CPQR$

by (b), (c), (3.48), and Theorem 3.1. In this case, (d) gives the desired result.

Theorem 4.13. Let $\alpha$ be a positive integer, let $n$ be a non-negative integer, and let $\beta$, $r$ ($0 \leq r \leq n$) be non-negative integers such that $\beta r \leq \alpha$ ($0 \leq r \leq n$). Let $\phi(P_0, \ldots, P_n)$ be a statement formula built up from $P_0, \ldots, P_n$ by means of $C$ and $N$. Let $\mu$ be that non-negative integer with $\mu \leq \alpha$ such that if $\mu$ is assigned the value $\beta r$, then $\phi(P_0, \ldots, P_n)$ takes the value $\mu$/$\alpha$. Then

$$V_\alpha R, W_\beta P_0 R, \ldots, W_\beta P_n R \vdash W_\mu \phi(P_0, \ldots, P_n) R.$$

Proof by induction on the structure of $\phi$, using Theorem 4.10 and Theorem 4.12.

Theorem 4.14. If $\alpha$ and $\beta$ are positive integers and $\beta \leq \alpha$, and $\gamma$ and $\delta$ are non-negative integers, then

$$LV_\beta SW_\delta PS, W_\gamma RP \vdash \sum_{r=0}^{\alpha} (W_r RS).$$

Proof. Assume $LV_\beta SW_\delta PS$ and $W_\gamma RP$. By (3.33), (3.34), Theorem 3.1, (4.8) and (4.14)

(a) $W_\gamma RS$,

(b) $B_\beta S$.

If $\gamma \delta \leq \alpha$, then we have

$$W_\gamma RS \vdash \sum_{r=0}^{\alpha} (W_r RS)$$

by (2.4) and (2.5), so that our theorem follows by (a). So let $\alpha < \gamma \delta$. Then $\beta < \gamma \delta$, so that by (b) and (4.9) we get $B_\gamma S$, while by (a) and (3.33) we get $CB_{\gamma \delta}SR$; thence we get

(c) $R$.

Then we conclude

(d) $W_\beta RS$

by (b), (c), (3.48), and Theorem 3.1. As $\beta \leq \alpha$, we have
by (2.4) and (2.5), so that our theorem follows from (d) in this case.

Theorem 4.15. If \( \alpha \) is a positive integer, then

\[
V_\alpha B_\beta P \vdash Q.
\]

Proof. By (4.1) and (3.53)

\[
\beta B_\beta P Q.
\]

By (4.8), \( \vdash B_\beta P \equiv B_\alpha B_\beta P \), and by (4.14), \( V_\alpha B_\beta P \vdash B_\alpha B_\beta P \), so that our theorem follows.

Theorem 4.16. If \( \alpha \) is a positive integer and \( \beta \) is a non-negative integer, then

\[
V_\alpha P, V_\beta Q, W_\beta P Q \vdash EPQ.
\]

Proof. Case 1. \( \beta = 0 \). Then by Theorem 3.1

\[
V_\alpha P, W_\beta P Q \vdash V_\alpha B_\beta Q.
\]

Then by Theorem 4.15,

\[
V_\alpha P, W_\beta P Q \vdash EPQ.
\]

Case 2. \( \beta = 1 \). Then by (4.4),

\[
W_\beta P Q \vdash EPQ.
\]

Case 3. \( \alpha = 1 \). Then by (1.29) and Theorem 3.1

\[
V_\alpha P \vdash P \equiv NB_0 P,
\]

\[
V_\alpha Q \vdash Q \equiv NB_0 Q.
\]

Then by (4.2), (3.6), and Theorem 3.1, \( V_\alpha P, V_\alpha Q \vdash EPQ \).

Case 4. \( \alpha \geq 2 \) and \( \beta \geq 2 \). Then \( (\alpha - 1)\beta \geq \alpha \), so that by (4.9) and (4.14)

\[
V_\alpha Q \vdash B_{(\alpha - 1)\beta} Q.
\]

However, by (4.8)

\[
W_\beta P Q \vdash B_{\alpha - 1} P \equiv B_{(\alpha - 1)\beta} Q.
\]

The last two results give

(a) \( V_\alpha Q, W_\beta P Q \vdash B_{\alpha - 1} P \).

By (4.18) and (4.4)

(b) \( V_\alpha P \vdash EB_{\alpha - 1} P N P \).

Then by (a), (b), and (3.34),

\[
V_\alpha P, V_\alpha Q, W_\beta P Q \vdash NP.
\]

From this by (3.49)

\[
V_\alpha P, V_\alpha Q, W_\beta P Q \vdash P \equiv B_\beta R.
\]
Thus

\[ V_\alpha P, V_\alpha Q, W_\beta PQ \vdash V_\alpha B_0 R, \]

so that by Theorem 4.15

\[ V_\alpha P, V_\alpha Q, W_\beta PQ \vdash EPQ. \]

**Theorem 4.17.** If \( \alpha \) is a positive integer and \( \beta \) is a non-negative integer, then

\( (4.27) \) \[ V_\alpha P, V_\alpha Q, \sum_{r=0}^{\beta} (W_r PQ) \vdash EPQ. \]

**Proof.** Use Theorem 4.16 and Theorem 2.3.

5. **The case when \( S \) has \( M \) members, \( S = 1 \), and \( C, N, \) and \( T \) are taken as undefined.** We make the assumptions just listed, and use Rule C, axiom schemes A1–A4 and also the two following axiom schemes:

AT1. \( V_{M-1} NTQ. \)

AT2. \( \sum_{r=0}^{M-1} (W_r PNTQ). \)

Since \( M \geq 2 \), we get by \( (4.27) \), Theorem 3.1, AT1, and AT2

\( (5.1) \) \[ V_{M-1} P \vdash P \equiv NTQ, \]

whence by AT1, (3.6), and (3.4)

\( (5.2) \) \[ \vdash TP = TQ. \]

By AT1, (3.40), and (3.4), we get \( \vdash W_{M-2} TPNTP \), whence we get

\( (5.3) \) \[ \vdash W_{M-2} TPNTQ \]

by (5.2). By (5.2) we can extend the equivalence and substitution theorems to the case where \( T \) is used as well as \( C \) and \( N \).

**Theorem 5.1.** Let \( n \) be a non-negative integer, and let \( \beta_r (0 \leq r \leq n) \) be non-negative integers such that \( \beta_r \leq M - 1 \) \((0 \leq r \leq n)\). Let \( \phi(P_0, \cdots, P_n) \) be a statement formula built up from \( P_0, \cdots, P_n \) by means of \( C, N, \) and \( T \). Let \( \mu \) be that non-negative integer with \( \mu \leq M - 1 \) such that if \( P_r \) is assigned the value \( \beta_r/(M-1) \) \((0 \leq r \leq n)\), then \( \phi(P_0, \cdots, P_n) \) takes the value \( \mu/(M-1) \). Then

\( (5.4) \) \[ W_{\beta_0} P_0 NTQ, \cdots, W_{\beta_n} P_n NTQ \vdash W_{\mu} \phi(P_0, \cdots, P_n) NTQ. \]

**Proof by induction on the structure of \( \phi \), using AT1, Theorem 4.10, Theorem 4.12, and (5.3).**

As a temporary definition, we introduce a generalized product by the following recursion:

\( (5.5) \) If \( \beta = \alpha \), then \( \prod_{i=\alpha}^{\beta} P_i \) denotes \( P_\alpha \).

\( (5.6) \) If \( \beta > \alpha \), then \( \prod_{i=\alpha}^{\beta} P_i \) denotes \( LP_\beta \prod_{i=\alpha}^{\beta-1} P_i \).

By (3.33) and (3.34), we can rewrite (5.4) as

\( (5.7) \) \[ \prod_{r=0}^{M-1} (W_{\beta_r} P_r NTQ) \vdash W_{\mu} \phi(P_0, \cdots, P_n) NTQ. \]

By (4.14) and AT1,

\( (5.8) \) \[ \vdash B_{M-1} NTQ. \]

Therefore, by (3.33)

\( (5.9) \) \[ W_{M-1} PNTQ \vdash P. \]

**Theorem 5.2.** Let \( \phi(P_0, \cdots, P_n) \) be a statement formula built up from
Then by (5.9)
\[ \prod_{r=0}^{n} (W_{\beta_r} P_r NTQ) \vdash \phi. \]

From this, by Theorem 2.3, AT2, and the distributive law for $A$ and $L$, we can infer $\vdash \phi$.

6. The case when $\beta$ has $M$ members, $\beta = 1$, and $C$ and $N$ are taken as undefined. With these assumptions, it follows from a theorem of McNaughton (see [3]) that one can define a function $F$ whose corresponding truth-value function $f(x, y)$ has the following property:

Let $b$ and $d$ be divisors of $M-1$. Let $x = a/b$ and $y = c/d$, where $(a, b) = (c, d) = 1$ (we regard 0 as being 0/1 for this purpose). Then $f(x, y) = 1/\{b, d\}$, where $\{b, d\}$ denotes the least common multiple of $b$ and $d$.

Clearly the definition of $F$ depends on the value of $M$.

We use Rule C, axiom schemes A1–A4, and also the three following axiom schemes:

AF1. $CFPFQFQP$,
AF2. $\sum_{r=0}^{M-1} (W_r PFPQ)$,
AF3. $\sum_{j=1}^{d} (V_{\alpha(j)} FPQ)$,

where $d$ denotes the number of positive divisors of $M-1$ and $\alpha(j)$ denotes the $j$th positive divisor of $M-1$, starting with the least and counting up.

Interchanging $P$ and $Q$ in AF1 gives

(6.1) $\vdash FPQ = FQP$,

so that by AF2

(6.2) $\vdash \sum_{r=0}^{M-1} (W_r PFPQ)$.

Theorem 6.1. If $\gamma$ is a non-negative integer, then

(6.3) $W_{\gamma} RP \vdash \sum_{r=0}^{M-1} (W_r RFQP)$.

Proof. If $1 \leq j \leq d$, then by (4.23)

\[ LV_{\alpha(j)} FQP W_{\gamma} PFPQ, W_{\gamma} RP \vdash \sum_{r=0}^{M-1} (W_r RFQP). \]

From this by Theorem 2.3, AF3, (6.2) and the distributive law for $A$ and $L$, we can infer (6.3).

Let us define $\Phi$ by the following recursion:
(6.4) If \( \alpha = \beta \), then \( \Phi_{i=\alpha}^{\beta} P_i \) denotes \( P_{\alpha} \).

(6.5) If \( \alpha < \beta \), then \( \Phi_{i=\alpha}^{\beta} P_i \) denotes \( FP_{\beta} \Phi_{i=\alpha}^{\beta-1} P_i \).

**Theorem 6.2.** Let \( \beta \) and \( n \) be non-negative integers with \( \beta \leq n \). Then

\[
(6.6) \quad \sum_{r=0}^{M-1} (W_r P_{\beta} \Phi_{i=0}^{\alpha} P_i).
\]

Proof by induction on \( n \). By (3.38) and (4.4),

\[
(6.7) \quad W_1 P_{\beta} \Phi_{i=0}^{n} P_i \equiv \sum_{r=0}^{M-1} (W_r P_{\beta} \Phi_{i=0}^{n+1} P_i).
\]

Then by Theorem 2.3

\[
\sum_{r=0}^{M-1} (W_r P_{\beta} \Phi_{i=0}^{n} P_i) \equiv \sum_{r=0}^{M-1} (W_r P_{\beta} \Phi_{i=0}^{n+1} P_i).
\]

Thus, since we are assuming (6.6) for \( n \), we get (6.6) for \( n + 1 \).

**Theorem 6.3.** Let \( \alpha \) be a positive integer and let \( \beta \) and \( n \) be non-negative integers with \( \beta \leq n \). Then

\[
(6.8) \quad V_\alpha \Phi_{i=0}^{n} P_i \equiv \sum_{r=0}^{\alpha} (W_r P_{\beta} \Phi_{i=0}^{n} P_i).
\]

Proof. In Theorem 4.14, take \( \beta = \alpha \), \( \delta = 1 \), \( P \) and \( S \) to be \( \Phi \), and \( R \) to be \( P_{\beta} \). Then by (6.7), we have

\[
V_\alpha \Phi, W_\gamma P_{\beta} \Phi \equiv \sum_{r=0}^{\alpha} (W_r P_{\beta} \Phi).
\]

Then by Theorem 2.3 and (6.6), we infer (6.8).

**Theorem 6.4.** Let \( \alpha \) be a positive integer, let \( \gamma \) be a non-negative integer with \( \gamma \leq \alpha \), and let \( n \) be a non-negative integer. Let \( \phi(P_0, \cdots, P_n) \) be a statement formula built up from \( P_0, \cdots, P_n \) by means of \( C \) and \( N \). Suppose that whenever \( \beta_r (0 \leq r \leq n) \) are non-negative integers with \( \beta_r \leq \alpha \), and \( P_r \) is given the value \( \beta_r/\alpha (0 \leq r \leq n) \), the corresponding value of \( \phi(P_0, \cdots, P_n) \) is greater than or equal to \( \gamma/\alpha \). Then

\[
(6.9) \quad V_\alpha \Phi_{i=0}^{n} P_i, B_\gamma \Phi_{i=0}^{n} P_i \equiv \phi(P_0, \cdots, P_n).
\]

Proof. Using the product notation of (5.5) and (5.6), we get by (4.22)

\[
V_\alpha \Phi, \prod_{r=0}^{n} (W_{\beta_r} P_r \Phi) \equiv W_\alpha \phi \Phi.
\]

Thus by (3.33)
(a) \( V_{\alpha} \Phi, \prod_{r=0}^{n} (W_{\beta_r} P_r \Phi) \vdash CB_{\mu} \Phi \).

By the hypothesis of the theorem, \( \gamma \leq \mu \). So by (4.9)

\[ B_{\gamma} \Phi \vdash B_{\mu} \Phi. \]

So by (a)

(b) \( V_{\alpha} \Phi, B_{\gamma} \Phi, \prod_{r=0}^{n} (W_{\beta_r} P_r \Phi) \vdash \phi \).

Since this holds for each choice of \( \beta_r \) with \( 0 \leq \beta_r \leq \alpha \) \((0 \leq r \leq n)\), we can use Theorem 2.3, Theorem 6.3, and the distributive law for \( A \) and \( L \) to infer (6.9).

**Theorem 6.5.** Let \( \phi(P_1, \cdots, P_n) \) be a statement formula built up from \( P_1, \cdots, P_n \) by means of \( C \) and \( N \). Then \( \vdash \phi \) if and only if the corresponding truth-value function takes only designated values.

**Proof.** Assume that the truth-value function corresponding to \( \phi \) takes only designated truth-values. Write \( \theta(P_0, \cdots, P_n) \) for \( CCP_0 P_0 \phi(P_1, \cdots, P_n) \).

Then \( \theta \) takes only designated truth-values for any assignment of values to \( P_0, P_1, \cdots, P_n \). Let \( \alpha(j) \) be a divisor of \( M-1 \). Then we may take both \( \alpha \) and \( \gamma \) equal to \( \alpha(j) \) in Theorem 6.4, so that by (4.14)

\[ V_{\alpha(j)} \Phi_{i=0}^{n} P_i \vdash \theta(P_0, \cdots, P_n). \]

Since \( n \geq 1 \), we may use (6.5), AF3, and Theorem 2.3 to infer

\[ \vdash \theta(P_0, \cdots, P_n). \]

Finally, by (2.10) and the definition of \( \theta \), we conclude \( \vdash \phi \).

7. **A fragment of the** C-N-J-D **calculus.** In this section we take \( C \) and \( N \) as undefined, and we assume that \( J \) and \( D \) are either undefined or are definable in terms of \( C \) and \( N \). We use Rule I and the axiom schemes:

- **AJ1.** \( JCPCQP \).
- **AJ2.** \( JCCPQCCQRCPR \).
- **AJ3.** \( JCAPQAQP \).
- **AJ4.** \( JCCNPQNCQP \).
- **AJ5.** \( IJCPQIQJPQ \).
- **AJ6.** \( IJCPQIQPQ \).
- **AJ7.** \( IICQRIAPQAPR \).

By Rule I and AJ5, we infer the following rule:

- Rule JC. If \( JP \) and \( JCPQ \), then \( JQ \).

Using this and AJ1–AJ4, we can easily prove the following theorem.

**Theorem 7.1.** If \( P_1, \cdots, P_n \vdash Q \) can be derived on the basis of Rule C and axiom schemes A1–A4, then \( JP_1, \cdots, JP_n \vdash JQ \).

From this by (2.16) and (2.5) we get
From these and AJ3 we get

\begin{align}
(7.1) & \vdash IAPP, \\
(7.2) & \vdash IPAPQ, \\
(7.3) & \vdash IAPQAQP,
\end{align}

by means of AJ6.

**Theorem 7.2.** If we count I, D, A, and & as the two-valued implication, negation, disjunction, and conjunction, we have the full two-valued statement calculus.

**Proof.** Rule I is the standard rule, and (7.1), (7.2), (7.3), and AJ7 are the standard axiom schemes for the two-valued calculus (for example, see [5]).

In particular, we can get such results as the two-valued commutativity and associativity of &, and we can get the two-valued distributive laws for & and A. Moreover, we can get such standard results as the following.

**Theorem 7.3.** If \( P_1, \ldots, P_p, R \vdash T \) and \( Q_1, \ldots, Q_q, S \vdash T \), then \( P_1, \ldots, P_p, Q_1, \ldots, Q_q, ARS \vdash T \).

By Theorem 7.1, (2.4), and (2.5), we have for \( \alpha \leq \gamma \leq \beta \)

\[ JP_\gamma \vdash J \sum_{i=\alpha}^{\beta} P_i. \]

Then by Theorem 7.3, we can infer the following theorem.

**Theorem 7.4.** If \( \alpha \) and \( \beta \) are integers with \( \alpha \leq \beta \), then

\[ \sum_{i=\alpha}^{\beta} (JP_i) \vdash J \sum_{i=\alpha}^{\beta} P_i. \]

By Rule JC and AJ1

\[ JP \vdash JCPP. \]

Then by Rule I and AJ6

\[ JP \vdash IJPP. \]

So by Rule I

\[ (7.5) \ JP \vdash P. \]

8. **The case when \( s \) has \( M \) members, \( s < 1 \), and \( C, N, \) and \( T \) are taken as undefined.** Let \( J \) and \( D \) be defined in terms of \( C \) and \( N \) (see [3] or [4]). Let \( H \) be the least integer such that \( H/(M-1) \) is designated. We use Rule I, axiom schemes AJ1–AJ5 and also the three following axiom schemes:

\begin{align}
ATJ1. & \ J(V_{M-1})NTP, \\
ATJ2. & \ J(\sum_{r=0}^{M-1} (W_rPNTQ). 
\end{align}
ATJ3. \( \text{IJCB}_n\text{NTQP} \).

Inasmuch as only Rule I and axiom schemes AJ1–AJ5 were used in proving Theorem 7.1, we see that we can prove a theorem analogous to Theorem 7.1 except that it refers to results derivable on the basis of Rule C and axiom schemes A1–A4 and axiom schemes AT1–AT2.

We now prove a theorem whose statement is identical with that of Theorem 5.2. We assume that \( \phi \) is a formula whose truth-value is always designated. Then Theorem 5.2 tells us that we can derive \( \text{CB}_n\text{NTQ}\phi \) from axiom schemes A1–A4 and AT1–AT2 by Rule C. So by our generalized Theorem 7.1, we get

\[
\vdash \text{JCB}_n\text{NTQ}\phi.
\]

Then \( \vdash \phi \) by axiom scheme ATJ3.

9. The case when \( 3 \) has \( M \) members, \( S < 1 \), and \( C \) and \( N \) are taken as undefined. As in §§6 and 8, we let \( F, J, \) and \( D \) be defined in terms of \( C \) and \( N \). We also take \( d \) and \( \alpha(j) \) as in §6, and if \( 1 \leq j \leq d \), we take \( \gamma(j) \) to be the least integer such that \( \gamma(j)/\alpha(j) \) is designated. We take \( J_n(P) \) as defined in [4] and use \( G(P) \) to designate

\[
\sum_{j=1}^{d} \text{K}_{J-M-\alpha(j)}(P)B_{\gamma(j)}P.
\]

We use Rule I, axiom schemes AJ5–AJ6, the following axiom scheme

AG. \( G(FPQ) \),

and a set of auxiliary axiom schemes built up as follows:

Choose a set of axiom schemes such that from them by means of Rule C one can derive exactly those statement formulas built up by means of \( C \) and \( N \) whose corresponding truth-value functions take only the truth-value 1. Then prefix a \( J \) to each of these axiom schemes. The resulting set of axiom schemes is the set of auxiliary axiom schemes.

In view of Theorem 6.5, the auxiliary axiom schemes could be got by prefixing a \( J \) to each of A1–A4 and AF1–AF3. Alternatively, the auxiliary axiom schemes could be got by prefixing a \( J \) to each of the five axiom schemes appearing in §14.

By Rule I and AJ5, we infer Rule JC. By Rule JC and the auxiliary axiom schemes, we can prove:

**Theorem 9.1.** Let \( \phi(P_0, \ldots, P_n) \) be a statement formula built up from \( P_0, \ldots, P_n \) by means of \( C \) and \( N \) such that the corresponding truth-value function takes only the truth-value 1. Then

\[
\vdash J\phi(P_0, \ldots, P_n).
\]

We now prove a theorem whose statement is identical with that of Theorem 6.5. We assume that \( \phi(P_0, \ldots, P_n) \) is a formula whose truth-value is always designated. Then
\[
CG(FP_0 \Phi_{i=0}^n P_i) \phi(P_0, \ldots, P_n)
\]
always takes the value unity. So by Theorem 9.1
\[
\vdash JCG(FP_0 \Phi_{i=0}^n P_i) \phi(P_0, \ldots, P_n).
\]
Now by axiom scheme AJ6,
\[
\vdash IG(FP_0 \Phi_{i=0}^n P_i) \phi(P_0, \ldots, P_n)
\]
so that we get \( \vdash \phi \) by axiom scheme AG.

10. Special results for use in the infinite-valued case. We adjoin an additional axiom scheme A5 to the four used in §§3 and 4. Actually, C. A. Meredith and later independently C. C. Chang discovered that axiom scheme A5 is a consequence of Rule C and axiom schemes A1–A4, so that it would suffice to assume the latter. The proofs of Meredith and Chang appear in notes after the end of the present paper, but for the present it is convenient merely to refer to the result in question as the fifth one of our axiom schemes. For the reader's convenience, we state in full the axiom schemes we will be using.

In this section, we use Rule C and the following axiom schemes:

A1. \( CPCQP \).
A2. \( CCPQCCQRCPR \).
A3. \( CAPQAQP \).
A4. \( CCNPNQCQP \).
A5. \( ACPQCQP \).

Theorem 10.1.
(10.1) \( \vdash LCCPQRCQP = LCCRQPCQR \).

Proof. Temporarily let us write
(a) \( V \) for \( LCCPQRCQP \)
and
(b) \( W \) for \( LCCRQPCQR \).

By (2.25), (3.35), and (a)
\[
\vdash CVCCRQP.
\]
Interchanging \( P \) and \( R \) in (3.51) gives
\[
\vdash CCCQRCQPCCRQCP.
\]
Then by (2.7)
(d) \( CRQ \vdash CCCQRCQPCRP \).

By A2, we have \( CPR \vdash CCRQCPQ \), whence, by A2 again, we get
\[
CPR \vdash CCCPQRCCRQR.
\]
Using this and \( \vdash CCRPCRP \) in (2.14) gives
\[
CPR \vdash CCCPQRCCRQCCRPP,
\]
whence by two uses of (2.7) we get
\[
CPR \vdash CCRPCCPQRCCCRQPR.
\]
Using this and (d) gives
\[
(e) \quad CPR, CRQ \vdash CCCQRCRQPCCCPQRCCRQP.
\]
By (2.8)
\[
(f) \quad CPR \vdash CCQPCQR.
\]
By (f), (e), (3.55), (a), and (b)
\[
CPR, CRQ \vdash CVW.
\]
By this, (c), A5, and Theorem 2.3
\[
(g) \quad CPR \vdash CVW.
\]
By (3.51) and (2.7)
\[
(h) \quad \vdash CCPCCPQRCCQPCQR.
\]
By A1, we have \( \vdash CPCCRQP \), whence by A2
\[
\vdash CCCCRQPCCPQRCPCPQ.
\]
By this and (h)
\[
(i) \quad \vdash CCCCRQPCCPQRCPCQPCQR.
\]
By (2.25) and (2.7)
\[
(j) \quad CQPCPQRCCRP.
\]
By A1, CRP \( \vdash CCRQCRP \), so that by (2.7)
\[
CRP \vdash CRCCRQP.
\]
Also by (2.5)
\[
CPQ \vdash CCCPQRR.
\]
By the last two results
\[
CPQ, CRP \vdash CCCPQRCCRQP.
\]
By this, (j), A5, and Theorem 2.3
\[
CRP \vdash CCCPQRCCRQP.
\]
By this, (i), and (3.55)
\[
CRP \vdash CLCQPCCPQRCLCQCRCCRQP.
\]
By the commutative law for \( L \) and (a) and (b)
\[
CRP \vdash CVW.
\]
By this, (g), A5, and Theorem 2.3
Interchanging $P$ and $R$ in this gives (10.1).

**Theorem 10.2.**

(10.2) \[ \neg LBLPQRBQP = LBLRQPBQR. \]

**Proof.** Replace $Q$ by $NQ$ in (10.1) and use (3.54).

In the succeeding theorems of this section, the letter $T$ will not denote the Słupecki operator characterized by (1.4), but will take the place of an unspecified statement, in the same role as $P, Q, R, \ldots$.

**Theorem 10.3.** If

(a) \[ \neg ANVW, \]
(b) \[ \neg R = LBVZW, \]
(c) \[ \neg T = LBVYY, \]

then

(d) \[ \neg LBRYBWZ = LBTZBWY. \]

**Proof.** By (3.45)

\[ NV \vdash Z \equiv BVZ, \]

so that by (b)

(e) \[ NV \vdash LBRYBWZ = LBLZBWZ. \]

Interchanging $Y$ and $Z$ in the above reasoning gives

(f) \[ NV \vdash LBZBWY = LBLYWZBWY. \]

From (e) and (f) by (10.2), we get

(g) \[ NV \vdash LBRYBWZ = LBTZBWY. \]

By (3.37), (b), and the commutativity of $L$

\[ W \vdash R \equiv BVZ. \]

Thus

\[ W \vdash BRY \equiv BBVZY, \]

so that by the associativity of $B$

(h) \[ W \vdash BRY \equiv BBVZY. \]

By (3.32), $W \vdash BWZ$, so that by (3.37) and the commutativity of $L$

\[ W \vdash LBRYBWZ \equiv BRY. \]

Thus by (h)

(i) \[ W \vdash LBRYBWZ \equiv BBVZY. \]

If we interchange $Y$ and $Z$ in the proof of (i), we get

(j) \[ W \vdash LBZBWY \equiv BVBYZ. \]

By (i), (j), and the commutativity of $B$,

(k) \[ W \vdash LBRYBWZ \equiv LBTZBWY. \]

By (g), (k), (a), and Theorem 2.3, we conclude (d).
Theorem 10.4. If
(a) \( \vdash ANVW \),
(b) \( \vdash R \equiv LBVZW \),
(c) \( \vdash S \equiv LBWZX \),
(d) \( \vdash T \equiv LBVYW \),
(e) \( \vdash U \equiv LBWYX \),
then
(f) \( \vdash LBRS \equiv LBTZU \).

Proof. By Theorem 10.3, we have
\( \vdash LBRYBWZ \equiv LBTZBWY \).

So
\( \vdash LLBRYBWZX \equiv LLBTZBXWY \).

Now use the associativity of \( L \), and (c) and (e).

Theorem 10.5. If
(a) \( \vdash ANRS \),
(b) \( \vdash ANST \),
(c) \( \vdash P \equiv LBRXS \),
(d) \( \vdash Q \equiv LBSXT \),
then
(e) \( \vdash ANPQ \).

Proof. By (3.56), (b), and (d), \( S \vdash Q \). So by (2.4)
(f) \( S \vdash ANPQ \).

By (c) and (3.33), \( \vdash CPS \), so that by (3.5), \( NS \vdash NP \). Then by (2.5)
(g) \( NS \vdash ANPQ \).

By (3.45) and (c)
\( NR \vdash P \equiv LXS \),
while by (3.37) and the commutativity of \( L \)
\( T \vdash Q \equiv BSX \).

Then (using (3.4)),
\( NR, T \vdash ANPQ \equiv ACXNSCNSX \).

So by A5
(h) \( NR, T \vdash ANPQ \).

Now by (f), (h), (a), and Theorem 2.3, we get
(i) \( T \vdash ANPQ \).

By (g), (i), (b), and Theorem 2.3, we get (e).

Theorem 10.6. If
(a) \( \vdash ANSM \),
(b) $\vdash A N U V$,
(c) $\vdash A N V W$,
(d) $\vdash A N Y Z$,
(e) $\vdash Q \equiv L B U X V$,
(f) $\vdash T \equiv L B V X W$,
(g) $\vdash R \equiv L B Y X Z$,
(h) $S \vdash C V C B P U Y$,
(i) $S \vdash C W C B P V Z$,
(j) $M \vdash C V C B S U Z$,

then
(k) $S \vdash C T C B P Q R$.

Proof. By (a) and (3.52), $\vdash C S M$. So by (j) and (3.32), $S, V \vdash Z$. Then by (3.37), (g), and the commutativity of $L$
(l) $S, V \vdash R = B Y X$.

By (e) and (3.34), $\vdash C Q B U X$. Then by (3.23), $\vdash C B P Q B P B U X$. So by the associativity of $B$,
(m) $\vdash C B P Q B P B U X$.

By (h) and (3.24)

$$S, V \vdash C B B P U X B Y X.$$  

So by (l) and (m),

$$S, V \vdash C B P Q R,$$

whence by A1
(n) $S, V \vdash C T C B P Q R$.

By (e), (3.33), and (3.5), $N V \vdash N Q$. So by (3.45) and the commutativity of $B$,

(o) $N V \vdash P \equiv B P Q$.

By (3.45) and (f)

(p) $N V \vdash T \equiv L X W$.

By (3.31), (3.27), and (g)

(q) $\vdash C L X Z R$.

By (i) and (2.7), $S \vdash C B P V C W Z$, so that by (3.32), $S \vdash C P C W Z$. Then by (3.26) and (2.7)

$$S \vdash C L X W C P L X Z.$$  

From this by (o) and (p)

$$S, N V \vdash C T C B P Q L X Z,$$

so that by (q) and (2.14)

(r) $S, N V \vdash C T C B P Q R$.

By (3.45) and (e)

(s) $N U \vdash Q \equiv L X V$. 

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By (3.37), (g), and the commutativity of $L$
(t) $Z \vdash R \equiv BYX$.

By (3.45), (h), and the commutativity of $B$
(u) $S, NU \vdash CVCPY$.

Then by (3.35) and the commutativity of $L$

$$S, NU \vdash CLPVY.$$  

Then by (3.24) and (t)

$$S, Z, NU \vdash CBLPVXR.$$  

Then by (3.34)

$$S, Z, NU \vdash CLBLPVXBVPR.$$  

Then by (10.2)

$$S, Z, NU \vdash CLBLXVPBVXR.$$  

So by (s) and the commutativity of $B$

$$S, Z, NU \vdash CLBPQBVXR.$$  

Then by the commutativity of $L$ and (3.35)

$$S, Z, NU \vdash CBVXCBPQR.$$  

Finally by (3.34) and (f)

(v) $S, Z, NU \vdash CTCPBQR$.

By (i) and the commutativity of $B$

(w) $S, W \vdash CBVPZ$.

By (u), (3.5), and (2.7)

$$S, NU, NY \vdash CVNP.$$  

Then by (2.5)

$$S, NU, NY \vdash CCCVNPXX.$$  

Then by (3.4) and the commutativity of $L$

$$S, NU, NY \vdash CBLPVXX.$$  

By applying (3.28) to this and (w), we get

$$S, W, NU, NY \vdash CBLPVXBVPLXZ.$$  

Then by (10.2)

$$S, W, NU, NY \vdash CLBLXVPBVXZ.$$  

Then by (s) and the commutativity of $B$

$$S, W, NU, NY \vdash CLBPQBVXZ.$$
Then by the commutativity of $L$ and (3.35)

$$S, W, NU, NY \vdash CBVXCBPQLXZ.$$  

Then by (q) and (2.14)

$$S, W, NU, NY \vdash CBVXCBPQR.$$  

Finally by (3.34) and (f)

(x) $S, W, NU, NY \vdash CTCBPQR.$  

We now make a succession of uses of Theorem 2.3. In particular, if we write $\phi$ for $CTCBPQR$, then by (v), (x), and (d), $S, W, NU \vdash \phi$. Then by (r) and (c), $S, NU \vdash \phi$. Then by (n) and (b), $S \vdash \phi$, which is the result we wish.

**Theorem 10.7.** If

(a) $\vdash ANUV,$
(b) $\vdash ANYZ,$
(c) $\vdash APW,$
(d) $\vdash Q=LBUXV,$
(e) $\vdash R=LBYXZ,$
(f) $\vdash CYBPV,$
(g) $\vdash CZBPV,$
(h) $W \vdash CZCCPYV,$

then

(i) $\vdash CRBPQ.$

**Proof.** By (f) and (2.7), $\vdash CNPCVU$. Then by (3.24), $\vdash CNPCBYXBUX$. Finally by (2.7)

(j) $\vdash CBYXBPBUX.$

Then by (e) and (3.34)

$\vdash CRBPBUX.$

However, by (d), (3.37) and the commutativity of $L$, we have $V \vdash BUX \equiv Q$, so that

(k) $V \vdash CRBPQ.$

From (g), by reasoning like that used to derive (j), we get

$\vdash CLXZBPLXV.$

However, by (3.45) and (e), $NY \vdash R \equiv LXZ$, so that

(l) $NY \vdash CRBPLXV.$

By (3.31), $\vdash CXBUX$, so that by (3.27) and (d)

$\vdash CLXVQ.$

By applying (2.14) to this and (l), keeping (1.8) in mind, we get

(m) $NY \vdash CRBPQ.$
From (f) by the commutativity of $B$ and (2.7), we get
(n) $NU \vdash CYP$.

By (3.45) and (d)
(o) $NU \vdash Q \equiv LXV$.

By (h), (n), and (2.25)
(p) $W, Z, NU \vdash CCVYP$.

By (2.8)
\[ \vdash CBXYCCVNXCVY. \]

By this, (p), and (2.14)
\[ W, Z, NU \vdash CBXYCCVNXP. \]

Then by (3.4)
\[ W, Z, NU \vdash CBXYBLVXP. \]

Then by (o) and the commutativity of $L$ and $B$
\[ W, Z, NU \vdash CBXYBPQ. \]

Finally by (e), (3.34), and the commutativity of $B$
(q) $W, Z, NU \vdash CRBPQ$.

By (3.32) and A1
(r) $P \vdash CRBPQ$.

We now use Theorem 2.3 with (a), (b), (c), (k), (m), (r), and (q) in order to infer (i).

**Theorem 10.8.**
(10.3) $LBLXQNXBQX \equiv Q$.

**Proof.** Temporarily let us write
(a) $V$ for $LBLXQNXBQX$.

By (a), (3.37), and the commutativity of $L$
\[ BQX \vdash V \equiv BLXQNX. \]

Then commutativity of $L$ gives
\[ BQX \vdash V \equiv BLQXNX. \]

By (3.54)
\[ BQX \vdash V \equiv AQNX. \]

Then commutativity of $A$ gives
\[ BQX \vdash V \equiv CBXQQ. \]

Finally by commutativity of $B$ and (2.15)
(b) $BQX \vdash V \equiv Q$.

By (3.4)
Thus by (a) and (3.45)

\[ CXNQ \vdash V \equiv LNXBQX. \]

By commutativity of \(L\),

\[ CXNQ \vdash V \equiv LBQXNX, \]

whence (3.4) gives

\[ CXNQ \vdash V \equiv NCCNQXX. \]

Then commutativity of \(A\) gives

\[ CXNQ \vdash V \equiv NAXNQ, \]

which is the same as

\[ CXNQ \vdash V \equiv LCXNQQ. \]

Finally by (3.37)

(c) \( CXNQ \vdash V \equiv Q. \)

Now we use Theorem 2.3 with (b), (c) and A5 to infer \( \vdash V \equiv Q \), which by (a) gives (10.3).

**Theorem 10.9**

(10.4) \( ANPQ, ANQR \vdash LBLBPXQNXLBQXR = Q. \)

**Proof.** Let us temporarily write

(a) \( W \) for \( LBLBPXQNXLBQXR \).

By (3.45) and (a)

(b) \( NP \vdash W = LBLXQNXLBQXR. \)

By (3.37) and the commutativity of \( L \), \( R \vdash BQX = LBQXR \), so that by (b)

\[ NP, R \vdash W = LBLXQNXBQX. \]

So by (10.3)

(c) \( NP, R \vdash W = Q. \)

By (3.5) and (3.33)

\[ NQ \vdash NLXQ. \]

Then by (b) and (3.45)

\[ NP, NQ \vdash W = LNXLXR. \]

By (3.4), this reduces to

(d) \( NP, NQ \vdash W = NCNXCXNR. \)

By (3.53) and (3.4)

\( \vdash NNCNXCXNR. \)
Then by (3.50)

\[ NQ \vdash NCNXCNXR \equiv Q. \]

So by (d)

\[ (e) \ NP, \ NQ \vdash W \equiv Q. \]

By Theorem 2.3, (c), and (e)

\[ (f) \ NP, \ ANQR \vdash W \equiv Q. \]

By (3.52),

\[ Q, \ ANQR \vdash R. \]

Then by (3.32) and (3.36)

\[ Q, \ ANQR \vdash LBQXR. \]

Consequently, by (a), (3.37), and the commutativity of \( L \)

\[ Q, \ ANQR \vdash W \equiv BBPXNX. \]

Commutativity and associativity of \( B \) gives

\[ Q, \ ANQR \vdash W \equiv BNXBXP, \]

which is the same as

\[ (g) \ Q, \ ANQR \vdash W \equiv CNNXCNXP. \]

By (3.53) and (3.48)

\[ Q \vdash CNNXCNXP \equiv Q. \]

So by (g)

\[ (h) \ Q, \ ANQR \vdash W \equiv Q. \]

By Theorem 2.3, (f), and (h)

\[ ANPQ, \ ANQR \vdash W \equiv Q, \]

which gives (10.4) by use of (a).

11. Some properties of inequalities for nonhomogeneous polynomials over the field of rationals. The results of this section were derived for us by Theodor Motzkin. They are based on a special case of the transposition theorem (see [6, §13]); we now state this special case.

**Theorem 11.1.** Let \( A \) and \( B \) be matrices of \( m \) rows, with rational components. Let \( x \) be a row vector of \( m \) components, each of which is a variable over the rationals. Let \( y_1 \) and \( y_2 \) be column vectors, each component of which is a variable over the rationals; let \( y_1 \) have as many rows as \( A \) has columns, and \( y_2 \) have as many rows as \( B \) has columns. Define two sets of conditions, as follows:

(I) Every component of \( xA \) is positive, and every component of \( xB \) is non-negative.

(II) \( Ay_1 + By_2 = 0 \), every component of \( y_1 \) or \( y_2 \) is non-negative, and at least one component of \( y_1 \) is positive.
Then we have the result that there is an \( x \) satisfying (I) if and only if there is no \( y_1 \) and \( y_2 \) satisfying (II).

To prove this, one merely follows the development of [6], noting that this development holds over any ordered field, and hence over the rationals.

**Theorem 11.2.** Let

\[
(11.1) \qquad f_i = a_i + \sum_{j=1}^{n} b_{ij}x_j \quad (1 \leq i \leq m),
\]

\[
(11.2) \qquad g = c + \sum_{j=1}^{n} d_jx_j,
\]

where the \( a \)'s, \( b \)'s, \( c \), and \( d \)'s are rationals. Suppose that there are sets of rational values of the \( x \)'s for which

\[
(11.3) \qquad f_i \geq 0 \quad (1 \leq i \leq m),
\]

and that \( g > 0 \) for all such sets of values of the \( x \)'s. Then there is a positive rational constant \( \mu \) such that whenever the \( x \)'s are rationals for which (11.3) holds, then

\[
(11.4) \qquad g \geq \mu.
\]

**Proof.** Assume the hypothesis of the theorem. Then (11.3) is inconsistent with \( -g \geq 0 \). Define

\[
\tilde{f}_i = a_i x_0 + \sum_{j=1}^{n} b_{ij}x_j,
\]

\[
\tilde{g} = c x_0 + \sum_{j=1}^{n} d_jx_j.
\]

Then in the field of rationals, the set of inequalities

\[
\tilde{f}_i \geq 0, \quad -\tilde{g} \geq 0, \quad x_0 > 0
\]

has no solution. Let us take \( x \) to be the row vector with components \( (x_0, x_1, \ldots, x_n) \), \( A \) to be the matrix of one column and \( n+1 \) rows with a 1 in the first row and 0's elsewhere, and \( B \) to be the matrix of \( m+1 \) columns and \( n+1 \) rows, whose last column consists of \( -c \) and the \( -d_j \)'s, and whose \( i \)th column \( (1 \leq i \leq m) \) consists of \( a_i \) and the \( b_{ij} \)'s. Then condition (I) of Theorem 11.1 cannot be fulfilled, so that condition (II) must be fulfilled. That is, there is a positive \( y_1 \) and non-negative \( y_2, \ldots, y_{m+2} \) such that

\[
(11.5) \quad y_1 + \sum_{i=1}^{m} y_{i+1}a_i - y_{m+2}c = 0,
\]
\[ \sum_{i=1}^{m} y_{i+1} b_{ij} - y_{m+2} d_j = 0 \quad (1 \leq j \leq n). \]

If we multiply (11.6) by \( x_j \), sum, and add (11.5), we conclude

\[ y_{m+2} g = y_1 + \sum_{i=1}^{m} y_{i+1} f_i \]

as an identity in the \( x \)'s. As \( y_1 > 0 \), and \( y_{i+1} \geq 0 \), and there is a set of \( x \)'s for which (11.3) holds, we may substitute this set of \( x \)'s into (11.7) and conclude \( y_{m+2} > 0 \). So, writing

\[ \mu = y_1/y_{m+2}, \]
\[ \lambda_i = y_{i+1}/y_{m+2} \]

we have

\[ g = \mu + \sum_{i=1}^{m} \lambda_i f_i, \]
\[ \mu > 0. \]

From these two results, our theorem follows.

**Theorem 11.3.** Let \( f_i \) and \( g \) be as in (11.1) and (11.2), with rational coefficients. Suppose that there are sets of rational values of the \( x \)'s for which (11.3) holds, and that \( g \geq 0 \) for all such sets of values. Then there are non-negative rational constants \( \mu, \lambda_1, \cdots, \lambda_m \) such that

\[ g = \mu + \sum_{i=1}^{m} \lambda_i f_i \]

is an identity in the \( x \)'s.

**Proof.** We modify slightly the proof of Theorem 11.2. We first note that the set of inequalities

\[ f_i \geq 0, \]
\[ -\bar{g} > 0, \]
\[ x_0 > 0 \]

has no solution. Then we use corresponding reasoning to conclude that (11.7) holds, except that now we have that all \( y \)'s are non-negative and at least one of \( y_1 \) or \( y_{m+2} \) must be positive. As before, we conclude that \( y_{m+2} \neq 0 \), and conclude (11.8), which is just the same as (11.10). We also have the required result that the \( \mu \) and \( \lambda_i \)'s are all non-negative.
12. Polynomial formulas. We shall make much use of linear polynomials such as

\[(12.1) \quad f = a + \sum_{j=1}^{n} b_j x_j.\]

Here \(a\) and the \(b_j\)'s are constant real numbers, and the \(x_j\)'s are variables. Since we permit some or all of the \(b_j\)'s to be zero, we cannot say unambiguously how many variables really occur in \(f\). Indeed, for our purposes, it is useful to consider the number of variables as indeterminate, but always finite. Thus if \(b_j = 0\) for \(n+1 \leq j \leq N\), then we consider the polynomial

\[g = a + \sum_{j=1}^{N} b_j x_j\]

to be identical with the \(f\) given by (12.1). Perhaps a better way to look at the situation is to say that we are considering forms such as

\[a + \sum_{j=1}^{\infty} b_j x_j,\]

where there is always to be a non-negative \(K\) such that \(b_j = 0\) for \(j > K\). Then we allow ourselves the convenience of using the form (12.1) as a shorthand provided that \(b_j = 0\) for \(j > n\). We assume that \(x_i\) is distinct from \(x_j\) if \(i \neq j\).

We now make some definitions.

Whenever we use the word "polynomial" throughout the remainder of the text, we shall mean a polynomial of the form (12.1) for which the constant term \(a\) and the coefficients \(b_j\) are integers.

We shall write \(\sigma(f)\) for the sum of the absolute values of the coefficients of the variables in \(f\). That is, with \(a\) as in (12.1),

\[(12.2) \quad \sigma(f) = \sum_{j=1}^{n} |b_j| .\]

If \(x\) is a real number, then we define

\[(12.3) \quad \tau(x) = \begin{cases} 1 & \text{if } 1 < x, \\ x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x < 0. \end{cases}\]

Let \(f\) be a polynomial. With \(f\) we wish to associate a class of statement formulas \(\text{PF}(f)\), called the polynomial formulas of \(f\). If \(f\) involves variables \(x_1, \ldots, x_n\), and \(P\) is in \(\text{PF}(f)\), then \(P\) is to depend on distinct statements \(X_1, \ldots, X_n\), correlated with the \(x_j\)'s. Just as \(f\) may not really depend on \(x_i\) (for instance, one may have \(b_i = 0\)), so \(P\) may not really depend on \(X_j\); indeed there need not even be occurrences of \(X_j\) in \(P\) in some cases. The definition of \(\text{PF}(f)\) is by induction on \(\sigma(f)\).
First let $\sigma(f) = 0$.

Case 1. $a \geq 1$. Then $P$ is in $\text{PF}(f)$ if and only if $P$ is $CX_jX_j$, where $x_j$ is one of the variables "occurring" in $f$.

Case 2. $a \leq 0$. Then $P$ is in $\text{PF}(f)$ if and only if $P$ is $NCX_jX_j$, where $x_j$ is one of the variables "occurring" in $f$.

Since $a$ is an integer, these cases cover the situation when $\sigma(f) = 0$.

Now let $\alpha$ be a positive integer and assume that $\text{PF}(f)$ has been defined for each $f$ for which $\sigma(f) < \alpha$. Let $f$ be a polynomial for which $\sigma(f) = \alpha$. There are two ways in which a statement formula $P$ can be in $\text{PF}(f)$.

Case 1. For some $j$, $b_j > 0$. Choose a $Q$ in $\text{PF}(f-x_j)$ and an $R$ in $\text{PF}(f+1-x_j)$, and take

$$\tag{12.4} P = LBQX_jR.$$

Case 2. For some $j$, $b_j < 0$. Choose a $Q$ in $\text{PF}(f+x_j-1)$ and an $R$ in $\text{PF}(f+x_j)$, and take

$$\tag{12.5} P = LBQNX_jR.$$

These two cases are intended to exhaust all $P$'s in $\text{PF}(f)$. Note that in Case 1, we allow ourselves to take any $j$ for which $b_j > 0$, any $Q$ in $\text{PF}(f-x_j)$, and any $R$ in $\text{PF}(f+1-x_j)$. Clearly, in this case $\sigma(f-x_j) = \sigma(f-1)$ and $\sigma(f+1-x_j) = \sigma(f)-1$, so that the classes from which we are to select $Q$ and $R$ have already been defined. Similar remarks hold relative to Case 2.

We say that $P$ is a polynomial formula if there is a polynomial $f$ such that $P$ is in $\text{PF}(f)$. More precisely, we take $\text{PF}$ to be the logical sum of all the $\text{PF}(f)$'s.

Clearly each $P$ in $\text{PF}$ is a statement formula of $X_1, X_2, \ldots$. If we assign the values $x_i$ to $X_i$, then there will be a value assigned to $P$, which we shall denote by $v(P)$.

**Theorem 12.1.** If $P$ is in $\text{PF}(f)$, then

$$\tag{12.6} v(P) = \tau(f)$$

whenever $0 \leq x_j \leq 1$ ($1 \leq j \leq n$).

Proof by induction on $\sigma(f)$. Clearly the theorem holds if $\sigma(f) = 0$. Let $\alpha$ be a positive integer, and assume that the theorem holds for each $f$ for which $\sigma(f) < \alpha$. Let $f$ be a polynomial for which $\sigma(f) = \alpha$. Let $P$ be in $\text{PF}(f)$.

Case 1. $b_j > 0$, $Q$ is in $\text{PF}(f-x_j)$, $R$ is in $\text{PF}(f+1-x_j)$, and $P = LBQX_jR$.

Subcase 1. $1 < f-x_j$. Then $\tau(f-x_j) = \tau(f+1-x_j) = \tau(f) = 1$. So by the hypothesis of the induction, $v(Q) = 1 = v(R)$. Then by (12.4), $v(P) = 1 = \tau(f)$.

Subcase 2. $0 \leq f-x_j \leq 1$. Then $v(Q) = \tau(f-x_j) = f-x_j$, and $v(R) = \tau(f+1-x_j) = 1$. Since the value $x_j$ is assigned to $X_j$, we see by (12.4) that $v(P) = \max (\min (f, 1), 0) = \min (f, 1) = \tau(f)$.

Subcase 3. $-1 \leq f-x_j < 0$. Then $v(Q) = \tau(f-x_j) = 0$, and $v(R) = \tau(f+1-x_j) = f+1-x_j$. Then $v(BQX_j) = x_j$, so that $v(P) = \max (0, f) = \tau(f)$.

Subcase 4. $f-x_j < -1$. Then $\tau(f-x_j) = \tau(f+1-x_j) = \tau(f) = 0$. So $v(Q) = 0 = v(R)$, whence $v(P) = 0 = \tau(f)$.
Case 2. \( b_j < 0 \), \( Q \) is in \( PF(f + x_j - 1) \), \( R \) is in \( PF(f + x_j) \), and \( P = LBQNX_jR \). This case proceeds similarly to Case 1, by considering the subcases \( 2 < f + x_j, 1 \leq f + x_j \leq 2, 0 \leq f + x_j < 1, f + x_j < 0 \).

It will be noted that Theorem 1 of [3] follows immediately from Theorem 12.1, so that we have incidentally furnished an alternative proof for Theorem 1 of [3]. This is probably just as well, inasmuch as the proof given in [3] for Theorem 1 is much more complicated than our proof of Theorem 12.1.

13. The case when \( S \) has an infinite number of members, \( s = 1 \), and \( C \) and \( N \) are taken as undefined. As in §10, we use Rule C and axiom schemes A1–A5. We remind the reader that Meredith and Chang have shown that axiom scheme A5 can be derived from the others.

**Theorem 13.1.** (a) If \( P \) and \( Q \) are both in \( PF(f) \), then \( \neg P \equiv Q \). (b) If \( P \) is in \( PF(f) \) and \( Q \) is in \( PF(f + 1) \), then \( \neg ANPQ \).

Proof by induction on \( \sigma(f) \). First let \( \sigma(f) = 0 \). If \( f \geq 1 \), then we infer part (a) by (2.10) and (3.48), while we infer part (b) by (2.10) and (2.4). If \( f \leq 0 \), then we infer part (a) by (2.10), (3.4), and (3.50), while we infer part (b) by (2.10), (3.4), and (2.5).

Let \( \alpha \) be a positive integer and assume that the theorem holds if \( \sigma(f) < \alpha \).

**Lemma.** Part (a) holds for every \( f \) with \( \sigma(f) = \alpha \).

Let \( \sigma(f) = \alpha \), and let both \( P \) and \( Q \) be in \( PF(f) \).

**Case 1.** \( b_j > 0 \), \( R \) and \( T \) are both in \( PF(f - x_j) \), \( S \) and \( U \) are both in \( PF(f + 1 - x_j) \), \( P = LBRX_jS \), and \( Q = LBTX_jU \). Then by the hypothesis of the induction, \( \neg R \equiv T \) and \( \neg S \equiv U \), so that we easily get \( \neg P \equiv Q \).

**Case 2.** \( b_j < 0 \), \( R \) and \( T \) are both in \( PF(f + x_j - 1) \), \( S \) and \( U \) are both in \( PF(f + x_j) \), \( P = LBRNX_jS \), and \( Q = LBTNX_jU \). Similar to Case 1.

**Case 3.** \( b_j > 0 \) and \( b_k > 0 \), \( R \) is in \( PF(f - x_j) \), \( S \) is in \( PF(f + 1 - x_j) \), \( T \) is in \( PF(f - x_k) \), \( U \) is in \( PF(f + 1 - x_k) \), \( P = LBRX_jS \), and \( Q = LBTX_kU \). Let \( V \), \( W \), and \( X \) be in \( PF(f - x_j - x_k) \), \( PF(f + 1 - x_j - x_k) \), and \( PF(f + 2 - x_j - x_k) \) respectively. By part (b) of our theorem for \( \alpha = 2 \)

\[ \neg ANVW. \]

Also \( LBVX_kW \) is in \( PF(f - x_j) \) so that by part (a) of our theorem for \( \alpha = 1 \)

\[ \neg R \equiv LBVX_kW. \]

Similarly

\[ \neg S \equiv LBWX_kX, \]

\[ \neg T \equiv LBVX_jW, \]

\[ \neg U \equiv LBWX_jX. \]

Then \( \neg P \equiv Q \) by Theorem 10.4.
Case 4. $b_j > 0$ and $b_k < 0$, $R$ is in $\text{PF}(f - x_j)$, $S$ is in $\text{PF}(f + 1 - x_j)$, $T$ is in $\text{PF}(f + x_k - 1)$, $U$ is in $\text{PF}(f + x_k)$, $P = LBRX_jS$, and $Q = LBTNX_kU$. Let $V$, $W$, and $X$ be in $\text{PF}(f - 1 - x_j + x_k)$, $\text{PF}(f - x_j + x_k)$, and $\text{PF}(f + 1 - x_j + x_k)$ respectively. By part (b) of our theorem for $\alpha - 2$

\[ \vdash \text{ANVW}. \]

By part (a) for $\alpha - 1$

\[ \vdash R = LBVNX_kW, \quad \vdash T = LBVX_jW, \]
\[ \vdash S = LBWNX_kX, \quad \vdash U = LBWX_jX. \]

Then $\vdash P \equiv Q$ by Theorem 10.4.

The two remaining cases, namely $b_j < 0$ and $b_k > 0$, or $b_j < 0$ and $b_k < 0$, are handled similarly.

This still leaves part (b) to be handled. So let $\sigma(f) = \alpha$, and let $P$ be in $\text{PF}(f)$ and $Q$ be in $\text{PF}(f + 1)$.

Case 1. There is a $b_j > 0$. Choose $R$, $S$, and $T$ in $\text{PF}(f - x_j)$, $\text{PF}(f + 1 - x_j)$, and $\text{PF}(f + 2 - x_j)$ respectively. Then by part (b) for $\alpha - 1$

\[ \vdash \text{ANRS}, \]
\[ \vdash \text{ANST}. \]

Also $LBRX_jS$ is in $\text{PF}(f)$, so that by our lemma

\[ \vdash P \equiv LBRX_jS. \]

Similarly

\[ \vdash Q \equiv LBSX_jT. \]

So $\vdash ANPQ$ by Theorem 10.5.

Case 2. There is a $b_j < 0$. Proceed as in Case 1.

Theorem 13.2. If $P$ is in $\text{PF}(f)$ and $Q$ is in $\text{PF}(1 - f)$, then $\vdash P \equiv NQ$.

Proof by induction on $\sigma(f)$. First let $\sigma(f) = 0$. If $1 \leq f$, then $\vdash P$ by (2.10) and $\vdash NQ$ by (2.10) and (3.4). So $\vdash P \equiv NQ$ by (3.48). If $f \leq 0$, then $\vdash NP$ and $\vdash NQ$ by (2.10) and (3.4). So $\vdash P \equiv NQ$ by (3.50).

Let $\alpha$ be a positive integer and assume the theorem holds if $\sigma(f) < \alpha$. Let $\sigma(f) = \alpha$ and let $P$ be in $\text{PF}(f)$ and $Q$ be in $\text{PF}(1 - f)$.

Case 1. There is a $b_j > 0$. Choose $R$, $S$, $T$, and $U$ in $\text{PF}(f - x_j)$, $\text{PF}(f + 1 - x_j)$, $\text{PF}(x_j - f)$, and $\text{PF}(x_j - f + 1)$ respectively. By our induction hypothesis

(a) \[ \vdash R \equiv NU, \]
(b) \[ \vdash S \equiv NT. \]

By Theorem 13.1(a),
By Theorem 13.1(b),

(e) \( \vdash ANRS. \)

By (3.37)

\[ S \vdash BRX_j = BRLSX_j, \]

and by (3.37), the commutativity of \( L \), and (c)

\[ S \vdash P = BRX_j. \]

So

(f) \( S \vdash P = BRLSX_j. \)

By (3.45)

\[ NR \vdash LSX_j = BRLSX_j, \]

and by (3.45) and (c)

\[ NR \vdash P = LX_jS. \]

So by the commutativity of \( L \)

(g) \( NR \vdash P = BRLSX_j. \)

Then by Theorem 2.3, (e), (f), and (g)

\( \vdash P = BRLSX_j. \)

By the commutativity of \( B \),

\( \vdash P = BLSX_jR. \)

Then by (3.8)

\( \vdash P = NLNLSX_jNR, \)

so that by (3.9)

\( \vdash P = NLNNBNSNX_jNR. \)

Then by (a), (b), and (3.4),

\( \vdash P = NLBTN_jU. \)

Thus we conclude finally by (d)

\[ \vdash P \equiv \neg Q. \]
Case 2. There is a \( b_j < 0 \). Interchange \( P \) and \( Q \) and replace \( f \) by \( 1 - f \). Then we are back to Case 1, and can conclude \( \vdash Q \equiv NP \). Then by (3.6) and (3.4), \( \vdash P \equiv NQ \).

**Theorem 13.3.** If \( P \) is in \( PF(f) \) and \( Q \) is in \( PF(2-f) \), then \( \vdash APQ \).

**Proof.** Take \( R \) in \( PF(1-f) \). Then \( \vdash P \equiv NR \) by Theorem 13.2 and \( \vdash ANRQ \) by Theorem 13.1(b).

**Theorem 13.4.** If \( \alpha \) is a non-negative integer, \( P \) is in \( PF(f) \) and \( Q \) is in \( PF(\alpha + f) \), then \( \vdash CPQ \).

Proof by induction on \( \alpha \). If \( \alpha = 0 \), use Theorem 13.1(a). So assume the theorem for \( \alpha \). Let \( P \) be in \( PF(f) \) and \( Q \) be in \( PF(\alpha + f) \). Choose \( R \) in \( PF(\alpha + f) \). Then \( \vdash CPR \) by the hypothesis of the induction, and \( \vdash ANRQ \) by Theorem 13.1(b). Then \( \vdash CRQ \) by (3.52), so that we can infer \( \vdash CPQ \).

**Theorem 13.5.** If \( \alpha \) is a non-negative integer, \( P \) is in \( PF(f) \) and \( Q \) is in \( PF(1-\alpha-f) \), then \( \forall f \), \( \vdash CFCQR \).

Proof by induction on \( \alpha \). If \( \alpha = 0 \), use Theorem 13.1(a). So assume the theorem for \( \alpha \). Let \( P \) be in \( PF(f) \) and \( Q \) be in \( PF(\alpha+1+f) \). Choose \( R \) in \( PF(\alpha+f) \). Then \( \vdash CPR \) by the hypothesis of the induction, and \( \vdash ANRQ \) by Theorem 13.1(b). Then \( \vdash CRQ \) by (3.52), so that we can infer \( \vdash CPQ \).

**Theorem 13.6.** Let \( f \) be a polynomial in which the coefficient of \( x_k \) is zero. Let \( \Phi(X_k) \) be in \( PF(f) \). Then \( \vdash \Phi(X_k) \equiv \Phi(R) \).

Proof by induction on \( \sigma(f) \). First let \( \sigma(f) = 0 \). If \( X_k \) does not occur in \( \Phi(X_k) \), then the theorem follows trivially by (2.11). If \( X_k \) does occur in \( \Phi(X_k) \) it must be because \( \Phi(X_k) \) is either \( CX_k X_k \) or \( NCX_k X_k \). In this case our theorem follows either by (3.48) or (3.50).

Assume the theorem for \( \sigma(f) < \alpha \), and let \( \sigma(f) = \alpha \).

**Case 1.** Some \( b_j > 0 \). Then \( j \neq k \). Choose a \( \Phi_1(X_k) \) in \( PF(f-x_j) \) and a \( \Phi_2(X_k) \) in \( PF(f+1-x_j) \). Then by Theorem 13.1(a), \( \vdash \Phi(X_k) \equiv LB\Phi_1(X_k)X_j\Phi_2(X_k) \). So \( \vdash \Phi(R) \equiv LB\Phi_1(R)X_j\Phi_2(R) \). However, by the hypothesis of the induction \( \vdash \Phi_1(X_k) \equiv \Phi_1(R) \) and \( \vdash \Phi_2(X_k) \equiv \Phi_2(R) \). So \( \vdash \Phi(X_k) \equiv \Phi(R) \).

**Case 2.** Some \( b_j < 0 \). Proceed similarly.

**Theorem 13.7.** Let \( f \) be a polynomial in which the coefficients of \( x_j \) and \( x_k \) are both zero. Let \( \Phi(X_k) \) be in \( PF(f+bx_k) \) and \( Q \) be in \( PF(f+b-bx_j) \). Then \( \vdash \Phi(NX_j) \equiv Q \).

Proof by induction on \( b \). First let \( b = 0 \). Then \( \vdash \Phi(X_k) \equiv \Phi(NX_j) \) by Theorem 13.6, while \( \vdash \Phi(X_k) \equiv Q \) by Theorem 13.1(a).
Assume the theorem for $b$, and let $\Phi(X_k)$ be in $PF(f+(b+1)x_k)$ and $Q$ be in $PF(f+b+1-(b+1)x_j)$. Choose a $\Phi_1(X_k)$ in $PF(f+bx_k)$ and a $\Phi_2(X_k)$ in $PF(f+1+bx_k)$. Then by Theorem 13.1(a),

$$\vdash \Phi(X_k) \equiv LB\Phi_1(X_k)X_k\Phi_2(X_k).$$

So

(a) $$\vdash \Phi(NX_j) \equiv LB\Phi_1(NX_j)NX_j\Phi_2(NX_j).$$

Similarly we choose an $R$ in $PF(f+b-bx_j)$ and an $S$ in $PF(f+1+b-bx_j)$ and have

(b) $$\vdash Q \equiv LBRNX_jS.$$

By the hypothesis of the induction

(c) $$\vdash \Phi_1(NX_j) \equiv R,$$

(d) $$\vdash \Phi_2(NX_j) \equiv S.$$

Then by (a), (b), (c), and (d), we get $\vdash \Phi(NX_j) \equiv Q$.

**Theorem 13.8.** Let $f$ be a polynomial in which the coefficients of $x_j$ and $x_k$ are both zero. Let $b$ and $c$ be non-negative integers. Let $\Phi(X_j, X_k)$ be in $PF(f+cx_j+(b+c)x_k)$ and $Q$ be in $PF(f+b+c-bx_j)$. Then $\vdash \Phi(X_j, NX_j) \equiv Q$.

Proof by induction on $c$. When $c = 0$, our theorem reduces to Theorem 13.7.

Assume the theorem for $c$. Let $\Phi(X_j, X_k)$ be in $PF(f+(c+1)x_j+(b+c+1)x_k)$ and $Q$ be in $PF(f+b+c+1-bx_j)$. Choose $\Phi_1(X_j, X_k)$, $\Phi_2(X_j, X_k)$, and $\Phi_3(X_j, X_k)$ in $PF(f+cx_j+(b+c)x_k)$, $PF(f+1+cx_j+(b+c)x_k)$, and $PF(f+2+cx_j+(b+c)x_k)$ respectively. Also choose $P$ and $R$ in $PF(f+b+c-bx_j)$ and $PF(f+b+c+2-bx_j)$ respectively. By the hypothesis of the induction

(a) $$\vdash \Phi_1(X_j, NX_j) \equiv P,$$

(b) $$\vdash \Phi_2(X_j, NX_j) \equiv Q,$$

(c) $$\vdash \Phi_3(X_j, NX_j) \equiv R.$$

Now $LB\Phi_1X_j\Phi_2$ is in $PF(f+(c+1)x_j+(b+c)x_k)$, and $LB\Phi_2X_j\Phi_3$ is in $PF(f+1+(c+1)x_j+(b+c)x_k)$. So $LB\Phi_1X_j\Phi_2X_kLB\Phi_2X_j\Phi_3$ is in $PF(f+(c+1)x_j+(b+c+1)x_k)$. So by Theorem 13.1(a)

$$\vdash \Phi(X_j, X_k) \equiv LB\Phi_1(X_j, X_k)X_k\Phi_2(X_j, X_k)X_k\Phi_3(X_j, X_k).$$

Then by (a), (b), and (c)

(d) $$\vdash \Phi(X_j, NX_j) \equiv LB\Phi_1X_jQNX_jLB\Phi_2X_jR.$$

Also, by Theorem 13.1(b)

(e) $$\vdash ANPQ,$$

(f) $$\vdash ANQR.$$
By (d), (e), (f), and Theorem 10.9, we conclude $\vdash \Phi(X_j, NX_j) \equiv Q$.

**Theorem 13.9.** Let $P$, $Q$, $R$, $S$, and $T$ be in $PF(f)$, $PF(g)$, $PF(f+g)$, $PF(f+1)$, and $PF(g+1)$ respectively. Then

$$(13.1) \quad S \vdash CTCBPQR.$$ 

Proof by induction on $\sigma(g)$. First let $\sigma(g) = 0$. If $g \geq 1$, then $S \vdash R$ by Theorem 13.4. So by two uses of A1, $S \vdash CTCBPQR$. If $g = 0$, then $\vdash NQ$ by (2.10) and (3.4), and $\vdash CPR$ by Theorem 13.1(a). So $\vdash CBPQR$ by (3.45) and the commutativity of $B$. Then $S \vdash CTCBPQR$ by A1. If $g \leq -1$, then $\vdash NT$. So $\vdash CTCBPQR$ by (3.53).

Assume the theorem for $\sigma(g) < \alpha$. Let $\sigma(g) = \alpha$. Let

(a) \quad \begin{align*}
    f &= a + \sum_{j=1}^{n} b_j x_j, \\
    g &= c + \sum_{j=1}^{n} d_j x_j.
\end{align*}

Case 1. There is a $j$ for which $d_j > 0$ and $b_j + d_j > 0$. Let $U$, $V$, $W$, $Y$, $Z$, and $M$ be in $PF(g-x_j)$, $PF(g+1-x_j)$, $PF(g+2-x_j)$, $PF(f+g-x_j)$, $PF(f+g+1-x_j)$, and $PF(f+2)$ respectively. By Theorem 13.1(a)

$\vdash Q = LBUX_jV,$

$\vdash T = LBVX_jW,$

$\vdash R = LBYX_jZ.$

By Theorem 13.1(b),

$\vdash ANSM,$

$\vdash ANUV,$

$\vdash ANVW,$

$\vdash ANYZ.$

By the hypothesis of the induction,

$S \vdash CVCBPUY,$

$S \vdash CWCBPVZ,$

$M \vdash CVCBSUZ.$

Then $S \vdash CTCBPQR$ by Theorem 10.6.

Case 2. There is a $j$ for which $d_j > 0$, but no $j$ for which both $d_j > 0$ and $b_j + d_j > 0$. Take a $j$ for which $d_j > 0$. Then $-b_j \geq d_j > 0$. Take a $k$ for which $b_k = d_k = 0$. If necessary, take $k > n$. Take $\Phi_1(X_j, X_k), \Phi_2(X_j, X_k), \Phi_3(X_j, X_k)$ in $PF(f+b_j-b_jx_j-b_jx_k)$, $PF(f+g+b_j-b_jx_j-b_jx_k)$, and $PF(f+1+b_j-b_jx_j-b_jx_k)$ respectively. By Case 1,
$\Phi_3(X_j, X_k) \vdash CTCB\Phi_1(X_j, X_k)Q\Phi_2(X_j, X_k)$.

So

(c) $\Phi_3(X_j, NX_j) \vdash CTCB\Phi_1(X_j, NX_j)Q\Phi_2(X_j, NX_j)$.

If $T$ or $Q$ contains occurrences of $X_k$, we can appeal to Theorem 13.6 to infer (c) from the preceding formula. Also by Theorem 13.8

\[ \vdash \Phi_1(X_j, NX_j) \equiv P, \]
\[ \vdash \Phi_2(X_j, NX_j) \equiv R, \]
\[ \vdash \Phi_3(X_j, NX_j) \equiv S. \]

Thus by (c), $S \vdash CTCBPQR$.

**Case 3.** For each $j$, $d_j \leq 0$. Then we can proceed as in Cases 1 and 2 if we replace $x_j$ by $1 - x_k$ and $X_j$ by $NX_k$ throughout; we conclude by appealing to Theorem 13.7.

**Theorem 13.10.** Let $P$, $Q$, $R$, $S$, and $T$ be in $PF(f)$, $PF(g)$, $PF(1-f+g)$, $PF(2-f)$, and $PF(g+1)$ respectively. Then

(13.2) $S \vdash CTCBPQR$.

**Proof.** Take $U$ to be in $PF(1-f)$. Then $\vdash P \equiv NU$ by Theorem 13.2. Also $S \vdash CTCBUQR$ by Theorem 13.9. So (13.2) follows.

**Theorem 13.11.** Let $P$, $Q$, and $R$ be in $PF(f)$, $PF(g)$, and $PF(f+g)$ respectively. Then

(13.3) $\vdash CRBPQ$.

**Proof by induction on $\sigma(f+g)$.** First let $\sigma(f+g) = 0$. Then $f+g \equiv \beta$, where $\beta$ is an integer. If $\beta \leq 0$, then $\vdash NR$ by (2.10) and (3.4). So $\vdash CRBPQ$ by (3.53). Now let $\beta \geq 1$. Then $g = \beta - f$. Take $S$ in $PF(1-f)$. Then $\vdash P \equiv NS$ by Theorem 13.2. As $\vdash BNSS$ by (3.1), we have $\vdash BPS$. But $\vdash CSQ$ by Theorem 13.4. So $\vdash BPQ$ by (3.23), whence $\vdash CRBPQ$ by A1.

Assume the theorem proved if $\sigma(f+g) < \alpha$. Let $\sigma(f+g) = \alpha$. Let

(a) $f = a + \sum_{j=1}^{n} b_jx_j,$

(b) $g = c + \sum_{j=1}^{n} d_jx_j.$

**Case 1.** There is a $j$ such that $b_j + d_j > 0$. Then either $b_j > 0$ or $d_j > 0$. Because of the commutativity of $B$, we can interchange $P$ and $Q$ if desired without affecting (13.3). So there is no loss of generality in assuming that $d_j > 0$. Choose $U$, $V$, $W$, $Y$, and $Z$ in $PF(g-x_j)$, $PF(g+1-x_j)$, $PF(2-f)$, $PF(f+g-x_j)$, and $PF(f+g+1-x_j)$ respectively. By Theorem 13.1(a),
\[ \vdash Q = LBUX_jV, \quad \vdash R = LBYY_jZ. \]

By Theorem 13.1(b),
\[ \vdash ANUV, \quad \vdash ANYZ. \]

By Theorem 13.3,
\[ \vdash APW. \]

By Theorem 13.10,
\[ W \vdash CZZCPYV. \]

By the hypothesis of the induction
\[ \vdash CYBPU, \quad \vdash CZBPV. \]

Then \( \vdash CRBPQ \) by Theorem 10.7.

**Case 2.** There is a \( j \) such that \( b_j + d_j < 0 \). Proceed as in Case 1.

**Theorem 13.12.** Let \( P, Q, \) and \( R \) be in \( PF(f), PF(g), \) and \( PF(1-f+g) \) respectively. Then
\[ (13.4) \quad \vdash CRCPQ. \]

**Proof.** Take \( S \) in \( PF(1-f) \). Then \( \vdash P = NS \) by Theorem 13.2, and \( \vdash CRBSQ \) by Theorem 13.11.

**Theorem 13.13.** If \( \alpha \) is a positive integer and \( P \) and \( Q \) are in \( PF(1+f) \) and \( PF(1+\alpha f) \) respectively, then \( \vdash CQP \).

**Proof by induction on \( \alpha \).** If \( \alpha = 1 \), use Theorem 13.1(a). So assume the theorem for \( \alpha \), and let \( P \) and \( Q \) be in \( PF(1+f) \) and \( PF(1+(\alpha + 1)f) \) respectively. Choose \( R \) and \( S \) in \( PF(1+\alpha f) \) and \( PF(1-\alpha f) \) respectively. By Theorem 13.3,

(a) \[ \vdash ARS. \]

By Theorem 13.12,

(b) \[ \vdash CSCQP. \]

By our induction hypothesis, \( \vdash CRP. \) But \( \vdash CPCQP \) by A1, so that

(c) \[ \vdash CRCQP. \]

Then we conclude \( \vdash CQP \) by (a), (b), (c), and (2.20).

**Theorem 13.14.** If \( m \) is a positive integer, \( P_i \) is in \( PF(1+f_i) \) \((1 \leq i \leq m)\) and \( Q \) is in \( PF(1 + \sum_{i=1}^{m} f_i) \), then
\[ (13.5) \quad P_1, \cdots, P_m \vdash Q. \]
Proof by induction on \( m \). If \( m = 1 \), use Theorem 13.1(a). So assume the theorem for \( m \). Let \( P_i \) be in \( \text{PF}(1 + f_i) \) \((1 \leq i \leq m + 1)\), and \( Q \) be in \( \text{PF}(1 + \sum_{i=1}^{m+1} f_i) \). Choose \( R \) in \( \text{PF}(1 + \sum_{i=1}^{m+1} f_i) \). By the hypothesis of the induction,

\[
\begin{align*}
(\text{a}) & \quad P_1, \ldots, P_m \vdash R. \\
(\text{b}) & \quad \vdash CRCP_{m+1}Q.
\end{align*}
\]

By (a) and (b), we readily infer (13.5).

**Theorem 13.15.** If \( P \) is in \( \text{PF}(1+x_j) \), then \( \vdash P \).

**Proof.** If \( P \) is in \( \text{PF}(1+x_j) \), then there must be a \( Q \) and \( R \), in \( \text{PF}(1) \) and \( \text{PF}(2) \) respectively, such that \( \vdash P = LBQX, R \). But \( \vdash Q \) and \( \vdash R \) by (2.10). Then \( \vdash P \) by (3.32) and (3.36).

**Theorem 13.16.** If \( P \) is in \( \text{PF}(2-x_j) \), then \( \vdash P \).

**Proof is similar to that of Theorem 13.15.**

**Theorem 13.17.** Let \( m \) be a positive integer. Let \( f_i \) and \( g \) be as in (11.1) and (11.2) with integer coefficients. Suppose that there are sets of rational values of the \( x \)'s for which

\[
\begin{align*}
(13.6) & \quad f_i \geq 0 \quad (1 \leq i \leq m), \\
(13.7) & \quad x_j \geq 0 \quad (1 \leq j \leq n), \\
(13.8) & \quad 1 - x_j \geq 0 \quad (1 \leq j \leq n).
\end{align*}
\]

Suppose that whenever (13.6), (13.7) and (13.8) hold and the \( x \)'s are rational, then \( g \geq 0 \). Let \( P_i \) be in \( \text{PF}(1+f_i) \) \((1 \leq i \leq m)\), and \( Q \) be in \( \text{PF}(1+g) \). Then

\[
\begin{align*}
(13.9) & \quad P_1, \ldots, P_m \vdash Q.
\end{align*}
\]

**Proof.** Let \( R_j \) be in \( \text{PF}(1+x_j) \) and \( S_j \) be in \( \text{PF}(2-x_j) \). Then by Theorem 13.15 and Theorem 13.16

\[
\begin{align*}
(\text{a}) & \quad \vdash R_j \quad (1 \leq j \leq n), \\
(\text{b}) & \quad \vdash S_j \quad (1 \leq j \leq n).
\end{align*}
\]

By Theorem 11.3, there are non-negative rationals \( \lambda_1, \ldots, \lambda_{m+2n}, \mu \) such that

\[
g = \mu + \sum_{i=1}^{m} \lambda_if_i + \sum_{j=1}^{n} \lambda_{m+j}x_j + \sum_{j=1}^{n} \lambda_{m+n+j}(1 - x_j).
\]

Multiplying through by the LCM of the denominators of the \( \lambda \)'s and \( \mu \), we find non-negative integers \( L_1, \ldots, L_{m+2n}, M \), and a positive integer \( K \) such that
Take $T$ and $U$ in $\text{PF}(1+Kg-M)$ and $\text{PF}(1+Kg)$ respectively. By Theorem 13.14, using each $P_i L_i$ times, each $R_j L_{m+j}$ times, and each $S_j L_{m+n+j}$ times, we conclude by (c)

\[ P_1, \ldots, P_m, R_1, \ldots, R_n, S_1, \ldots, S_n \vdash T, \]

so that by (a) and (b)

\[ P_1, \ldots, P_M \vdash T. \]

By Theorem 13.4,

\[ \vdash CTU. \]

By Theorem 13.13

\[ \vdash CQ. \]

Then we infer (13.9), by (d), (e), and (f).

**Theorem 13.18.** Let $m$ and the $f_i$ be as in Theorem 13.17, but suppose that there is no set of rational values of the $x$'s for which (13.6), (13.7), and (13.8) all hold. Let $P_i$ be in $\text{PF}(1+f_i)$ ($1 \leq i \leq m$), and let $Q$ be any statement whatever. Then

\[ P_1, \ldots, P_m \vdash Q. \]

**Proof.** Let $\alpha$ be the least integer for which there is no set of rational values of the $x$'s satisfying (13.7), (13.8) and

\[ f_i \geq 0 \quad (1 \leq i \leq \alpha + 1). \]

Then there is a rational set of $x$'s satisfying (13.7), (13.8), and

\[ f_i \geq 0 \quad (1 \leq i \leq \alpha). \]

Also, for each such set of $x$'s, \(-f_{\alpha+1}>0\). Then by Theorem 11.2 there is a positive rational $\mu$ such that \(-f_{\alpha+1} \geq \mu\) whenever the $x$'s are rational and satisfy (13.7), (13.8), and (a). Let $\mu = M/K$, where $M$ and $K$ are positive integers. Then \(-M-Kf_{\alpha+1} \geq 0\) whenever \(-f_{\alpha+1} \geq \mu\); that is, whenever the $x$'s are rational and satisfy (13.7), (13.8), and (a). Choose $R$ and $S$ in $\text{PF}(1-M-Kf_{\alpha+1})$ and $\text{PF}(1+Kf_{\alpha+1})$ respectively. By Theorem 13.17

\[ P_1, \ldots, P_\alpha \vdash R, \]

by Theorem 13.14, using $P_{\alpha+1} K$ times,

\[ P_{\alpha+1} \vdash S, \]

and by Theorem 13.5
(d) \[ \vdash CRCSQ. \]

Then we infer (13.10) by (b), (c), and (d).

**Theorem 13.19.** Let \( m \) be a positive integer. Let \( f_i \) and \( g \) be as in (11.1) and (11.2) with integer coefficients. Suppose that \( g \geq 0 \) whenever the \( x \)'s are rationals in the range \( 0 \leq x \leq 1 \) such that each \( f_i \geq 0 \). Let \( P_i \) be in \( PF(1+f_i) \) (\( 1 \leq i \leq m \)) and \( Q \) be in \( PF(1+g) \). Then

(13.11)
\[ P_1, \ldots, P_m \vdash Q. \]

**Proof.** If there are sets of rational \( x \)'s in the range \( 0 \leq x \leq 1 \) for which each \( f_i \geq 0 \), use Theorem 13.17. Otherwise, use Theorem 13.18.

**Definition.** If \( P \) is a statement, and \( f \) is a polynomial, we define \( VfP \) as follows. Choose \( P_1, P_2, \) and \( P_3 \) in \( PF(f), PF(2-f) \), and \( PF(1+f) \) respectively. Then we set

(13.12)
\[ VfP = LLEPP_1P_2P_3. \]

By Theorem 13.1(a), the exact choice of \( P_1, P_2, \) and \( P_3 \) is immaterial.

**Theorem 13.20.** Let \( P \) be a statement formula of \( X_1, X_2, \ldots \). Then there is a non-negative integer \( p \), there are \( PF's \) \( P_1, \ldots, P_p \), \( P* \) \( P* \) \( \ldots, \) \( P* \), and there are polynomials \( f_i (1 \leq i \leq 2p) \), with the following properties:

(13.13)
\[ \vdash A\overline{P_i}P_i^* \quad (1 \leq i \leq p). \]

If \( j_1, \ldots, j_m \) constitute some subset (possibly empty) of the positive integers \( \leq p \), and \( j_{m+1}, \ldots, j_p \) constitute the remaining positive integers \( \leq p \) (if any), then there is a \( k (1 \leq k \leq 2p) \) such that

(13.14)
\[ P_{j_1}, \ldots, P_{j_m}, \overline{P_{j_{m+1}}}, \ldots, \overline{P_{j_p}} \vdash Vf_kP. \]

Proof by induction on the number of occurrences of symbols in \( P \). First let \( P \) have a single symbol. Then it must be \( X_j \). We take \( p = 0 \), and \( f_1 = x_j \). Let us take \( P_1, P_2, \) and \( P_3 \) in \( PF(x_j), PF(2-x_j) \), and \( PF(1+x_j) \) respectively. By Theorem 13.15 and Theorem 13.16

(a) \[ \vdash P_2, \]
(b) \[ \vdash P_3. \]

Also, by Theorem 13.1(a)

\[ \vdash P_1 \equiv LBNCYXYPCYY. \]

That is

\[ \vdash P_1 \equiv LBNCYXYPCYY. \]

By (2.10), (3.37), and the commutativity of \( L \)
\[ \vdash P_1 \equiv BNCVYP. \]

So by (2.10), (3.4), and (3.45)

\[ \vdash P_1 \equiv P. \]

Then \( \vdash EPP_1 \) by Theorem 3.1, whence we get \( \vdash Vf_1P \) by (a), (b), (3.36), and (13.12).

Assume the theorem for all \( P \)'s with fewer than \( \alpha \) symbols, and let \( P \) have \( \alpha \) symbols.

**Case 1.** \( P \) is of the form \( NQ \). Then there are \( q, \overline{Q}'s, Q^*\)'s, and \( g \)'s for \( Q \) with the stated properties. We take \( p = q, \overline{P}_i = \overline{Q}_i, P_i^* = Q_i^* \), and \( f_i = 1 - g_i \). Now consider any set of \( j \)'s. We have by the hypothesis of the induction

\[ P_{j_1}, \ldots, P_{j_m}, P_{m+1}^*, \ldots, P_{j_p}^* \vdash Vg_kQ. \]

Now \( Q_1, Q_2, \) and \( Q_3 \) are in \( PF(g_k) \), \( PF(2 - g_k) \), and \( PF(1 + g_k) \) respectively. Take \( P_1, P_2, P_3 \) in \( PF(f_k) \), \( PF(2 - f_k) \), and \( PF(1 + f_k) \) respectively. Since \( g_k = 1 - f_k \), we have

\[ P_2 \equiv Q_3, \]
\[ P_3 \equiv Q_2 \]

by Theorem 13.1(a), and

\[ \vdash P_1 \equiv NQ_1 \]

by Theorem 13.2. As \( EQQ_1 \vdash ENQNQ_1 \) by (3.40), we have \( EQQ_1 \vdash EPP_1 \). Then by (d), (e), (3.33), (3.34), and (3.36), \( Vg_kQ \vdash Vf_1P \). Then (c) gives (13.14).

**Case 2.** \( P \) is of the form \( CQR \). Then there are \( q, \overline{Q}'s, Q^*\)'s, and \( g \)'s for \( Q \) with the stated properties, and there are \( r, \overline{R}'s, R^*\)'s, and \( h \)'s for \( R \) with the stated properties. We take

\[ p = q + r + 2^q + r. \]

For \( 1 \leq l \leq 2^q \) and \( 1 \leq m \leq 2^r \), take \( S_{lm} \) and \( S_{lm}^* \) to be in \( PF(1 - g_l + h_m) \) and \( PF(1 + g_l - h_m) \) respectively. Then

\[ \vdash AS_{lm}S_{lm}^* \]

by Theorem 13.3. We take the \( P \)'s to consist of the \( \overline{Q}'s, \overline{R}'s, \) and \( S \)'s, and we take the \( P^* \)'s to consist of the \( Q^*\)'s, \( R^*\)'s, \) and \( S^*\)'s. Then (13.13) holds. We choose the \( f \)'s as follows. Let \( j_1, \ldots, j_m \) be some subset of the positive integers \( \leq p \). Among the formulas

\[ P_{j_1}, \ldots, P_{j_m}, P_{j_{m+1}}^*, \ldots, P_{j_p}^* \]

will be a subset \( Q \) of the \( \overline{Q}'\)'s and \( Q^*\)'s, corresponding to which there is an \( l \) such that

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(i) \[ Q \vdash Vg_i \tilde{Q}. \]

Also among the formulas of (h) there will be a subset \( \mathfrak{A} \) of the \( \mathfrak{R} \)'s and \( \mathcal{R}^* \)'s, corresponding to which there is an \( m \) such that

(j) \[ \mathfrak{A} \vdash Vh_m \mathcal{R}. \]

By (i), (j), (13.12), (3.33), and (3.34),

(k) \[ Q, \mathfrak{A} \vdash W \]

where \( W \) is any of \( E\mathcal{Q}_1, Q_2, Q_3, ERR_1, R_2, \) or \( R_3 \). Then by (3.41), (k) also holds when \( W \) is \( EPCQ_1 R_1 \). By Theorem 13.12

(l) \[ \vdash CS_{im} CQ_1 R_1. \]

By Theorem 13.10

(m) \[ Q_2 \vdash CR_3 CCQ_1 R_1 \tilde{S}_{im}. \]

Then by (k) with \( Q_2, R_3, \) and \( EPCQ_1 R_1 \) successively for \( W \), we infer

(n) \[ Q, \mathfrak{A} \vdash ESP_{im}. \]

We still have to define \( f_k \) and prove (13.14).

Subcase 1. \( \tilde{S}_{im} \) is among the formulas of (h). In this case, we take \( f_k = 1 \). By (2.10) and (3.48),

\[ \tilde{S}_{im} \vdash E\tilde{S}_{im} CYY. \]

Then by (n) and (3.39)

\[ Q, \mathfrak{A}, \tilde{S}_{im} \vdash EPCYY. \]

As \( f_k = 1 \), we have by Theorem 13.1(a), \( \vdash P_1 = CYY, \vdash P_2 = CYY, \) and \( \vdash P_3 = CYY. \) So we easily conclude by (2.10) and (3.36) that

\[ Q, \mathfrak{A}, \tilde{S}_{im} \vdash Vf_k P, \]

so that (13.14) holds.

Subcase 2. \( \tilde{S}_{im} \) is not among the formulas of (h), so that \( S^*_{im} \) must be among the formulas of (h). In this case we take \( f_k = 1 - g_i + h_m. \) Then by Theorem 13.1(a), we have \( \vdash P_1 = \tilde{S}_{im} \) and \( \vdash P_2 = \tilde{S}^*_{im}. \) So by (n) and (3.36),

(o) \[ Q, \mathfrak{A}, \tilde{S}^*_{im} \vdash LEPP_1 P_2. \]

By Theorem 13.14

\[ Q_2, R_3 \vdash P_3. \]

So by taking \( W \) to be \( Q_2 \) and \( R_3 \) in (k), we conclude from (o) that

\[ Q, \mathfrak{A}, \tilde{S}^*_{im} \vdash Vf_k P, \]

so that (13.14) holds.
Theorem 13.21. If \( \vdash P \), then \( P \) takes only the value unity.

Usual proof.

Theorem 13.22. If \( P \) takes the value unity exclusively, then \( \vdash \neg P \).

Proof. Clearly it suffices to restrict attention to statements \( P \) which are statement formulas of \( X_1, X_2, \ldots, \), since any other formula \( P \) can be handled by changing the \( X_j's \) to the constituents of \( P \). By Theorem 13.20, there are \( \rho \), \( \overline{P} \)'s, \( \rho^* \)'s, and \( f \)'s such that

\[(a) \quad \vdash A \overline{P}_i \rho^*_i \quad (1 \leq i \leq \rho)\]

and for each choice of \( j_1, \ldots, j_m \) there is an \( f_k \) such that

\[(b) \quad \overline{P}_{j_1}, \ldots, \overline{P}_{j_m}, \rho^*_{j_{m+1}}, \ldots, \rho^*_\rho \vdash \rho \overline{P}_1 P,\]

\[(c) \quad \overline{P}_{j_1}, \ldots, \overline{P}_{j_m}, \rho^*_{j_{m+1}}, \ldots, \rho^*_\rho, P \vdash P_1,\]

where \( P_1 \) is in \( \rho \overline{P}(f_k) \). Let \( g_1, \ldots, g_\rho \) be the polynomials such that \( \overline{P}_{j_i} \) is in \( \rho \overline{P}(g_i) \) \((1 \leq i \leq m)\) and \( \rho^*_j \) is in \( \rho \overline{P}(g_i) \) \((m+1 \leq i \leq \rho)\). Since \( P \) takes the value unity exclusively, we may apply to (c) the same sort of reasoning used in the proof of Theorem 13.21, and conclude by use of Theorem 12.1 that whenever the \( x \)'s are rationals in the range \( 0 \leq x \leq 1 \) such that each \( g_i \geq 1 \), then \( f_k \geq 1 \). Then by Theorem 13.19,

\[\overline{P}_{j_1}, \ldots, \overline{P}_{j_m}, \rho^*_{j_{m+1}}, \ldots, \rho^*_\rho \vdash P_1.\]

Then by (b),

\[(d) \quad \overline{P}_{j_1}, \ldots, \overline{P}_{j_m}, \rho^*_{j_{m+1}}, \ldots, \rho^*_\rho \vdash P.\]

Since (d) holds for each choice of \( j_1, \ldots, j_m \), we may use Theorem 2.3 and (a) to conclude that \( \vdash \neg P \).

14. The case when \( \exists \) has \( M \) members, \( s = 1 \), and \( C \) and \( N \) are taken as undefined. It suffices to add a single axiom scheme to those used in the preceding section. To describe this axiom scheme, we make some definitions.

Let \( i \) be a non-negative integer, and take \( \Phi_i(X_1) \) and \( \Psi_i(X_1) \) to be in \( \rho \overline{P}(1+i-(M-1)x_1) \) and \( \rho \overline{P}(1-i+(M-1)x_1) \) respectively. Define

\[M(P) = \sum_{i=0}^{M-1} L \Phi_i(P) \Psi_i(P).\]

We take \( M(P) \) as the sixth axiom scheme.

We note that \( M(P) \) takes the value 1 if and only if \( P \) is assigned one of the values \( \alpha/(M-1) \), where \( \alpha \) is an integer with \( 0 \leq \alpha \leq M-1 \). As these are the only values in \( \exists \), \( M(P) \) takes the value unity exclusively.

Let \( Q \) be a statement formula of \( P_1, \ldots, P_n \), and let \( Q \) take the value unity whenever each of the \( P \)'s is assigned a value \( \alpha/(M-1) \). This says that
if we assign rational values between 0 and 1 inclusive to the $P$'s, then $Q$ takes the value unity whenever

$$\prod_{j=1}^{n} M(P_j)$$

does. Then by Lemma 1 of [7], we conclude that there is a non-negative integer $\beta$ such that

$$(a) \quad (C \prod_{j=1}^{n} M(P_j))^\beta Q$$

takes the value unity whenever we assign rational values between 0 and 1 inclusive to the $P$'s. So by Theorem 13.22, we can derive (a) from A1–A5 by means of Rule C. But since each of $M(P_j)$ is an instance of our sixth axiom scheme, we can deduce $Q$ from (a).

References

6. T. Motzkin, Beiträge zur Theorie der Linear Ungleichungen, Doctoral dissertation, University of Basel, 1933 (Jerusalem, Azriel Printers, 1936). An English translation of this, under the title The theory of linear inequalities and with the reference number T-22, was published on March 7, 1952 by the RAND Corp., Santa Monica, California.