ON THE FREQUENCY OF SMALL FRACTIONAL PARTS IN CERTAIN REAL SEQUENCES

BY
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1. Introduction. Let \( X_1, X_2, \ldots \) be a sequence of independent random variables, each uniformly distributed on \([0, 1/2]\). If \( f \) is an arbitrary function from the positive integers to \([0, 1/2]\), the equation

\[
\Pr \{ X_k < f(k) \} = 2f(k)
\]

holds, and it is a consequence of the Borel-Cantelli lemmas \([3]\) that the probability that the inequality \( X_k < f(k) \) is satisfied for infinitely many \( k \) is zero or one, according as the series

\[
\sum_{k=1}^{\infty} f(k)
\]

is convergent or divergent. While it is well known that no such general assertion can be made when the \( X_k \) are dependent, Khinchin \([6]\) has found a direct analogue in an important case. His theorem is usually stated in measure-theoretic language: the inequality \( |kx - p| < f(k) \) has infinitely many integral solutions \( k, p \) for almost all \( x \) or almost no \( x \), according as (2) diverges or converges. We may, however, consider \( x \) as a random variable uniformly distributed over some interval, and define the quantity \( U_k \) \((k = 1, 2, \ldots)\) as the distance \( (kx) \) between \( kx \) and the nearest integer to \( kx \). Then the \( U_k \) form a sequence of dependent random variables uniformly distributed on \([0, 1/2]\); Khinchin’s theorem shows that the nature of the dependence is not such as to affect the finiteness of the number of solutions of the inequality \( U_k < f(k) \).

From a probabilistic standpoint the Borel-Cantelli lemmas yield very crude information about a sequence of random variables, and it is of some interest to know whether the \( U_k \) also resemble the \( X_k \) in their finer structure. We consider here the case in which (2) diverges, so that there are almost surely infinitely many solutions of \( |kx - p| < f(k) \), and investigate in §§2–3 the number \( T_n \) of such solutions with \( k \leq n \). The result is not quite what would be expected from the case of independent variables. For if we put \( Y_k \) equal to 1 or 0 according as the inequality \( X_k < f(k) \) does or does not hold, then \( S_n = Y_1 + \cdots + Y_n \) is the number of \( k \leq n \) such that \( X_k < f(k) \). Since

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\[ E(Y_k) = 1 \cdot 2f(k) + 0 \cdot (1 - 2f(k)) = 2f(k), \]

\[ \text{Var } Y_k = E(Y_k^2) - E^2(Y_k) = 2f(k) - 4f^2(k), \]

\[ E(S_n) = 2 \sum_{k=1}^{n} f(k), \]

\[ \text{Var } S_n = 2 \sum_{k=1}^{n} f(k) - 4 \sum_{k=1}^{n} f^2(k), \]

we deduce from the central limit theorem that if \( \sum_{k=0}^{\infty} f^2(k) \) converges, then

\[ \lim_{n \to \infty} \Pr \left\{ S_n < 2 \sum_{k=1}^{n} f(k) + \omega \left( 2 \sum_{k=1}^{n} f(k) \right)^{1/2} \right\} = \phi(\omega), \]

where

\[ \phi(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\omega} e^{-u^2/2} du \]

is the normal distribution function.

The law of the iterated logarithm yields the closely related result that

\[ \Pr \left\{ \limsup_{n \to \infty} \left| \frac{S_n - 2 \sum_{k=1}^{n} f(k)}{4 \left( \sum_{k=1}^{n} f(k) \log \log \sum_{k=1}^{n} f(k) \right)^{1/2}} \right| = 1 \right\} = 1 \]

and so in particular

\[ \Pr \left\{ S_n \sim 2 \sum_{k=1}^{n} f(k) \right\} = 1. \]

Theorem 1 exhibits the result corresponding to (3) for \( T_n \); it differs from (3) in that the coefficient 2 is replaced by \( 12\pi^{-2} \).

In §§4–6 we consider the much less strongly dependent sequence \( \langle r_1 r_2 \cdots r_k x \rangle \), where \( r_1, r_2, \cdots \) is a fixed increasing sequence of positive integers, and show that here the situation is again as described in (3) and (4).

2. A lemma. Let \( f \) be a function with the following properties:

\[ f(x) \text{ is positive and decreasing for } x \geq 0; \]

\[ f(x) = O(x^{-1}) \text{ and } f'(x) = O(x^{-2}) \text{ as } x \to \infty; \]

\[ \sum_{k=1}^{\infty} f(k) = \infty. \]
We shall need some further properties of $f$, which we collect in the following lemma.

**Lemma 1.** If $f$ satisfies (5)–(7) and if $c$ and $\delta$ are positive constants, then

(a) \[ \sum_{k=1}^{n} f(k) = \int_{1}^{n} f(u) du + O(1); \]

(b) \[ f(k + O(k^{1-\delta})) = f(k) + O(k^{-1-\delta}); \]

(c) \[ \sum_{k=1}^{n} f(k) = \sum_{k=1}^{cn} f(k) + O(1); \]

(d) \[ \sum_{k=1}^{n} cf(ck) = \sum_{k=1}^{n} f(k) + O(1); \]

(e) \[ \sum_{k=1}^{n} f(k) = c \sum_{k=1}^{n} \frac{f(c \log k)}{k} + O(1), \]

(f) if $a_1, a_2, \cdots$ and $\alpha$ are such that \[ \sum_{k=1}^{n} a_k \sim n\alpha \]

as $n \to \infty$, then

\[ \sum_{k=1}^{n} a_k f(k) = \alpha \sum_{k=1}^{n} f(k) + O(1). \]

Part (a) is trivial, and (b) follows from (6) and the law of the mean. Part (c) follows from the estimate

\[ \sum_{k=n}^{cn} f(k) = \sum_{k=n}^{cn} O(k^{-1}) = O(\log cn - \log n) = O(1), \]

and (d) from the fact that

\[ \sum_{k=1}^{n} cf(ck) = \int_{1}^{n} cf(cu) du + O(1) = \int_{c}^{cn} f(t) dt + O(1) = \sum_{k=c}^{cn} f(k) + O(1). \]

The substitution $u = c \log v$ in (a) gives (e). To obtain (f), write

\[ \sum_{k=1}^{n} (a_k - \alpha)f(k) = f(n) \sum_{k=1}^{n} (a_k - \alpha) + \sum_{k=1}^{n-1} \left( \sum_{l=1}^{k} (a_l - \alpha) \right) (f(k) - f(k+1)) \]

and note that

\[ f(n) \sum_{k=1}^{n} (a_k - \alpha) = O(n^{-1})o(n) = o(1) \]
and
\[
\sum_{k=1}^{n-1} \left( \sum_{l=1}^{k} (a_l - \alpha) \right) (f(k) - f(k + 1)) = \sum_{k=1}^{n-1} o(k)(f(k) - f(k + 1))
\]
\[
= O(n) \sum_{k=1}^{n-1} (f(k) - f(k + 1)) = O(nf(n)) = O(1).
\]

We shall use the following notation: \( \mathfrak{M}\{A\} \) means the measure of the set of \( x \in [0, 1] \) such that \( A \), if \( A \) is a sentence, and it means the measure of \( A \) if \( A \) is a set.

\( \text{No}\{m \leq n\} \cdots \) means the number of positive integers \( m \leq n \) such that . . . .

\( E_x\{\cdots\} \) or \( \{x|\cdots\} \) means the set of \( x \in [0, 1] \) such that . . . .

3. The fractional part of \( mx \). We prove the following theorem:

**Theorem 1.** Suppose that \( f \) satisfies conditions (5)–(7) and put
\[
g(x) = f(\log x)/x.
\]

Let
\[
T_n = T_n(x) = \text{No}\{m \leq n|\langle mx\rangle < g(m)\}.
\]

Then for fixed \( \omega \),
\[
\lim_{n \to \infty} \mathfrak{M}\left\{T_n < \frac{12}{\pi^2} \sum_{k=1}^{n} g(k) + \omega \left( \frac{12}{\pi^2} \sum_{k=1}^{n} g(k) \right)^{1/2} \right\} = \phi(\omega).
\]

If \( x \) is a real number with continued fraction expansion
\[
x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} = a_0 + \frac{1}{a_1 + \cdots \frac{1}{a_k + \frac{1}{x_{k+1}}}}
\]
and convergents
\[
\frac{p_k}{q_k} = a_0 + \frac{1}{a_1 + \cdots \frac{1}{a_k}}
\]
then
\[
x = \frac{p_k x_{k+1} + p_{k-1}}{q_k x_{k+1} + q_{k-1}}
\]
and
\[
|q_k x - p_k| = \frac{1}{q_k x_{k+1} + q_{k-1}}.
\]
Lemma 2. Put

\[ W_n = \text{No}\left\{ k \leq n \mid q_k x - p_k < \frac{f(k)}{q_k} \right\}. \]

Then

\[
\lim_{n \to \infty} \Pr \left\{ W_n < \frac{1}{\log 2} \sum_{k=1}^{n} f(k) + \omega \left( \frac{1}{\log 2} \sum_{k=1}^{n} f(k) \right)^{1/2} \right\} = \phi(\omega).
\]

We take \( x \) as a random variable uniformly distributed on \([0, 1]\), and use \( \Pr_k, E_k \) and \( \text{Var}_k \) to denote conditional probability, expectation and variance when \( a_0, \cdots, a_k \) are given. We suppose throughout this section that \( f \) satisfies conditions (5)-(7), and we put \( \alpha_k = f(k)(1+q_{k-1}/q_k) \) and

\[ V_k = \begin{cases} 
1 - \alpha_k & \text{if } |q_k x - p_k| < \frac{f(k)}{q_k}, \\
-\alpha_k & \text{otherwise}.
\end{cases} \]

Then

\[
\Pr_k \{ V_k = 1 - \alpha_k \} = \Pr_k \left\{ \frac{1}{(q_k x_{k+1} + q_{k-1})} < \frac{f(k)}{q_k} \right\} = \Pr_k \left\{ x_{k+1} > \frac{1}{f(k)} - \frac{q_{k-1}}{q_k} \right\} = \frac{p_k q_k / f(k) \left( \frac{p_k}{q_k} + \frac{p_k - 1}{q_k} \right)}{q_k + q_{k-1}} = f(k) \left( 1 + \frac{q_{k-1}}{q_k} \right) = \alpha_k.
\]

Hence

\[
E_k(V_k) = (1 - \alpha_k)\alpha_k + (-\alpha_k)(1 - \alpha_k) = 0,
\]

(8)

\[
\mu_k^2 = E_k(V_k^2) = f(k) \left( 1 + \frac{q_{k-1}}{q_k} \right) + O(f^2(k)).
\]

P. Lévy [9; 10, p. 321] has shown that
and it follows from (f) of Lemma 1 that for almost all \( x \),

\[
\sum_{k=1}^{n} f(k) \left(1 + \frac{q_{k-1}}{q_k} \right) = \frac{1}{\log 2} \sum_{k=1}^{n} f(k) + O(1).
\]

Combining (8) and (9), we see that for almost all \( x \),

\[
\mu_1^2 + \cdots + \mu_n^2 = \frac{1}{\log 2} \sum_{k=1}^{n} f(k) + O(1).
\]

We now use a form of the central limit theorem for dependent variables due to Lévy [10, p. 246] (and later extended by J. L. Doob [2, p. 383] as a theorem on martingales):

**Lemma 3.** Let \( Z_1, Z_2, \cdots \) be a sequence of bounded random variables, and let \( E_{n-1} \) denote conditional expectation for given \( Z_1, \cdots, Z_{n-1} \). Suppose that \( E_{n-1}(Z_n) = 0 \) for \( n \geq 2 \), and put

\[
\mu_n^2 = E_{n-1}(Z_n^2) = \text{Var}_{n-1}(Z_n).
\]

For \( t > 0 \), determine \( N = N(t) \) so that

\[
\mu_1^2 + \cdots + \mu_N^2 \sim t,
\]

and put

\[
S(t) = Z_1 + \cdots + Z_N.
\]

Then if

\[
\Pr \left\{ \sum_{n=1}^{\infty} \mu_n^2 < \infty \right\} = 0,
\]

we have

\[
\lim_{t \to \infty} \Pr \left\{ \frac{S(t)}{t^{1/2}} < \omega \right\} = \phi(\omega).
\]

If \( Z_k = V_k \), it follows from (10) that aside from a set of measure 0, the functions \( N(t) \) corresponding to various \( x \)'s are asymptotically equal, and that

\[
\lim_{n \to \infty} \Pr \left\{ \frac{V_1 + \cdots + V_n}{\left( \frac{1}{\log 2} \sum_{k=1}^{n} f(k) \right)^{1/2}} < \omega \right\} = \phi(\omega).
\]

But
\[ W_n = \sum_{k=1}^{n} V_k + \sum_{k=1}^{n} f(k) \left( 1 + \frac{q_{k-1}}{q_k} \right), \]

and hence for almost all \( x \),

\[ W_n = \sum_{k=1}^{n} V_k + \frac{1}{\log 2} \sum_{k=1}^{n} f(k) + O(1). \]

Thus

\[ \lim_{n \to \infty} \Pr \left\{ W_n < \frac{1}{\log 2} \sum_{k=1}^{n} f(k) + \omega \left( \frac{1}{\log 2} \sum_{k=1}^{n} f(k) \right)^{1/2} \right\} = \phi(\omega), \]

which completes the proof of the lemma.

The remainder of the proof of Theorem 1 consists in transforming (11) into a statement not involving continued fractions. For this we need an estimate of \( q_k \).

**Lemma 4.** If \( \delta < 1/2 \), then for almost every \( x \) there is a constant \( k = k(x, \delta) \) such that

\[ \log q_k - \frac{\pi^2}{12 \log 2} k < k^{1-\delta}. \]

This results from an extension of the following theorem of Khinchin [7]: Let \( F \) be a function of \( k \) positive integral arguments, such that for \( n \geq k \),

\[ \int_0^1 F^2(a_n, \ldots, a_{n-k+1}) dx < C, \]

where \( a_m = a_m(x) \) denotes the \( m \)th denominator in the continued fraction expansion of \( x \). Then

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=k}^{n} F(a_i, a_{i-1}, \ldots, a_{i-k+1}) \]

exists and is constant almost everywhere.

Examination of the proof shows that the theorem may be modified in two ways. The function \( F \) may be replaced by a quantity depending on a slowly increasing number of the \( a_m \); we write

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} F_i(a_i, a_{i-1}, \ldots, a_{i-k_i+1}), \]

and require that \( i-k_i+1 \) be positive for \( i \geq 1 \). Secondly, the rapidity of approach of the sum in (12) to its limiting value can be estimated by replacing the \( \epsilon \) occurring in Khinchin's proof by \( n^{-\epsilon} \), where \( \epsilon \) is now a sufficiently small
positive constant. In this way the following theorem can be proved:

Let \( \{ F_i(r_1, \cdots, r_{k_i}) \} \) be non-negative functions of the positive integral arguments \( r_1, r_2, \cdots \), and suppose that the integrals

\[
\int_0^1 F_i^2(a_i, a_{i-1}, \cdots, a_{i-k_i+1}) \, dx
\]

are uniformly bounded. Suppose further that \( \delta < 1/2 \) and that

\[
k_i = O(\log^\sigma i)
\]

for some constant \( \sigma > 0 \). Then there is a constant \( B \) such that

\[
\frac{1}{n} \sum_{i=1}^n F_i(a_i, \cdots, a_{i-k_i+1}) = B + O(n^{-\delta})
\]

for almost all \( x \).

We put

\[
\phi_i(x) = a_i + \frac{1}{a_{i-1} + \cdots + \frac{1}{a_{i-k_i+1}}}
\]

and

\[
F_i(a_i, \cdots, a_{i-k_i+1}) = \log \phi_i(x).
\]

Since \( \phi_i(x) \leq a_i + 1 \) and \( \mathcal{M} \{ a_i = r \} = \mathcal{M} \{ r \leq x_i < r + 1 \} < 1/r^2 \), we have

\[
\int_0^1 F_i \, dx \leq \int_0^1 \log^2 (a_i + 1) \, dx \leq \sum_{r=1}^\infty \frac{\log^2 (r + 1)}{r^2}.
\]

Thus for

\[
k_i = 1 + \lfloor 2 \log i \rfloor
\]

there is a \( B_0 \) such that for almost all \( x \),

\[
\sum_{i=1}^n \log \phi_i(x) = B_0 n + O(n^{1-\delta}).
\]

On the other hand, if \( \phi_i(x) = q_i/q_{i-1} \), then by the law of the mean,

\[
| \log \phi_i(x) - \log \bar{\phi}_i(x) | = \xi | \phi_i(x) - \bar{\phi}_i(x) |
\]

where \( \xi < 1 \). Since

\[
(13) \quad \bar{\phi}_i(x) = a_i + \frac{1}{a_{i-1} + \cdots + \frac{1}{a_1}},
\]

this implies that
\[
| \log \phi_i(x) - \log \Phi_i(x) | < \left| \left( a_i + \frac{1}{a_{i-1}} + \cdots + \frac{1}{a_{i-k} + 1} \right) - \left( a_i + \frac{1}{a_{i-1}} + \cdots + \frac{1}{a_{i-k} + 1} \right) \right| < 1/Q_{k_i},
\]

where \( P_l/Q_l \) is the \( l \)th convergent in the expansion (13). Since

\[
Q_l \geq Q_{l-1} + Q_{l-2} > 2Q_{l-2} > \cdots > 2^{l/2},
\]

we see that

\[
| \log \phi_i(x) - \log \Phi_i(x) | < 2^{1-k_i} < i^{-2} \log 2.
\]

Thus for almost all \( x \),

\[
\sum_{i=1}^{n} \log \phi_i(x) = \log q_n = B_0 n + O(n^{1-\delta}).
\]

Lévy [10, p. 320] showed that \( B_0 = \pi^2/12 \log 2 \). The proof of Lemma 4 is complete.

Now let

\[
s_n = \text{No} \left\{ k \leq n \mid q_k x - p_k < \frac{f(B_0^{-1} \log q_k)}{q_k} \right\},
\]

\[
l_n(\kappa) = \text{No} \left\{ k \leq n \mid q_k x - p_k < \frac{f(k - \kappa k^{1-\delta})}{q_k} \right\}.
\]

By (11),

\[
\lim_{n \to \infty} \mathfrak{R} \left\{ l_n(\kappa) < \frac{1}{\log 2} \sum_{k=1}^{n} f(k - \kappa k^{1-\delta}) + \omega \left( \frac{1}{\log 2} \sum_{k=1}^{n} f(k - \kappa k^{1-\delta}) \right)^{1/2} \right\} = \phi(\omega).
\]

Putting

\[
A_n = \frac{1}{\log 2} \sum_{k=1}^{n} f(k),
\]

it follows from (b) of Lemma 1 that for each \( \kappa \),

\[
(14) \quad \lim_{n \to \infty} \mathfrak{R} \left\{ l_n(\kappa) < A_n + \omega A_n^{1/2} \right\} = \phi(\omega).
\]

Let

\[
F_n = \left\{ x \mid s_n < A_n + \omega A_n^{1/2} \right\},
\]

\[
G(\kappa) = \left\{ x \mid \log q_k - B_0 k < \kappa k^{1-\delta} \text{ for every } k \geq 1 \right\},
\]

\[
H_n(\kappa) = \left\{ x \mid l_n(\kappa) < A_n + \omega A_n^{1/2} \right\}.
\]
Then by Lemma 2 and Equation (14), to each $\epsilon > 0$ there corresponds a $\kappa_0 = \kappa_0(\epsilon)$ and an $n_0 = n_0(\kappa_0, \epsilon) = n_0(\epsilon)$ such that

$$\mathfrak{M}\{G(\kappa)\} > 1 - \epsilon \quad \text{for } \kappa \geq \kappa_0$$

and

$$\left| \mathfrak{M}\{H_n(\pm \kappa_0)\} - \phi(\omega) \right| < \epsilon \quad \text{for } n \geq n_0.$$ 

Clearly

$$G(\kappa_0)H_n(\kappa_0) \subset F_n,$$

and since $\mathfrak{M}(AB) \geq \mathfrak{M}(A) + \mathfrak{M}(B) - 1$ if $A$ and $B$ are subsets of $[0, 1]$, we have that for $n \geq n_0$,

$$\mathfrak{M}\{F_n\} \geq 1 - \epsilon + \phi(\omega) - \epsilon - 1 = \phi(\omega) - 2\epsilon.$$ 

Similarly, since $G(\kappa_0)F_n \subset H_n(-\kappa_0)$,

$$\mathfrak{M}\{F_n\} \leq \phi(\omega) + 2\epsilon.$$ 

Hence

$$\lim_{n \to \infty} \mathfrak{M}\{F_n\} = \lim_{n \to \infty} \text{Pr}\{s_n < A_n + \omega A_n^{1/2}\} = \phi(\omega).$$ 

By the same reasoning we can use (d) of Lemma 1 to show that if

$$r_n = \text{No} \left\{ k \leq n \mid q_kx - \rho_k \mid < \frac{B_0f(\log q_k)}{q_k} \right\},$$

then

$$\lim_{n \to \infty} \text{Pr}\{r_n < A_n + \omega A_n^{1/2}\} = \phi(\omega).$$ 

Replacing $f$ by $f/B_0$, it follows immediately that

$$\lim_{n \to \infty} \text{Pr}\left\{ \text{No} \left\{ k \leq n \mid q_kx - \rho_k \mid < \frac{f(\log q_k)}{q_k} \right\} < \frac{12}{\pi^2} \sum_{k=1}^{n} f(k) + \omega \left( \frac{12}{\pi^2} \sum_{k=1}^{n} f(k) \right)^{1/2} \right\} = \phi(\omega).$$

If $|mx - l| < 1/2m$, then $l/m$ is a convergent to $x$. Since $f(x) = o(1)$,

$$\text{No} \left\{ k \leq n \mid q_kx - \rho_k \mid < \frac{f(\log q_k)}{q_k} \right\} = \text{No} \left\{ m \leq q_n \mid mx < \frac{f(\log m)}{m} \right\} + O(1),$$

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the error term being uniformly bounded for all \( x \). Putting

\[
A(n) = \frac{12}{\pi^2} \sum_{k=1}^{n} \frac{f(\log k)}{k}
\]

and using (e) of Lemma 1 with \( c = 1 \), it follows that

\[
(15) \quad \lim_{n \to \infty} \Pr \left\{ \text{No } m \leq q_n \left| \langle mx \rangle < \frac{f(\log m)}{m} \right. \right\} < A(n) + \omega A(n)^{1/2} = \phi(\omega).
\]

There is now a final set-theoretic argument required to eliminate \( q_n \) entirely. Put

\[
F(n, \omega) = E_x \left\{ \text{No } m \leq q_n \left| \langle mx \rangle < \frac{f(\log m)}{m} \right. \right\} < A(n) + \omega A(n)^{1/2},
\]

\[
G(n, \beta, \omega) = E_x \left\{ \text{No } m \leq e^{\beta n} \left| \langle mx \rangle < \frac{f(\log m)}{m} \right. \right\} < A(n) + \omega A(n)^{1/2},
\]

\[
H_N(\epsilon) = E_x \left\{ e^{B_0 (\epsilon + 1) n} < q_n < e^{B_0 (\epsilon + 1) n} \text{ for all } n \geq N \right\}.
\]

It is easily seen that

\[
(16) \quad H_N(\epsilon) G(n, B_0 (1 + \epsilon), \omega) \subset F(n, \omega), \quad H_N(\epsilon) F(n, \omega) \subset H_N(\epsilon) G(n, B_0 (1 - \epsilon), \omega)
\]

for \( 0 < \epsilon < 1, \ n \geq N \). On the other hand, we have

\[
(17) \quad A \left( \frac{1 - \epsilon}{1 + \epsilon} n \right) + \frac{\eta A^{1/2} \left( \frac{1 - \epsilon}{1 + \epsilon} n \right)}{} > A(n) + \omega A^{1/2}(n),
\]

then

\[
(18) \quad G \left( \frac{1 - \epsilon}{1 + \epsilon} n, B_0 (1 + \epsilon), \eta \right) \subset G(n, B_0 (1 - \epsilon), \omega).
\]

By (c) of Lemma 1, \( A(cn) = A(n) + O(1) \), so

\[
A \left( \frac{1 - \epsilon}{1 + \epsilon} n \right) + \eta A^{1/2} \left( \frac{1 - \epsilon}{1 + \epsilon} n \right) = A(n) + (\eta + O(A^{-1/2}(n))) A^{1/2}(n).
\]

Since \( A(n) \to \infty \) as \( n \to \infty \), it follows that if \( \delta > 0 \) is arbitrary, (17) holds with \( \eta = \omega + \delta \), if \( n > n_0(\epsilon, \delta) \). But then by (16) and (18),
\[ H_N(\epsilon) F(n, \omega) \subseteq H_N(\epsilon) G(n, B_0(1 - \epsilon), \omega) \subseteq F\left(\frac{1 - \epsilon}{1 + \epsilon} n, \omega + \delta\right) \]

for

\[ n > \min\left(\frac{1 + \epsilon}{1 - \epsilon} N, n_0\right). \]

By Lemma 4, \( \mathcal{M}\{H_N(\epsilon)\} \rightarrow 1 \) as \( N \rightarrow \infty \), and by (15), \( \mathcal{M}\{F(n, \omega)\} \rightarrow \phi(\omega) \) as \( n \rightarrow \infty \). Hence, if we allow \( n \) and \( N \) to increase in such a way that

\[ N(1 + \epsilon)/(1 - \epsilon) < n, \]

we obtain the inequality

\[ \phi(\omega) \leq \lim_{n \rightarrow \infty} \mathcal{M}\{G(n, B_0(1 - \epsilon), \omega)\} \leq \phi(\omega + \delta). \]

Since \( \delta \) is arbitrary and \( \phi \) is continuous,

\[ \lim_{n \rightarrow \infty} \mathcal{M}\{G(n, B_0(1 - \epsilon), \omega)\} = \phi(\omega). \]

Since \( \epsilon \) is arbitrary (in \([0, 1]\)), we can choose \( \epsilon = 1 - B_0^{-1} \), and obtain

\[ \lim_{n \rightarrow \infty} \mathcal{M}\{G(n, 1, \omega)\} = \phi(\omega), \]

or

\[ \lim_{n \rightarrow \infty} \Pr\left\{\text{No } \left\{m \leq e^n \mid \langle mx \rangle < \frac{f(\log m)}{m}\right\}\right. \]

\[ < \frac{12}{\pi^2} \sum_{k=1}^{e^n} \frac{f(\log k)}{k} + \omega\left(\frac{12}{\pi^2} \sum_{k=1}^{e^n} \frac{f(\log k)}{k}\right)^{1/2} \}

\[ = \phi(\omega). \]

Using (c) of Lemma 1 again (with \( 1 \leq c \leq (n+1)/n \)) and the fact that there are at most three denominators \( q_k \) lying between \( e^n \) and \( e^{n+1} \), we obtain Theorem 1.

4. The small values of \( \langle r_1 r_2 \cdots r_n x \rangle \). We now consider sequences of the form \( \langle r_1 r_2 \cdots r_n x \rangle \), where \( x \) is again uniformly distributed on \([0, 1]\) and \( r_1, r_2, \cdots \) is a fixed nondecreasing sequence of integers larger than 1, not depending on \( x \), with \( \lim r_n = \infty \). Let the sequences \( \{x_n\} \) and \( \{a_n\} \) of real numbers and integers, respectively, be determined by the following conditions:

\[ r_1 x = a_1 + x_1, \quad -1/2 \leq x_1 < 1/2, \]

\[ r_2 x_1 = a_2 + x_2, \quad -1/2 \leq x_2 < 1/2, \]

\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]

\[ r_n x_{n-1} = a_n + x_n, \quad -1/2 \leq x_n < 1/2, \]

\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]
Then

\[ a_n = \left[ r_n x_{n-1} + 1/2 \right], \quad |x_n| = \langle r_n x_{n-1} \rangle, \]

for \( n = 1, 2, \ldots \), and

\[ x = \sum_{n=1}^{\infty} \frac{a_n}{r_1 \cdots r_n}. \]

The series (21) bears an obvious relation to the expansion of \( x \) to the base \( r \) if, contrary to assumption, we take all \( r_n = r \), and to the Cantor factorial expansion if \( r_n = n \) for all \( n \). In any case, the expansion is unique except for a set of measure zero.

Since \( x \) is a random variable, so is every element of \( \{x_n\} \) and \( \{a_n\} \), and it is easily seen that each \( x_n \) is uniformly distributed on \([-1/2, 1/2]\), and that each \( a_n \) is discretely uniformly distributed, in the sense that

\[ \Pr\{a_n = j\} = \frac{1}{r_n} \quad \text{for} \quad -\left\lfloor \frac{r_n}{2} \right\rfloor \leq j < \left\lceil \frac{r_n}{2} \right\rceil. \]

There is a significant difference between the two sets of variables, however, in that the \( a_n \) are statistically independent, while the \( x_n \) are not, as the Equations (19) show. Dependence makes the sequence \( \{x_n\} \) difficult to analyze probabilistically, but a considerable amount of information can be gained indirectly by transferring results about \( \{a_n\} \) via the relation

\[ x_{n-1} = \frac{a_n}{r_n} + O\left(\frac{1}{r_n}\right). \]

**Theorem 2.** Suppose that \( r_1, r_2, \ldots \) is a nondecreasing sequence of positive integers such that \( r_m^n > n \) for some fixed integer \( m \). Let \( R_n = r_1 r_2 \cdots r_n \), and let \( f \) be a positive function. Let \( S \) be an increasing sequence of positive integers. Then the inequality

\[ (R_n x) < f(n) \]

has infinitely many solutions \( n \in S \) for almost all \( x \) or almost no \( x \), according as the series

\[ \sum_{n \in S} f(n) \]

diverges or converges.

We note first that it suffices to consider functions \( f \) such that \( f(n) \geq n^{-2} \) for all \( n \in S \). For if (24) converges, then so does the series
\[ \sum_{n \in S} f^*(n), \]

where

\[ f^*(n) = \begin{cases} f(n) & \text{if } f(n) \geq n^{-2}, \\ n^{-2} & \text{otherwise,} \end{cases} \]

and if the inequality \( \langle R_n x \rangle < f^*(n) \) has only finitely many solutions in \( S \), the same is surely true of (23). Suppose on the other hand that (24) diverges. Then so also does

\[ \sum_{n \in S} f(n), \]

the summation being extended over the integers \( n_j \in S \) such that \( f(n_j) \geq n_j^{-2} \). These integers constitute a subsequence \( S' \) of \( S \), and the truth of the theorem for \( S' \) implies its truth for \( S \).

We suppose throughout the proof that \( n \in S \). If we put

\[ P_n = R_n \sum_{j=1}^{n} \frac{a_j}{R_j}, \]

then

\[ | R_n x - P_n | = | x_n | \leq 1/2, \]

so

\[ | R_n x - P_n | = \langle R_n x \rangle. \]

For each \( n \) let \( k_n \) be the unique positive integer such that

\[ (25) \quad [r_{n+1} \cdots r_{n+k_n-1} f(n) + 1/2] = 0, \quad [r_{n+1} \cdots r_{n+k_n} f(n) + 1/2] \neq 0; \]

in particular, if \( [r_{n+1} f(n) + 1/2] \neq 0 \) then \( k_n = 1 \). Then

\[ (26) \quad \frac{1}{r_{n+1} \cdots r_{n+k_n}} \leq 2f(n). \]

Let \( \varepsilon_n \) be the event that (i.e., the set of \( x \in [0, 1] \) such that)

\[ a_{n+1} = \cdots = a_{n+k_n-1} = 0, \quad | a_{n+k_n} | < r_{n+1} \cdots r_{n+k_n} f(n) + 1 = \frac{R_{n+k_n}}{R_n} f(n) + 1, \]

and for \( c > 0 \) let \( \mathcal{F}_n(c) \) be the event that \( \langle R_n x \rangle < cf(n) \).

Suppose that \( x \in \mathcal{F}_n(1) \). If \( k_n = 1 \), then we have

\[ | x_n | < f(n), \]

\[ | a_{n+1} | = | a_{n+k_n} | = | [r_{n+1} x_n + 1/2] | \leq r_{n+1} | x_n | + 1/2 < r_{n+k_n} f(n) + 1, \]

so \( x \in \varepsilon_n \). If \( k_n > 1 \), then
\[ |a_{n+1}| \leq |r_{n+1}f(n) + 1/2| = 0, \quad x_{n+1} = r_{n+1}x, \]
\[ |a_{n+2}| \leq |r_{n+1}r_{n+2}f(n) + 1/2| = 0, \quad x_{n+2} = r_{n+1}r_{n+2}x_n, \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ |a_{n+k_n-1}| \leq |r_{n+1} \cdots r_{n+k_n-1}f(n) + 1/2| = 0, \quad x_{n+k_n-1} = r_{n+1} \cdots r_{n+k_n-1}x_n, \]
\[ |a_{n+k_n}| \leq |r_{n+1} \cdots r_{n+k_n}f(n) + 1/2| < r_{n+1} \cdots r_{n+k_n}f(n) + 1, \]

and again \(x \in \varepsilon_n\). Hence \(\mathcal{F}_n(1) \subset \varepsilon_n\).

On the other hand, if \(x \in \varepsilon_n\), then

\[
x = \sum_{j=1}^{n} \frac{a_j}{R_j} + \sum_{j=n+k_n}^{\infty} \frac{a_j}{R_j},
\]

so

\[
(R_nx) = |R_nx - P_n| < \left( \frac{|a_{n+k_n}| + 1)R_n}{R_{n+k_n}} \right) \left( \frac{R_{n+k_n}}{R_n} \right) f(n) + 2
\]

(27)

\[
= f(n) + \frac{2R_n}{R_{n+k_n}}
\]

and it follows from (26) that \(\varepsilon_n \subset \mathcal{F}_n(3)\).

Thus if \(\varepsilon_n\) occurs for only finitely many \(n \in S\), the same is true of \(\mathcal{F}_n(1)\); while if \(\varepsilon_n\) occurs for infinitely many \(n \in S\), the same is true of \(\mathcal{F}_n(3)\). Since the convergence of (24) is unaffected by replacing \(f(n)\) by \(3f(n)\), there remains only the task of showing that \(\varepsilon_n\) occurs for infinitely many \(n \in S\), or only finitely many \(n \in S\), for almost all \(x\), according as (24) diverges or converges.

Since \(r_n^m > n\) and \(f(n) > n^{-2}\), we have \(r_{n+1} \cdots r_{n+2m}f(n) > 1\). Hence \(k_n \leq 2m\), and the event \(\varepsilon_n\) depends on at most the \(2m\) random variables \(a_{n+1}, \ldots, a_{n+2m}\). Hence for fixed \(l (0 \leq l < 2m)\), the events \(\varepsilon_{2m+l} (\nu=0, 1, \ldots)\) are independent. By (22),

\[
\Pr \{ |a_n| = j \} = \begin{cases} 
\frac{1}{r_n} & \text{if } j = 0, \\
\frac{2}{r_n} & \text{if } 0 < j < \frac{r_n}{2}, \\
\frac{1}{r_n} & \text{if } j = \frac{r_n}{2}, \quad r_n \text{ even.}
\end{cases}
\]

Hence for arbitrary real \(u \in [0, r_n/2)\),

\[
\Pr \{ |a_n| \leq u \} = \frac{2[u] + 1}{r_n} \begin{cases} 
(2u + 1)/r_n, \\
(2u - 1)/r_n.
\end{cases}
\]

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Thus, because of the independence of the $a_n$, we have

$$\Pr \{ \mathcal{E}_n \} \leq \frac{1}{r_{n+1}} \cdots \frac{1}{r_{n+k_n-1}} \cdot \frac{2 \frac{R_{n+k_n}}{R_n} f(n) + 3}{r_{n+k_n}} = 2f(n) + \frac{3R_n}{R_{n+k_n}},$$

and by (26),

$$\Pr \{ \mathcal{E}_n \} < 8f(n).$$

Also

$$\Pr \{ \mathcal{E}_n \} \geq \frac{2 \frac{R_{n+k_n}}{R_n} f(n) + 1}{r_{n+1} \cdots r_{n+k_n}} > 2f(n).$$

Hence for each $l$ the series

$$\sum_{n:2r_m+l \in S} \Pr \{ \mathcal{E}_{2r_m+l} \}$$

converges or diverges with the series

$$\sum_{n:2r_m+l \in S} f(2r_m + l).$$

But if the series

$$\sum_{n \in S} f(n)$$

diverges, at least one of the series (30), for $0 \leq l < 2m$, must diverge, while if (31) converges, all the series (30) converge. The theorem therefore follows from the Borel-Cantelli lemmas.

5. We now consider the case in which (23) has infinitely many solutions for almost all $x$, and investigate the number of such solutions with $n \leq N$. For simplicity we suppose that $S$ is the full set of positive integers.

**Theorem 3.** Let $\{ r_n \}$ and $\{ R_n \}$ be as described in Theorem 2. Let $f$ be a positive function such that

$$\sum_{n=1}^\infty f(n) = \infty, \quad f(n) = O(n^{-1/2-\epsilon}).$$

Let $k_n$ be the positive integer defined in (25), and suppose that

$$\sum_{n=1}^\infty (r_{n+1} \cdots r_{n+k_n})^{-1} < \infty.$$

---

(1) The symbol $\sum_{r_i \cdots}$ means summation over those $r$ such that $\cdots$. 
Then

\[
\lim_{N \to \infty} \Pr \left\{ \text{No } n \leq N \mid \langle R_n x \rangle < f(n) \right\} < 2 \sum_{n=1}^{N} f(n) + \omega \left(\frac{1}{2} \sum_{n=1}^{N} f(n)\right)^{1/2} = \phi(\omega).
\]

According to Theorem 2, the \( n \) for which \( f(n) < n^{-2} \) contribute only a bounded number of solutions of the inequality (23), so we may suppose that \( f(n) \geq n^{-2} \). Put

\[
X_n = \begin{cases} 
1 & \text{if } \langle R_n x \rangle < f(n), \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
S_N = \sum_{n=1}^{N} X_n.
\]

Similarly, put

\[
Y_n = \begin{cases} 
1 & \text{if } \varepsilon_n \text{ occurs,} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
T_N = \sum_{n=1}^{N} Y_n,
\]

where \( \varepsilon_n \) has the same meaning as before. Since \( \mathfrak{F}(1) \subseteq \mathfrak{C} \), we have

\[
S_N < T_N.
\]

On the other hand, if \( Y_n = 1 \) then either \( X_n = 1 \) or

\[
\langle R_n x \rangle \in \left[ f(n) \right] \left( f(n) + \frac{2R_n}{R_n + k_n} \right),
\]

by (27). Because of the uniform distribution of the \( x_n \), the probability of the event (35) is \( 2R_n/R_n + k_n \), and by (32) and the first Borel-Cantelli lemma, the event (35) occurs only finitely many times, for almost all \( x \). Thus given \( \epsilon > 0 \), there is a constant \( M \) so large that

\[
T_N < S_N + M
\]

for all \( N \) and all \( x \) not in a set of measure at most \( \epsilon \). Combining (34) and (36), we see that (33) will follow if it can be shown that
To this end we first prove a general lemma, suggested by work of Hoeffding and Robbins [5]. A set of random variables $Z_1, Z_2, \cdots$ is said to be $m$-dependent if for every $r,s$ and $n$ for which $n > s > r + m$, the sets $Z_1, \cdots, Z_r$ and $Z_s, \cdots, Z_n$ are independent. (The variables $Y_n$ above are $2m$-dependent.)

**Theorem 4.** Let $Z_1, Z_2, \cdots$ be a sequence of $m$-dependent random variables such that

\[
Z_n = \begin{cases} 
1 \text{ with probability } p_n, \\
0 \text{ with probability } 1 - p_n.
\end{cases}
\]

Suppose that

\begin{align*}
(38) & \quad \sum_{n=1}^{\infty} p_n = \infty, \\
(39) & \quad p_n = O(n^{-1/2 - \epsilon}), \quad \epsilon > 0, \\
(40) & \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\text{Cov}(Z_i, Z_{i+j})| < \infty.
\end{align*}

Then

\[
\lim_{n \to \infty} \Pr \left\{ Z_1 + \cdots + Z_n < \sum_{k=1}^{n} p_k + \omega \left( \sum_{k=1}^{n} p_k \right)^{1/2} \right\} = \phi(\omega).
\]

We decompose the finite sequence $1, 2, \cdots, n$ into blocks, in the following way. Choose $\eta$ smaller than $\epsilon$, and find an integer $l_0$ such that

\[
(41) \quad (l_0 + 1)^{2+\eta} - l_0^{2+\eta} > 2m.
\]

For $q \geq 1$ put

\[
l_q = \left[ (l_0 + q)^{2+\eta} \right],
\]

and define $\kappa = \kappa(n)$ by the inequality

\[
l_{\kappa} \leq n < l_{\kappa+1}.
\]

For $1 \leq q < \kappa - 1$, let $I_{q+1}$ be the set of integers $j$ such that $l_q < j \leq l_{q+1} - m$, and let $J_{q+1}$ be the set of integers $j$ such that $l_{q+1} - m < j \leq l_{q+1}$. Finally, put

\[
U_q = \sum_{r \in I_q} Z_r = \sum_{l_q} Z_r, \\
V_q = \sum_{J_q} Z_r,
\]
for \( q = 2, \cdots, \kappa \), so that

\[
Q_n = \sum_{r=1}^{n} Z_r = \sum_{r=1}^{l_1} Z_r + \sum_{q=2}^{\kappa} U_q + \sum_{q=2}^{\kappa} V_q + \sum_{r=l_\kappa+1}^{n} Z_r.
\]

By the definitions of \( l_0 \) and \( m \)-dependence, the variables \( U_2, \cdots, U_\kappa \) are independent, as are \( V_2, \cdots, V_\kappa \). We shall show that the limiting behavior of \( Q_n \) is determined by that of \( \sum U_q \), and then apply a standard version of the central limit theorem.

Since \( l_1 \) is fixed and the \( Z \)'s are bounded, the sum

\[
\sum_{r=1}^{l_1} Z_r
\]

is clearly negligible in the limit, if \( \text{Var} (S_n) \to \infty \). By (40), (39), and (38),

\[
\text{Var} \left( \sum_{q=2}^{\kappa} V_q \right) = \sum_{q=2}^{\kappa} \sum_{q'=2}^{\kappa} \text{Var} (Z_{q'}) + 2 \sum_{q=2}^{\kappa} \sum_{q'=2}^{\kappa} \text{Cov} (Z_{q}, Z_{q'})
\]

\[
= \sum_{q=2}^{\kappa} \sum_{q'=2}^{\kappa} (p_{q'} - p_q)^2 + O(1)
\]

\[
= \sum_{q=2}^{\kappa} \sum_{q'=2}^{\kappa} p_{q'} + O(1)
\]

\[
= \sum_{q=2}^{\kappa} \sum_{r=1}^{m} O(I_q^{-1.5}) + O(1)
\]

\[
= O \left( \sum_{q=2}^{\kappa} q^{-1.5} \right) + O(1),
\]

so that

\[
\text{(42) Var} \left( \sum_{q=2}^{\kappa} V_q \right) = O(1).
\]

Turning to \( U_q \), we see that

\[
\text{E} (U_q) = \sum_{I_q} p_q = \epsilon_q,
\]

and

\[
\text{Var} (U_q) = \sum_{I_q} \text{Var} (Z_q) + 2 \sum_{\mu, \nu \in I_q, \mu < \nu < \mu + m} \text{Cov} (Z_{\mu}, Z_{\nu}),
\]

so that

\[
\text{(43) } \sigma_{2}^{2} = \text{Var} (U_2 + \cdots + U_\kappa) = \sum_{q=2}^{\kappa} \epsilon_q + O(1).
\]
Now
\[ e_q < c \sum_{I_q} \frac{1}{\nu^{1/2+\varepsilon}} < c(l_q^{1/2-\varepsilon} - l_{q-1}^{1/2-\varepsilon}) \]
\[ < a_{l_{q-1}}^{1/2-\varepsilon} \left\{ \left( 1 + \frac{1}{q} \right)^{(1/2-\varepsilon)(2+\eta)} - 1 \right\} \]
\[ = O \left( q^{(2+\eta)(1/2-\varepsilon)} \frac{1}{q} \right) \]
and so
\[ e_q = O(1). \]

This implies in particular that
\[ \text{Var} \left( \sum_{r=1}^{n} Z_r \right) = O(1), \]
and hence, since
\[ \sum_{q=2}^{\infty} \sum_j p_q < \infty, \]
that
\[ \sigma^2_k = \sum_{r=1}^{n} p_r + O(1), \quad E(U_2 + \cdots + U_k) = \sum_{r=1}^{n} p_r + O(1). \]
If we put
\[ \pi_n = \sum_{r=1}^{n} p_r, \]
then (42) shows that
\[ \text{Var} \left( \sum_{q=2}^{k} V_q \right) = O(1), \]
and it follows from Chebyshev's inequality that the random variable \( \pi_n^{-1/2} \sum_{q=2}^{k} V_q \) approaches zero in probability. By the same reasoning this is true also of \( \pi_n^{-1/2} \sum_{r=1}^{k} Z_r \). Combining these facts with (46), we see [1, p. 254] that the limiting distribution of \( (Q_n - \pi_n) / \pi_n^{1/2} \) is identical with that of
\[ (U_2 + \cdots + U_k - \pi_n) / \pi_n^{1/2}. \]

We now wish to apply Lyapunov's criterion [1, p. 213], according to which the normalized sum (48) is asymptotically normal, with mean zero and variance 1, if
\[
\left( \sum_{q=2}^{k} \rho_q \right)^{1/3} = O(\sigma_\kappa),
\]

where
\[ \rho_q^3 = E( | U_q - E(U_q) |^3 ). \]

This will complete the proof of Theorem 4. We have
\[
\rho_q^3 \leq E \left\{ \left( \sum_{\nu \in I_q} | Z_\nu - p_\nu | \right)^3 \right\}
\]
\[
< 6 E \left\{ \sum_{\nu \in I_q} | Z_\nu - p_\nu |^3 + \sum_{\mu, \nu \in I_q} | Z_\mu - p_\mu | \cdot | Z_\nu - p_\nu |^2 + \sum_{\mu, \nu, \lambda \in I_q} | Z_\mu - p_\mu | \cdot | Z_\nu - p_\nu | \cdot | Z_\lambda - p_\lambda | \right\}.
\]

Now
\[
\sum_{I_q} E( | Z_\nu - p_\nu |^3 ) = \sum_{I_q} (1 - p_\nu)^3 p_\nu + \sum_{I_q} p_\nu^3 (1 - p_\nu)
\]
\[
= e_q + O \left( \sum_{I_q} p_\nu^2 \right).
\]

Since \(| Z_\nu - p_\nu | < 1\), we have, by the generalized Hölder inequality [4, p. 140],
\[
\sum_{\mu, \nu \in I_q} E( | Z_\mu - p_\mu | \cdot | Z_\nu - p_\nu |^2 ) \leq \sum_{\mu, \nu \in I_q} E( | Z_\mu - p_\mu | \cdot | Z_\nu - p_\nu | )
\]
\[
\leq \left( \sum_{\mu, \nu \in I_q} \text{Var}( Z_\mu ) \text{Var}( Z_\nu ) \right)^{1/2} \leq \sum_{\mu \in I_q} \text{Var}( Z_\mu )
\]
\[
= \sum_{\mu \in I_q} (p_\mu - p_\mu^2) = e_q + O \left( \sum_{I_q} p_\mu^2 \right).
\]

Similarly,
\[
\sum_{\mu, \nu, \lambda \in I_q} E( | Z_\mu - p_\mu | \cdot | Z_\nu - p_\nu | \cdot | Z_\lambda - p_\lambda | )
\]
\[
\leq \left\{ \sum_{I_q} E( | Z_\mu - p_\mu |^3 ) E( | Z_\nu - p_\nu |^3 ) E( | Z_\lambda - p_\lambda |^3 ) \right\}^{1/3}
\]
\[
\leq \sum_{I_q} E( | Z_\mu - p_\mu |^3 ) = e_q + O \left( \sum_{I_q} p_\mu^2 \right).
\]

Thus (49) reduces to the triviality
\[
\sum_{q=2}^{k} e_q + O(1) = o \left\{ \left( \sum_{q=2}^{k} e_q \right)^{3/2} \right\}.
\]
To complete the proof of Theorem 3, we must show that the hypotheses of Theorem 4 are satisfied when $Z_n = Y_n$, $p_n = \Pr \{ \varepsilon_n \}$. We know that

$$2f(n) \leq p_n \leq 8f(n),$$

and hence, from the hypotheses of Theorem 3, we obtain (38) and (39). Since the $Y_n$ are $2m$-dependent, we can rewrite (40) in the form

$$\sum_{i=1}^{\infty} \sum_{j=1}^{2m} | \text{Cov} (Y_i, Y_{i+j}) | < \infty.$$

Now if $j > k_n$, then $Y_i$ and $Y_{i+j}$ are independent, and their covariance is 0. If $i \leq j \leq k_n$, then

$$| \text{Cov} (Y_i, Y_{i+j}) | = | E(Y_i Y_{i+j}) - E(Y_i)E(Y_{i+j}) |$$

$$= | \Pr \{ Y_i = 1, Y_{i+j} = 1 \} - \Pr \{ Y_i = 1 \} \cdot \Pr \{ Y_{i+j} = 1 \} |$$

$$\leq \left( r_{n+1} \cdots r_{n+k_n} \right)^{-1} + 8f(i)f(i+j),$$

and the convergence of (50) follows from (32).

6. A strong theorem.

**Theorem 5.** Let $\{ R_n \}$ and $f(n)$ satisfy the hypotheses of Theorem 3. Then for almost all $x$, the number of integers $m \leq n$, for which $(R_{mx}) < f(m)$, is asymptotic to

$$2 \sum_{k=1}^{n} f(k).$$

As in the proof of Theorem 3, it suffices to prove the theorem with $S_n$ replaced by $T_n = \sum_i Y_i$, and to suppose that $f(n) > n^{-2}$, so that the $Y_k$ are $2m$-dependent. We write

$$T_n = \sum_{1}^{2m} Y_{2m\nu+1} + \sum_{2}^{2m+2} Y_{2m\nu+2} + \cdots + \sum_{2m}^{2m+2m} Y_{2m\nu+2m},$$

where each summation extends over those $\nu$ for which the subscripts are not larger than $n$. The terms in $T_n^{(j)}$ are independent and uniformly bounded, and

$$E(T_n^{(j)}) = 2 \sum_{1}^{2m} f(2m\nu + j), \quad \text{Var} (T_n^{(j)}) = 2 \sum_{1}^{2m} f(2m\nu + j) + O(1).$$

Hence Kolmogorov's version of the law of the iterated logarithm [8] implies that for $1 \leq j \leq 2m$,

$$\Pr \left\{ \limsup_{n \to \infty} \frac{| T_n^{(j)} - 2 \sum_{1}^{2m} f(2m\nu + j) |}{2(\sum_{1}^{2m} f(2m\nu + j) \cdot \log \log \sum_{1}^{2m} f(2m\nu + j))^{1/2}} = 1 \right\} = 1;$$

and it follows from these equations that
Pr\{ \left| T_n - 2 \sum_{k=1}^{n} f(k) \right| = O\left( \sum_{j=1}^{2m} \left( \sum_{p=1}^{\star} f(2mv + j) \cdot \log_2(\sum_{p=1}^{\star} f(2mv + j)) \right)^{1/2} \right) \} = 1,

and the theorem is a weak consequence of this result.

Note added in proof.

I. There is a strong version of Theorem 1:

Under the hypotheses of Theorem 1, the number of solutions \( m \leq n \) of the inequality \( mx < g(m) \) is asymptotic to

\[
\frac{12}{\pi^2} \sum_{k=1}^{n} g(k),
\]

for almost all \( x \).

The proof depends on a strong law of large numbers for dependent variables, due to Lévy [10, p. 253]: Under the hypotheses of Lemma 3,

\[
\Pr \left\{ \lim_{t \to \infty} \frac{S(t)}{\log t^{1/2+\epsilon}} = 0 \right\} = 1
\]

for every positive constant \( \epsilon \). Using this in place of Lemma 3, we obtain a strong analogue of Lemma 2, to the effect that for \( \epsilon > 0 \),

\[
\Pr \left\{ W_n - (\log 2)^{-1} \sum_{1}^{n} f(k) = o((\sum_{1}^{n} f(k))^{1/2+\epsilon}) \right\} = 1,
\]

and thereafter the proof parallels that of Theorem 1.

II. It has been pointed out to me that Lemma 3 is not immediately applicable in the proof of Lemma 2, since \( E_k(V_k) \), in the equation preceding (8), means \( E(V_k, \text{given} \, a_0, \ldots, a_k) \) and not \( E(V_k, \text{given} \, V_0, \ldots, V_{k-1}) \), and it is possible that \( V_{k-1} \), for example, is not uniquely determined by \( a_0, \ldots, a_k \).

But in order for this to be the case it is necessary, since \( |q_{k-1}x - p_{k-1}| = (q_{k-1}a_k)^{-1} \) and \( a_k = [x_k] \), that

\[
\frac{1}{q_{k-1}(a_k + 1) + q_{k-2}} < \frac{f(k - 1)}{q_{k-1}} < \frac{1}{q_{k-1}a_k + q_{k-2}}.
\]

This happens only if

\[
a_k = \left[ \frac{1}{f(k - 1)} - \frac{q_{k-2}}{q_{k-1}} \right].
\]

The difficulty vanishes, therefore, if we prove the following theorem, and exclude from the beginning the exceptional set mentioned in it (taking \( b = 1 \) and \( h(k) = 1/f(k - 1) \)):

Let \( h \) be a real-valued function on the positive integers, with \( h(k) > ck \) for some positive constant \( c \). Then for every positive constant \( b \), the set of \( x \), for which the inequality \( |ak - h(k)| < b \) has infinitely many solutions, has measure zero.

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Put $F_k(t) = \Pr \{ x_k < t \}$; then Lévy's form of the Gauss-Kuzmin theorem [10, pp. 298–306] asserts that for some $g$ with $0 < g < 1,$

$$\left| F_k(t) - \frac{1}{\log 2} \log \frac{2t}{t + 1} \right| < g^{k-1}$$

for all $t > 1$ and all positive integers $k$. Now the inequality $|a_k - h(k)| < b$ is equivalent to

$$h(k) - b < x_k < h(k) + b + 1,$$

and we have

$$\lim_{k \to \infty} \Pr \{ h(k) - b < x_k < h(k) + b + 1 \}$$

$$< \frac{1}{\log 2} \log \left( \frac{2(h(k) + b + 1) \cdot h(k) - b + 1}{h(k) + b + 2 \cdot 2(h(k) - b)} \right) + 2g^{k-1}$$

$$= \frac{1}{\log 2} \log \left( \frac{2h^2(k) + 4h(k) - 2(b^2 - 1)}{2h^2(k) + 4h(k) - 2(b^2 + b)} \right) + 2g^{k-1}$$

$$= \frac{1}{\log 2} \log (1 + O(h^{-2}(k))) + 2g^{k-1} = O(h^{-2}(k)) + 2g^{k-1}.$$ 

Hence the probabilities of the inequalities in question form the terms of a convergent series, and the required result follows from the Borel-Cantelli lemma.

**References**