AUTOMORPHISMS OF THE GAUSSIAN UNIMODULAR GROUP

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1. Introduction. Let $G$ be the ring of Gaussian integers, and $G_n$ the group of $n \times n$ unimodular matrices over $G$. Define $G_+^n = \{ X \in G_n : \det X = +1 \}$, and likewise define $G_-^n$, $G_0^n$, $G_1^n$. Let $X'$ = transpose of $X$, $X^\sim$ = conjugate of $X$, $I^{(n)}$ = identity matrix in $G_n$, $0$ = null matrix of appropriate size, and $A + B$ = direct sum of $A$ and $B$. For $X, Y \in G_n$, write $X \sim Y$ if $X$ and $Y$ are conjugate in $G_n$. We assume throughout that $n \geq 2$. For $a$ = unit in $G_n$ and $A \in G_{n-1}$ define $(a) +^1 A$ to be the matrix $B$ for which $b_{rr} = a$, $b_{ij} = b_{ji} = 0$ for $j \neq r$, and such that the submatrix obtained by deleting the $r$th row and $r$th column from $B$ coincides with $A$. Thus $(a)^{-1} A$ coincides with the ordinary direct sum $(a) + A$. We use $[a_1, \ldots, a_n]$ to denote the diagonal matrix with diagonal elements $a_1, \ldots, a_n$.

The generators of $G_n$ are [1, p. 425]

$$\begin{align*}
T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + I^{(n-2)}, \\
S &= \begin{pmatrix} 0 & \cdots & 0 & (-1)^{n-1} \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \\
P &= (i) + I^{(n-1)}. 
\end{align*}$$

For the case $n = 2$ we shall use $T_0, S_0, P_0$ as symbols for the generators, where $S_0$ now denotes the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

In this paper we prove the following

**Main Theorem.** Let $\mathfrak{A}_n$ be the automorphism group of $G_n$. Then $\mathfrak{A}_n$ is generated by

1. $X \rightarrow AXA^{-1}$, $A \in G_n$;
2. $X \rightarrow X'^{-1}$ (may be omitted when $n = 2$);
3. $X \rightarrow X$;
4. $X \rightarrow (\det X)^k X$, where $k = 1$ if $n$ is even, and $k = 2$ if $n$ is odd;
5. For $n = 2$ only, $(P_0, S_0, T_0) \rightarrow (P_0, -S_0, -T_0)$.

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We may remark that 1., 2. and 3. are obviously automorphisms. We shall prove later (Lemmas 3.2 and 3.3) that 4. and 5. are also automorphisms. For \( n = 2 \), it is easily verified that 2. is expressible in terms of the other automorphisms in the list, hence may be omitted.

2. **Involutions in** \( G_n \). We begin by giving a canonical form for involutions in \( G_n \) under conjugacy. Throughout this paper let \( \xi \) denote \( 1 + i \), and set

\[
J_a = \begin{pmatrix} -1 & 0 \\ \alpha & 1 \end{pmatrix},
\]

\[
W(a, b, c, d) = (J_1 + \cdots + J_d) + (J_2 + \cdots + J_\xi) + (-I)^{(c)} + I^{(d)},
\]

where \( a J_1 \)'s and \( b J_\xi \)'s occur in (2.2).

**Theorem 2.1.** As \( (a, b, c, d) \) range over all non-negative integers for which \( 2a + 2b + c + d = n \), the matrices \( W(a, b, c, d) \) give a full set of non-conjugate involutions in \( G_n \).

**Proof.** The proof given in [2, pp. 336–337] can be used with a few modifications, due to the fact that \( G \) is a principal ideal ring. From the reasoning there, it is easily established that for any involution \( X \in G_n \), we have

\[
X \sim \begin{pmatrix} -I^{(q)} & 0 \\ T & I^{(p)} \end{pmatrix}
\]

where \( T \) is a diagonal matrix with entries 0, 1 or \( \xi \). The right-hand side of (2.3) is conjugate to some \( W(a, b, c, d) \), and it is not hard to verify that two distinct \( W(a_j, b_j, c_j, d_j) \) \( (j = 1, 2) \) cannot be conjugate in \( G_n \).

We may remark that \( p, q \) in (2.3) are the dimensions of the plus-space \( X^+ \), and the minus-space \( X^- \), respectively, of the involution \( X \). Call \( X \) a \((p, q)\) involution in such case. We find at once that \( W(a, b, c, d) \) is an \((a + b + d, a + b + c)\) involution.

Our next step will be to characterize the \( \pm (1, n - 1) \) involutions in \( G_n \). Let \( \mathfrak{C}(S) \) denote the centralizer in \( G_n \) of a set \( S \) of elements in \( G_n \).

**Lemma 2.1.** Let \( X \in G_n \) be an involution and let

\[
\mathfrak{M} = \left\{ M \in \mathfrak{C}(X) : M \equiv X \pmod{2} \right\}.
\]

Then the only involutions in \( \mathfrak{C}(\mathfrak{M}) \) are \( \pm I^{(n)}, \pm X \).

**Proof.** For fixed \( B \in G_n \) we note that \( M \in \mathfrak{M} \) implies \( BMB^{-1} \in \mathfrak{C}(BXB^{-1}) \) and \( BMB^{-1} \equiv BXB^{-1} \pmod{2} \), and conversely. Without loss of generality, we may therefore take \( X \) in the form of the right-hand side of (2.3). In that case, \( \mathfrak{C}(X) \) consists of all elements \( K \in G_n \) given by

\[
K = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad A \in G_n, \quad D \in G_p, \quad C = (DT - TA)/2.
\]
Since \( A \equiv -I \pmod{4} \) and \( D \equiv I \pmod{4} \) imply \( C \equiv T \pmod{2} \) we see that \( \mathfrak{M} \) contains all elements \( K \) satisfying
\[
(2.5) \quad A \equiv -I \pmod{4}, \quad D \equiv I \pmod{4}.
\]

Now if \( Y \in \mathfrak{C}(\mathfrak{M}) \), then \( Y \) commutes with all \( K \) satisfying (2.5). Set
\[
Y = \begin{pmatrix}
Y_1 & 0 \\
Y_2 & Y_3
\end{pmatrix}.
\]

In that case, \( Y_1 \) commutes with all \( A \in G_q \) for which \( A \equiv -I \pmod{4} \), and \( Y_3 \) commutes with all \( D \in G_p \) for which \( D \equiv I \pmod{4} \). This shows at once that \( Y_1 = u \cdot I, \) \( Y_3 = v \cdot I, \) \( u, v \) units. If, further, \( Y \) is to be an involution, it follows that \( Y_1 = \pm I, \) \( Y_3 = \pm I. \) Since \( Y \in \mathfrak{C}(\mathfrak{M}) \) implies \( Y \in \mathfrak{C}(X) \), therefore \( Y_2 = (Y_3 T - T Y_1)/2, \) and \( Y_2 \) is uniquely determined by \( Y_1 \) and \( Y_3 \). Hence \( \mathfrak{C}(\mathfrak{M}) \) contains at most four involutions.

**Theorem 2.2.** The image of any \((1, n-1)\) involution in \( G_n \) under any \( \tau \in \mathfrak{A}_n \) must be either a \((1, n-1)\) or an \((n-1, 1)\) involution.

**Proof.** The result is trivial for \( n = 2 \) and \( n = 3 \). Assume hereafter that \( n > 3 \). We shall characterize the \( \pm (1, n-1) \) involutions in \( G_n \) by intrinsic properties using a method due to Mackey [5]; see also Rickart [7]. Letting \( \mathfrak{C}^2(\cdots) \) denote \( \mathfrak{C}(\mathfrak{C}(\cdots)) \), define for an involution \( X \in G_n \)
\[
(2.6) \quad \nu(X) = \text{Max (number of involutions in } \mathfrak{C}^2(X, X_1)),
\]
where \( X_1 \) ranges over all involutions in \( \mathfrak{C}(X) \). Taking \( X \) to be a \((p, q)\) involution we shall show that \( \nu(X) \geq 16 \) if \( \text{Min } (p, q) > 1 \), while \( \nu(X) = 8 \) if \( \text{Min } (p, q) = 1 \); this will imply the theorem.

To begin with, note that \( \nu(X) \) depends only upon the conjugate class of the involution \( X \). We may therefore take \( X \) as the right-hand side of (2.3), and then \( \mathfrak{C}(X) \) is given by all \( K \) satisfying (2.4). For \( \text{Min } (p, q) > 1 \) define
\[
(2.7) \quad X_1 = \begin{cases} -1 & I \\ I & -1 \end{cases}, \quad W = \begin{cases} 1 & I \\ -t & -1 \end{cases}
\]
where we have set \( T \) (occurring in (2.3)) = \((t) + T_1. \) Then both \( X_1 \) and \( W \) are involutions in \( \mathfrak{C}(X). \) Since every \( K \in \mathfrak{C}(X_1) \) is of the form
\[
K = \begin{cases} a & b \\ A_1 & B_1 \\ c & d \end{cases},
\]
\[
\begin{cases} C_1 & D_1 \end{cases}
\]
we see that the general element of \( \mathfrak{C}(X, X_1) \) is given by

\[
K = \begin{pmatrix}
a & 0 \\
A_1 & c \\
c & d \\
C_1 & D_1
\end{pmatrix},
\]

where

\[
c = (d - a)t/2, \quad C_1 = (D_1T_1 - T_1A_1)/2.
\]

But then the involution \( W \) defined above commutes with all such \( K \), so that \( \mathfrak{C}^2(X, X_1) \) contains the 16 distinct involutions \( \pm X^aX_1^bW^c \), \( a, b, c = 0, 1 \), and \( \nu(X) \geq 16 \) for \( \text{Min}(p, q) > 1 \). (Indeed, \( \nu(X) = 16 \) for this case although we do not need the stronger result here.)

We now show that \( \nu(X) = 8 \) for \( p = 1 \). We may choose

\[
X = \begin{pmatrix}
-1 & 0 \\
t & I^{(n-1)}
\end{pmatrix},
\]

t = (t, 0, \ldots, 0)'. Then \( \mathfrak{C}(X) \) consists of all elements

\[
K = \begin{pmatrix}
a & 0 \\
c & D
\end{pmatrix}, \quad a \in G_1, \quad D \in G_{n-1}, \quad c = (D - 2aI)t/2.
\]

To compute \( \nu(X) \) we may assume that \( X_1 \in \mathfrak{C}(X) \) is given by

\[
X_1 = \begin{pmatrix}
1 & 0 \\
I & U
\end{pmatrix}, \quad u = (U - I)t/2,
\]

where \( U \in G_{n-1} \) is an involution. Then \( \mathfrak{C}(X, X_1) \) consists of all \( K \) given by (2.11) for which \( D \in \mathfrak{C}(U) \). In particular, whenever \( D \in \mathfrak{C}(U) \) and \( D \equiv U \) (mod 2) then (2.11), with \( a = 1 \), defines an element of \( \mathfrak{C}(X, X_1) \). If now \( L \in \mathfrak{C}^2(X, X_1) \) is an involution, then \( L \) has the form

\[
L = \begin{pmatrix}
a^* & 0 \\
c^* & D^*
\end{pmatrix}, \quad c^* = (D^* - a^*I)t^*/2
\]

and \( L \) commutes with every \( K \) for which \( a = 1 \), \( D \in \mathfrak{C}(U) \) and \( D \equiv U \) (mod 2). Then \( D^* \) commutes with all such \( D \), whence by Lemma 2.1, \( D^* = \pm I \) or \( \pm U \). Certainly \( a^* = \pm 1 \). Since \( a^* \) and \( D^* \) uniquely determine \( c^* \) it follows that \( \mathfrak{C}^2(X, X_1) \) contains at most 8 involutions. Since \( \pm I, \pm X, \pm X_1, \pm XX_1 \) are all in \( \mathfrak{C}^2(X, X_1) \) we have established that \( \nu(X) = 8 \) if \( \text{Min}(p, q) = 1 \).

Let us now set

\[
L_\alpha = J_\alpha + I^{(n-2)}, \quad \alpha = 0, 1 \text{ or } \xi.
\]

Then every \((1, n-1)\) involution in \( G_n \) is conjugate to one of \( L_0 \), \( L_1 \), \( L_\xi \), and
hence any \( \tau \in \mathfrak{A}_n \) maps \( L_0 \) onto \( \pm AL_0 A^{-1} \) for some \( A \in G_n \), where \( \alpha = 0, 1 \) or \( \xi \).

**Theorem 2.3.** For \( \tau \in \mathfrak{A}_n \) there exists \( A \in G_n \) such that \( L_0^* = \pm AL_0 A^{-1} \).

**Proof.** For \( n \geq 3 \) we shall use the method of maximal sets of involutions (see [6]). By a maximal set in \( G_n \) we mean an abelian group of \( 2^n \) involutions in \( G_n \). As in [6] we may at once establish the following results:

(i) The number of elements in any abelian group of involutions is \( \leq 2^n \).

(ii) A maximal set contains precisely \( C_{n,p} \) involutions of type \( (p, q) \).

(iii) Any maximal set may be obtained from a generating matrix \( M^n \) (whose columns are primitive vectors with components in \( G \)) by choosing any \( p \) columns of \( M \) as basis for the plus-space \( W^+ \) of an involution, and the remaining \( q \) columns as basis for \( W^- \). Each such choice defines a unique involution \( \omega \), and this process gives rise to \( C_{n,p} \) involutions of type \( (p, q) \). If this process is carried out for \( p = 0, 1, \ldots, n \), an abelian group of \( 2^n \) involutions is obtained. Furthermore, if each of the invariant factors of \( M \) is either 1, \( \xi \) or 2, then each of the \( 2^n \) involutions will lie in \( G_n \). In this case we call \( M \) a permissible generating matrix.

(iv) Two permissible generators \( M_1, M_2 \) give rise to conjugate maximal sets if and only if there exist \( A, B \in G_n \), where \( B \) is obtained from \( I \) by permuting columns and multiplying them by units, such that \( M_2 = AM_1 B \). In such case call \( M_1, M_2 \) equivalent.

(v) Every permissible generating matrix is equivalent to one of the form

\[
M^{(n)} = \begin{bmatrix}
I^{(r)} & A & B \\
0 & \xi I^{(s)} & C \\
0 & 0 & 2I^{(t)}
\end{bmatrix},
\]

where the columns of \( M \) are primitive, the elements of \( A \) are 0's and 1's, those of \( B \) are 0, 1 or \( \xi \), and those of \( C \) are 0 or \( \xi \).

Now define \( M_1 \) by: \( s = 1, t = 0 \), all entries in \( A \) are 1's; \( M_2 \) by: \( r = 1, t = 0 \), all entries in \( A \) are 1's; \( M_3 \) by: \( s = 0, t = 1 \), all entries in \( B \) are 1's; \( M_4 \) by: \( r = 1, s = 0 \), all entries in \( B \) are 1's. The maximal sets generated by \( M_1 \) and \( M_2 \) are nonconjugate (since \( n \geq 3 \)), and each contains \( n \) involutions which are conjugate to \( L_0 \). The maximal sets generated by \( M_3 \) and \( M_4 \) are nonconjugate, and each contains \( n \) involutions conjugate to \( L_1 \).

On the other hand, it is easy to show that any two maximal sets, each of which contains \( n \) involutions conjugate to \( L_0 \), must be conjugate. Hence for \( n \geq 3 \) the class of \( L_0 \) is characterized by intrinsic properties, and the theorem holds. We postpone until later the proof for \( n = 2 \).

3. **General remarks.** Before we turn to the question of determining all automorphisms of \( G_n \), it is desirable to state several lemmas.

**Lemma 3.1.** For any automorphism \( \tau \) of \( G_n \), either \( \det X^\tau = \det X \) for all \( X \in G_n \) or \( \det X^\tau = \text{conjugate of } \det X \) for all \( X \in G_n \).
Proof. Let \( S^{(k)} = P^{-k}SP^k, T^{(k)} = P^{-k}TP^k \). Since every \( X \in G_n \) is expressible as a power product of \( P, S \) and \( T \) we find that every \( X \) can be written as
\[
X = P^n \Pi(S, T, S^{(k)}, T^{(k)}),
\]
and then \( \det X = i^m \). Exactly as in [2, Corollary 1 of Theorem 1], we deduce that \( \det S^r = \det T^r = 1 \), whence also \( \det (S^{(k)})^r = \det (T^{(k)})^r = 1 \). Hence we have
\[
\det X^r = (\det P^r)^m.
\]
But \( \det P^r = \pm i \), since if \( \det P^r = \pm 1 \), then \( G_n \subseteq (G_n^+ \cup G_n^-) \), which is impossible. Hence \( \det X^r = (\pm i)^m \), where \( \det X = i^m \), whence the result follows.

Lemma 3.2. For \( n \) even the mapping \( X \rightarrow (\det X)X \) is an automorphism of \( G_n \). For \( n \) odd \( X \rightarrow (\det X)^2X \) is an automorphism of \( G_n \).

Proof. Consider first the case where \( n \) is even. The mapping is clearly an endomorphism of \( G_n \). If \( X^r = I \), then \( (\det X)X = I \) whence \( X = u \cdot I, u = (\det X)^{-1} \). But then \( \det X = u^n \), so \( u^n = u^{-1} \), whence \( u = 1 \) (because \( n \) is even). Therefore \( r \) is one-to-one.

To show that \( r \) is onto, we observe that \( S^r = S, T^r = T \); set \( Q = -iP \) for \( n = 0 \) (mod 4), and \( Q = iP \) for \( n = 2 \) (mod 4). In either case \( Q^r = P \), whence \( r \) is onto.

A similar proof is valid for odd \( n \).

Lemma 3.3. For \( n = 2 \), the mapping \( \tau \) defined by \( P_0^r = P_0, S_0^r = -S_0, T_0^r = -T_0 \) is an automorphism of \( G_2 \).

Proof. To begin with, we must show that \( \tau \) induces a well-defined mapping of \( G_2 \) into itself. This will be so if we can show that if a power product \( \Pi(P_0, S_0, T_0) = I \) in \( G_2 \) then the total number of factors of \( S_0 \) and \( T_0 \) is even.

Letting \( \xi = 1 + i \) as usual, we remark that since
\[
\Pi(P_0, S_0, T_0) = I
\]
implies
\[
(3.1) \quad \Pi(S_0, T_0) \equiv I \pmod{\xi}.
\]
However, there are only 6 elements in \( G_2 \) mod \( \xi \), represented by \( I, S_0, T_0, S_0T_0, T_0S_0, S_0T_0S_0 \), since \( S_0^2 \equiv T_0^2 \equiv I \pmod{\xi} \). Any power product \( \Pi(S_0, T_0) \) can be brought into one of these 6 forms by repeated use of \( S_0^2 \equiv T_0^2 \equiv (S_0T_0)^3 \equiv I \pmod{\xi} \). Hence in the left-hand side of (3.1), the total number of \( S_0 \)'s and \( T_0 \)'s must be even.

Now that \( \tau \) has been shown to be well-defined we see at once that \( \tau \) is onto. Further \( \tau^2 = 1 \) implies \( \tau \) is one-to-one, whence \( \tau \) is indeed an automorphism of \( G_2 \).

4. Generators of \( \mathfrak{X}_2 \). We shall obtain here the generators of \( \mathfrak{X}_2 \), the automorphism group of \( G_2 \). As before, define
\[ J_\alpha = \begin{pmatrix} -1 & 0 \\ \alpha & 1 \end{pmatrix}. \]

Our previous discussion shows that \( \pm I \) and \( J_\alpha, \alpha = 0, 1, \xi \) constitute a full set of nonconjugate involutions in \( G_2 \).

**Lemma 4.1.** For any \( \tau \in \mathfrak{A}_2 \), there exists an \( A \in G_2 \) such that \( J_\tau^r = AJ_\tau A^{-1} \).

**Proof.** By Theorem 2.2, to within an inner automorphism we have \( J_\tau^r = \pm J_\alpha, \alpha = 0, 1 \) or \( \xi \). However, the centralizer \( \mathcal{C}(J_\tau) \) contains 8 elements, whereas \( \mathcal{C}(J_\alpha) \) contains 16 elements for \( \alpha = 0 \) or \( \xi \). This completes the proof since \( -J_1 \) is conjugate to \( J_1 \).

**Theorem 4.1.** \( \mathfrak{A}_2 \) is generated by the automorphisms

1. \( X \mapsto AXA^{-1} \),
2. \( X \mapsto X'^{r-1} \),
3. \( X \mapsto \overline{X} \),
4. \( X \mapsto (\det X)X \),
5. \((P_0, S_0, T_0) \mapsto (P_0, -S_0, -T_0)\).

**Proof.** Let \( \tau \in \mathfrak{A}_2 \); changing \( \tau \) by an inner automorphism if necessary, we may assume hereafter that \( J_\tau^r = J_1 \). Let \( K = S_0J_0 \), then

\[ (4.1) \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad KJ_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (KJ_1)^3 = -I, \quad K^2 = I. \]

Let us put

\[ K^r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

Using the fact \( J_\tau^r = J_1 \), (4.1) implies that

\[-a + b + d = 1, \quad b(a + d) = c(a + d) = 0, \quad a^2 + bc = d^2 + bc = 1.\]

These imply that \( d = -a, b = 2a + 1, \) and

\[ a^2 + (2a + 1)c = 1, \]

that is

\[ 4(a + c)^2 - (2c - 1)^2 = 3. \]

There are only 4 solutions in Gaussian integers of this equation, and therefore \( K^r \) has only 4 possible expressions given by \( K, K_1, K_2, K_3 \) where

\[ K_1 = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad K_3 = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}. \]

A further inner automorphism by a factor of \( J_1 \) leaves \( J_\tau^r \) unaltered, but takes
$K_2$ into $K$, and $K_3$ into $K_1$. We may therefore assume from now on that $J'_i = J_i$ and either $K^r = K$ or $K^r = K_1$. In the latter case, replace $\tau$ by the automorphism

$$X \rightarrow (V^{-1}X^rV)^{-1}, \quad \text{where} \quad V = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$ 

This new automorphism leaves $J_1$ and $K$ invariant. Hence in all cases, after changing $\tau$ by automorphisms chosen from the list in Theorem 4.1, we may assume that $J'_i = J_i$ and $K^r = K$.

**Lemma 4.2.** If $\tau \in \mathfrak{U}_2$ is such that $J'_i = J_i$ and $K^r = K$, then $J'_0 = \pm J_0$.

**Proof.** We have $J_0K = -KJ_0$, $J'_0 = I$. Setting

$$J'_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have $d = -a$, $c = -b$, $a^2 - b^2 = 1$. Solving this last equation in Gaussian integers, we find that there are only 4 possibilities for $J'_0$, namely $\pm J_0$ or $\pm iS_0$.

Suppose now that $J_0 = +iS_0$. Since $J_0 = P^0_0$, setting

$$P^r_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we obtain

$$\begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{pmatrix} = \pm \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$ 

Hence $a + d$ is a unit. On the other hand,

$$a^2 + d^2 = -2bc, \quad (a + d)^2 = 2(ad - bc),$$

so $(a + d)^2$ is a nonunit. This is a contradiction, whence $J'_0$ must be $\pm J_0$.

We have now shown that by changing the given $\tau$ by automorphisms in the list, we can assume that

$$J'_0 = \pm J_0, \quad J'_1 = J_1, \quad K^r = K.$$

**Case I.** $J'_0 = J_0$. Then $S_0 = KJ_0$ implies $S'_0 = S_0$, and $T_0 = KJ_0J_1K$ implies $T'_0 = T_0$. From $P^0_0 = J_0$, setting

$$P^r_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have
\[ a^2 + bc = -1, \quad d^2 + bc = 1, \quad b(a + d) = c(a + d) = 0. \]

These imply that \( b = c = 0 \), and \( a = \pm i, \ d = \pm 1 \). Hence \( P_0^* = \pm P_0 \) or \( \pm \overline{P}_0 \). In the latter case, changing \( \tau \) by \( X \rightarrow \overline{X} \), we may assume \( P_0^* = \pm P_0 \). This new \( \tau \) is the automorphism \( X \rightarrow (\det X)^m X \) where \( m = 4 \) or \( 2 \), and hence is a product of automorphisms on the list.

**Case II.** \( J_0 = -J_0 \). As above, we find that \( S_0^* = -S_0, \ T_0^* = -T_0, \ P_0^* = \pm iP_0 \) or \( \pm i\overline{P}_0 \). In the latter case, change \( \tau \) by \( X \rightarrow \overline{X} \) to obtain

\[
S_0^* = -S_0, \quad T_0^* = -T_0, \quad P_0^* = \pm iP_0.
\]

This automorphism is an obvious product of automorphisms on the list.

This completes the proof of Theorem 4.1. We may remark that \( X \rightarrow X'^{-1} \) can be omitted from the list, since it can be expressed as a product of the other automorphisms on the list. Further, Theorem 4.1 implies Theorem 2.3 for the case \( n = 2 \).

5. Generators of \( \mathfrak{N}_3 \). In this section we prove the main theorem for the case \( n = 3 \).

**Step 1.** Let \( D_j \) be obtained from \( \mathfrak{N}_3 \) by changing the \( j \)th diagonal element to \( -1 \). Given any automorphism \( \tau \in \mathfrak{N}_3 \), we may assume by Theorem 2.3 (after changing \( \tau \) by an inner automorphism) that \( D_1 = \pm D_1 \). In that case \( \tau \) maps \( \mathfrak{C}(D_1) \) onto itself, that is,

\[
\begin{pmatrix}
a & 0 \\
0 & A
\end{pmatrix}^\tau = \begin{pmatrix}
b & 0 \\
0 & B
\end{pmatrix}
\]

where \( a \) is a unit in \( G \), \( A \in G_2 \). By Lemma 3.1 \( \tau : G_3^+ \rightarrow G_3^+ \) so that \( \tau : \mathfrak{C}(D_1) \cap G_3^+ \rightarrow \mathfrak{C}(D_1) \cap G_3^+ \). For each \( A \in G_3 \) choose \( a \) to be a unit for which \( a + A \in G_3^+ \). Then \( b \) and \( B \) in (5.1) are uniquely determined by \( a \). Set

\[
\begin{pmatrix}
a & 0 \\
0 & A
\end{pmatrix}^\tau = \begin{pmatrix}
\lambda(A) & 0 \\
0 & A^\sigma
\end{pmatrix}, \quad \text{where} \ a \cdot \det A = 1, \ A \in G_2.
\]

Then \( \lambda : G_2 \rightarrow G_1 \) is a homomorphism, as is \( \sigma : G_2 \rightarrow G_2 \). Since \( \lambda(A) \cdot \det A^\sigma = 1 \) we see that if \( A^\sigma = I \) then \( \lambda(A) = 1 \), and so \( A = I \). Hence \( \sigma \) is one-to-one, and from this we see that \( \sigma \) is an automorphism of \( G_2 \). Consequently \( \det A^\sigma = \det A \) always or conjugate of \( \det A \) always, whence \( \lambda(A) = a \) always or \( a \) always. Therefore \( \lambda(A) = 1 \) for \( A \in G_3^+ \) and \( \lambda(A) = -1 \) for \( A \in G_2^+ \).

Using the results of the preceding section, we deduce that there exists a \( Y \in G_2 \) such that

\[
A^\sigma = (\det A)^m Y A^* Y^{-1}
\]

for all \( A \in G_2 \) where \( A^* \) is obtained from \( A \) by applying neither, or one, or both of automorphisms 3., 5. (§4). If we change \( \tau \) by an inner automorphism with a factor of \( 1 + Y^{-1} \), we may then assume that \( A^\sigma = (\det A)^m A^* \), and that \( D_1^* = \pm D_1 \) is still valid.
Let us apply the above results to evaluate $D_j$, $j = 2, 3$. We have

$$(D_1D_2)^r = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^r = \begin{bmatrix} \lambda(A) & 0 \\ 0 & A^\sigma \end{bmatrix},$$

where $A = (-1) + (1)$. Then $\lambda(A) = -1$ by the above remarks since $A \in G_2$. Further, $A = P_0^*$, so $A^* = A$ for any choice of $\ast$. Therefore $A^\sigma = \pm A$, whence $(D_1D_2)^r = D_1D_2$ or $D_1D_3$. The latter case is reduced to the former by changing $\tau$ by an inner automorphism with factor $(1) + K$ where $K$ is given in (4.1). Therefore we obtain $D_1' = \pm D_1$, $D_2' = \pm D_2$. Since $(-I)^r = -I$ we also have $D_3' = \pm D_3$. Thus, starting with any $\tau \in \mathfrak{A}_3$ and changing $\tau$ by inner automorphisms, we arrive at a new $\tau$ for which $D_j' = \pm D_j$, $j = 1, 2, 3$.

**Step 2.** Now let $\tau \in \mathfrak{A}_3$ satisfy $D_j' = \pm D_j$, where $r = 1, 2, 3$. By the preceding discussion we may set

$$(5.3) \ ( (a) + \tau A)^r = (\lambda_r(A) + \tau A^\sigma r),$$

where $A \in G_2$ is arbitrary, $a$ is a unit such that $a \cdot \det A = 1$, $\lambda_r : G_2 \rightarrow G_1$ is a homomorphism such that either $\lambda_r(A) = a$ for all $A \in G_2$ or $\lambda_r(A) = \bar{a}$ for all $A \in G_2$, and $\sigma_r \in \mathfrak{A}_2$ is expressible as

$$(5.4) \quad A^\sigma_r = (\det A)^{\omega r} Y_r A^{\omega r} Y_r^{-1},$$

for all $A \in G_2$, where $Y_r \in G_2$, and $A^{\omega r}$ is obtained by applying to $A$ neither, or one, or both of automorphisms 3. and 5. ($\S 4$).

Now we evaluate $(( -1) + A)^r$ where $A = (-1) + (1)$. By the above this yields

$$Y_1 A Y_1^{-1} = \pm A$$

whence $Y_1$ is either diagonal or anti-diagonal, that is

$$Y_1 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \quad \text{or} \quad Y_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$u$ and $v$ units. A similar argument shows that each $Y_r$ is either diagonal or anti-diagonal.

**Case 1.** Suppose to begin with that at least one $Y_r$ is diagonal; without loss of generality we may assume that $Y_1$ is diagonal. After an inner automorphism with factor $(1) + Y_1^{-1}$ we may assume that $Y_1 = I$ in (5.4); $Y_2$ and $Y_3$ will now be different, but $D_j' = \pm D_j$ is still valid. We may again deduce that $Y_3$ is either diagonal or anti-diagonal.

**Case 1(a).** Suppose $Y_3$ is diagonal, say $Y_3 = [u, v]$. Then changing $\tau$ by an inner automorphism with factor $[u^{-1}, v^{-1}, v^{-1}]$ we still have $Y_1 = I$, $D_j' = \pm D_j$, and now also $Y_3 = I$. Therefore
\[ T^r = (T_0 + (1))^r = T_0^{\omega_3} + (1), \]

where now \( T_0^{\omega_3} = \pm T_0 \); the minus sign occurs if and only if automorphism 5. (§4) is one of the factors of \( \omega_3 \). We show next that \( T_0^{\omega_3} = -T_0 \) is impossible.

For, if \( T^r = -T_0 + (1) \), then \( (S_0 + (1))^r = -S_0 + (1) \). Set

\[
U = ((1) + S_0)^\cdot (S_0 + (1)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Then \( U^r = ((1) + (\pm S_0))(-S_0 + (1)) = U_1 \) or \( U_2 \), according to the sign, where

\[
U_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.
\]

Set \( Z = T U_2^2 \); then

\[
T' = T_0' + (1) = (UZ^{-1})^2 UZ^2
\]

and \( T_0' = S_0^{-1} T_s S_0 \). Therefore \( (T_0')^{\omega_3} = -T_0' \). Applying \( \tau \) to both sides of (5.6), and using \( U^r = U_1 \) or \( U_2 \) we obtain a contradiction. Hence \( T_0^{\omega_3} = -T_0 \) cannot occur.

We may now assume that both \( T \) and \( S_0 + (1) \) are invariant under \( \tau \). In that case, again defining \( U \) by (5.5), \( U^r \) has the two possible values \( U_1 \), \( U_2 \), where \( U_3 = D_s U D_3 \). But \( S = U^2 \) so that either \( S^r = S \) or \( S^r = D_s D_3 \).

In order to find \( P^r \), we observe that

\[
V = D_1 D_2 \cdot i P = [1, -i, i] = (1) + V_1
\]

where \( V_1 = [-i, i] \); hence \( V = (1) + V_1^\tau \). But \( V_1 = P_0^3 S_0^{-1} P_0 S_0 \), whence \( V_1^\tau = V_1 \) or \( \overline{V}_1 \). Using the fact that \( (iI)^r = \pm iI \) we obtain \( P^r = \pm P \) or \( \pm \overline{P} \). In the latter case, change \( \tau \) by the automorphism 3. to get \( P^r = \pm P \). If \( P^r = -P \) change \( \tau \) by the automorphism 4. to get \( P^r = P \). Hence after changing \( \tau \) by automorphisms on the list, we may assume \( T^r = T \), \( S^r = D_s D_3 \), \( P^r = P \). But then \( \tau \) is just an inner automorphism by a factor of \( D_3 \), and therefore is on our list. This completes the proof for the present case.

**Case I(b).** Suppose next that \( Y_3 \) is anti-diagonal, say

\[
Y_3 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.
\]

After changing \( \tau \) by an inner automorphism with factor \( [u^{-1}, -v^{-1}, -v^{-1}] \), we may assume that \( Y_1 = I, D_1^\tau = \pm D_r \) and \( Y_3 = S_0 \). Then \( T^r = \pm S_0^{-1} T_0 S_0 + (1) \); the same type of argument as above shows that the minus sign is impossible. Hence \( T^r = S_0^{-1} T_0 S_0 + (1) = T^r \), and we find again that either \( U^r = U \) or \( U^r = U_3 \), whence \( S^r = S \) or \( S^r = D_s D_3^{-1} \). Furthermore, we obtain \( P^r = \pm P \) or
As before, and changing \( \tau \) by 3. and 4. as needed, we get \( P^\tau = P \). Now change \( \tau \) by 2. Since \( S = S^{-1}, P = P^{-1} \) we find that \( T^\tau = T, S^\tau = S \) or \( D_3SD_3, P^\tau = P \) which is clearly a product of automorphisms on the list. We have completed the proof for this case.

**Case II.** Suppose that in (5.4) each \( Y_r \) is anti-diagonal. After an inner automorphism by a suitably chosen diagonal matrix, we may assume that \( Y_1 = Y_3 = S_0 \). Then we find that \( T^\tau = (\pm T_0)^{-1} + (1), \) and the same reasoning as before shows that the minus sign cannot occur. Hence we obtain \((S_0 + (1))^\tau = S_0 + (1), \) and \( S^\tau = S \) or \( D_3SD_3 \). In the latter case, an inner automorphism by a factor of \( D_3 \) gives a new \( \tau \) with \( T^\tau = T'^{-1}, S^\tau = S \). Changing this \( \tau \) by \( X^X^{-1} \) we arrive at an automorphism \( \tau \) which leaves \( U, S \) and \( T \) invariant. The same reasoning as in Case I(a) shows that \( P^\tau = \pm P \) or \( \pm P \), and the remainder of the proof is as before.

6. **Generators of** \( \mathfrak{A}_n \). We are now ready to prove the main theorem by induction on \( n \).

We suppose \( n \geq 4 \), and that the result holds for \( n-1 \). Let \( D_i \) be the diagonal matrix \([1, \ldots, 1, -1, 1, \ldots, 1]\) with \(-1\) occurring in the \( j \)th position. By Theorem 2.3, given any \( \tau \in \mathfrak{A}_n \), we may change \( \tau \) by an inner automorphism so as to achieve \( D_i^\tau = \pm D_i \). Therefore \( \tau \) maps \( \mathcal{C}(D_1) \cap G_1^+ \) onto itself. Hence if \( A \in G_{n-1} \) and \( a \cdot \text{det} A = 1 \), we have

\[
\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix}^\tau = \begin{pmatrix} \lambda_1(A) & 0 \\ 0 & A^\sigma_1 \end{pmatrix}
\]

where \( \lambda_1: G_{n-1} \to G_1 \) is a homomorphism and \( \sigma_1 \) is an automorphism of \( G_{n-1} \). As before, \( \lambda_1(a) = a \) always or \( \bar{a} \) always. Using the induction hypothesis, we may write

\[
A^\sigma_1 = (\text{det} A)^{\omega_1} Y_1 A^{\omega_1} Y^{-1}_1, \quad Y_1 \in G_{n-1},
\]

where \( \omega_1 \) is a product of automorphisms chosen from 2. or 3. After an inner automorphism with factor \( (1) + Y^{-1}_1 \), we may take \( Y_1 = 1 \). Now \( D_1D_2 = [-1, -1, 1, \ldots, 1] \); by computing \( (D_1D_2)^\tau \) we find that \( D_1^\tau = \pm D_1 \). Likewise \( D_1 = \pm D_1, 1 \leq r \leq n \). We may therefore write

\[
(a + r A)^\tau = \lambda_r(A) + r (\text{det} A)^{\omega_r} Y_r A^{\omega_r} Y^{-1}_r,
\]

where \( A \in G_{n-1} \) is arbitrary, \( a \cdot \text{det} A = 1 \), \( \lambda_r: G_{n-1} \to G_1 \) is a homomorphism such that either \( \lambda_r(A) = a \) always or \( \bar{a} \) always, where \( Y_r \in G_{n-1} \), and \( \omega_r \) is a product of some (of none) of the automorphisms 2., 3. (Further, we have already seen that we may choose \( Y_1 = 1 \).)

Now let \( Z \in G_{n-2} \); since \( (I^2 + Z) \in \mathcal{C}(D_1) \cap \mathcal{C}(D_2) \) we can compute \((I^2 + Z)^\tau\) in two ways. This gives

\[
Y_1 \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} Y_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & Z_1 \end{pmatrix}
\]
for some $Z_1 \in G_{n-2}$. Since such a relation holds for all $Z \in G_{n-2}$, it follows that $Y_1 = (y_1) + \tilde{Y}$. By a similar argument we see that $Y_1$, and indeed each $Y_r$, must be diagonal, and that all $Y_r$ are sections of a single diagonal matrix $D$. Following $\tau$ with an inner automorphism with factor $D^{-1}$, we see that we may assume each $Y_r = I$. The same type of argument shows that the various $m_r$ are all the same, and that all the $\omega_r$ coincide. Hence we have

$$X^r = X^\omega$$

for all decomposable $X \in G_n^+$, where $\omega$ is the common automorphism $\omega_1 = \omega_2 = \cdots$, i.e., $\omega$ is a product of automorphisms chosen from 2. or 3. Changing $\tau$ by the automorphisms 2., 3. as needed we may thus assume that $X^r = X$ for all decomposable $X \in G_n^+$. For $n \geq 4$, these decomposable matrices generate $G_n^+$, and so $X^r = X$ for all $X \in G_n^+$.

We now determine the effect of $\tau$ on $G_n^-$ and $G_n^{\pm 1}$. Let $Y, Z \in G_n^-$ where $Z$ is fixed. Then

$$Y^*Z^r = (YZ)^r$$

implies

$$Y^* = YB$$

for all $Y \in G_n^-$, where $B$ is independent of $Y$. Using $(Y^2)^r = (Y^r)^2$, we obtain

$$BYB = Y$$

for all $Y \in G_n^-$. This implies that $B = \pm I$ or $\pm iI$. However, $B = \pm iI$ is impossible, and therefore $B = \pm I$, whence

$$Y^r = \pm Y$$

for all $Y \in G_n^-$. If $n$ is odd, $\tau: G_n^- \to G_n^-$ shows that only the plus sign can hold. If $n$ is even, then changing $\tau$ by the automorphism $X \mapsto (\det X)X$, if necessary, we may assume that $X^r = X$ for all $X \in G_n^+ \cup G_n^-$. The same argument as above shows that $Y^r = \pm Y$ for all $Y \in G_n^{\pm 1}$. If the plus sign occurs, $\tau$ is the identity; if the minus sign occurs, then $\tau$ is simply the automorphism $X^r = (\det X)^2X$. This concludes the proof of the Main Theorem.

Another approach to the proof of the Main Theorem, which is less computational than that given here, is contained in references [3] and [4].

References


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