SEMIGROUP OF ENDMORPHISMS OF A LOCALLY
COMPACT GROUP

BY

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Introduction. A homomorphism of a group \( \mathcal{G} \) into itself is called an endomorphism of \( \mathcal{G} \). The set of all endomorphisms of \( \mathcal{G} \) forms a semigroup containing all automorphisms of \( \mathcal{G} \). If \( \mathcal{G} \) has a topology in which it is a topological group, then it will be natural to limit ourselves to the study of the set of all continuous endomorphisms of \( \mathcal{G} \), which set is also a semigroup containing the group of all bicontinuous automorphisms of \( \mathcal{G} \). In the present paper we are interested in these semigroups for locally compact topological groups.

Let \( G \) be a locally compact group, and let \( \mathcal{E}(G) \) be the semigroup of all continuous endomorphisms of \( G \). The first step of our investigation is to topologize \( \mathcal{E}(G) \) in such a way that \( \mathcal{E}(G) \) becomes a topological semigroup, which will be done in \( \S 1 \). The results in \( \S 1 \) and in \( \S 2 \) will indicate that the topology introduced here is the unique natural one.

\( \S 2 \) will be devoted to the special case where \( G \) is a Lie group, mainly for the purpose of later use. The fact that any continuous endomorphism of a Lie group induces an endomorphism of the Lie algebra (infinitesimal group) is useful to us.

In \( \S 3 \) we shall consider the case of compact groups. Let us denote by \( \mathcal{C}(G) \) the connected component containing the identity of \( \mathcal{E}(G) \). Then in the case in which \( G \) is a compact group, \( \mathcal{C}(G) \) is composed of bicontinuous automorphisms (Theorem 1), and the structure of the group \( \mathcal{C}(G) \) will be determined completely.

In \( \S 4 \) we consider the group \( \mathcal{A}(G) \) of all bicontinuous automorphisms of a locally compact group \( G \). \( \mathcal{A}(G) \) will be topologized relatively as a subspace of \( \mathcal{E}(G) \). Then \( \mathcal{A}(G) \) is, of course, a topological semigroup. Moreover, under a certain additional condition, we may prove the continuity of the operation of taking inverses in \( \mathcal{A}(G) \) so that \( \mathcal{A}(G) \) will also be a topological group (Theorem 2). This proof uses certain properties of endomorphisms.

In \( \S 5 \) we shall discuss in detail the structure of \( \mathcal{E}(G) \) for a locally compact connected group \( G \) under the structure theorem of locally compact groups. Most of the theorems and arguments used here are quite analogous to those

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(1) This work was done under the support of the National Science Foundation.

(2) A set with a binary operation (multiplication) is called a semigroup if the operation is associative.

(3) A semigroup is called a topological semigroup if it is a topological space and if multiplication is continuous with respect to both variables.
in a paper by one of the authors on the automorphism group of $G$. The main results are stated in Theorem 3, Theorem 4 and Theorem 5.

1. **Topology of the semigroup of endomorphisms.** Let $G$ be a locally compact topological group. By an endomorphism of $G$ we mean a continuous homomorphism from $G$ into itself. Let $\sigma$ and $\tau$ be endomorphisms of $G$. Then the product $\sigma \tau$ defined by $(\sigma \tau)(x) = \sigma(\tau(x))$, $x \in G$, is also an endomorphism. So the set $\mathcal{E}(G)$ of all endomorphisms of $G$ forms a semigroup. Let now $\mathcal{A}(G)$ be the subset of $\mathcal{E}(G)$ composed of all bicontinuous automorphisms of $G$. Then $\mathcal{A}(G)$ clearly forms a group.

In §1 we shall consider a natural topology of $\mathcal{E}(G)$. Although most of the results here seem to be known, we shall discuss the basic properties of the topology of $\mathcal{E}(G)$ for the sake of completeness.

Let $G$ be a locally compact group. By a nucleus we mean here an open neighborhood $U$ of the identity $e$ whose closure is compact and satisfies $U = U^{-1}$. As is known, all the nuclei form a base of neighborhoods of $e$. For a given triple of an endomorphism $\rho$, a compact subset $S$, and a nucleus $U$ of $G$, we define a subset $(\rho; S, U)$ of $\mathcal{E}(G)$ in the following way:

$$(\rho; S, U) = \{ \sigma \in \mathcal{E}(G), \rho(x)^{-1}\sigma(x) \in U \text{ for all } x \in S \}.$$

**Proposition 1.** By considering the set of all possible $(\rho; S, U)$'s as a base of neighborhoods of $\rho$, $\mathcal{E}(G)$ becomes a topological space satisfying the separation axiom of Hausdorff. Moreover the multiplication of $\mathcal{E}(G): \mathcal{E}(G) \times \mathcal{E}(G) \ni (\rho, \sigma) \mapsto \rho \sigma \in \mathcal{E}(G)$ is continuous with respect to the topology.

**Proof.** (i) Clearly we have $(\rho; S_1 \cup S_2, U_1 \cap U_2) \subseteq (\rho; S_1, U_1) \cap (\rho; S_2, U_2)$. Now let $\sigma$ be in $(\rho; S, U)$. Then the set $S' = \{ \rho(x)^{-1}\sigma(x) \mid x \in S \}$ is compact as a continuous image of $S$. Since $S'$ is a subset of $U$, we can find a nucleus $U'$ so that $S' \subseteq U$, and we have $(\sigma; S', U') \subseteq (\rho; S, U)$. Therefore the set of all $(\rho; S, U)$'s defines a topology.

(ii) In order to show that $\mathcal{E}(G)$ is a Hausdorff space we take two distinct endomorphisms $\rho_1$ and $\rho_2$ of $G$. Take an element $x$ of $G$ so that $\rho_1(x) \neq \rho_2(x)$. Then for a nucleus $U$ satisfying $\rho_1(x)^{-1}\rho_2(x) \in U^2$ we have $(\rho_1; \{x\}, U) \cap (\rho_2; \{x\}, U) = \emptyset$, where $\{x\}$ denotes the set composed of a single element $x$, and $\emptyset$ is the empty set.

(iii) For a given neighborhood $(\sigma \tau; S, U)$ of $\sigma \tau$, we take a nucleus $V$ satisfying $\sigma(V) \subset U$. Then we have

$$(\sigma; \tau(\mathcal{V}), V)(\tau; S, V) \subseteq (\sigma \tau; S, U),$$

(4) On topological groups and especially on locally compact topological groups, see Pontrjagin [14] and Weil [15].

(5) A continuous automorphism is not bicontinuous in general. However if the space of $G$ is a countable union of compact sets, then the continuity of an automorphism implies openness. Cf. Goto [8].

(6) $\{*, \ldots\}$ is the set of $*$'s satisfying $\ldots$.

(7) In general, $\mathcal{A} \mathcal{B} = \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ for two subsets $\mathcal{A}$ and $\mathcal{B}$ of a semigroup $G$.

(8) $\overline{V}$ denotes the closure of $V$. 

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which implies the continuity of the multiplication.

Hereafter by a topology of \( \mathcal{G}(G) \) we mean the topology introduced above. Here we mention two propositions concerning \( \mathcal{G}(G) \) without proof, because these are direct consequences of the definition of the topology.

**Proposition 2.** The mapping \( \mu \) from \( \mathcal{G}(G) \times G \) into \( G \) defined by \( \mu(\rho, x) = \rho(x) \) is continuous. Moreover the above topology of \( \mathcal{G}(G) \) is the weakest of the topologies of \( \mathcal{G}(G) \) having the properties in Proposition 1 and making \( \mu \) continuous.

**Proposition 3.** Let \( G \) be a locally compact group, and let \( N \) be a closed normal subgroup of \( G \). Let \( \mathcal{G}^N(G) \) be the subset (semigroup) of \( \mathcal{G}(G) \) composed of endomorphisms which map \( N \) into itself. Let \( \sigma \) be in \( \mathcal{G}^N(G) \). Then \( \sigma \) induces an endomorphism \( \psi(\sigma) \) of \( G/N \), and the mapping \( \psi: \mathcal{G}^N(G) \to \mathcal{G}(G/N) \) is continuous.

**Proposition 4.** Let \( G \) be a locally compact group generated by a nucleus \( W \), then the set of neighborhoods of \( \rho \) given by \{ \( (\rho; W, V) \mid V \) runs over all nuclei \} forms a base of \( \rho \) in \( \mathcal{G}(G) \).

**Proof.** Let \( (\rho; \mathfrak{K}, U) \) be a neighborhood of \( \rho \). Then we can find a positive integer \( n \) so that \( \mathfrak{K} \subset W^n \). Since \( W \) is compact and \( \rho \) is continuous, \( \rho(W) \) is compact. Hence we can find a nucleus \( V \) so that \( V^n \subset U \) and \( zV = Vz \) for \( z \in \rho(W) \).

Let us prove that \( (\rho; W, V) \subset (\rho; \mathfrak{K}, U) \).

Let \( y = x_1x_2 \cdots x_n, x_i \in W \), be an element in \( W^n \), and let \( \sigma \) be in \( (\rho; W, V) \). Since \( \rho(x_i)^{-1} \in V \) and \( \rho(x_i)^{-1}V = V \rho(x_i)^{-1} \), we have \( \rho(y)^{-1}\sigma(y) = \rho(x_n)^{-1}\rho(x_{n-1})^{-1} \cdots \rho(x_1)^{-1}\sigma(x_1) \cdots \sigma(x_n) \subset V^n \subset U \), whence \( (\rho; W, V) \subset (\rho; W^n, U) \subset (\rho; \mathfrak{K}, U) \).

Next, let us consider some further concepts related to \( \mathcal{G}(G) \) which shall be necessary for our purposes later.

1°. **Semigroup of compact subsets of \( G \).** Let \( G \) be a locally compact group, and let \( 2^G \) be the set of all compact nonempty subsets of \( G \). \( 2^G \) forms a semigroup with respect to the product \( \mathfrak{K}_1 \mathfrak{K}_2 = \{ k_1k_2 \mid k_1 \in \mathfrak{K}_1 \text{ and } k_2 \in \mathfrak{K}_2 \} \), (Gleason [4]). Let \( U \) be a nucleus, and \( \mathfrak{K} \) an element of \( 2^G \). Define the set \( (\mathfrak{K}; U) \) by \( (\mathfrak{K}; U) = \{ \mathfrak{K}' \mid \mathfrak{K}' \subset 2^G, \mathfrak{K}' \subset \mathfrak{K}U \text{ and } \mathfrak{K} \subset \mathfrak{K}'U \} \). Now it is easy to prove the following propositions:

**Proposition 5.** By considering all \( (\mathfrak{K}; U) \)'s, where \( U \) runs over all nuclei, as a base of the neighborhoods of \( \mathfrak{K} \) in \( 2^G \), \( 2^G \) becomes a locally compact Hausdorff space so that the multiplication is continuous.

**Proposition 6.** The mapping \( \nu \) from \( \mathcal{G}(G) \times 2^G \) into \( 2^G \) defined by \( \nu(\rho, \mathfrak{K}) = \rho(\mathfrak{K}) \) is continuous.

2°. **Automorphism group of \( G \).** Since \( \mathfrak{K}(G) \) is a subset of \( \mathcal{G}(G) \), we may consider the relative topology of \( \mathfrak{K}(G) \) as a subspace of \( \mathcal{G}(G) \), which we shall use as the topology of \( \mathfrak{K}(G) \) hereafter, (Goto [7], Hochschild [9], Iwasawa [10])
To show that $\mathfrak{A}(G)$ is a topological group under some additional conditions, we need the structure theorem of locally compact groups, and the proof will be given in §4.

3°. Component group (semigroup). Let $G$ be a topological semigroup with the identity $e$. Then the connected component $G^0$ containing $e$ is a closed sub-semigroup. We call $G^0$ the component semigroup of $G$. If in particular $G$ is a topological group, then $G^0$ becomes a subgroup called the component group of $G$.

We shall use the notation $\mathfrak{E}(G)$ for the component semigroup of $\mathfrak{A}(G)$ and $\mathfrak{E}(G)$ for the component semigroup of $\mathfrak{A}(G)$.

4°. Group of inner automorphisms. Let $G$ be a locally compact group and $a$ an element of $G$. The mapping $\tau_a(x) = a^{-1}xa (x \in G)$ defines a bicontinuous automorphism of $G$. $\tau_a$ is called an inner automorphism of $G$. The correspondence $a \mapsto \tau_a$ gives a continuous homomorphism of $G$ into $\mathfrak{A}(G)$, and the kernel coincides with the center of $G$.

We denote the group of all inner automorphisms by $\mathfrak{I}(G)$. If $G$ is connected, then $\mathfrak{I}(G) \subset \mathfrak{A}(G)$. Let $H$ be a subgroup of $G$. The set $\{\tau_a | a \in H\}$ forms a subgroup of $\mathfrak{I}(G)$. We shall use the notation $\mathfrak{I}_{G}(H)$ for this group.

2. Lie group case. In §2 we shall be concerned with $\mathfrak{E}(L)$ where $L$ is a Lie group. Some of the arguments here are quite similar to those for $\mathfrak{A}(L)$ in §XV, Chapter IV of Chevalley: Lie groups I.

Let $\mathfrak{L}$ be an $r$-dimensional Lie algebra over the field of real numbers. A linear homomorphic mapping of $\mathfrak{L}$ into itself is called an endomorphism of $\mathfrak{L}$, and the set of all endomorphisms of $\mathfrak{L}$ will be denoted by $\mathfrak{E}(\mathfrak{L})$. Let now $GL(r)$ be the semigroup of all linear transformations of $r$-dimensional vector space. Then $\mathfrak{E}(\mathfrak{L})$ is a sub-semigroup of $GL(r)$. As the topology of $GL(r)$, we use the usual euclidean one of $r^2$-dimensional vector space.

Let $e_1, e_2, \cdots, e_r$ be a basis of $\mathfrak{L}$. Assume the commutator multiplication of $\mathfrak{L}$ is expressed by $[e_i, e_j] = \sum \gamma_{ij} e_s$ with respect to the basis, where $\gamma_{ij}$'s are real numbers called structure constants. Let $\sigma$ be a linear transformation of the vector space $\mathfrak{L}$ so that $\sigma(e_i) = \sum u_e e_e$. If $\sigma \in \mathfrak{E}(\mathfrak{L})$ then we have $[\sigma(e_i), \sigma(e_j)] = \sigma([e_i, e_j])$, and vice versa. The condition that a linear transformation $\sigma$ be an endomorphism is written in the form

$$\sum_{u,v} \gamma_{uv}^k = \sum_s \gamma_{ij}^k \xi_{ij},$$

Therefore $\mathfrak{E}(\mathfrak{L})$ is an algebraic set and is closed in $GL(r)$.

Now let $L$ be a connected Lie group, and let $\mathfrak{L}$ be the Lie algebra of $L$. We can identify a sufficiently small nucleus of $L$ with a neighborhood of 0 in $\mathfrak{L}$ by taking a canonical system of coordinates of the first kind (Pontrjagin [14]), and for any local endomorphism of $L$ we have a corresponding endomorphism of $\mathfrak{L}$. Hence for any endomorphism $\sigma$ of $L$ we have the correspond-
ing endomorphism $\psi(\sigma)$ of $\mathfrak{g}$. The correspondence $\sigma \mapsto \psi(\sigma)$ is clearly a continuous homomorphism, and since $L$ is generated by any nucleus, the homomorphism $\psi$ is one-to-one.

In particular, let $L$ be a connected simply connected Lie group. Then any local endomorphism generates an endomorphism of $L$, and we obtain a bicontinuous isomorphism $\mathfrak{g}(L) \cong \mathfrak{g}(\mathfrak{g})$.

Next, let us consider a connected Lie group $L$ which is not necessarily simply connected, and let $\tilde{L}$ be the universal covering group of $L$. Then there exists a discrete normal, hence central, subgroup $F$ of $\tilde{L}$, which is isomorphic to the fundamental group of the space of $L$, so that $\tilde{L}/F \cong L$. Consider the sub-semigroup $\mathfrak{g}_F(\tilde{L})$ of $\mathfrak{g}(\tilde{L}) = \{ \sigma | \sigma \in \mathfrak{g}(\tilde{L}), \sigma(F) \subset F \}$. Then $\mathfrak{g}_F(\tilde{L})$ is clearly a closed sub-semigroup of $\mathfrak{g}(\tilde{L})$, and an element of $\mathfrak{g}_F(\tilde{L})$ induces an endomorphism $\phi(\sigma)$ of $\tilde{L}/F \cong L$. On the other hand for a given endomorphism $\sigma'$ of $L$, we have a corresponding local endomorphism of $\tilde{L}$, which induces $\sigma$ in $\mathfrak{g}(\tilde{L})$ so that $\sigma(F) \subset F$ and $\phi(\sigma) = \sigma'$. So $\mathfrak{g}(L)$ is naturally identified with $\mathfrak{g}_F(\tilde{L})$, which is closed in $\mathfrak{g}(\tilde{L})$. Thus we get

**Proposition 7.** Let $L$ be an $r$-dimensional connected Lie group. Then we have a bicontinuous isomorphism of $\mathfrak{g}(L)$ onto a closed sub-semigroup of $\text{GL}(r)$. Hence $\mathfrak{g}(L)$ is locally compact.

**Remark.** Although $\mathfrak{g}(L)$ is a Lie group, $\mathfrak{g}(L)$ is not locally euclidean in general.

In a manner analogous to the proof of Proposition 7, we have the following

**Proposition 8.** Let $L$ be a connected Lie group of $r$-dimension, and let $D$ be a discrete normal subgroup of $L$. Let $\mathfrak{g}_D(L)$ be the sub-semigroup composed of all endomorphisms of $L$ which leave every element of $D$ fixed, and let $\mathfrak{g}_D(L)$ be the component semigroup of $\mathfrak{g}_D(L)$. Then $\mathfrak{g}_D(L)$, and accordingly $\mathfrak{g}_D(L)$ also, is a closed sub-semigroup of $\text{GL}(r)$.

Let $\sigma$ be an endomorphism of a connected Lie group $L$, and let $|\sigma|$ be the determinant of the corresponding linear transformation. Then the mapping $\sigma \mapsto |\sigma|$ clearly defines a continuous homomorphism of $\mathfrak{g}(L)$ into the multiplicative semigroup of real numbers. Since $|\sigma| \neq 0$ implies $\sigma$ is a local automorphism and vice versa, we have the following

**Proposition 9.** Let $\sigma$ be an endomorphism of a connected Lie group $L$. Then $\sigma$ is onto if and only if $|\sigma| \neq 0$.

In general, an endomorphism of $L$ onto itself is not an automorphism. However it is obviously true for simply connected groups, and is also true in the following case.

**Proposition 10.** Let $L$ be a connected Lie group whose center is discrete. Then any onto endomorphism is an automorphism.
Proof. Let \( \sigma \) be an endomorphism of \( L \) onto \( L \), and let \( D \) be the kernel of the homomorphism \( \sigma \). Then \( L \cong L/D \). So \( \dim L = \dim L/D \), whence \( D \) is discrete. Therefore \( D \) is contained in the center \( C \) of \( L \). Now, it is clear that \( C/D \) is the center of \( L/D \). Hence \( C \) is isomorphic with \( C/D \). On the other hand, since \( C \) has a finite system of generators (Goto [5]), \( C \) cannot be isomorphic with any proper factor group. Hence \( D = \{ e \} \).

Corollary. Let \( S \) be a connected semi-simple Lie group. Then \( \mathfrak{C}(S) = \mathfrak{A}(S) \), which is open in \( \mathfrak{E}(S) \).

Proof. It is well-known\(^{(10)}\) that \( |\sigma| = \pm 1 \) for \( \sigma \in \mathfrak{E}(S) \). On the other hand a semi-simple Lie group has a discrete center. Hence \( |\sigma| \) can take only the values \( \pm 1 \) or \( 0 \) for \( \sigma \in \mathfrak{E}(S) \). Since the mapping \( \sigma \to |\sigma| \) is continuous and \( |1| = 1 \), \( \mathfrak{E}(S) \subset \{ \sigma \mid |\sigma| = 1 \} \subset \mathfrak{A}(S) \), which is open and closed in \( \mathfrak{E}(S) \). Hence \( \mathfrak{E}(S) = \mathfrak{A}(S) \). On the other hand \( \mathfrak{A}(S) \) is known to be open in \( \mathfrak{E}(S) \)\(^{(10)}\).

Proposition 11. Let \( L \) be a connected Lie group. Then \( \mathfrak{E}'(L) \) is open in \( \mathfrak{E}(L) \).

Proof. (i) Let \( Z \) be a compact connected abelian Lie group. Take a canonical coordinate system of the first kind in \( Z \) so that \( (x_i, y_i) = (x_i + y_i \pmod{1}) \). Then with respect to the coordinate system, any endomorphism of \( Z \) has a matrix with integer coefficients. So \( \mathfrak{E}(Z) \) is discrete. Hence \( \mathfrak{E}'(Z) \) is, of course, open in \( \mathfrak{E}(Z) \).

Let now \( L \) be a semi-simple Lie group or a vector group of finite dimension. Then \( \mathfrak{E}'(L) \) coincides with \( \mathfrak{E}' = \{ \sigma \mid \sigma \in \mathfrak{E}(L), |\sigma| \neq 0 \} \).

(ii) Let \( L \) be a connected Lie group and let \( \mathfrak{E}'(L) \) be the sub-semigroup of \( \mathfrak{E}(L) \) composed of all onto endomorphisms. \( \mathfrak{E}'(L) \) is clearly open in \( \mathfrak{E}(L) \).

Now in \( L \) we shall construct a sequence of closed normal subgroups

\[
L = L_0 \supset L_1 \supset \cdots \supset L_n = e,
\]

so that

\[
\mathfrak{E}'(L)L_i \subset L_i,
\]

and each factor group \( L_i/L_{i+1} \) is semi-simple, compact abelian, or is a vector group. For that we take as \( L_1 \) the radical (maximal connected closed solvable normal subgroup), take as \( L_3 \) the closure of the commutator subgroup of \( L_1 \), and the closure of the commutator subgroup of \( L_3 \) as \( L_5 \), and so on. \( L_{2m} \) will be given so that \( L_{2m}/L_{2m+1} \) is the maximal compact subgroup of \( L_{2m-1}/L_{2m+1} \) for \( m = 1, 2, 3, \ldots \).

Then since any element of \( \mathfrak{E}'(L) \) is a local automorphism of \( L \) we have \( \mathfrak{E}'(L)L_i \subset L_i \). Then it is clear that \( \mathfrak{E}'(L)L_i \subset L_i \) for \( i = 2, 3, \ldots \). Also, from

\footnote{\( \ast \)}

\footnote{\( \ast \)} We may use the notation \( \epsilon \) sometimes instead of \( \{ e \} \).

\footnote{\( \ast \)} It is known that \( \mathfrak{A}(S) = \mathfrak{N}(S) \) and \( \mathfrak{N}(S) \) has a finite index in \( \mathfrak{A}(S) \). (See Gantmacher \([2]\)). Since \( \mathfrak{N}(S) \) is a connected Lie group with no nontrivial abelian factor groups, \( |\sigma| = 1 \) for \( \sigma \in \mathfrak{N}(S) \). So \( |\sigma| \) is a representation of the finite group \( \mathfrak{A}(S)/\mathfrak{N}(S) \) into the multiplicative group of real numbers. Hence \( |\sigma| = \pm 1 \).

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the construction, \(L_0/L_1\) is semi-simple, \(L_{2m-1}/L_{2m}\) is a vector group, and \(L_{2m}/L_{2m+1}\) is compact and abelian.

(iii) Let \(\sigma\) be an endomorphism of \(L\) onto itself. Then \(\sigma\) induces an endomorphism \(\sigma_i\) of \(L_i/L_{i+1}\), and \(\sigma\) is an automorphism if and only if all \(\sigma_i\)'s are so. Since the mapping \(\sigma \rightarrow \sigma_i\) is continuous by Proposition 3, the problem is reduced to the special cases of (i).

**Corollary.** Let \(L\) be a connected Lie group. If the radical of \(L\) is simply connected, then \(\mathfrak{g}'(L) = \mathfrak{h}(L)\).

**Proof.** The Corollary is true for vector groups and semi-simple Lie groups. On the other hand if the radical of \(L\) is simply connected, then in the sequence of factor groups in the proof above, each \(L_i/L_{i+1}\) can be either a vector group or a semi-simple group.

3. **Compact group case** \((^{11})\). In §3 the following theorem and some related results shall be obtained.

**Theorem 1.** Let \(G\) be a compact group. Then \(\mathfrak{g}(G) = \mathfrak{a}(G)\).

Before proving the theorem, we shall obtain the following preliminary proposition.

**Proposition 12.** Let \(G\) be a locally compact group, and \(U\) a nucleus of \(G\). If \(U\) contains a closed subgroup \(A\) containing all subgroups of \(G\) in \(U\), then

\[
\mathfrak{g}^*(G) = \{\sigma | \sigma \in \mathfrak{g}(G), \sigma(A) \subset A\}
\]

forms an open and closed sub-semigroup of \(\mathfrak{g}(G)\).

Hence we have \(\mathfrak{g}(G) \subset \mathfrak{g}^*(G)\).

**Proof.** Let \(T\) be the set of all compact subgroups of \(G\), and let \(T'\) be the set of all closed subgroups of \(K\). Then \(T\) is a closed subset of \(2^G\), and \(T'\) is clearly an open and compact subset of \(T\). Hence by Proposition 5

\[
\mathfrak{g}^*(G) = \{\sigma | \sigma \in \mathfrak{g}(G), \sigma(T') \subset T'\}
\]

is closed and open.

**Proof of Theorem 1.** (i) Let \(L^0\) be a compact connected Lie group, let \(S\) be the commutator subgroup of \(L^0\), and let \(Z\) be the component group of the center of \(L^0\). Then \(S\) is a closed semi-simple normal subgroup, and \(L^0 = SZ\) is a locally direct decomposition.

Let \(\sigma\) be an endomorphism of \(L^0\). Since any endomorphism leaves the commutator subgroup fixed, \(\sigma\) induces an endomorphism of \(S\). If in particular \(\sigma \in \mathfrak{g}(L^0)\), then the induced endomorphism \(\sigma_i\) of \(S\) is an automorphism by the corollary to Proposition 10.

Since \((^{12})\) \([Z, S] = e\), we have \([\sigma(Z), \sigma(S)] = e\), namely \([\sigma(Z), S] = e\). There-

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\(^{11}\) About the theory of compact groups, see Pontrjagin \([14]\) and Weil \([15]\).

\(^{12}\) Let \(\mathfrak{A}\) and \(\mathfrak{B}\) be nonempty subsets of a group \(\mathfrak{G}\). \([\mathfrak{A}, \mathfrak{B}]\) denotes the subgroup generated by \(\{aba^{-1}b^{-1} | a \in \mathfrak{A} \text{ and } b \in \mathfrak{B}\}\). For example \([\mathfrak{G}, \mathfrak{G}]\) is the commutator subgroup of \(\mathfrak{G}\).
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Therefore \([\sigma(Z), L^0] = e\). On the other hand \(\sigma(Z)\) is connected. Hence \(\sigma(Z) \subset Z\). So \(\sigma\) induces an endomorphism of \(Z\). By using the connectedness of \(G(Z^0)\) again, we have \(\sigma = 1\) on \(Z\).

These show that \(\sigma \in H(L^0)\), so \(\sigma \in \mathfrak{H}(L^0)\).

(ii) Let \(L\) be a compact Lie group, and \(L^0\) the component group of \(L\). Let \(\sigma\) be an endomorphism of \(L\). Then \(\sigma\) leaves \(L^0\) invariant and induces an endomorphism \(\psi(\sigma)\) of \(L^0\). Since \(\psi\) is continuous, we have \(\psi(G(L)) \subset G(L^0) = \mathfrak{H}(L^0)\).

Let \(L = a_1 L^0 + \cdots + a_m L^0\) be the coset decomposition of \(L\) with respect to \(L^0\). Then the set \(\{\sigma \in \mathfrak{H}(L), \sigma(a_1) \in a_1 L^0, \cdots, \sigma(a_m) \in a_m L^0\}\) forms an open and closed sub-semigroup of \(\mathfrak{H}(L)\), and contains \(\mathfrak{H}(L)\). Hence \(\sigma(a_i) \in a_i L^0\) for \(\sigma \in \mathfrak{H}(L)\).

Therefore \(\sigma\) is an automorphism of \(L\).

(iii) Let \(G\) be a compact group. Then we can find a set \(\{U_a\}\) of nuclei, forming a base at \(e\), so that

\[
U_a = L'_a \times N_a,
\]

namely \(U_a = L'_a \cap N_a = e\), \([L'_a, N_a] = e\), where \(N_a\) is a closed normal subgroup of \(G\), \(L_a = G/N_a\) is a Lie group, and \(L'_a\) is a local Lie group isomorphic to a nucleus of \(L_a\). We may assume that \(L'_a\) does not contain any subgroup other than \(e\). Then \(U_a\) and \(N_a\) clearly satisfy the assumptions in Proposition 12. Therefore an element \(\sigma\) in \(G(G)\) induces an endomorphism \(\sigma_a\) of \(L_a\), which is obviously in \(G(L_a) = \mathfrak{H}(L_a)\).

Let \(D\) be the kernel of \(\sigma\). Since \(\sigma_a\) is one-to-one on \(L_a\), \(D\) should be contained in \(N_a\). On the other hand \(\bigcap N_a \cap \bigcup U_a = e\). Hence \(D = e\), that is, \(\sigma\) is one-to-one.

Now, let us show that \(\sigma\) is an onto mapping. Let \(a\) be an element of \(G\), and let \(M_a\) be the set of all elements \(a_a\) so that \(\sigma(a_a) \in a N_a\). Since \(\sigma_a\) is an automorphism, \(M_a\) is not empty. Moreover, for given \(\alpha_1, \alpha_2, \cdots, \alpha_n\), we can find \(\alpha_0\) so that \(\bigcap U_{a_{\alpha_i}} \supset U_{\alpha_0}\). For such \(\alpha_0\) we have \(N_{\alpha_1} \cap \cdots \cap N_{\alpha_n} \supset N_{\alpha_0}\) whence \(M_{\alpha_1} \cap \cdots \cap M_{\alpha_n} \supset M_{\alpha_0}\). Therefore the set \(\{M_a\}\) of compact sets has the finite intersection property. So we can take \(b\) in \(\bigcap M_a\). Then \(\sigma(b) = a N_a\) for all \(\alpha\). Hence \(\sigma(b) = a\).

**Corollary 1.** Let \(G\) be a compact group and \(G^0\) the component group of \(G\). Then

\[
\mathfrak{H}(G) = \mathfrak{H}(G) \cong \mathfrak{H}(G^0) \cong \Pi \mathfrak{H}(S^a),
\]

where \(\mathfrak{H}(G^0)\) is the group of all inner automorphisms of \(G\), induced by elements of \(G^0\) and \(S^a\)'s are compact connected simple Lie groups with no centers (Iwasawa [10] and Goto [7]).

Next, let us consider a locally compact connected group \(G\). By the radical of \(G\) we mean also the uniquely determined maximal solvable closed con-
nected normal subgroup $R$ (Gleason [3], Iwasawa [10] and Matsushima [11]). If $R$ is compact, we can find a closed connected normal semi-simple Lie group $L$ and a compact connected normal subgroup $K$ so that $G = LK$, $[L, K] = e$, and $L \cap K$ is a finite group (Goto [6]). Then the following corollary is a direct consequence of Proposition 10 and Theorem 2.

**Corollary 2.** Let $G$ be a locally compact connected group whose radical is compact. Then $\mathfrak{g}(G) = \mathfrak{A}(G) = \mathfrak{f}(G)$.

Here the last equality follows from the corresponding theorem for connected semi-simple Lie groups [10].

4. Automorphism group. First, let us remember the already known structure theorem of locally compact groups.

**Structure Theorem.** Let $G$ be a locally compact group. Then we can find a base at $e$, composed of nuclei of the form $W = L^* \times K$, where $L^*$ is a local Lie group, and $K$ is a compact subgroup (Montgomery-Zippin [12], Iwasawa [10] and Yamabe [16; 17]).

Let us consider the group $L^*$ which is generated by $L_i^*$ in $G$. Then we can find a uniquely determined connected Lie group $L$ which maps continuously and isomorphically onto $L^*$. The subgroup $L^* K$ is open in $G$.

If in particular $G$ is connected, then $K$ is a normal subgroup and $G = L^* K$, $[L^*, K] = e$. Denote by $D^*$ the intersection of $L^*$ and $K$, and let $D$ be the inverse image of $D^*$ in $L$. Then the fact $L_i^* \cap K = e$ implies that $D$ is a discrete normal, accordingly central, subgroup of $L$.

Let us call a local decomposition of $G$, as in the structure theorem, a *canonical decomposition*, including the notions of $L, L^*, D$ and $D^*$, in the connected case.

**Theorem 2.** Let $G$ be a locally compact group, and let $W_0 = L_0^* \times K$ be a local canonical decomposition of $G$. If $G$ is generated by $W_0$, then $\mathfrak{h}(G)$ is a topological group, (i.e., $\sigma^{-1}$ is a continuous function on $\mathfrak{h}(G)$).

**Proof.** (i) Let $L$ be a connected Lie group and $L_0$ a nucleus of $L$. As we saw, $\mathfrak{h}(L)$ has a linear representation obtained by taking a canonical system of coordinates of the first kind. Let $L_1$ and $L_2$ be open spheres of radius $r_1$ and $r_2$, respectively, with respect to the coordinate system. If $r_1 < r_2$ and if $r_2$ is sufficiently small, then we can find a neighborhood $U$ of the identity 1 of $\mathfrak{h}(L)$ so that $U \supset \tau$ implies $L_1 \subset \tau(L_2) \subset L_0$. Since $\mathfrak{h}(L)$ is open in $\mathfrak{h}(L)$ by Proposition 11 we may take $U$ in $\mathfrak{h}(L)$.

(ii) Let $G$ be a locally compact group satisfying the assumptions in Theorem 2. Since $G$ is generated by $W_0$, $K$ is a normal subgroup of $G$. We use the notation $L$ here for the factor group $G/K$. By (i) we get nuclei $L_i^*$ and $L_0^*$ of $L_0^*$ so that $L_i^* \subset \tau(L_0^*) \subset L^*$ for $\tau \in U$ where $U$ is a neighborhood of 1 in $\mathfrak{h}(L)$. The nucleus $W_1 = L_1^* \times K$ also generates $G$. 

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Let us consider the sub-semigroup $\tilde{\mathcal{E}}^K(G)$ of $\tilde{\mathcal{E}}(G)$ composed of all endomorphisms leaving $K$ invariant. Then we have a continuous homomorphism $\psi$ from $\tilde{\mathcal{E}}^K(G)$ into $\tilde{\mathcal{E}}(L)$. Since $\tilde{\mathcal{E}}^K(G)$ is open in $\tilde{\mathcal{E}}(G)$, by Proposition 12, and any neighborhood of 1 contains a neighborhood of the form $(1, \overline{W_1}, V)$, we may find a nucleus $V$ of $G$ so that $\psi((1, \overline{W_1}, V)) \subseteq \mathcal{U}$.

We use the notation $[1; \mathfrak{K}, V]$ for the intersection of $(1; \mathfrak{K}, V)$ and $\tilde{\mathcal{E}}(G)$. Since $\mathcal{U} \subseteq \tilde{\mathcal{E}}(L)$, any element $\sigma$ of $[1; W_1, V]$ induces an automorphism of $L$, whence $\sigma(K) = K$.

Let $\sigma$ be in $[1; W_2, V]$ where $W_2 = L_2^* \times K$. Then for $x \in \overline{W_2}$, $x^{-1}\sigma(x) \in V$, namely $(\sigma^{-1}(y))^{-1}y \in V$ for $y \in \sigma(\overline{W_2})$. Since $V = V^{-1}$, we have $y^{-1}(\sigma^{-1}(y)) \in V$ for $y \in \sigma(\overline{W_2})$. On the other hand since $\psi(\sigma) \in \mathcal{U}$ we have $L_i^* \subseteq \psi(\sigma)L_i^* \subseteq L_i^*$. Therefore $\sigma(W_2) = \sigma(L_2^*) \times \sigma(K) = \sigma(L_2^*) \times K = \psi(\sigma)L_2^* \times K \supset L_i^* \times K = W_1$. Hence for $y \in \overline{W_1}$, we have $y^{-1}(\sigma^{-1}(y)) \in V$, namely $[1; W_2, V]^{-1} \subseteq [1; \overline{W_1}, V]$ which proves the continuity of $\sigma^{-1}$.

5. Structure of $\tilde{\mathcal{E}}(G)$ of a locally compact connected group $G$. Let $G = L^* K$ be a canonical decomposition of a locally compact connected group $G$. In §5 we will study the structure of $\tilde{\mathcal{E}}(G)$ with respect to the decomposition (Goto [7]).

Let $K^0$ be the component group of $K$. Then $K^0$ is a normal subgroup. Let $k_1$ and $k_2$ be elements of $K^0$. If $k_1^{-1}\cdot k_2 = k_2^{-1}\cdot k_1$ for every $x \in K$, then $k_1k_2^{-1}$ is contained in the center of $K^0$. On the other hand, in a connected topological group any compact abelian normal subgroup is central, because $\mathfrak{A}(Z)$ for a compact abelian group $Z$ consists only of the identity (Iwasawa [10]). Hence $k_1k_2^{-1}$ is contained in the center of $G$. Let $\mathfrak{F}_0(K^0)$ and $\mathfrak{F}_K(K^0)$ be the subgroups of $\mathfrak{A}(G)$ and $\tilde{\mathcal{E}}(K)$ composed of all inner automorphisms induced by elements of $K^0$ respectively. Then by the above argument we have a natural bicontinuous isomorphism between $\mathfrak{F}_0(K^0)$ and $\mathfrak{F}_K(K^0)$.

Let $\sigma$ be an element in $\tilde{\mathcal{E}}(G)$. Then by Proposition 12 we have $\sigma(K) \subseteq K$. Hence $\sigma$ induces an endomorphism of $K$. Since $\tilde{\mathcal{E}}(K)$ is connected, $\sigma \in \tilde{\mathcal{E}}(K) = \mathfrak{F}_K(K^0)$ on $K$ by Theorem 2. So we obtain a natural homomorphism $\phi$ from $\tilde{\mathcal{E}}(G)$ into $\mathfrak{F}_0(K^0)$. $\phi$ is clearly an onto mapping.

Let now $\mathfrak{B}$ be the set of all elements of $\tilde{\mathcal{E}}(G)$ which go to the identity by $\phi$: 

$$\mathfrak{B} = \{\sigma \mid \sigma \in \tilde{\mathcal{E}}(G), \phi(\sigma) = 1\} = \{\sigma \mid \sigma \in \tilde{\mathcal{E}}(G), \sigma(x) = x \text{ for all } x \in K^0\}.$$ 

We shall prove that $\mathfrak{F}_0(K^0)$ and $\mathfrak{B}$ are elementwise commutative. Let $\rho$ be in $\mathfrak{F}_0(K^0)$. Then there is an element $a$ of $K^0$ such that $\rho(x) = a^{-1}\cdot xa$ for $x \in G$. Now let $\sigma$ be in $\mathfrak{B}$. Then for $x$ in $G$ we have $(\sigma \rho)(x) = \sigma(a^{-1}\cdot xa) = \sigma(a)^{-1}\cdot \sigma(x) \cdot \sigma(a) = a^{-1}\cdot \sigma(x) \cdot a = (\rho \sigma)(x)$, namely $\sigma \rho = \rho \sigma$.

Let $\sigma$ be an element in $\tilde{\mathcal{E}}(G)$. Then 

$$\psi(\sigma) = \phi(\sigma)^{-1} \sigma \in \mathfrak{B},$$
because $\phi(\psi(\sigma)) = \phi(\phi(\sigma)^{-1}) = \phi(\sigma)^{-1} = 1$. So $\sigma$ has a decomposition

\[ \sigma = \phi(\sigma)\psi(\sigma) \quad \text{where} \quad \phi(\sigma) \in \mathbb{G}(K^0) \quad \text{and} \quad \psi(\sigma) \in \mathbb{B}. \]

Let $\sigma = \rho \tau (\rho, \in \mathbb{G}(K^0), \tau \in \mathbb{B})$ be another decomposition. Then $\phi(\sigma) = \phi(\rho)\phi(\tau) = \phi(\phi(\rho))\phi(\psi(\sigma)))$, so $\rho = \phi(\sigma)$. Hence $\tau = \rho^{-1}\psi(\sigma)$. Therefore the decomposition (*) is unique.

Since $\phi$ and $\psi$ are clearly both continuous, the mapping $\theta$:

\[ \mathbb{E}(G) \ni \sigma \rightarrow \theta(\sigma) = (\phi(\sigma), \psi(\sigma)) \in \mathbb{G}(K^0) \times \mathbb{B}, \]

is continuous. Moreover $\theta$ is an onto mapping because $\mathbb{E}(G)$ contains both $\mathbb{G}(K^0)$ and $\mathbb{B}$.

Next, let us prove the continuity of $\theta^{-1}$. Let $(\sigma; S, U)$ be a neighborhood of $\sigma$ in $\mathbb{E}(G)$. Take nuclei $U_1$ and $U_2$ of $G$ so that $U_1 \subseteq U$ and $\phi(\sigma)(U_2) \subseteq U_1$. Then it is easy to show that

\[ \theta(\sigma; S, U) \supset (\phi(\sigma); (\psi(\sigma)S)U_2, U_1) \times (\psi(\sigma); S, U_2), \]

which implies the openness of $\theta$.

Thus we get the following theorem.

**Theorem 3.** Let $G = L * K$ be a canonical decomposition of a locally compact connected group $G$, and let $K^0$ be the component group of $K$. Then $\mathbb{E}(G)$ is a topological direct product of $\mathbb{A}$ and $\mathbb{B}$:

\[ \mathbb{E}(G) = \mathbb{A} \times \mathbb{B}, \]

where $\mathbb{A} = \mathbb{G}(K^0) =$ the group of inner automorphisms induced by $K^0$, and $\mathbb{B} =$ semigroup \( \{ \sigma \in \mathbb{E}(G), \sigma(x) = x \quad \text{for all} \quad x \in K^0 \} \).

Now, about the structure of the semigroup $\mathbb{B}$, we can prove the following theorem.

**Theorem 4.** In Theorem 3,

(1) we have a bicontinuous isomorphism $f$ from $\mathbb{E}_D(L)$ into $\mathbb{B}$:

\[ f(\mathbb{E}_D(L)) \subseteq \mathbb{B}. \]

(2) If $L^*$ is mapped into itself by $\mathbb{E}(G)$, then

\[ f(\mathbb{E}_D(L)) = \mathbb{B} \]

and $\mathbb{E}(G)$ is locally compact.

**Proof.** (1) Let $x$ be an element of $L$. We use the notation $x^*$ for the corresponding element of $L^*$. Let $\sigma$ be in $\mathbb{E}_D(L)$. Define the function $f$ by

\[ \begin{cases} f(\sigma)x^* = (\sigma(x))^* & x^* \in L^*, \\ f(\sigma)y = y & y \in K, \end{cases} \]

then $f(\sigma)$ can easily be extended uniquely into a homomorphism of $G = L^* K$.
into itself. Since \( f(\sigma) \) is continuous on a nucleus of \( G \), \( f(\sigma) \) is an endomorphism of \( G \). Thus we get a homomorphism \( f \) from \( \mathfrak{D}(L) \) into \( \mathfrak{D}(G) \). Clearly \( f \) is one-to-one.

Now it is enough for our purposes to show the bicontinuity of \( f \).

Take a neighborhood \((f(\sigma); L^* \times K, U)\) of \( f(\sigma) \). Then we can find a nucleus \( V \) of \( L \) so small that \( k^{-1}V^*k \subset U \) for any \( k \) in \( K \). If \( \rho \in (\sigma; L_l, V) \), then \( \sigma(x)^{-1}\rho(x) \in V \) for \( x \in L_l \). Hence for any \( x^* \in L^* \), \( k \in K \),

\[
((f(\sigma))(x^*k))^{-1}(f(\rho))(x^*k) = (f(\sigma)(k))^{-1}(f(\sigma)(x^*))^{-1}(f(\rho)(x^*))(f(\rho)(k)) = k^{-1}(\sigma(x)^{-1}\rho(x))^*k \in k^{-1}V^*k \subset U,
\]

namely \( f(\sigma; L_l, V) \subset (f(\sigma); L^* \times K, U) \), which implies the continuity of \( f \).

Next, let \((\sigma; \bar{L}, U)\) be a neighborhood of \( \sigma \) in \( \mathfrak{D}(L) \). Then there exists a nucleus \( L^* \times K \) in \( G \) such that \( L_l \subset U \). Take any element \( \rho \) of \( \mathfrak{D}(L) \) such that \( f(\rho) \in (f(\sigma), K^*, L^* \times K) \). Then for \( x \in \bar{L} \), \( (f(\sigma)(x^*))^{-1}(f(\rho)(x^*)) = (\sigma(x)^{-1}\rho(x))^* \in L^* \times K \), whence \( \sigma(x)^{-1}\rho(x) \in U \), namely \( \rho \in (\sigma; \bar{L}, U) \). This shows that \( f^{-1} \) is continuous.

(2) If \( \mathfrak{D}(G)L^* \subset L^* \), then every element of \( \mathfrak{D}(G) \) induces an endomorphism of \( L^* \). On the other hand every element of \( \mathfrak{D}(G) \) induces an inner automorphism of \( K \), and leaves every element of \( D^* \) fixed. So we can define a homomorphism \( f' \) from \( \mathfrak{D}(G) \) into \( \mathfrak{D}(L) \). \( f' \) is clearly one-to-one on \( \mathfrak{B} \), and coincides with \( f^{-1} \). Therefore \( f(\mathfrak{D}(L)) = \mathfrak{B} \).

Since \( L \) is a connected Lie group, \( \mathfrak{D}(L) \) is locally compact by Proposition 8, and so is \( \mathfrak{B} \). Since \( \mathfrak{A} \) is a compact group, \( \mathfrak{G}(G) = \mathfrak{A} \times \mathfrak{B} \) is locally compact.

Remark. \( \mathfrak{G}(G)L^* \subset L^* \) is not valid in general. However we may prove the following

**Theorem 5.** Let \( G = L^*K \) be a canonical decomposition of a locally compact connected group. Then \( L^* \) is mapped into itself by \( \mathfrak{G}(G) : \mathfrak{G}(G)L^* \subset L^* \), if either

(a) the center of \( K \) is totally disconnected, or

(b) \( L \) is perfect (i.e. \( [L, L] = L \)).

**Proof.** Let \( \sigma \) be an element of \( \mathfrak{G}(G) \). We may find a connected nucleus \( L_\rho \) of \( L \) such that

\[
\sigma(L_\rho^*) \subset L^*_l \times K,
\]

where \( L_\rho^* \) denotes the image of \( L_\rho \) in \( L^* \). Then for \( x \in L_\rho^* \) we have a continuous decomposition

\[
\sigma(x) = l(x)k(x) \quad \text{where} \quad l(x) \in L^*_l, \quad k(x) \in K.
\]

Let \( y \) be an element of \( K \). Since \( xy = yx \), we have \( \sigma(x)\sigma(y) = \sigma(y)\sigma(x) \), and therefore

\[
l(x)k(x)\sigma(y) = \sigma(y)l(x)k(x) = l(x)\sigma(y)k(x),
\]

whence \( k(x)\sigma(y) = \sigma(y)k(x) \). Since \( \sigma \) induces an automorphism of \( K \), \( \sigma(y) \) can be an arbitrary element of \( K \). Hence \( k(x) \) is contained in the center of \( K \).
Now we consider the two cases (a) and (b) separately:

(a) Because \( \{ k(x) \mid x \in L^*_\rho \} \) is connected, and is contained in the totally disconnected center of \( K \),
(b) because \( k(x) \) gives a continuous homomorphism from a perfect local Lie group \( L^*_\rho \) into a commutative group,

\( k(x) = e \) in both cases.

Namely \( \sigma(L^*_\rho) \subseteq L^*_I \), whence \( \sigma(L^*_\rho) \subseteq L^*_I \).

**Corollary.** Let \( G \) be a locally compact connected group. If \( G \) is perfect or the center of \( G \) is totally disconnected, then \( \mathcal{E}(G) \) is locally compact.

**Proof.** (a) If \( G \) is perfect, so is the factor group \( G/K \). On the other hand, every connected locally isomorphic group of a connected perfect Lie group is perfect. Since \( L \) is locally isomorphic with \( G/K \), \( L \) is perfect.

(b) The center of \( G \) contains the center of \( K \). So if the former is totally disconnected, then so is the latter.

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