CONNECTED ORDERED TOPOLOGICAL SEMIGROUPS WITH IDEMPOTENT ENDPOINTS. I[1]

BY

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The semigroups described in the title will be called *threads*. More specifically, a thread is a system $S(\circ, <)$ with the following properties:

(1) S is a semigroup with respect to $\circ$, i.e. $\circ$ is an associative binary operation in S;

(2) $<$ is a total (i.e. linear, or simple) order relation in S;

(3) the mapping $p(x, y) = x \circ y$ of $S \times S$ into S is continuous in the order topology;

(4) S is connected in the order topology;

(5) S has a greatest and a least element with respect to $<$, and these elements (which will be called the upper and lower endpoints of S, respectively) are idempotent.

Condition (5) is included only for present purposes, to save endless repetition of the phrase "with idempotent endpoints." Let us refer to a system $S(\circ, <)$ satisfying (1)–(4) as a "thread in the wider sense." In the terminology of Wallace [12], a thread in the wider sense is a connected mob in which the topology is that induced by an order relation.

The systematic study of threads (in the narrower sense) was initiated by Faucett, [5] and [6]. The purpose of the present undertaking is to continue his investigation to the point of obtaining a complete determination of all possible threads. In the present paper this is done for threads having a zero element; threads without zero will be considered in a later paper.

With any thread S we can associate its *order-dual*, obtained from S by reversing the direction of the order relation, and its *product-dual*, obtained from S by writing $y \circ x$ for $x \circ y$. Two threads will be called *isomorphic* if there is a one-to-one correspondence between them *preserving both order and product.*

Aczél [1] has shown effectively that if S is a thread in the wider sense for which the underlying space is a real interval, and which satisfies the *strict monotony condition*,

$$\text{(SMC)} \quad x < y \quad \text{implies } x \circ z < y \circ z \quad \text{and} \quad z \circ x < z \circ y, \quad \text{for all } z \text{ in } S,$$

then S is isomorphic with a subthread of the additive thread of all real numbers. Tamari [13] showed that SMC followed from cancellation and the other

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assumptions. Moreover it is easy to see that their arguments are valid for S any connected ordered set, not necessarily a real interval, and hence *any thread S in the wider sense, obeying the cancellation law, is isomorphic with a subthread of the additive thread of all real numbers.*

We shall call a thread S *standard* if its lower endpoint is the zero and its upper endpoint is the identity element of S. Two basic examples are (1) the *real unit interval* $[0, 1]$ with the usual product and order, and (2) the *real nil interval* $[1/2, 1]$ with the usual order, and with product defined by

$$x \circ y = \max(1/2, xy).$$

The latter was found by Calabi (Faucett [5, p. 747]). Let us define a *unit thread* to be a standard thread having no interior idempotent element and no nonzero nilpotent element. Faucett's Theorem 2 in [5] states that every unit thread is isomorphic with the real unit interval. We shall show (Theorem 1) that this can be derived from a fundamental theorem due to O. Hölder [7].

Let us define a *nil thread* to be a standard thread having no interior idempotent but having at least one nonzero nilpotent element. We shall show (Theorem 2) that every nil thread is isomorphic with the real nil interval.

In Theorem 3 we show how to construct any standard thread: take any compact totally ordered set and fill the gaps with *ligaments* which are either unit threads or nil threads. Theorems 2 and 3 were found by Mostert and Shields [8] for the case in which the underlying space of the given standard thread is the real interval $[0, 1]$ ("I-semigroup"). They were also found independently by Gleason, but not published; see the concluding remark in [3].

Let S be any thread with zero; let e and f be its idempotent endpoints, and suppose $e>f$. If $f=0$ then it is immediate (by the corollary to Lemma 1 below) that e is the identity element of S, so that S is a standard thread. If $e=0$, then the order-dual of S is standard. If $f<0<e$, Faucett's Lemma 4 in [5] shows that the subintervals $[f, 0]$ and $[0, e]$ are subthreads of S. The determination of all threads with zero, which occupies the remainder of this paper, thus reduces to the problem of how to put two standard threads back to back.

The main tool used to solve this problem is the notion of linear extension. If J is an ideal of a thread S, and is also a closed interval in S, then we can order the Rees [10] factor semigroup $T=S/J$ in the obvious way, and T becomes thereby a thread with zero. We call S a *linear extension of J by T.*

Cohen and Wade [4] have investigated threads in the wider sense, having endpoints one of which is the identity element. Their work overlaps the present investigation precisely in the case where the other endpoint is idempotent. This is just the case treated in Theorem 4 below, which was found independently by Cohen and Wade. If S is a thread $[f, u]$ with identity ele-

ment u and with interior zero 0, then $[f, 0]$ is an ideal of S so that S is a linear extension of the order-dual of a standard thread by another standard thread $[0, u]$. Theorem 4 shows that these are determined by homomorphisms of $[0, u]$ onto $[f, 0]$.

In [9], Mostert and Shields determine the structure of all threads (in the wider sense) on the real interval $[0, \infty)$ with 0 and 1 playing their usual algebraic roles.

The threads discussed above are all commutative. To construct a noncommutative one, let S_+ and S_- be two copies of a standard thread S. Let $+x$ and $-x$ denote the elements of S_+ and S_- corresponding to the element x of S. Let T be the union of S_+ and S_-, except that we identify $+0$ and -0. Define order in T by retaining that in S_+, reversing that in S_-, and declaring every element of S_- to be less than every nonzero element of S_+. Define product in T as follows:

$$(+x)(+y) = +(xy), \qquad (+x)(-y) = -(xy),$$
$$(-x)(+y) = +(xy), \qquad (-x)(-y) = -(xy).$$

The resulting thread T we call the *january thread associated with* S, the name suggested by pictures of the god Janus, with two identical faces back to back. Faucett's Example 3 on p. 747 of [5] is the january thread associated with the real unit interval. Actually we have defined what we should call the *right-handed january thread* associated with S; the left-handed one differs only in that the "sign" of a product is that of its left instead of its right factor.

There is an almost trivial kind of linear extension which we call a *contact extension*. The remaining results of the paper can be summed up in the assertion that every thread with zero is a contact extension of a thread of one of the types discussed above by one or two standard threads.

NOTATION AND TERMINOLOGY. If A and B are sets, $A \backslash B$ will mean the set of elements of A not belonging to B. All order relations considered in this paper are total (=linear=simple). *Intervals* in an ordered set T are defined and denoted as usual, e.g.

$$(a, b] = \{x \mid x \in T, a < x \leqq b\}.$$

By *neighborhood* of an element a of T we shall mean an open interval containing a. If e and f are the endpoints of T, and $f<e$, then $[f, b)$ and $(c, e]$ are considered to be open intervals. Neighborhoods so defined form a basis for the open sets in the *order topology* of T. T is *complete* if every subset of T bounded from above has a least upper bound (LUB); the order-dual is logically equivalent. T is *dense* if $a<b$ implies (a, b) is not empty. T is (topologically) connected if and only if it is both dense and complete.

1. Preliminary results due to Faucett. We begin with seven lemmas contained either implicitly or explicitly in Faucett's paper [5].

LEMMA 1. *Let a, b, c be elements of a thread S. Then*

$$[ac, bc] \subseteq [a, b]c,$$
$$[ca, ca] \subseteq c[a, b].$$

The same holds for open and half-open intervals.

Proof. The continuous mapping $x \to xc$ carries the connected set $[a, b]$ into $[a, b]c$, which is therefore also connected. Since $[a, b]c$ contains ac and bc, it contains every element of S between them.

COROLLARY. *If $S = [0, e]$ is a thread with zero 0 at one end, the other endpoint e is the identity element of S, and so S is a standard thread.*

Proof. $S = [0, e] = [0e, ee] \subseteq [0, e]e = Se$, and similarly $S = eS$.

LEMMA 2. *Let S be a thread with zero 0, and let a and b be elements of S. If $a < 0$, then $a \leqq ab$ and $a \leqq ba$. If $a > 0$, then $a \geqq ab$ and $a \geqq ba$.*

Proof. This is a consequence of Faucett's Lemma 1 in [5, p. 742]. Suppose $a < 0$. Let $A(B)$ be the set of all elements x of S such that $x > a$ $(x < a)$. The kernel K of S consists of the single element 0 of S, and so $K \subseteq A$. $S \setminus \{a\} = A \cup B$ and $A \mid B$ (i.e. neither intersects the closure of the other). We conclude from Faucett's Lemma 1 that Sa and aS are both contained in the closure of A, and so all elements thereof are $\geqq a$. The proof of the case $a > 0$ is similar.

COROLLARY. *If a thread S with zero has an identity element u, then u must be an endpoint of S.*

Proof. If $u = 0$, $S = \{u\}$, and the result is trivially true. If $u > 0$ and $a \in S$, then $u \geqq ua = a$, by Lemma 2, so that u is the upper endpoint of S. Similarly, if $u < 0$ then u is the lower endpoint of S.

LEMMA 3. *Let a be an element of a thread S with zero 0. If $a > 0$, then* (1) $a \geqq a^2 \geqq a^3 \geqq \cdots$, (2) $z = \lim_{n \to \infty} a^n$ *exists*, (3) $az = za = z$, *and* (4) $z^2 = z$.

Proof. (1) is immediate from Lemma 2. In a compact, connected, ordered set every monotone sequence has a limit, whence (2) holds. $az = a \lim a^n = \lim a^{n+1} = z$, and similarly $za = z$, proving (3). Hence $a^n z = za^n = z$ for every n, and (4) follows: $z^2 = z \lim a^n = \lim za^n = \lim z = z$.

We shall say that a semigroup S is *naturally totally ordered* (n.t.o.) by a (total) order relation $<$ in S, if:

(1) $ab \leqq a$ and $ab \leqq b$ for every a, b in S;

(2) if $a < b$ then $a = bx = yb$ for some x, y in S. These imply the *monotony condition*:

(MC) if $a < b$ then $ac \leqq bc$ and $ca \leqq cb$.

We shall say that S is *strictly* n.t.o. if it satisfies (1′) below and (2):

(1′) $ab<a$ and $ab<b$ for every a, b in S. Then the strict monotony condition holds (stated in the introduction), whence we see that an n.t.o. semigroup is strictly n.t.o. if and only if the cancellation laws holds ($ac=bc$ or $ca=cb$ implies $a=b$).

We remark that product in an n.t.o. semigroup need not be continuous in the order topology. This is the case for the semigroup P_1^* defined on p. 632 of [3].

We combine Faucett's Lemmas 2 and 5 of [5] in the following.

LEMMA 4. *A standard thread is naturally totally ordered and commutative.*

COROLLARY. *Let $S=[0, e]$ be a standard thread. Let g be any idempotent element of S, and let a be any element of S. Then*

$$ag = ga = \begin{cases} a & \text{if } a \leqq g, \\ g & \text{if } a \geqq g. \end{cases}$$

Proof. If $a<g$ then $a=gx=xg$ for some x in S, by Lemma 4 and (2) above, whence $ga=ag=a$. If $a>g$ then $ag\geqq gg=g$, by (MC). But $ag\leqq g$ by (1) above, or by Lemma 2.

The following is Faucett's Lemma 4 in [5].

LEMMA 5. *Let S be a thread with zero, and let e and f be idempotent elements of S with $f<e$. If f acts as a zero on $[f, e]$, then $[f, e]$ is a standard subthread of S.*

The following is extracted from the proof of Faucett's Theorem 3 in [5].

LEMMA 6. *Let $T=[f, u]$ be a thread with interior zero 0 and identity element u (which must be an endpoint by the corollary to Lemma 2). Let $S=[0, u]$ and $S'=[f, 0]$. Then $fS=Sf=S'$.*

Faucett's Theorem 4 in [5] asserts that a thread is commutative if and only if it has a zero and its endpoints commute. The following is extracted from the proof of this theorem.

LEMMA 7. *Let $S=[f, e]$ be a thread with interior zero, $f<0<e$, and let $fe=ef\leqq 0$. Then $[fe, e]=eS=Se$.*

2. **Hölder's theorem; unit threads.** In his fundamental paper [7] of 1901, Hölder laid down the following axioms, and showed that they characterize the system of positive real numbers under the usual operation of addition and the usual order relation.

I. S is (totally) ordered by a relation $<$.
II. S has no least element.
III. $+$ is a binary operation in S.
IV. $a<a+b$ and $b<a+b$ for every a, b in S.
V. If $a<b$, $a+x=y+a=b$ for some x, y in S.

VI. $+$ is associative.

VII. S is complete.

(Incidentally, Hölder also shows that if VII be replaced by the Archimedean Axiom, then S can be embedded in the additive ordered semigroup of positive reals. An immediate consequence of this is the theorem that every archimedean ordered group is commutative.)

Except for the fact that IV and V are the order-duals of conditions (1) and (2) in the above definition of n.t.o. semigroups, these axioms assert precisely that S is a connected, strictly n.t.o. semigroup without least (or greatest) element. (Commutativity and continuity of product are consequences of these axioms.) Now the additive n.t.o. semigroup of reals is isomorphic with the order-dual of the multiplicative n.t.o. semigroup of the open interval (0, 1). We may thus express Hölder's theorem as follows: there exists to within isomorphism only one connected, strictly n.t.o. semigroup without least or greatest element, and this is the open interval (0, 1) under the usual multiplication and order relation. Finally, adjoining 0 and 1, we arrive at the following expression of his result.

HÖLDER'S THEOREM (CLOSED MULTIPLICATIVE VERSION). *The real unit interval* $[0, 1]$ *under the usual multiplication and order relation is, to within isomorphism, the only connected, naturally totally ordered semigroup* S *with the following additional properties*:

(1) S *has a zero element* 0 *and an identity element* 1 *and these are respectively the least and greatest elements of* S;

(2) *if* $ab=a$ *or* $ba=a$ *then* $a=0$ *or* $b=1$.

The following is Faucett's Theorem 2 of [5]. We shall deduce it from Hölder's Theorem.

THEOREM 1. *Every standard thread without interior idempotent or nilpotent elements is isomorphic with the real unit interval.*

Proof. Let $S=[0, u]$ be a standard thread without interior idempotent or nilpotent elements. S is n.t.o. by Lemma 4 (in fact we do not need to use the commutativity, but rather deduce it from Hölder's Theorem). S is connected by definition, and condition (1) of Hölder's Theorem is evident (with u in place of 1). To show that (2) also holds, let $ab=a$ and assume $b\neq u$. Then, by Lemma 3, the sequence $\{b^n\}$ converges to an idempotent element $<u$, hence $=0$, since S contains no interior idempotent. But $ab^n=a$ for every n, whence $a=\lim ab^n=a \lim b^n=a0=0$. Similarly, $ba=a$ and $b\neq u$ imply $a=0$.

3. **Nil threads.** Let us recall the notion of *segment*, introduced in [3]. As remarked on p. 632 of [3], a segment may be characterized as an n.t.o. commutative semigroup with zero, in which every element is nilpotent. If P is the additive semigroup of all positive real numbers, then the Rees factor

semigroups $P_1 = P/[1, \infty)$ and $P_1^* = P/(1, \infty)$ are complete, nondiscrete segments. The corollary to Theorem 4 of [3], on p. 643, asserts that *every complete nondiscrete segment S is isomorphic either to* P_1 *or to* P_1^*. (Amendment: if the zero element of S is at the lower end of S, then the order-dual of S is isomorphic to P_1 or P_1^*.)

Let $P_1' = P_1 \cup \{0\}$. P_1' is in an evident one-to-one correspondence with the real interval $[0, 1]$, and when so ordered is evidently a thread (with 0 the identity element and 1 the zero element!). The mapping $x \to (1/2)^x$, x in P_1', is an order-reversing isomorphism of P_1' onto the real nil interval $[1/2, 1]$ mentioned in the introduction (Calabi example).

THEOREM 2. *Any nil thread, i.e. any standard thread having no interior idempotent elements, but having at least one interior nilpotent element, is isomorphic with the real nil interval* $[1/2, 1]$.

Proof. Let $S = [0, u]$ be a nil thread. We first show that $S' = S \setminus \{u\}$ is a segment. That S, and hence S', is n.t.o. and commutative follows from Lemma 4. We must show that any element a of S' is nilpotent. Let b be any interior nilpotent element of S. By Lemma 3, $a^n \to z$ with z idempotent and $z < u$, whence $z = 0$ by hypothesis on S. But then $a^n < b$ for some n. If $b^r = 0$ then, since (MC) holds in the n.t.o. semigroup S, $a^{nr} \leqq b^r = 0$. Thus $a^{nr} = 0$, and a is nilpotent. Hence S' is a segment.

Moreover S' is complete and nondiscrete, since it is connected. Applying the amendment, stated above, to the corollary to Theorem 4 in [3], we conclude that the order-dual of S' is isomorphic with P_1 or P_1^*. The latter is impossible, since P_1^* is not connected. After adjoining the identity element u to S' to obtain S, and the identity element 0 to P_1 to obtain P_1', we see that S is isomorphic with the order-dual of P_1'. But we observed above that P_1' is isomorphic with the order-dual of the real nil interval.

4. Standard threads. For convenience in what follows, we shall use the term *ligament* to mean either a unit thread or a nil thread; it will be called *ordinary* or *nil* if it is a unit thread or a nil thread, respectively.

Let E be a compact totally ordered set. Two distinct elements of E will be called *adjacent* if there is no element of E between them. The lesser of two adjacent elements will be called an *initial element*. An element which is not an initial element, i.e. which has no immediate successor, will be called a *limit element*. With each initial element e of E, let there be associated a ligament S_e, ordinary or nil. We identify e with the zero element of S_e, and denote by S_e' the semigroup obtained by deleting from S_e its identity element. If e is a limit element, we define both S_e and S_e' to be $\{e\}$.

Let $S = \bigcup_{e \in E} S_e'$. Define the product ab of two elements a, b of S as follows. If a and b belong to the same S_e', ab shall be the product of a and b as already defined in S_e'. If $a \in S_e'$, $b \in S_f'$, and $e < f$, then let $ab = ba = a$. Except that we have reversed the direction of the order relation, S is the ordinal

sum of the S_e', as defined on p. 631 of [3]. Hence S is a commutative n.t.o. semigroup.

Being compact, E must have a least element e_0 and a greatest element e_1, and it is evident that e_0 is the zero and e_1 the identity element of S. If e and f are adjacent elements of E, with $e<f$, then f acts as identity element on S_e', so that $S_e' \cup \{f\} \cong S_e$. We may thus identify S_e with the closed interval $[e, f]$ of S. Evidently the set of idempotent elements of S is precisely E.

We summarize the above construction by saying that *S is the ordinal sum of a compact totally ordered set of half-open ligaments and one-element semigroups.*

THEOREM 3. *Every standard thread S is the ordinal sum of a compact totally ordered set of half-open ligaments and one-element semigroups; this set is order-isomorphic with the set E of idempotents of S. Conversely, every such ordinal sum is a standard thread.*

Proof. Let S be a standard thread. By Lemma 4, S is n.t.o. and commutative. But Theorem 1 of [3] asserts that every n.t.o. commutative semigroup is the ordinal sum of a totally ordered set $\{S_\alpha\}_{\alpha \in I}$ of ordinally irreducible n.t.o commutative semigroups S_α.

By Theorem 2 of [3], each S_α contains at most one idempotent, which—if its exists—is the zero element of S_α, and will be denoted by f_α. If S_α contains no idempotent, and if $a \in S_\alpha$, then, by Lemma 3, $a \geqq a^2 \geqq a^3 \geqq \cdots$ and $z = \lim a^n$ is an idempotent element which is a zero for a. z must evidently be the GLB of S_α, hence independent of a, so that z acts as a zero for S_α; again we shall denote it by f_α. In either case, $S_\alpha \cup \{f_\alpha\}$ is an n.t.o. commutative semigroup with zero f_α and no other idempotent.

Let e_α be the LUB of S_α. Then e_α is idempotent. If not, since the set E of idempotent elements of S is closed, there exists $c > e_\alpha$ such that $[e_\alpha, c] \cap E$ is empty. $c \in S_\beta$ for some $\beta > \alpha$ in I. $c^n \in S_\beta$ for every positive integer n, so that $c \geqq c^2 \geqq c^3 \geqq \cdots \geqq e_\alpha$. By Lemma 3, $\lim c^n$ is an idempotent in $[e_\alpha, c]$, a contradiction.

By the corollary to Lemma 4, e_α acts as an identity element for S_α. Hence $[f_\alpha, e_\alpha] = S_\alpha \cup \{f_\alpha\} \cup \{e_\alpha\}$ is a standard thread with no interior idempotents. If $e_\alpha = f_\alpha$, then $S_\alpha = \{f_\alpha\}$. Otherwise, by Theorems 1 and 2, f_α and e_α are adjacent idempotents, and $\{f_\alpha, e_\alpha]$ is a ligament, ordinary or nil.

Turning to the converse, let $S = \bigcup_{e \in E} S_e'$ be an ordinal sum of a set of semigroups S_e', indexed by a compact, totally ordered set E, where S_e' is either a ligament with identity element removed, or else a one-element semigroup, depending on whether e is an initial or a limit element of E. We have already seen that S is an n.t.o. commutative semigroup with endpoints e_0 and e_1, serving as zero and identity elements for S, respectively. In order to show that S is a standard thread, we must show that product is continuous and that S is connected in the order topology.

Let A be a nonempty subset of S. Let e be the LUB of $E \cap A$. Then no element of A is greater than every element of S'_e. If $A \cap S'_e$ is empty, then e is the LUB of A. Otherwise, the LUB of $A \cap S'_e$ in the connected set S'_e is also the LUB of A. Thus S is complete. Since S is evidently dense, it is connected.

To show that product is continuous, let $ab = c$ and let V be a neighborhood of c. If a and b are interior points of the same ligament S'_e, continuity follows from that in S'_e. Suppose that a and b are interior points of the ligaments S'_e and S'_f, respectively, with $e < f$. Then $c = a$. $V_a = (S'_e \setminus \{e\}) \cap V$ and $V_b = S'_f \setminus \{f\}$ are neighborhoods of a and b such that $V_a V_b = V_a \subseteq V$, since every element of V_b acts as an identity on V_a. The various other cases, in which $a \in E$ or $b \in E$ or both, are easy but tedious, and we omit the details.

5. **Homomorphisms of standard threads.** The construction of nonstandard threads involves that of linear extensions of one standard thread by another, and this in turn involves the construction of homomorphisms of one standard thread into another. We consider the latter in this section.

LEMMA 8. *Let S and S' be naturally totally ordered semigroups, and let ϕ be an (algebraic) homomorphism of S into S'. Then ϕ is order-preserving (i.e. monotone nondecreasing). If S' is dense and $\phi(S) = S'$, then ϕ is continuous. In particular, if S and S' are standard threads, any homomorphism of S onto S' is continuous and order-preserving.*

Proof. If $a \leqq b$ then $ax = b$ for some x in S, since S is n.t.o. by hypothesis. Then $\phi(a)\phi(x) = \phi(ax) = \phi(b)$, so that $\phi(a) \leqq \phi(b)$, since S' is also n.t.o.

Now assume that S' is dense and $\phi(S) = S'$. Let $a' = \phi(a)$, and let (b', c') be any neighborhood of a'. Since S' is dense, there exist b'_0 and c'_0 in S' such that $b' < b'_0 < a' < c'_0 < c'$. Since $\phi(S) = S'$, there exist b and c in S such that $\phi(b) = b'_0$ and $\phi(c) = c'_0$. Evidently $b < a < c$, since ϕ has been shown to be order-preserving. If $x \in (b, c)$, then $b'_0 = \phi(b) \leqq \phi(x) \leqq \phi(c) = c'_0$, so that $b' < \phi(x) < c'$, as desired.

The last statement of the lemma is immediate from Lemma 4.

We recall that a congruence relation in a semigroup S is an equivalence relation ρ such that $a\rho b$ implies $ac\rho bc$ and $ca\rho cb$ for all c in S. The congruence (or residue) classes mod ρ can be multiplied in the usual way, and constitute a semigroup S/ρ. S is homomorphic to S/ρ under the canonical mapping, which sends each element of S into the congruence class to which it belongs. Conversely, if ϕ is a homomorphism of S onto a semigroup S', and we define ρ by $a\rho b$ if $\phi(a) = \phi(b)$, then S' is isomorphic with S/ρ.

We shall say that ρ is a *closed convex congruence relation* in a standard thread S if the congruence classes mod ρ are closed subintervals of S. We proceed to show that these correspond to those homomorphic images of S which are also standard threads.

Let ρ be a closed convex congruence relation in the thread S. Order S/ρ

in the obvious way. All the axioms for a thread, in fact a standard thread, evidently hold for S/ρ, with the possible exception of continuity of product. To prove this, let $AB=C$ (A, B, C in the interior of S/ρ) and let (P, Q) be a neighborhood of C in S/ρ. Let p be the upper endpoint of P, and q the lower endpoint of Q. Since A and B are compact subsets of S such that AB is contained in the open set (p, q), it follows (see, for example, A. D. Wallace [11, Lemma 2]) that there exist open sets V_A, V_B containing A and B, respectively, such that $V_A V_B \subseteq (p, q)$. Then there exist elements r_1, r_2 in $V_A \backslash A$ and r_3, r_4 in $V_B \backslash B$ such that $A \subseteq (r_1, r_2)$, $B \subseteq (r_3, r_4)$, and $(r_1, r_2)(r_3, r_4) \subseteq (p, q)$. Let R_i be the congruence class to which r_i belongs. It is then evident that $A \in (R_1, R_2)$, $B \in (R_3, R_4)$, and $(R_1, R_2)(R_3, R_4) \subseteq (P, Q)$, as desired. The modifications necessary in the above proof when A, B, or C is an endpoint of S/ρ are straightforward.

Conversely, let ϕ be a homomorphism of a standard thread S onto another standard thread S', and let ρ be the corresponding congruence relation in S. By Lemma 8, ϕ is continuous and order-preserving. Hence, for each a' in S', the congruence class $\phi^{-1}(a')$ is closed. If $a \leqq x \leqq b$ with $\phi(a)=\phi(b)$, then $\phi(a) \leqq \phi(x) \leqq \phi(b) = \phi(a)$, so that $\phi(x)=\phi(a)$. This shows that the congruence class to which a and b belong is convex. But closed convex subsets of S are precisely closed intervals.

We include the foregoing as the first part of the following lemma.

LEMMA 9. *Let S be a standard thread.*

(1) *If ρ is a closed convex congruence relation in S, then S/ρ, ordered in the obvious way, is a standard thread. Conversely, every homomorphic image S' of S which is also a standard thread is isomorphic to such an S/ρ.*

(2) *A partition of S into a set $\mathcal{I}$ of mutually disjoint closed intervals is that of a closed convex congruence relation if and only if the lower endpoint of each nondegenerate interval in $\mathcal{I}$ is idempotent.*

Proof. (1) having been proved, we proceed to (2). First let ρ be a closed convex congruence relation in S. Let I be a nondegenerate congruence class mod ρ. Let $I=[a, b]$ with $a<b$, and suppose, by way of contradiction, that a is not idempotent. Let S_e be the ligament to which a belongs (representing S as the ordinal sum of ligaments and one-element semigroups, as established by Theorem 3). There exists c in S_e such that $a<c \leqq b$, and there exists d in S_e such that $a=cd$. Since $c \in I$, $a\rho c$. Hence $ad\rho cd=a$. But $ad \leqq a$, while from $ad\rho a$ we have $ad \in I$; hence $ad=a$. By the usual argument, $ad^n=a$, $d^n \to e$, and $e=ae=a$, a contradiction.

Conversely, let there be given a partition of S into a set $\mathcal{I}$ of mutually disjoint closed intervals with the stated property. Let $x\rho y$ mean that x and y belong to the same interval in $\mathcal{I}$. Let x, y, $z \in S$, and let $x\rho y$. If $x=y$, then $xz\rho yz$ trivially. Assume $x \neq y$, and let I be the interval in $\mathcal{I}$ to which x and y belong. Since I is nondegenerate, $I=[e, a]$ with e idempotent. If $z \geqq e$ then

xz and yz both belong to I; for I is an ideal in the subthread $[e, u]$ of S (u the upper endpoint of S), by Lemmas 2 and 4. Hence $xz\rho yz$. If $z<e$, then $xz=z=yz$, so that $xz\rho yz$ trivially. Hence ρ is a congruence relation in S, closed and convex since $\mathcal{I}$ consists of closed intervals.

Part (2) of Lemma 9 enables us to describe all possible closed convex congruence relations ρ, and hence all homomorphisms of S onto standard threads. For to get all partitions of the type described in the lemma, we need only pick an arbitrary set $\mathcal{E}$ of initial idempotents e_i, then associate with each e_i a closed nondegenerate interval $[e_i, a_i]$ not containing any $e_j \neq e_i$ in $\mathcal{E}$, and let $\mathcal{I}$ consist of all these and all the degenerate intervals consisting of elements of S not in any $[e_i, a_i]$.

On the other hand, the foregoing does not tell us how to find all possible homomorphisms of a standard thread S onto a given standard thread S'. Since this is of some interest, although not needed for our present purposes, we shall state the following results without proof.

First let S and S' be ligaments, ordinary or nil. We shall represent ligaments by the real intervals $[0, 1]$ or $[1/2, 1]$, as the case may be, in accordance with Theorems 1 and 2. If S is nil and S' is ordinary, there exists no homomorphism of S onto S'. In the other three cases, all possible homomorphisms ϕ are described as follows.

CASE 1. $S=S'=[0, 1]$. $\phi(x)=x^\alpha$ (x in S), where α is a real, positive number (Cauchy's Theorem).

CASE 2. $S=[0, 1]$, $S'=[1/2, 1]$. $\phi(x)=\max(1/2, x^\alpha)$, ($x$ in S), where α is a real, positive number.

CASE 3. $S=S'=[1/2, 1]$. $\phi(x)=\max(1/2, x^\alpha)$, ($x$ in S), where α is a real number $\geqq 1$.

Now let S and S' be arbitrary standard threads. Let E and E' be their sets of idempotent elements, respectively. Any homomorphism ϕ of S onto S' maps E onto E', and preserves order (by Lemma 8). If e, f ($e<f$) is an adjacent pair of idempotents in E, then either $\phi(e)=\phi(f)$, and ϕ maps $[e, f]$ onto $\phi(e)$, or else $\phi(e)$, $\phi(f)$ is an adjacent pair of idempotents in E', and ϕ induces a homomorphism of the ligament $[e, f]$ onto the ligament $[\phi(e), \phi(f)]$. The nature of the latter has already been described.

6. **Linear extensions.** Let $S=[0, u]$ be a standard thread, and let $S'=[f', e']$ be any thread. Let ϕ be a continuous homomorphism of S into S' such that $\phi(0)=e'$. According to Theorem 2 of [2, p. 167], ϕ determines an extension T of S' by S as follows. Let $S^0=S\setminus\{0\}$, and let $T=S^0\cup S'$. Define product $\circ$ in T as follows ($x, y\in S^0$; $x', y'\in S'$):

$$x\circ y = \begin{cases} xy & \text{if } xy \neq 0, \\ e' & \text{if } xy = 0; \end{cases} \tag{M1}$$

$$x\circ y' = \phi(x)y'; \tag{M2}$$

(M3) $$x' \circ y = x'\phi(y);$$

(M4) $$x' \circ y' = x'y'.$$

In the theorem cited, the second alternative in M1 reads: $x \circ y = \phi(x)\phi(y)$ if $xy = 0$. But in the present case we have $\phi(x)\phi(y) = \phi(xy)$ whether or not $xy = 0$, and if $xy = 0$ then $\phi(x)\phi(y) = \phi(0) = e'$.

We note that $S^0 \cup \{e'\}$ is a subsemigroup of T isomorphic with S, so that we have effectively amalgamated e' and 0. For by M2 we have

$$x \circ e' = \phi(x)e' = \phi(x)\phi(0) = \phi(x0) = \phi(0) = e';$$

similarly, we have $e' \circ y = e'$ by M3; by M4 we have $e' \circ e' = e'$; and by M1 we have $x \circ y = e'$ if and only if $xy = 0$. $S^0 \cup \phi(S)$ is also a subsemigroup of T; for example,

$$x \circ \phi(y) = \phi(x)\phi(y) = \phi(xy) \in \phi(S).$$

e' is a zero and u an identity element for $S^0 \cup \phi(S)$, but perhaps not for all of T.

We give T the order which retains that already present in S^0 and in S', and in which we declare every element of S' to be less than every element of S^0. T is evidently connected, and has the idempotent endpoints f' and u. In order to show that T is a thread, all that remains is to show that $\circ$ is a continuous mapping of $T \times T$ into T.

Let $a \circ b = c$, and let V be a neighborhood of c. If a and b both belong to S^0 or both to $S' \setminus \{e'\}$, then continuity follows from that in S or S', respectively. If $a \in S' \setminus \{e'\}$ and $b \in S^0$, we have $c = a \circ b = a\phi(b)$. There exist neighborhoods V_a of a and V' of $\phi(b)$ in S' such that $V_a V' \subseteq V$. V_a may be chosen to be open in T as well as S, while if $\phi(b) = e'$, V' may have to contain e' as upper endpoint. By the continuity of ϕ, there exists a neighborhood V_b of b in S, which may also be chosen to be open in T, such that $\phi(V_b) \subseteq V'$. Then

$$V_a \circ V_b = V_a\phi(V_b) \subseteq V_a V' \subseteq V,$$

as desired. There are three other cases to be considered, according to the possibilities $a = e'$ or $b = e'$ or both. These occasion no real difficulty, and we omit the tedious details.

Now $\phi(S)$ is compact and connected, hence a closed interval in T. ϕ maps $[e', u]$ in order-reversing fashion onto $\phi(S)$, and hence $\phi(S) = [\phi(u), e']$. Thus $S^0 \cup \phi(S)$ is the interval $[\phi(u), u]$. We have already observed that it is a subsemigroup of T with zero e' and identity u; we now see that it is a subthread.

We formulate the foregoing results in the following lemma.

LEMMA 10. *Let $S = [0, u]$ be a standard thread, and let $S' = [f', e']$ be any thread. Let ϕ be a continuous homomorphism of S into S' such that $\phi(0) = e'$. Let $S^0 = S \setminus \{0\}$ and let $T = S^0 \cup S'$. Define a product $\circ$ in T by* M1–4. *Order*

T so as to retain the given orderings in S^0 and S', and so that every element of S' precedes every element of S^0. Then T is a thread $[f', u]$ containing S' as an ideal and as a closed interval $[f', e']$ at the lower end of T, and the Rees quotient T/S', ordered in the obvious way, is a thread isomorphic with S. In fact $S' \cup \{e'\}$ is a subthread of T isomorphic with S. $S^0 \cup \phi(S)$ is a subthread $[\phi(u), u]$ of T having e' as zero and u as identity element.

We call T a *right linear extension of S' by S*. In particular, the constant mapping $\phi(x) = e'$ for all x in S is a continuous homomorphism of S into S' such that $\phi(0) = e'$. This particular right linear extension we call *the right contact extension of S' by S*. The reason for the term "contact" is more algebraic than topological; thus, to multiply an element of S' by an element x of S, in either order, we simply replace x by the point of contact e'.

The left contact extension of S' by S is obtained by taking $\phi(x) = f'$, reversing order in S^0, and declaring $x < y'$ for every x in S^0, y' in S'.

7. **Threads with identity element and interior zero.**

THEOREM 4. *Let $S = [0, u]$ be a nontrivial standard thread, and let $S' = [u', 0']$ be the order-dual of a standard thread $(u' < 0')$. Let ϕ be a homomorphism of S onto S' such that $\phi(0) = 0'$. Let T be the right linear extension of S' by S determined by ϕ as in Lemma* 10. *Then T is a thread with identity element u and interior zero element* $0'$.

Conversely, every thread T with identity element u and interior zero 0 *has this structure or its order-dual. More precisely, by passing to the order-dual of T if necessary, $T = [f, u]$ with $f < 0 < u$, and T is the right linear extension of $S' = [f, 0]$ by $S = [0, u]$ determined by $\phi(x) = fx = xf$.*

Proof. By Lemma 8, ϕ is continuous and order-reversing, since S' is the order-dual of a standard thread. The first part of the theorem now follows from Lemma 10, which asserts that T is a thread and that u is the identity element and $0'$ the zero element of the subthread $S^0 \cup \phi(S)$. The latter is T, since $\phi(S) = S'$ by hypothesis.

Passing to the converse, let T be a thread with identity element u and interior zero element 0. By the corollary to Lemma 2, u is one of the endpoints of T. By passing to the order-dual of T if necessary, we can assume that u is the upper endpoint of T. Let f be the lower endpoint. Let $S = [0, u]$ and $S' = [f, 0]$. By Lemma 5, S and S' are (standard) subthreads of T. By the corollary to Lemma 4, f is the identity element of S'. Thus $fS' = S'f = S'$. By Lemma 6, $fS = Sf = S'$. Since $T = S \cup S'$, we conclude that $fT = Tf = S'$, so that S' is an ideal of T (with idempotent generator f). The Rees factor thread T/S' is evidently isomorphic with S, and S' is a closed subinterval of T at the lower end. Thus T is a right linear extension of S' by S.

S' contains an identity element, namely f. Hence, by Theorem 2 of [2], every extension of S' by S is determined by a "ramified homomorphism" ϕ of S into S'. By this we mean a mapping ϕ of $S^0 = (0, e]$ into S' such that

$\phi(xy)=\phi(x)\phi(y)$ if $xy\neq 0$. In the proof of that theorem it is shown that $\phi(x)=fx=xf$ (x in S^0). In the present case, the zero element of S is also that of S', and we may include 0 in the domain of definition of ϕ, with $\phi(0)=0$. This ϕ is an ordinary homomorphism of S into S', evidently continuous. Moreover $\phi(u)=fu=f$. $\phi(S)$ is connected and contains 0 and f, hence must be all of S'.

Theorem 4 and Lemma 9 put us in a position to describe all threads with identity element and zero. First construct an arbitrary standard thread $S=[0, u]$ as prescribed in Theorem 3. Now take an arbitrary closed convex congruence relation ρ in S; Lemma 9 (2) tells us how to find all these. Let $S'=S/\rho$, and let ϕ be the canonical homomorphism of S onto S'. ϕ is also a homomorphism of S onto the order-dual S'' of S'. Let T be the right linear extension of S'' by S determined by ϕ. That T is a thread with identity u and zero $0'=\phi(0)$ then follows from the first part of Theorem 4.

The second part of Theorem 4 and Lemma 9 (1) tell us that every thread $T=[f, u]$ with identity u and interior zero 0 is isomorphic to one so constructed. In the notation of Theorem 4, let ρ be the (closed convex) congruence relation in $S=[0, u]$ corresponding to $\phi(\phi(x)=fx=xf)$. Let η be the canonical homomorphism of S onto S/ρ, and let T' be the right linear extension of the order-dual of S/ρ by S determined by η. Define $\theta(x)=x$, $\theta(\eta(x))=\phi(x)$, for every x in S. It is a routine matter to see that θ is an isomorphism of T' onto T.

8. **Threads with zero and commuting endpoints.** According to Faucett's Theorem 4 of [5, p. 746], a thread with zero and commuting endpoints is commutative. We can deduce this theorem as a corollary of the following, which, together with Theorem 4, gives explicitly the structure of all such threads.

THEOREM 5. *Let $T=[f, e]$ be a thread with zero and commuting endpoints. By passing to the order-dual of T if necessary, we can assume ef $(=fe)\leqq 0$. Then $S'=[ef, e]$ is a thread with zero and identity element e, and T is the left contact extension of S' by the (dual) standard thread $S=[f, ef]$. Conversely, every contact extension of a thread with zero and identity by a standard thread is a thread with zero and commuting endpoints.*

Proof. Let $T=[f, e]$ be a thread with zero 0 and commuting endpoints such that $ef=fe\leqq 0$. By Lemma 7, $S'=[ef, e]=eT=Te$, so that S' is a thread with zero 0 and identity element e, and is in fact an ideal of T (with generator e). By Lemma 5, $[f, 0]$ is a (dual) standard subthread of T. Since $ef=fe$, ef is idempotent. By the corollary to Lemma 4, ef is a zero element for $[f, ef]$. By Lemma 5 again, $S=[f, ef]$ is a subthread of T with identity f and zero ef. With x in S and y' in S', we have

$$xy' = (xf)(ey') = x(fe)y' = (fe)y' = (ef)y'\cdot$$
$$y'x = (y'e)(fx) = y'(ef)x = y'(ef).$$

Since ef is the point of contact of S and S', we see that T is the left contact extension of S' by S.

The converse is immediate.

COROLLARY. *Let $T=[f, e]$ be any thread with zero 0 and commuting endpoints. Let $S'=[f, 0]$ and $S=[0, e]$. Then either S or S' is an ideal of T. They are both ideals if and only if $ef=fe=0$, in which case $SS'=S'S=\{0\}$.*

Proof. If $ef\leqq 0$, then, by the theorem, $[ef, e]$ is a thread with identity e and zero 0. By Theorem 4, $[ef, 0]$ is an ideal in $[ef, e]$. But the product xy' of any element x of $[f, ef]$ by any element y' of $[ef, e]$ is equal to $(ef)y'$, and so belongs to $[ef, 0]$. From this it is clear that S' is an ideal of T. If $ef\geqq 0$ then S is an ideal of T. Conversely, if S' is an ideal, then $ef\in S'$, i.e. $ef\leqq 0$, and if S is an ideal then $ef\geqq 0$. Hence they are both ideals if and only if $ef=0$. If $x\in S$ and $y'\in S'$ in that case, we have

$$xy' = (xe)(fy') = x(ef)y' = x0y' = 0.$$

9. **January threads.** The notion of the january thread $T=S_+\cup S_-$ associated with a standard thread S was defined in the introduction.

THEOREM 6. *Let S be a standard thread. Then the (right-handed) january thread I associated with S is a thread $[f, e]$ with zero such that $ef=f$ and $fe=e$. Conversely, if T is any thread $[f, e]$ with zero 0 such that $ef=f$ and $fe=e$, then the subthread $[f, 0]$ of T is isomorphic with the order-dual of the standard subthread $S=[0, e]$ of T, and T is the (right-handed) january thread associated with S. Moreover, e and f are left identity elements of T.*

Proof. To assure ourselves that T really is a thread, we note first of all that it is a semigroup. In fact, if we take the direct product $S\times Q$ of S with the semigroup Q consisting of two symbols $+$, $-$ and with product defined by $qq'=q'$ for every q, q' in Q, and then take the Rees factor

$$S\times Q/\{(0, +), (0, -)\},$$

we get T. T is certainly totally ordered and connected by the prescription given, and has the idempotent endpoints

$$e = +u = (u, +), \qquad f = -u = (u, -),$$

with the additional property $ef=f$, $fe=e$. Product is continuous in the direct product $S\times Q$, hence also in T, since $\{+0, -0\}$ is a closed ideal.

Conversely, let $T=[f, e]$ be a thread with 0 and such that $ef=f$ and $fe=e$. By Lemma 1 we have

$$T = [f, e] = [ef, ee] \subseteq e[f, e] = eT,$$
$$T = [f, e] = [ff, fe] \subseteq f[f, e] = fT.$$

Hence e and f are left identity elements of T. Of course e is also a right identity element for $[0, e]$, and f for $[f, 0]$.

We show next that $0<x<e$ implies $f<xf<0$. If $xf=0$, then

$$x = xe = x(fe) = (xf)e = 0e = 0.$$

If $xf=f$, then $x=(xf)e=fe=e$. If $xf>0$, then $xf=(xf)e=x(fe)=xe=x$. Now, by Lemma 1,

$$[0, e] = [0e, fe] \subseteq [f, 0]e,$$

so that $x=x'e$ with x' in $[f, 0]$, and evidently $f<x'<0$. But then

$$x = xf = x'ef = x'f = x',$$

which is absurd, since $x>0$ and $x'<0$.

Dually, $f<x'<0$ implies $0<x'e<e$. Thus $x\to xf$ maps $[0, e]$ into $[f, 0]$, $x'\to x'e$ maps $[f, 0]$ into $[0, e]$, and they are mutual inverses:

$$(xf)e = x(fe) = xe = x,$$
$$(x'e)f = x'(ef) = x'f = x'.$$

We may therefore label the elements of T as ordered pairs (x, q) with x in $S=[0, e]$ and q in $Q=\{+, -\}$ as follows:

$$(x, +) = x, \qquad (x, -) = xf.$$

Then

$$(x, +)(y, +) = xy = (xy, +),$$
$$(x, -)(y, -) = xfyf = xyf = (xy, -),$$
$$(x, +)(y, -) = xyf = (xy, -),$$
$$(x, -)(y, +) = (xf)y = x(fy) = xy = (xy, +).$$

Thus T has the structure claimed.

10. Noncommutative threads with zero. To save circumlocution in the following theorem, let us agree that a contact extension of a thread T by a degenerate thread is just T itself, and that a standard thread $[0, u]$ may be degenerate (i.e. $u=0$).

THEOREM 7. *Every noncommutative thread $T=[f, e]$ with zero is a two-sided contact extension of a january thread by two standard threads.*

More precisely, ef and fe are idempotent, and are separated by 0. *By passing to the product dual of T if necessary, we can assume that $ef<0<fe$. $[ef, fe]$ is a january subthread of T, and T is obtained therefrom by contact extension by $[f, ef]$ on the left and $[fe, e]$ on the right, the order of formation being immaterial.*

Proof. Let $T=[f, e]$ be a noncommutative thread with zero 0. By Faucett's Theorem 4 of [5], we see that $ef\neq fe$; for $ef=fe$ would imply that T is commutative. It follows that ef and fe cannot both belong to $[0, e]$. For, if they did, we would have

$$ef = (ef)e = e(fe) = fe,$$

since, by Lemma 5 and the corollary to Lemma 4, e is the identity element of the subthread $[0, e]$. Similarly we can show that ef and fe cannot both belong to $[f, 0]$. Consequently neither ef nor fe can be 0, and 0 lies between them. By passing to the product-dual of T if necessary, we can assume that $ef<0<fe$. Moreover, $e(fe)=fe$ and $f(ef)=ef$, from which we conclude that ef and fe are idempotent.

$[ef, fe]$ is a subthread of T. For $[ef, 0]$ and $[0, fe]$ are subthreads, by Lemma 5, and if $x\in[ef, 0]$, $y\in[0, fe]$, then xy and yx are $\geqq x$ and $\leqq y$ by Lemma 2. Since

$$(ef)(fe) = efe = fe,$$
$$(fe)(ef) = fef = ef,$$

it follows from Theorem 6 that $[ef, fe]$ is a january thread.

$[ef, e]$ and $[f, fe]$ are also subthreads of T, by the same argument as above. Assume $ef\neq f$. In order to show that $[f, fe]$ is the left contact extension of $[ef, fe]$ by $[f, ef]$, all we need to show is that if $x\in[f, ef]$ and $y\in[ef, fe]$, then $xy=(ef)y$ and $yx=y(ef)$.

By Theorem 6, ef is a left identity for $[ef, fe]$, and, by the corollary to Lemma 4, it is a zero for $[f, ef]$. Hence

$$xy = x(efy) = (xef)y = (ef)y,$$

as desired. If $y\in[ef, 0]$,

$$yx = (yef)x = y(efx) = y(ef).$$

If $y\in[0, fe]$, then

$$yx = (ye)(fx) = y(efx) = y(ef).$$

The proof that $[ef, e]$ is the right contact extension of $[ef, fe]$ by $[fe, e]$ if $fe\neq e$ is similar.

Assume $ef\neq f$ and $fe\neq e$. In order to prove that T is the left contact extension of $[ef, e]$ by $[f, ef]$, we must show that if $x\in[f, ef]$ and $y\in[ef, e]$, then $xy=(ef)y$ and $yx=y(ef)$. This has already been shown above if $y\in[ef, fe]$ and hence we may assume that $y\in[fe, e]$. Since we have already established the fact that $[ef, e]$ is the right contact extension of $[ef, fe]$ by $[fe, e]$, we have

$$(ef)y = (ef)(fe) = fe, \qquad y(ef) = (fe)(ef) = ef.$$

Hence the equations we are to prove reduce to $xy=fe$ and $yx=ef$, which show us, incidentally, how the two "tails" multiply. Since we also have $(ef)x=ef$ and $x(fe)=(ef)(fe)=fe$, we conclude as follows:

$$xy = (xf)(ey) = x(fey) = x(fe) = fe,$$
$$yx = (ye)(fx) = y(efx) = y(ef) = ef.$$

The proof that T is also the right contact extension of $[f, fe]$ by $[fe, e]$ is similar.

COROLLARY. *Let $T=[f, e]$ be a thread with zero 0. Let $S=[0, e]$ and $S'=[f, 0]$. S and S' are subthreads of T. If e and f commute, then either*

$$SS' \subseteq S' \quad \text{and} \quad S'S \subseteq S',$$

or

$$SS' \subseteq S \quad \text{and} \quad S'S \subseteq S.$$

If e and f do not commute, then either $ef<0<fe$ or $fe<0<ef$, and we have respectively either

$$SS' \subseteq S' \quad \text{and} \quad S'S \subseteq S,$$

or

$$SS' \subseteq S \quad \text{and} \quad S'S \subseteq S'.$$

In either event, if we denote by S_a whichever of S or S' the nonzero element a of T belongs to, and if $a\neq 0$, $b\neq 0$, then either $S_aS_b\subseteq S_a$ or $S_aS_b\subseteq S_b$.

Proof. That S and S' are subthreads follows from Lemma 5. The assertion made when e and f commute is just the first part of the corollary to Theorem 5.

Let $ef\neq fe$. By Theorem 7, either $ef<0<fe$ or $fe<0<ef$. Assume the former. Then $[ef, fe]$ is a (right-handed) january thread, and from its very definition we see that

$$[ef, 0][0, fe] \subseteq [0, fe],$$
$$[0, fe][ef, 0] \subseteq [ef, 0].$$

T is a contact extension of $[ef, fe]$ by $[f, ef]$ on the left (if $ef\neq f$) and by $[fe, e]$ on the right (if $fe\neq e$). But the elements in these "tails" behave like the elements ef and fe, respectively, and hence

$$SS' \subseteq [0, fe] \subseteq S,$$
$$S'S \subseteq [ef, 0] \subseteq S'.$$

If $fe<0<ef$ then the product-dual of the foregoing holds, and we are led to the second alternative.

The last assertion (included only for future convenience in Part II) is immediate. If $S_a=S_b$ it holds because S and S' are subthreads of T. If $S_a\neq S_b$, it merely asserts that the set-products SS' and $S'S$ are wholly contained in either S or S'.

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