PROPERTIES OF PRIMARY NONCOMMUTATIVE RINGS

BY

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Introduction. In a series of papers [7; 8; 9; 10], E. Snapper discussed extensively the properties of completely primary rings, primary rings, and associated topics for the commutative case. This paper extends some of the theory of [7] to the noncommutative case. Most of [7] extends intact for duo rings which are rings where every right ideal is a left ideal and every left ideal is a right ideal. Some of this theory for general noncommutative rings is proved here.

An example in §5 of this paper answers Snapper's tempting conjecture [7, p. 678].

1. Preliminary discussion and definitions. Let $R$ be a ring with identity.

Definition 1.1. Let $a$ be a two sided ideal of $R$. The union of all ideals $b$ such that $b^n \subseteq a$ for some positive integer $n$ is a two sided ideal of $R$ and is called the radical of $a$ which we shall denote by $N(a)$.

Definition 1.2. Let $a$ be a two sided ideal of $R$. The set of all elements $x \in R$ such that $x^n \in a$ for some positive integer $n$ is said to be the nil-radical of $a$ which we shall denote by $P(a)$.

If $a$ is 0 in the previous definitions we use the symbols $N$ and $P$ for the radicals of 0. Although $P$ is not always a two sided ideal, we shall be interested in $P$ only when it is a two sided ideal.

Definition 1.3. The set of all elements $x \in R$ such that $yx + 1$ is a unit of $R$ for all $y \in R$ is a two sided ideal of $R$ and is called the Jacobson radical of $R$ which we denote by $J$.

If $x \in J$ then $xy + 1$ will also be a unit of $R$ for all $y \in R$.

Definition 1.4. A ring $R$ is called duo provided every right ideal is a left ideal and every left ideal is a right ideal.

If follows immediately from the above definition that if $R$ is a duo ring then $Ra = aR$ for all $a \in R$, i.e., for $a, b \in R$ there exists an element $c \in R$ such that $ab = ca$. Hence in this case $P$ is a two sided ideal.

Definition 1.5. A ring $R$ is said to be completely $N$ primary provided $R/N$ is a division ring(1).

In this definition if we substitute for $N$ the symbols $J$ and $P$ we have the definitions for a completely $J$ primary ring and a completely $P$ primary ring when $P$ is a two sided ideal.

Definition 1.6. A two sided ideal $q$ of $R$ is said to be completely prime

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(1) Every Grassmann algebra with finite basis is a completely $N$ primary duo ring.
provided \( ab \in q, \ a \in R, \ b \in R, \) implies at least one of these elements is contained in \( q. \)

**Definition 1.7.** A two sided ideal \( q \) is called **right \( N \) primary** provided \( ab \in q, \ b \in q \) implies \( a \in N(q). \) The ideal \( q \) is called **left \( N \) primary** provided \( ab \in q, \ a \in q \) implies \( b \in N(q). \) The ideal \( q \) is said to be **\( N \) primary** provided it is both right and left \( N \) primary.

If, in Definition 1.7 we substitute the symbols \( P \) and \( J \) for \( N \) we have the definitions of **right \( P \) primary**, **left \( P \) primary**, **\( P \) primary**, **right \( J \) primary**, **left \( J \) primary**, and **\( J \) primary**.

**Definition 1.8.** A ring \( R \) is said to be a **right \( N \) primary ring** provided \( 0 \) is a right \( N \) primary ideal. Similarly, a **left \( N \) primary ring** is defined. A ring \( R \) is **\( N \) primary** provided it is both left and right \( N \) primary.

If, in Definition 1.8 we substitute the symbols \( P \) and \( J \) for \( N \) we have the definition for **right \( P \) primary ring**, **left \( P \) primary ring**, **\( P \) primary ring**, **right \( J \) primary ring**, **left \( J \) primary ring**, and **\( J \) primary ring**.

Suppose \( P \) is a two sided ideal. Then \( NC \subseteq P \subseteq J \) and it follows that **\( N \) primary** \( \Rightarrow \) **\( P \) primary** \( \Rightarrow \) **\( J \) primary**. In addition, suppose \( R \) is \( N \) primary. Then since the elements of \( P \) are nilpotent \( P = N \) and \( P \) is a two sided ideal. Further, since \( R \) is then \( P \) primary, it follows from the proof of Theorem 2.2 of [1] that \( P \), which is equal to \( N \), is a completely prime ideal, i.e., \( R/N \) is an integral domain.

If \( R \) is a completely \( N \) primary ring certainly \( N \) is a two sided ideal which is a maximal right ideal of \( R \). Hence \( N = P = J \). In this case suppose \( ab = 0 \), then if \( a \neq 0 \), \( b \) must be contained in \( N \) otherwise it would have an inverse. Similarly if \( ab = 0 \), \( b \neq 0 \) then \( a \in N \). Hence, completely \( N \) primary implies \( N \) primary.

If \( R \) satisfies the \( A.C.C. \) for right ideals and is \( P \) primary, one can easily show from the proof of Theorem 2.2 of [1] and from [5] that \( P \) is a two sided ideal and \( P^n = 0 \) for some positive integer \( n \). Hence in this case \( P \) primary \( \Rightarrow \) **\( N \) primary**.

Many of the properties of \( N \) primary rings follow from the fact that divisors of zero are contained in \( J \).

**Statement 1.1.** If \( R \) is \( J \) primary and if \( x \in R \) is a right unit then \( x \) is a unit. If \( xy \) is a right unit both \( x \) and \( y \) are units.

**Proof.** Suppose \( xz = 1, \) then \( (1 - zx)z = 0. \) Hence \( 1 - zx \in J, \) which implies that \( zx = 1 + w \) for \( w \in J. \) Hence \( zx \) is a unit. The proof of the second statement follows from the first.

**Statement 1.2.** Let \( R \) be \( J \) primary and \( x \neq 0 \) and \( y \) be elements of \( R, \) then \( xy = x \) or \( yx = x \) implies \( y \) is a unit. Hence 1 is the only nonzero idempotent.

**Proof.** If \( xy = x, \) then \( x(y - 1) = 0 \Rightarrow y - 1 \in J. \) Hence \( y = 1 + z \) for \( z \in J. \) Thus \( y \) is a unit. The proof for \( yx = x \) is the same. To prove the second part suppose \( x^2 = x, \) then \( x \) is a unit. Consequently \( x = (x^{-1}x)x = x^{-1}x^2 = x^{-1}x = 1. \)
If $R$ is a completely $N$ primary ring then of course $N = J$. However, there exist rings which are not completely $N$ primary and where $N = J$. (See [7, p. 668]).

**Statement 1.3.** Let $R$ be a ring with identity. Then the following conditions are equivalent.

1. The Jacobson radical $J$ is a maximal right ideal.
2. The set of nonunits of $R$ coincide with $J$.
3. The set of nonunits of $R$ is a two sided ideal.
4. $R$ contains a unique maximal right ideal.

**Proof.** If $J$ is a maximal right ideal and $x \in J$, then $J + xR = R$. This implies that there exists $y \in J$, $z \in R$ such that $y + xz = 1$. Hence $xz = 1 - y$, which implies that $xz$ is a unit and therefore $x$ is a right unit. Similarly $x$ is a left unit since $J$ is also a maximal left ideal. Thus (1) $\Rightarrow$ (2) while it is trivial that (2) $\Rightarrow$ (3) $\Rightarrow$ (4). To show that (4) $\Rightarrow$ (1) let $q$ be the unique maximal right ideal. We will show that $q \subseteq J$. If $x \in q$ then $xy + 1 \in q$ for all $y \in R$. Suppose $xy + 1$ is not a right unit, then $(xy + 1) R \subset R$ and hence $xy + 1$ would be contained in maximal right ideal, namely $q$, which is impossible. Hence $xy + 1$ is a right unit for all $y$. Thus $x \in J$.

2. **Relationship between $R$ and $R/N$.** Let $R$ be a ring with identity and radicals $N$, $P$ and $J$ as defined in §1. Let $H$ denote the natural homomorphism of $R$ onto $R/N$. If $q$ is any subset of $R$ let $\bar{q}$ denote the subset $qH$ of $R/N$. Thus $R = R/N$.

2a. **Units.** One can easily see that $H$ maps units of $R$ on units of $R/N$. The following statement shows the converse to be true.

**Statement 2.1.** If $m$ is a two sided ideal of $R$ the following conditions are equivalent.

1. An element $x \in R$ is a unit if and only if the coset of $x$ modulo $m$ is a unit of $R/m$.
2. $m \subseteq J$.

**Proof.** (1) $\Rightarrow$ (2). If $x \in m$ then certainly $xy + 1 \mod m$ is a unit. Hence $xy + 1$ is a unit of $R$ for all $y \in R$. Thus $x \in J$. To show now that (2) $\Rightarrow$ (1). Let $x$ be an element of $R$ whose coset is a unit of $R/m$. Then $xy + 1 \in R$ for $u \in J$. Consequently $x$ is a right unit of $R$. Similarly $xz = 1 + u$ for $w \in J$ and $z$ is a left unit.

Hence unit $\iff$ unit mod $N$ and unit ideal $\iff$ unit ideal mod $N$.

2b. **Relatively prime.**

**Definition 2.1.** Two right ideals $q_1$ and $q_2$ are termed relatively prime provided $q_1 + q_2 = R$. Two elements $x$ and $y$ are relatively prime if $xR + yR = R$.

Clearly if $q_1 + q_2 = R$ then $\bar{q_1} + \bar{q_2} = \bar{R}$ and from 2a the converse follows immediately. Hence relatively prime $\iff$ relatively prime mod $N$ where we used only the fact that $N \subseteq J$.

2c. **Divisors and associates.**

**Definition 2.2.** If $x$ and $y$ are nonzero elements of $R$, then $x$ is termed
either a left factor of \( y \) or a right divisor of \( y \) if \( yR \subseteq xR \), i.e., \( y = xa \). Similarly we define right factor and left divisor. The element \( x \) is a factor of \( y \) if it is both a right and left factor of \( y \).

**Definition 2.3.** Two elements \( x \) and \( y \) of \( R \) are called right associates if \( xR = yR \), left associates if \( Rx = Ry \), and associates if they are both right and left associates.

Clearly if \( x \) and \( y \) are right (left) associates then \( x \) and \( y \) are right (left) associates. Hence right (left) associates \( \equiv \) right (left) associates mod \( N \). If two elements are right (left) associates, they are clearly both zero or nonzero and either both left (right) divisor of zero or not left (right) divisors of zero.

**Statement 2.2.** If \( R \) is a \( J \) primary ring and if \( x \) and \( y \) are nonzero right associates, then \( x = yv \) where \( v \) is a unit of \( R \). If \( x \) and \( y \) are left associates then \( x = vy \) where \( v \) is a unit of \( R \).

**Proof.** If \( xR = yR \) then \( x = yv \) and \( y = xv \). Consequently \( x = xwv \) and from Statements 1.1 and 1.2 we conclude that \( v \) is a unit.

2d. Proper divisors.

**Definition 2.4.** An element \( x \in R \) is called a proper right divisor of \( y \in R \) if \( yR \subseteq xR \), and a proper left divisor of \( y \in R \) if \( Ry \subseteq Rx \). The element \( x \) is a proper divisor if it is both a proper left and right divisor.

Clearly proper right (left) divisor \( \equiv \) proper right (left) divisor mod \( N \). In addition, we have the following converse for a \( J \) primary ring.

**Statement 2.3.** Let \( m \) be an ideal contained in \( J \) and let \( x \in R \), \( x \notin m \). Let \( R/m \) be \( J \) primary. Then, if \( y \in R \), the following are equivalent.

1. \( y \) is a proper right (left) divisor of \( x \).
2. \( y \) is a right (left) divisor of \( x \) and \( y \) is a proper (left) right divisor of \( x \).

**Proof.** Obviously (2) \( \Rightarrow \) (1). If (1) holds then \( x = yz \) where \( z \) is not a unit. Hence from 2.1 we conclude that \( x = \tilde{y}z \) where \( \tilde{z} \) is not a unit of \( R/m \), while \( x \notin m \) implies that \( \tilde{x} \) is not the zero element of \( R/m \). Statement 2.2 implies that \( \tilde{x} \) and \( \tilde{y} \) are not associates and hence \( \tilde{y} \) is a proper divisor of \( \tilde{x} \).

2e. Irreducibles and fundamental irreducibles.

**Definition 2.5.** An element \( x \) is called irreducible provided \( x = yz \) implies that either \( y \) or \( z \) is a unit.

From this definition, it follows for a \( J \) primary ring or a duo ring that a unit is necessarily an irreducible.

**Statement 2.4.** If \( m \) is an ideal contained in \( J \) and \( x \) is irreducible in \( R/m \), then \( x \) is irreducible in \( R \).

**Proof.** If \( x = yz \) then \( \tilde{x} = \tilde{y}z \) in \( R/m \). Hence since \( \tilde{x} \) is irreducible we can assume that \( \tilde{y} \) is a unit. From Statement 2.1 then \( y \) is a unit. Hence \( x \) is irreducible.

This shows that irreducible mod \( N \Rightarrow \) irreducible. In view of the direction of this implication we made the following definition.

**Definition 2.6.** An element \( x \in R \) is called a fundamental irreducible if its coset \( \tilde{x} \) is irreducible in \( R/N \).
Accordingly the fundamental irreducibles are irreducible and the units are fundamental irreducibles for \( J \) primary or duo rings.

2f. Completely prime ideals and prime elements in duo rings.

**Statement 2.5.** A two sided ideal \( p \) is completely prime if and only if \( N \subseteq p \) and \( p/N \) is a completely prime ideal of \( R/N \).

**Proof.** If \( x \in N \) we have \( x^n = 0 \in p \) for some positive integer \( n \). Hence \( x \in p \) which implies that \( N \subseteq p \). If now \( N \subseteq p \) then \( \overline{R}/\overline{p} \cong R/p \). From this we can conclude that \( p \) is completely prime in \( R \) if and only if \( \overline{p} \) is completely prime in \( \overline{R} \).

If \( N \) is a completely prime ideal then \( N \) is the only nil completely prime ideal and if \( N \) is not completely prime than \( R \) contains no completely prime nilpotent ideals. If \( N = P \) and \( N \) is not completely prime, then \( R \) contains no nil completely prime ideals.

We make the following definition for duo rings.

**Definition 2.7.** An element \( x \) of a duo ring \( R \) is termed prime if \( xR = Rx \) is a completely prime ideal.

This definition is equivalent to the statement that \( x \) is prime if \( x = yz \) implies that either \( y \) or \( z \) is an associate of \( x \).

**Statement 2.6.** If \( R \) is a \( J \) primary duo ring then every nonzero prime element is irreducible.

**Proof.** If \( x \) is prime and \( x = yz \) then either \( yR = xR \) or \( zR = xR \) which implies by Statement 2.2 that either \( y \) or \( z \) must be a unit.

If a two sided ideal \( p \) is a maximal right ideal then \( R/p \) is a division ring.

We call such an ideal a right maximal two sided ideal. It is obvious that a two sided ideal \( p \) is right maximal if and only if (1). \( N \subseteq p \) and (2). \( \overline{p} \) is right maximal in \( \overline{R} \).

2g. Radical \( N \) of a two sided ideal. If \( a \) is a two sided ideal then as in §1 let \( N(a) \) denote the union of all nilpotent two sided ideals mod \( a \). Clearly \( N(a + N) = N(a) + N = N(a) \), where \( N \) is \( N(0) \).

**Statement 2.7.** If \( a \) and \( b \) are two sided ideals of \( R \), then:\n
\[(1) N(a) = \{ N(a) \} - , \]
\[(2) N(a) \subseteq N(b) \iff N(a) \subseteq N(b) \],
\[(3) N(a) = N(b) \iff N(a) = N(b) \],
\[(4) a = b \iff N(a) = N(b) \].

**Proof.** (1). If \( q \) is a two sided ideal of \( R \) and \( \bar{q} \) is contained in \( N(\bar{a}) \), then \( \bar{q}^n \subseteq a + N \). Hence \( q \subseteq N(a + N) = N(a) \). Conversely, if \( q \subseteq \{ N(a) \} - \), \( q \subseteq N + N(a) = N(a) \). Hence \( q^n \subseteq a \) and therefore \( \bar{q}^n \subseteq \bar{a} \) which shows that \( q \subseteq N(\bar{a}) \).

(2). If \( N(a) \subseteq N(b) \) then \( \{ N(a) \} - \subseteq \{ N(b) \} - \) and by (1) \( N(\bar{a}) \subseteq N(\bar{b}) \). Conversely, if \( N(\bar{a}) \subseteq N(\bar{b}) \) let \( q \subseteq N(\bar{a}) \). Then \( q^n \subseteq a \) and hence \( \bar{q}^n \subseteq \bar{a} \). It follows
that \( q \subseteq N(a) \subseteq N(b) = \{ N(b) \} \) and hence \( q \subseteq N+N(b) = N(b) \). Conversely, \( N(a) \subseteq N(b) \) and (2) is proved.

It follows immediately that (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4).

2h. Residue class rings. If \( q \) is a two sided ideal of \( R \) then \( N(R/q) \) is equal to \( N(q)/q \), i.e., \( N(R/q) \) consists of the images of \( N(q) \) under the natural homomorphism from \( R \) to \( R/q \). We denote again by \( \{ R/q \} \) the residue class ring of \( R/q \) modulo its radical \( N(R/q) \), i.e., \( \{ R/q \} = (R/q)/(N/q) = [R/q]/[N(q)/q] \). Since \( q \subseteq N(q) \subseteq R \) then \( R/N(q) \cong [R/q]/[N(q)/q] = \{ R/q \} \). Since \( N \subseteq N(q) \subseteq R \) then \( R/N(q) \cong [R/N]/[N(q)/N] \cong \overline{R}/[N(q)] \). Hence we have \( \{ R/q \} \cong R/N(q) \cong \overline{R}/[N(q)] \). In particular we see that if \( b \) is a nilpotent two sided ideal, then \( N(b) = N \) and \( \{ R/b \} \cong R/N \). Thus the residue class ring \( R/b \) modulo its radical is isomorphic to \( R \) modulo its radical.

3. Right primary ideals. We defined a two sided ideal \( q \) to be right \( P \) primary provided \( ab \in q \) and \( b \in q \) implies \( a^n \in q \) for some positive integer \( n \). In order to discuss the right primary ideals we impose three conditions, namely:

(i) \( P(q) \) is a two sided ideal where \( q \) is any right \( P \) primary two sided ideal.

(ii) \( P = N \).

(iii) The nontrivial completely prime two sided ideals of \( R/N \) are maximal right ideals.

3a. Right primary nil ideals. It is easy to discuss the right \( P \) primary nil two sided ideals. For if \( q \) is such an ideal then \( P(q) \) is \( P \) and in this case \( P \) is a completely prime ideal. Consequently if \( P \) is not completely prime there are no right \( P \) primary nil two sided ideals. If however \( P \) is completely prime \( R \) may have many such primary ideals. In fact if \( R \) is completely \( P \) primary every nil two sided ideal is right \( P \) primary.

3b. Not nil primary ideals.

Statement 3.1. Let \( R \) be a ring satisfying (i), (ii), and (iii). Then a nontrivial not nil two sided ideal \( p \) is a completely prime ideal if and only if \( p \) is a maximal right ideal. A not nil two sided ideal \( q \) is \( P \) primary if and only if \( P(q) \) is a completely prime ideal.

Proof. Let \( \bar{p} \) be a nontrivial not nil completely prime two sided ideal of \( R \). If \( \bar{p} \) is completely prime we conclude from 2f that \( \bar{p} \) is completely prime and nontrivial by 2a. Hence \( \bar{p} \) is a maximal right ideal of \( \overline{R} \). We conclude that \( p \) is a two sided ideal which is a maximal right ideal by 2f. This proves the first part of the statement.

It follows from [1] and (i) that if \( q \) is a \( P \) primary two sided ideal then \( P(q) \) is a completely prime ideal. If \( P(q) = R \) then \( q = R \) since \( R \in 1 \) and in this case \( q \) is completely prime. If \( P(q) \subset R \) then \( P \subset P(q) \subset R \). Consequently \( P(q) \) is a maximal right ideal. Hence \( R/P(q) \) is a division ring which implies that \( q \) is \( P \) primary.

\(^{(3)}\) This is true if \( R \) satisfies the A.C.C. for right ideals or if \( R \) is duo.

\(^{(4)}\) In this case all the right (left) \( P \) primary two sided ideals are \( P \) primary.
We can now easily show that if \( R \) satisfies the conditions (i), (ii), (iii), then right \( P \) primary \( \Rightarrow \) right \( P \) primary mod \( N \). For if \( q \) is a right \( P \) primary two sided ideal then \( P(q) \) is completely prime and by 2f and 2g it follows that \( \{ P(q) \} ^{-1} = P(\bar{q}) \) is a completely prime ideal. Then since \( \bar{R} \) will necessarily satisfy conditions (i), (ii), (iii), it follows that \( \bar{q} \) will be right \( P \) primary.

**Definition 3.1.** An element \( x \) contained in a duo ring \( R \) is said to be \( P \) primary provided \( xR = Rx \) is \( P \) primary.

From the preceding discussion in this section we can conclude that an element \( x \) contained in a duo ring \( R \) which satisfies (i), (ii), (iii), is \( P \) primary and not nilpotent if and only if \( \bar{x} \) is a \( P \) primary, nonzero element of \( \bar{R} \).

Note that if \( q \) is a nonzero completely prime ideal of \( R \) we can only conclude that \( q \) is \( P \) primary in \( R \). (See [7, p. 673].)

For our case, when \( q \) is a \( P \) primary, not nil, nontrivial, two sided ideal of \( R \), then \( P(q) = p \) is a maximal right ideal of \( R \). Hence from 2h, we have \( \{ R/q \} ^{-1} \cong R/p \cong \{ R/p \} ^{-1} \) and therefore \( \{ R/q \} ^{-1} \) is a division ring. Consequently if \( R \) is a ring satisfying (i), (ii), (iii) and if \( q \) is a \( P \) primary two sided ideal which is not nil or trivial with \( P \) radical \( p \), then \( R/q \) is a completely \( P \) primary ring whose residue class is isomorphic to \( R/p \cong \bar{R}/p \).

This section depended heavily on conditions (i), (ii), (iii). One should note that if these conditions are satisfied in \( R \) then they will be satisfied in \( R/n \) where \( n \) is a two sided ideal contained in \( P = N \).

4. **Factorization in duo rings.** In this section \( R \) will be a duo ring with identity. For such a ring we have the following property. If \( x \in R \) then for every element \( y_1, y_2 \in R \) there exists \( z_1, z_2 \in R \) such that \( xy_1 = z_1x \) and \( y_2x = xz_2 \). Most of the theorems for commutative rings on factorization in primary rings will be valid for \( N \) primary duo rings(6).

We want to discuss the factorization of a duo ring \( R \) when \( R/N \) is a unique factorization domain whose nontrivial completely prime ideals are maximal.

**Definition 4.1.** A duo ring \( A \) with identity is called a unique factorization domain if the following two conditions are satisfied.

1. Every nonzero element of \( A \) is a product of a finite number of irreducible elements.

2. If \( a_1a_2 \cdots a_s = b_1b_2 \cdots b_t \) where \( a_1, a_2, \ldots, a_s \) and \( b_1, b_2, \ldots, b_t \) are nontrivial irreducibles, then \( s = t \) and for a suitable ordering of the subscripts \( a_i \) is an associate of \( b_i \), for \( i = 1, 2, \ldots, s \).

For a duo ring the following three properties are valid.

1. If \( x \) is irreducible then \( xv \) and \( vx \) are irreducible where \( v \) is a unit.

2. If \( x \) is irreducible and \( v \) is a unit then \( vx = xv \) implies that \( w \) is a unit.

3. If \( x = ab \) and \( b = vd \) where \( a, b, \) and \( d \) are irreducible then \( x = uad \) where \( u \) is a unit.

(6) The factorization theorems for commutative rings are given in [7, pp. 674–678].
Lemma 4.1. A duo ring $A$ which is a unique factorization domain is an integral domain.

Proof. Let $a_1 \cdots a_n$ be a product of irreducibles elements of $A$. We shall show that $a_1 \cdots a_n \neq 0$. If $n = 1$ this statement follows from the fact that 0 is reducible, i.e., $0 = 0 \cdot 0$. We assume that the statement is valid for less than $n$ irreducibles. If $a_1 \cdots a_n = 0$ then none of the irreducibles is zero. For in that case less than the product of $n$ irreducibles would be zero. Furthermore $a_1 a_1 \cdots a_n = 0 \cdot a_1 = 0 a_1 \cdots a_n$ which contradicts Definition 4.1. Thus $a_1 \cdots a_n \neq 0$. Since every two elements $x$ and $y$ of $A$ can be factored into irreducibles certainly $xy \neq 0$ if $x \neq 0$ and $y \neq 0$.

If $A$ is a unique factorization domain then since it is an integral domain it follows that an irreducible element is prime. For suppose $bc \in aR = Ra$ where $a$ is irreducible. Then $bc = ad$ and hence since $a$ is irreducible $a$ must appear in the factorization of either $b$ or $c$. Say $c = g_1 g_2 \cdots g_{a} g_{i+2} \cdots g_n$. Then since $g a = a h_i$, $i = 1, 2, \cdots, n$ where the $h_i$ must be irreducible since $A$ is a unique factorization domain, we have $a = af$, i.e., $c$ is contained in $aR$.

We refer to a duo ring which is at the same time a principal ideal ring and an integral domain as a duo principal ideal domain.

Theorem 4.1. A duo ring $A$ is a unique factorization domain whose completely prime ideals are maximal if and only if $A$ is a principal ideal domain.

Proof. One can using the classical method of [4, pp. 114–122] and the fact that $R$ is duo show quite easily that a duo principal ideal domain is a unique factorization domain (8). Certainly the completely prime ideals are maximal. For from Statement 2.6 we can conclude that prime elements are irreducible. In a duo principal ideal ring an irreducible element will generate a maximal ideal.

To prove the converse let the duo ring $A$ be a unique factorization domain whose prime ideals are maximal. We first show that if $x$ and $y$ are two elements of an ideal $q$, then any greatest common divisor of $x$ and $y$ belongs to $q(7)$. If $x$ or $y$ is zero, this is obvious. Let the g.c.d. of $x \neq 0$ and $y \neq 0$ be $d$ and then $x = x_1 d$ and $y = y_1 d$. If either $x_1$ or $y_1$ is a unit, obviously $d \in q$. Hence we can assume that $x_1$ and $y_1$ are not units and hence $x_1 = a_1 \cdots a_n$ and $y_1 = b_1 \cdots b_m$ are the factorizations of $x_1$ and $y_1$ into nontrivial irreducibles. None of the $a_i$, $i = 1, 2, \cdots, n$ could be associates of the $b_i$, $i = 1, 2, \cdots, m$ since $d$ is the g.c.d. of $x$ and $y$. Then since the irreducible elements are prime the ideals $a_i R$ and $b_j R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$ are maximal. Consequently the $a_i$ and $b_j$ are relatively prime. Hence $x_1$ and $y_1$ are relatively

(8) This statement also follows from [3, p. 34]. For similar elements of duo rings are associates.

(7) One can easily show using the classical methods of [4, Chapter IV] that every two elements have a g.c.d. which is unique up to the multiplication by a unit.
prime. Hence there exists \( \lambda_1 \) and \( \lambda_2 \) in \( A \) such that \( \lambda_1 x + \lambda_2 y = 1 \). From this it follows that \( \lambda_1 x + \lambda_2 y = d \). Therefore \( d \in q \). We shall now show that every ideal of \( A \) is principal. If an ideal \( q \) is zero it is principal. If \( q \) is not the zero ideal we can choose from the element in \( q \) an element \( x \) with the least number of irreducible factors. We prove that \( q = xR \). Let \( y \in q \) and let \( d \) be the g.c.d. of \( x \) and \( y \). Then \( x = xd \) and \( y = yd \) and we know that \( d \in q \). Hence \( d \) is a divisor of \( x \) with the same number of nontrivial irreducibles. Consequently \( dR = xR \) and \( y \in xR \). This proves our theorem.

**Principal lemma for duo rings.** Let \( x \) be an element of the duo ring \( R \) and let \( x = a_1 \cdots a_m \) mod \( N \) where \( a_1, \cdots, a_m \) are elements of \( R \) which are relatively prime in pairs. Then there exists elements \( b_1, \cdots, b_m \) such that:

1. \( b_i = a_i \) mod \( N \) for \( i = 1, \cdots, m \) and hence \( b_1, \cdots, b_m \) are also relatively prime in pairs.
2. \( x = b_1 \cdots b_m \).

The proof of this lemma follows closely the proof of the similar lemma for the commutative case (see [7, p. 672]) and is therefore omitted.

If \( R \) is a duo ring where \( R/N \) is a principal ideal domain then \( R \) satisfies the conditions (i), (ii), (iii) of §3 and therefore \( N = P \), the nontrivial prime ideal of \( R/N \) are maximal, and \( P(q) \) is an ideal where \( q \) is an ideal of \( R \).

**Theorem 4.2.** Let \( R \) be a duo ring with identity and let \( R/N \) be a principal ideal domain then:

1. Every not nilpotent element \( x \) of \( R \) can be factored as \( x = va_1 \cdots a_n \), where \( v \) is a unit and where \( a_1, \cdots, a_n \) are \( P \) primary, not nilpotent, nonunits, which are relatively prime in pairs.
2. If \( va_1 \cdots a_n = ub_1 \cdots b_m \), where \( v \) and \( u \) are units, \( a_1, \cdots, a_n \) are \( P \) primary nonunits which are relatively prime in pairs, and where the same is true for \( b_1, \cdots, b_m \); then \( n = m \) and for a suitable rearrangement of the subscripts \( a_i \) is associated with \( b_i, \) \( i = 1, 2, \cdots, n \). If \( n > 1 \), the elements \( a_1, \cdots, a_n, b_1, \cdots, b_n \) are not nilpotent.

**Proof.** We first show that (2) is valid. Since \( R \) is duo it follows that \( (a_1R) \cdot (a_2R) \cdots (a_nR) = (b_1R) \cdot (b_2R) \cdots (b_mR) \). In addition since these are relatively prime in pairs and since \( R \) is duo it follows that \( (a_1R) \cap (a_2R) \cap \cdots \cap (a_nR) = (b_1R) \cap (b_2R) \cap \cdots \cap (b_mR) \). From [2] it follows that these ideals in some order are equal and \( n = m \). If \( n > 1 \), none of the \( a_i \)'s or \( b_i \)'s can be nilpotent. For if \( a_1 \) is nilpotent and \( a_2 \) and \( a_1 \) are relatively prime then \( 2a \) implies that \( a_2 \) is a unit which is not the case.

To prove (1) let \( x \) be a not nilpotent element of \( R \). Then \( x \) is a nonzero element of \( \bar{R} \) and hence \( \bar{x} \) can be factored as \( \bar{x} = \bar{v}\bar{a}_1^h \cdots \bar{a}_n^h \) where \( \bar{v} \) is a unit of \( \bar{R} \) and \( \bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n \) are irreducible, not associated, nonunits of \( \bar{R} \). Consequently \( a_1^h, a_2^h, \cdots, a_n^h \) are \( P \) primary, not zero, nonunits of \( R \) which
are relatively prime in pairs. It follows then from the principal lemma for
duo rings that \( x=ub_1\cdots b_n \) where \( v=u \mod N \), and \( a_i=b_i \mod N \), \( i=1, 2, \ldots, n \). We conclude from 3b that \( b_1, \ldots, b_n \) are \( P \) primary and not nilpotent elements of \( R \) and from 2a and 2b that \( u \) is a unit of \( R \) and \( b_1, \ldots, b_n \) are relatively prime in pairs.

We shall now discuss some of the implications of this theorem for a duo
ring \( R \) where \( R/P \) is a principal ideal domain.

4a. The primary not nilpotent elements of \( R \). From §3, since the conditions
(i), (ii), and (iii) are satisfied, and element \( a \in R \) is \( P \) primary, not nilpotent
nonunit if and only if \( a \) is a \( P \) primary, nonzero, nonunit of \( R \). Therefore
\( a=\tilde{v}\pi^h \) where \( \tilde{v} \) is a unit of \( \bar{R} \) and \( \pi \) is a nontrivial irreducible element of \( \bar{R} \).

Thus we have an element \( a \in R \) is a \( P \) primary, nonnilpotent, nonunit, if and
only if \( a=\nu\pi^h+d \) where \( \nu \) is a unit and \( \pi \) is a nontrivial fundamental irreducible
while \( h \geq 1 \) and \( d \in P \). If \( a=\nu_1\pi_1^{h_1}+d_1=\nu_2\pi_2^{h_2}+d_2 \) then \( \nu_1\pi_1^{h_1}=\nu_2\pi_2^{h_2} \). Hence \( h_1=h_2 \)
and \( \pi_1R=\pi_2R \). Conversely if \( a=\nu\pi^h+d_1 \) and \( \pi_2=\nu_1\pi_1+d' \) then \( a=\nu_2\pi_2^{h_2}+d_2 \)
for suitable \( \nu_2 \) and \( d_2 \). Thus if \( a=\nu\pi^h+d \) then \( a=\nu_1(\nu_2\pi+d_2)^h+d_1 \) for all units
\( \nu_1 \) of \( R \) and all \( d_2 \in P \) where \( \nu_1 \) and \( d_1 \) are suitably chosen and these include all
such representations. The nontrivial fundamental irreducibles \( \nu\pi+d \) for all \( \nu \)
and \( d \) are called the fundamental irreducibles of the \( P \) primary element \( a \).

4b. The not nilpotent elements of \( R \). Let \( x \) be a not nilpotent element of \( R \).
Then \( x=\nu a_1\cdot a_2\cdots a_n \) where \( \nu \) is a unit and the \( a_i \) are \( P \) primary. Then if
\( \nu\pi_i+d_i \) is a nontrivial fundamental irreducible of \( a_i \) of multiplicity \( h_i \), we
say that \( \nu\pi_i+d_i \) is a nontrivial fundamental irreducible of \( x \) of multiplicity
\( h_i \). From Theorem 4.2 and §4a the nontrivial fundamental irreducibles of \( x \)
and there unique multiplicities are determined by \( x \) and do not depend upon
any factorization of \( x \). Consequently we have that two not nilpotent elements
of \( R \) are relatively prime if and only if they have no nontrivial fundamental
irreducibles in common.

4c. Irreducible elements of \( R \). We shall show that if \( x \) is an irreducible not
nilpotent element of \( R \) then \( x \) is a \( P \) primary element of \( R \). From §3 all we need
show is that \( x \) is a nonzero \( P \) primary element of \( \bar{R} \). If \( x \) is not primary then
\( x=\tilde{a}b \) where \( \tilde{a} \) and \( b \) are nonunits which are relatively prime. Then from the
principal lemma for duo rings we have \( x=ab \) where \( a \) and \( b \) are nonunits
which is a contradiction. Hence \( x=\nu\pi^h+d \).

5. Factorization in general noncommutative rings where \( R/P \) is a prin-
cipal ideal domain. Let \( R \) be a ring with identity where \( R/P \) is a principal
ideal domain. If \( u \) is a right unit of \( R \) then by 2a, \( \bar{u} \) is a right unit of \( R/P \).
Consequently \( \bar{u} \) is also a left unit of \( R/P \) and again by 2a \( u \) is a left unit of \( R \). Consequently every right or left unit of \( R \) is a unit. In this way the units of
\( R \) are trivial irreducibles. From [3, p. 34] we know that every nonzero element
\( \tilde{a} \) of \( R/P \) which is not a unit may be written as \( b_1\cdots b_m \), where \( b_i \) are non-
trivial irreducible; and if \( \tilde{a}=\tilde{c}_1\cdots\tilde{c}_m \), where \( \tilde{c}_i \) are nontrivial irreducible
then \( m=n \) and the \( b_i \)'s and \( \tilde{c}_i \)'s may be arranged into similar pairs in \( R/P \).
Suppose \(a\) is a nonnilpotent, nonunit of \(R/P\) and \(a = b_1 \cdots b_n\) where \(b_i\) are nontrivial irreducible in \(R/P\). Suppose \(a = c_1c_2 \cdots c_m\) where \(c_i\) are nonunits, then \(a = \tilde{c}_1\tilde{c}_2 \cdots \tilde{c}_m\) and consequently \(m \leq n\). Thus we have:

**Theorem 5.1.** If \(R\) is a ring with identity where \(R/P\) is a principal ideal domain then every nonnilpotent, nonunit element of \(R\) can be expressed as a product of nontrivial irreducible elements. The number of irreducible elements in such a product will be less than or equal to the unique number of irreducible elements in \(R/P\) whose product is equal to \(a\).

It would be tempting to conjecture as in [7, p. 678] that the number of irreducible elements is unique in \(R\). However, this is not the case as the following example shows.

**Example.** Consider the following commutative ring. Let \(R\) be the set of all elements \(a+bx\) where \(a, b \in I\), the ring of integers. Equality is defined by: \(a_1+b_1x = a_2+b_2x\) if and only if \(a_1 = a_2\) and \(b_1 = b_2\). Addition is defined by \((a_1+b_1x)+(a_2+b_2x) = (a_1+a_2)+(b_1+b_2)x\). Define \(x^2 = 0\). Then multiplication is defined by the usual polynomial multiplication, i.e., \((a_1+b_1x)(a_2+b_2x) = a_1a_2 + (a_1b_2+a_2b_1)x\). Hence \(xR = N\) is the radical of \(R\). In \(R\) we have:

1. \(16 = 2 \cdot 2 \cdot 2 \cdot 2\),
2. \(16 = (4-x)(4+x)\).

We assert in (1) and (2) that 16 is a product of nonnilpotent, nonunits which are irreducible in \(R\). Obviously this is true for (1). In (2), \((4+x)\) is obviously not nilpotent and is a nonunit by 2a. We shall now show that \((4 \pm x)\) is irreducible. Suppose:

3. \((a_1+b_1x)(a_2+b_2x) = (4 \pm x)\) where \((a_1+b_1x)\) and \((a_2+b_2x)\) are nonnilpotent, nonunits.

Then \(a_1\) and \(a_2\) are not zero, nor can they be \(\pm 1\). Equation (3) implies that \(a_1a_2 = 4\), which in turn implies that \(a_1 = a_2 = \pm 2\). From this equation (3) implies that \(\pm 2(b_1+b_2) = \pm 1\), which is impossible.

Thus, as this example displays, the number of irreducible elements in a factorization in \(R\) is not unique.

**6. Polynomial ring over a duo ring.** Suppose \(R\) is a duo ring with identity. Let \(R[x]\) denote the ring of polynomials with coefficients in \(R\) where \(ax = xa\) for all \(a\) contained in \(R\). The ring \(R[x]\) will not necessarily be duo but does have the following property. If \(f\) is any polynomial there exist polynomials \(g\) and \(h\) such that \(af = ga\) and \(fa = ah\) for all \(a \in R\). The degree of a nonzero polynomial \(f\) in \(R[x]\) is the exponent of the highest power of \(x\) which occurs in \(f\) with a nonzero coefficient. The degree of \(f\) is denoted by \(D(f)\).

For each ring \(R\) in this section we shall denote the set of nilpotent elements by \(P(R)\).

**Lemma 6.1.** If \(R\) is a duo ring with identity the following conditions are equivalent:

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(1) The radical \( P(R) = 0 \).

(2) If \( f \) is any nonzero polynomial and \( g \) is a regular polynomial of \( R[x] \), then \( D(fg) \geq D(f) \).

Hence if \( P(R) = 0 \) and \( f \) is regular then \( D(f) \) is the minimum degree of the regular polynomials of \( fR[x] \) and \( R[x]f \).

**Proof.** \((2) \Rightarrow (1)\). Let \( a \) be a nonzero nilpotent element of \( R \) and say \( a^s = 0 \), \( a^{s-1} \neq 0 \) where \( s \geq 2 \). If \( b \) is any regular element of \( R \) let \( f = a^{s-1}x - c \) and \( g = a^{s-1}x + b \), where \( ca^{s-1}x = a^{s-1}xb \). Then \( g \) is regular and \( fg = -cb \) which has degree zero. This contradicts \((2)\).

We shall now show that \((1) \Rightarrow (2)\). Suppose \( P(R) = 0 \). Let \( f = a_nx^n + \cdots + a_1x + a_0 \) and \( g = b_nx^n + \cdots + b_1x + b_0 \), where \( a_n \neq 0 \). In addition suppose \( D(fg) < n \). We need only show that \( g \) is not regular. We shall prove that \( a_nb_i = a_{n-1}b_{i-1} = \cdots = a_1b_0 = 0 \). Certainly \( a_nb_s = 0 \) since \( D(fg) < n \), and hence we shall make the induction hypothesis that \( a_nb_i = \cdots = a_{n-i+1}b_i = 0 \) for some \( i \), \( i = 1, 2, \cdots, s \). Since \( R \) is a duo ring there exists a polynomial \( h \) of degree \( n \) such that \( a_n^tb = ha_n^t+1 \) where the leading coefficient of \( h \) is \( a_n \). Then \( ha_n^t+1g = ha_n^t+1(b_n-tx^{t-1} + \cdots + b_0) \) where \( D(ha_n^t+1g) = D(a_n^t+1f) \leq D(fg) < D(f) = n \). Thus \( a_n^t+1b_{t+1} \), which is the coefficient of \( x^{n+i-1} \) in \( ha_n^t+1 \), must be zero. This completes the inductive portion of the proof. Consequently the statement is true for all integers \( i \). Hence \( a_nb_i = \cdots = a_1b_0 = 0 \). It follows that \( a_n^tb_i = \cdots = a_1b_0 = 0 \) and hence since \( a_n \neq 0 \) and \( P(R) = 0 \), we have \( a_n^t+1g = 0 \). Thus \( g \) is not regular.

**Lemma 6.2.** Let \( f \) be a polynomial in \( R[x] \) where \( R \) is a duo ring. If \( fg = 0 \), where \( g \) is also a polynomial in \( R[x] \), there exists an element \( c \neq 0 \) of \( R \) such that \( f(x)c = 0 \).

The proof of this lemma is very similar to the proof for the commutative case and is therefore omitted. (See [6, p. 34].)

Since a polynomial \( f \) is contained in \( P(R[x]) \) if and only if its coefficients are contained in \( P(R) \), we have that \( P(R[x]) \) is a two sided ideal of \( R[x] \) and that \( \sum \bar{a}_i x^i = 0 \) if and only if each \( \bar{a}_i = 0 \). (The single bar denotes the image under the natural homomorphism from \( R[x] \) to \( R[x]/P(R[x]) \)). Hence we can consider \( R[x]/P(R[x]) \) as the polynomial ring \( (R/P)[x] \) where \( P \) is the radical \( P(R) \).

**Theorem 6.1.** A polynomial \( f = a_nx^n + \cdots + a_1x + a_0 \) is a unit of \( R[x] \), where \( R \) is a duo ring, if and only if \( a_0 \) is a unit of \( R \) and the other coefficients are nilpotent elements of \( R \).

**Proof.** If \( a_0 \) is a unit and \( a_1, \cdots, a_n \) are nilpotent, then \( f \) is a unit of \( (R/P)[x] \). Consequently by 2a \( f \) is a unit of \( R[x] \). Conversely, if \( f \) is a unit of

\[(^*) \text{An element } g \text{ is regular if it is neither a left nor right divisor of zero.}\]
$R[x]$, then $\tilde{f} = \tilde{a}_n\tilde{x}^n + \cdots + \tilde{a}_0$ is a unit of $(R/P)[x]$ and hence $\tilde{f}(R/P)[x]$ contains a regular polynomial of degree zero. We conclude from Lemma 6.1, since $P(R/P) = 0$, that $f$ has degree zero. Consequently $\tilde{a}_n = \cdots = \tilde{a}_1 = 0$ and $\tilde{a}_0$ is a unit of $R/P$. Hence $a_n, \cdots, a_1$ are nilpotent and from $2a \ a_0$ is a unit of $R$.

**Theorem 6.2.** The Jacobson radical $J(R[x])$ and the radical $P(R[x])$ of a polynomial ring $R[x]$, where $R$ is a duo ring, are the same.

**Proof.** We need only show that $J(R[x]) \subseteq P(R[x])$. If $f = a_nx^n + \cdots + a_1x + a_0 \in J(R[x])$, then $xf + 1 = a_nx^{n+1} + \cdots + a_1x^2 + a_0x + 1$ is a unit. Hence $a_n, \cdots, a_0$ are nilpotent.

**Theorem 6.3.** If $R$ is a duo ring the following statements are equivalent.

1. $R$ is a right $P$ primary ring.
2. $R[x]$ is a right $P$ primary ring.
3. $R[x]$ is a right $J$ primary ring.

**Proof.** Obviously, by Theorem 6.2, (2) and (3) are equivalent. (2) immediately implies (1). We shall now show that (1) $\Rightarrow$ (2). Suppose $fg = 0$ and $g \neq 0$. Then, by Lemma 6.2, there exists a nonzero element $c$ in $R$ such that $fc = 0$. Hence the coefficients of $R$ are left divisor of zero and therefore are contained in $P(R)$. Consequently $f$ is contained in $P(R[x])$.

From the preceding discussion it would seem natural to investigate algebraic extensions of $R$ where $R$ is a completely $N$ primary ring or a duo completely $N$ primary ring.

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