LINEAR OPERATORS ON QUASI-CONTINUOUS FUNCTIONS

BY
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1. Introduction. In this paper we study a class of linear transformations for each of which the transform of a function is a function. This special class of transformations has such applications as the smoothing of experimental data, the prediction of outputs of a physical system for various inputs, and the estimation of the velocities and accelerations of an object from observations of its positions at various times.

By the statement that \( f \) is a function, we imply that if \( t \) is a real number, then \( f(t) \) is a number. By the statement that \( f \) is quasi-continuous\(^{(1)}\), we mean that \( f \) is a function such that if \( t \) is a real number then the limits \( f(t-) \) and \( f(t+) \) exist. Some lemmas on quasi-continuous functions appear in §2.

**Definition 1.1.** The statement that \( T \) is a Q operator over the interval \([a, b]\) means that \( T \) is a transformation such that

(i) if \( y \) is quasi-continuous, then \( Ty \) is a function; if \( g = Ty \) and \( s \) is a real number, then we denote the number \( g(s) \) by \( Ty(s) \),

(ii) if \( y_1 \) is quasi-continuous and \( y_2 \) is quasi-continuous, then \( T(y_1 + y_2) = Ty_1 + Ty_2 \),

(iii) if \( y \) is quasi-continuous and \( k \) is a number, then \( T(ky) = k(Ty) \),

(iv) if \( y \) is quasi-continuous, \( c \) is a real number, and \( z(t) = y(t+c) \) for each real number \( t \), then \( Tz(s) = Ty(s+c) \) for each real number \( s \), and

(v) if \( s \) is a real number, then there is a positive number \( B_s \) such that if \( y \) is quasi-continuous and \( M > |y(s-t)| \) for each number \( t \) in \([a, b]\), then \( |Ty(s)| \leq MB_s \); by the norm, \( |T(s)| \), of \( T \) at \( s \) we mean the greatest lower bound of the set of all such numbers \( B_s \).

It will be observed that if \( T \) is a Q operator over the interval \([a, b]\) and \( T_0y = Ty(0) \) for each quasi-continuous function \( y \), then \( T_0 \) is a bounded linear transformation from the set of all quasi-continuous functions to the set of all numbers (i.e., \( T_0 \) is a bounded linear functional operation as defined in [2] and [3]). We give the following example of a Q operator.

**Example 1.1.** Suppose that if \( y \) is quasi-continuous and \( s \) is a real number, then

\[
Ty(s) = \frac{[-y(s-2) - 3y(s-1) + 76y(s) + 76y(s+1) - 3y(s+2) - y(s+3)]}{144}.
\]

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\(^{(1)}\) Except for the use of the word "quasi-continuous," we use the terminology and notation of [1]. In particular, "integral" is defined as in [1].

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It follows that $T$ is a $Q$ operator over the interval $[-3, 2]$ and that $|T(s)| = 10/9$ for each real number $s$. Moreover, if $y$ is a polynomial of degree 3 or less, then $Ty(s) = \int_{s^-}^{s^+} y(t) \, dt$. This operator is designed for use with experimental data, for which the values used for $y$ in the formula may include errors of observation. In effect, the operator smooths the raw data, interpolates, and gives the integral of the smoothed and interpolated data; it is derived from Jenkins' modified osculatory interpolation formula [4].

In §3 we show that if $T$ is a $Q$ operator over the interval $[a, b]$, then $Ty$ is quasi-continuous, of bounded variation, or continuous, according as $y$ is quasi-continuous, of bounded variation, or continuous. In §4 we show that if $T$ is a $Q$ operator and $y$ is a quasi-continuous function then $Ty$ is the sum of two integrals.

We adopt the following notation. If $y$ is quasi-continuous, then $y_L$ and $y_R$ denote the functions such that $y_L(t) = y(t^-)$ and $y_R(t) = y(t^+)$ for each real number $t$.

**Definition 1.2.** The statement that $T$ is a $Q_1$ operator over the interval $[a, b]$ means that $T$ is a $Q$ operator over $[a, b]$ such that if $y$ is quasi-continuous then $Ty(s^+) - Ty(s^-) = 2[Ty_R(s) - Ty_L(s)]$ for each real number $s$; i.e., if $Ty = x$, $Ty_L = u$ and $Ty_R = v$, then $x(s^+) - x(s^-) = 2[v(s) - u(s)]$ for each real number $s$.

In §5 we show that a $Q_1$ operator $T$ is a $Q$ operator such that if $y$ is a quasi-continuous function then $Ty$ is an integral. In §6 we find conditions sufficient to assure that a $Q_1$ operator $T$ has various properties which may be desirable in applications. For example, we find a condition sufficient to assure that if $y$ is quasi-continuous then $Ty$ has a derivative, and we exhibit a $Q_1$ operator $T'$ such that $T'y$ is the derivative of $Ty$. In §7 we give a family of $Q_1$ operators, one of which is a limit of the "most powerful" smoothing operators given in [4].

2. **Lemmas concerning quasi-continuous functions.** The following results will be used later in this paper.

**Lemma 2.1.** For the function $f$ to be quasi-continuous, it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a step-function $s$ such that $|f(t) - s(t)| < \epsilon$ for each number $t$ in $[a, b]$.

For a proof, see Lemma 4.1b of [1]; see [5] also.

**Lemma 2.2.** For the function $f$ to be quasi-continuous it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a subdivision $t_0, t_1, \ldots, t_n$ of $[a, b]$ such that if $p$ and $q$ are in one of the segments $(t_i, t_{i+1})$ then $|f(p) - f(q)| < \epsilon$.

Proof is omitted.

**Lemma 2.3.** If $f_1, f_2, f_3, \ldots$ is a sequence of quasi-continuous functions which converges uniformly to a function $f$, then $f$ is quasi-continuous. Moreover,
if \( f_n(t+) = f_n(t) \) for each positive integer \( n \) and each real number \( t \), then \( f_R = f \); likewise, if \( f_n(t-) = f_n(t) \) for each positive integer \( n \) and each real number \( t \), then \( f_L = f \).

Proof is omitted.

**Lemma 2.4.** If \( f_1, f_2, \ldots \) is a sequence of functions which converges uniformly to a function \( f \), and \( [a, b] \) is an interval, and \( V > 0 \), and \( V_a^b(f_n) \leq V \) for each positive integer \( n \), then \( V_a^b(f) \leq V \).

For a proof, see Lemma 4.2a of [1].

**Lemma 2.5.** If \( f \) is a quasi-continuous function, then there are a quasi-continuous function \( g \) and a quasi-continuous function \( h \) such that

(i) \( g_L = g \) and \( h_R = h \),
(ii) \( g + h = f \), and
(iii) if \( [a, b] \) is an interval and \( |f(t)| \leq M < M_1 \) for each number \( t \) in \( [a, b] \), then \( |g(t)| \leq 1.5M_1 \) and \( |h(t)| \leq 1.5M_1 \) for each number \( t \) in \( [a, b] \).

**Proof.** We introduce the following notation. Suppose that \( [a, b] \) is an interval, \( z \) is quasi-continuous, and \( D \) is a subdivision \( t_0, t_1, \ldots, t_m \) of \( [a, b] \). Then \( L_Dz \) and \( R_Dz \) denote the pair of functions such that \( 2L_Dz(a) = z(a) \), \( 2R_Dz(b) = z(b) \), and for \( i = 0, 1, \ldots, m - 1 \),

\[
2(t_{i+1} - t_i)L_Dz(t) = (t_{i+1} - t_i)[2z(t_{i+1}) - z(t_i)] + (t - t_i)z(t_{i+1}) \quad \text{if} \quad t_i < t \leq t_{i+1},
\]

and

\[
2(t_{i+1} - t_i)R_Dz(t) = (t_{i+1} - t_i)z(t_i) + (t - t_i)[2z(t_{i+1}) - z(t_i)] \quad \text{if} \quad t_i \leq t < t_{i+1}.
\]

We make the following three observations. First, if \( |z(t)| \leq M \) for each number \( t \) in \( [a, b] \), then \( |L_Dz(t)| \leq 1.5M \) and \( |R_Dz(t)| \leq 1.5M \) for \( a \leq t \leq b \). Second, if \( \epsilon > 0 \) and it is true that \( |z(p) - z(q)| \leq \epsilon \) if \( p \) and \( q \) are in one of the segments \( (t_i, t_{i+1}) \), then \( |z(t) - L_Dz(t) - R_Dz(t)| \leq \epsilon \) for \( a \leq t \leq b \). Finally, \( L_Dz(t-) = L_Dz(t) \) if \( a < t \leq b \), and \( R_Dz(t+) = R_Dz(t) \) if \( a \leq t < b \).

Now suppose that \( [a, b] \) is an interval and \( |f(t)| \leq M < M_1 \) if \( a \leq t \leq b \). Let \( \epsilon \) denote a positive number less than \( M_1 - M \). We now construct a sequence \( f_1, f_2, f_3, \ldots \) of functions and a sequence \( D_1, D_2, D_3, \ldots \) of subdivisions of \( [a, b] \) in the following manner. Let \( f_1 \) denote the function \( f \), and let \( D_1 \) denote a subdivision \( t_0, t_1, \ldots, t_m \) of \( [a, b] \) such that if \( p \) and \( q \) are in one of the segments \( (t_i, t_{i+1}) \) then \( |f_1(p) - f_1(q)| < \epsilon/2 \). For each positive integer \( n \), let \( f_{n+1} = f_n - L_{D_n}f_n - R_{D_n}f_n \), where \( D_n \) denotes a subdivision \( t_0, t_1, \ldots, t_m \) of \( [a, b] \) such that if \( p \) and \( q \) are in one of the segments \( (t_i, t_{i+1}) \) then \( |f_n(p) - f_n(q)| \leq \epsilon/2^n \). Suppose that \( a \leq t \leq b \). Then \( |f_1(t)| \leq M \), and therefore \( |L_{D_1}f_1(t)| \leq 1.5M \) and \( |R_{D_1}f_1(t)| \leq 1.5M \); moreover, \( |f_2(t)| = |f_1(t) - L_{D_1}f_1(t) - R_{D_1}f_1(t)| \leq \epsilon/2 \). By induction, if \( n \) is an integer greater than 1, and \( a \leq t \leq b \), then \( |L_{D_n}f_n(t)| \leq 1.5\epsilon/2^{n-1} \), \( |R_{D_n}f_n(t)| \leq 1.5\epsilon/2^{n-1} \), and
\[
\left| f_1(t) - \sum_{p=1}^{n} L_{D_p}f_p(t) - \sum_{p=1}^{n} R_{D_p}f_p(t) \right|
= \left| f_{n+1}(t) \right| = \left| f_n(t) - L_{D_n}f_n(t) - R_{D_n}f_n(t) \right| \leq \varepsilon/2^n.
\]
But \( \sum_{p=1}^{n} L_{D_p}f_p \) converges uniformly in \([a, b]\) to a function \( g \); and if \( a \leq t \leq b \), then \( |g(t)| \leq \sum_{p=1}^{n} |L_{D_p}f_p(t)| \leq 1.5(M+\varepsilon) < 1.5M_1 \). Similarly, \( \sum_{p=1}^{n} R_{D_p}f_p \) converges uniformly in \([a, b]\) to a function \( h \); and if \( a \leq t \leq b \) then \( |h(t)| < 1.5M_1 \). Moreover, if \( a \leq t \leq b \) then \( g(t) + h(t) = f(t) \). From Lemma 2.3, it follows that if \( a < t \leq b \) then \( h(t-) \) exists and \( g(t-) = g(t) \), and that if \( a \leq t < b \) then \( g(t+) \) exists and \( h(t+) = h(t) \). This completes the proof.

**Lemma 2.6.** Suppose that \([t_0, t_1]\) is an interval, \( \varepsilon > 0 \), and \( f \) is a function such that

(i) if \( t_0 \leq t \leq t_1 \), then the derivative \( f'(t) \) exists, and

(ii) if \( s_1 \) and \( s_2 \) are in \([t_0, t_1]\), then \( |f'(s_1) - f'(s_2)| < \varepsilon. \) Then

\[
\left| \frac{f(t_1) - f(t_0)}{(t_1 - t_0)} - f'(t) \right| < \varepsilon(2)^{1/2}
\]

for each number \( t \) in \([t_0, t_1]\).

Proof is omitted, since this lemma can be obtained by applying the theorem to the real part and the imaginary part of \( f \).

**Lemma 2.7.** If \( f \) has a quasi-continuous derivative \( f' \), then \( f' \) is continuous.

Proof is omitted, since the lemma follows readily from well-known results and can be derived from Lemma 2.6.

**Lemma 2.8.** Suppose that \([a, b]\) is an interval, \( y \) is a function which is bounded in \([a, b]\), and \( x \) is a function whose derivative, \( x' \), is continuous in \([a, b]\). If \( \int_a^b y(t)x'(t)dt = I \) or \( \int_a^b y(t)x'(t)dt = I \), then \( \int_a^b y(t)x'(t)dt = \int_a^b y(t)x'(t)dt \).

Proof is omitted, since this lemma follows with little difficulty from Lemma 2.6.

**Lemma 2.9.** Suppose that \( f \) is a function whose derivative, \( f' \), is of bounded variation in the interval \([a, b]\), \( h \) is a real number other than zero, and \([c, d]\) is an interval such that if \( t \) is in \([c, d]\) then \( t \) and \( t+h \) are in \([a, b]\). If \( g(t) = \frac{[f(t+h) - f(t)]}{h} \) for each number \( t \) in \([c, d]\), then \( g \) is of bounded variation in \([c, d]\), and \( V'_f(g) \leq V^b_f(f') \).

Proof. Suppose that \( t_0, t_1, \ldots, t_n \) is a subdivision of \([c, d]\), and let \( S \) denote the sum \( \sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)| \). Now

\[
h[g(t_{i+1}) - g(t_i)] = \int_0^h 1df(t_{i+1} + t) - \int_0^h 1df(t_i + t)
= \int_0^h [f'(t_{i+1} + t) - f'(t_i + t)]dt.
\]
It follows that if $h > 0$, then
\[ \frac{h}{n} \sum_{i=0}^{n-1} \left| f'(t_{i+1}) - f'(t_i + h) \right| dt \]
and
\[ hS \leq \int_0^h \sum_{i=0}^{n-1} \left| f'(t_{i+1} + h) - f'(t_i + h) \right| dt \leq \int_0^h V_a^b(f')dt = hV_a^b(f'), \]
so that $S \leq V_a^b(f')$. By a similar argument, if $h < 0$, then $S \leq V_a^b(f')$. Hence $g$ is of bounded variation in $[c, d]$, and $V_a^b(g) \leq V_a^b(f')$. This completes the proof.

3. Some properties of $Q$ operators. In this section we suppose that there are given an interval $[a, b]$ and a $Q$ operator $T$ over $[a, b]$.

**Lemma 3.1a.** Suppose that $f$ is a real number and $y$ and $z$ are quasi-continuous functions such that $y(s - t) = z(s - t)$ if $a \leq t \leq b$. Then $Ty(s) = Tz(s)$.

**Proof.** Let $w = y - z$. By (ii) and (iii) of Definition 1.1, $Ty(s) - Tz(s) = Tw(s)$. But $w(s - t) = 0$ if $a \leq t \leq b$; and by (v) of Definition 1.1, if $\epsilon > 0$ then $\| Tw(s) \| \leq \epsilon \| T(s) \|$; hence $Tw(s) = 0$, or $Ty(s) = Tz(s)$. This completes the proof.

**Theorem 3.1.** If $s$ is a real number, then $\| T(s) \| = \| T(0) \|$. 

**Proof.** Suppose that $s$ is a real number. Now if $y$ is quasi-continuous and $z(t) = y(t - s)$ for each real number $t$, then $z(s - t) = y(-t)$ for each number $t$ in $[a, b]$; by Lemma 3.1a and (iv) of Definition 1.1, $Tz(s) = Ty(0)$. It follows from (v) of Definition 1.1 that $\| T(s) \| = \| T(0) \|$. This completes the proof.

**Remark.** In view of Theorem 3.1, we shall hereafter refer to the norm of $T$ as $\| T \|$; i.e., if $s$ is a real number, then $\| T \| = \| T(s) \|$.

**Theorem 3.2.** If $a < c < b$, then there are a $Q$ operator $T_1$ over $[a, c]$ and a $Q$ operator $T_2$ over $[c, b]$ such that

(i) if $y$ is quasi-continuous then $Ty = T_1y + T_2y$, and

(ii) $\| T_1 \| + \| T_2 \| \leq \| T \|$. 

**Proof.** If $y$ is quasi-continuous and $s$ is a real number, then we define numbers $T_1y(s)$ and $T_2y(s)$ in the following manner. Let $u(s - t) = 0$ if $t > c$ and $u(s - t) = y(s - t)$ if $t \leq c$ and let $T_1y(s) = Tu(s)$. Let $v(s - t) = y(s - t)$ if $t > c$ and $v(s - t) = 0$ if $t \leq c$; and let $T_2y(s) = Tv(s)$. By Definition 1.1, $T_1$ is a $Q$ operator over $[a, c]$, and $T_2$ is a $Q$ operator over $[c, b]$; moreover, if $y$ is quasi-continuous, then $T_1y + T_2y = Ty$.

We now show that $\| T_1 \| + \| T_2 \| \leq \| T \|$. Suppose that $\epsilon > 0$, $s$ is a real number and $y$ is a quasi-continuous function such that $\| T_y(s) \| > \| T \| - \epsilon$ and $\| y(s - t) \| \leq 1$ for each number $t$ in $[a, b]$. Now $\| T_1 \| + \| T_2 \| \leq \| T_y(s) \| + \| T_2y(s) \| \leq \| T_1y(s) + T_2y(s) \| = \| Ty(s) \| > \| T \| - \epsilon$. Hence $\| T_1 \| = T_1 + \| T_2 \| \leq \| T \|$. 

Finally we show that $\| T_1 \| + \| T_2 \| \leq \| T \|$. Suppose that $\epsilon > 0$, $s$ is a real
number, and \( z \) is a quasi-continuous function such that \( T_1 z(s) = |T_1 z(s)| > |T_1| - \epsilon, T_2 z(s) = |T_2 z(s)| > |T_2| - \epsilon, \) and \( |z(s - t)| \leq 1 \) if \( t \) is in \([a, b]\). Now \( |T| \geq |T_1| + |T_2| \). This completes the proof.

**Lemma 3.3a.** There is a number sequence \( c_1, c_2, c_3 \) such that if \( s \) is a real number and \( y \) is a step-function such that \( y(s - p) = y(s - q) \) for each pair \( p, q \) of numbers between \( a \) and \( b \), then \( Ty(s) = c_1 y(s - b) + c_2 y(s - b +) + c_3 y(s - a) \); moreover, \( |T| \geq |c_1| + |c_2| + |c_3| \).

**Proof.** Let functions \( u, v, w \) be defined as follows:

\[
 u(-t) = 1 \text{ if } t \geq b \text{ and } u(-t) = 0 \text{ if } t < b;
\]

\[
 v(-t) = 1 \text{ if } a < t < b \text{ and } v(-t) = 0 \text{ if } t \geq b \text{ or } t \leq a; \text{ and}
\]

\[
 w(-t) = 1 \text{ if } t \leq a \text{ and } w(-t) = 0 \text{ if } t > a.
\]

Let \( c_1 = Tu(0), c_2 = Tv(0), \) and \( c_3 = Tw(0) \). Now if \( s \) is a real number, \( a \leq t \leq b \), and \( y \) is a step-function such that \( y(s - p) = y(s - q) \) for each pair \( p, q \) of numbers between \( a \) and \( b \), then \( y(s - t) = y(s - b)u(-t) + y(s - b +)v(-t) + y(s - a)w(-t) \), and therefore \( Ty(s) = c_1 y(s - b) + c_2 y(s - b +) + c_3 y(s - a) \).

Now let \( d_1, d_2, d_3 \) denote a number sequence such that \( |d_1| = |d_2| = |d_3| = 1 \) and \( c_1 d_1 \geq 0, c_2 d_2 \geq 0, \) and \( c_3 d_3 \geq 0 \). For each real number \( t \), let \( z(t) = d_1 u(t) + d_2 v(t) + d_3 w(t) \). Now if \( a \leq t \leq b \), then \( |z(-t)| = 1 \), so that \( |Tz(0)| \leq |T| \); but \( Tz(0) = c_1 d_1 + c_2 d_2 + c_3 d_3 = |c_1| + |c_2| + |c_3| \); so \( |T| \geq |c_1| + |c_2| + |c_3| \). This completes the proof.

**Lemma 3.3b.** Suppose that \( t_0, t_1, \ldots, t_{2n} \) is a subdivision of \([a, b]\). There is a number sequence \( c_0, c_1, \ldots, c_{2n} \) such that if \( s \) is a real number and \( y \) is a step-function such that \( y(s - p) = y(s - q) \) for each pair \( p, q \) of numbers between \( t_i \) and \( t_{i+2}, i = 0, 1, \ldots, n - 1 \), then \( Ty(s) = \sum_{i=0}^{2n} c_i y(s - t_i) \); moreover, \( |T| \geq \sum_{i=0}^{2n} |c_i| \).

**Proof.** is omitted, since this lemma follows from Theorem 3.2 and Lemma 3.3a.

**Lemma 3.3c.** If \( y \) is a step-function and \([c, d]\) is an interval, then \( Ty \) is of bounded variation in \([c, d]\), and \( V^*_c(Ty) \leq |T| \cdot V^*_{c-y}(y) \).

**Proof.** Let \( s_0, s_1, \ldots, s_m \) denote a subdivision of \([c, d]\). Let \( t_0, t_1, \ldots, t_{2n} \) denote a subdivision of \([a, b]\) such that if \( j \) is one of the integers \( 0, 1, \ldots, m \), and \( i \) is one of the integers \( 0, 1, \ldots, n - 1 \), and \( p \) and \( q \) are numbers between \( t_i \) and \( t_{i+2}, n - 1 \), then \( y(s_j - p) = y(s_j - q) \). By Lemma 3.3b, there is a number sequence \( c_0, c_1, \ldots, c_{2n} \) such that \( Ty(s_j) = \sum_{i=0}^{2n} c_i y(s_j - t_i) \) for \( j = 0, 1, \ldots, m \), and \( |T| \geq \sum_{i=0}^{2n} |c_i| \). Now if \( j \) is one of the integers \( 0, 1, \ldots, m - 1 \), then \( Ty(s_j + 1) - Ty(s_j) = \sum_{i=0}^{2n} c_i [y(s_{j+1} - t_i) - y(s_j - t_i)] \). Hence
\[ \sum_{j=0}^{m-1} | Ty(s_{j+1}) - Ty(s_j) | = \sum_{j=0}^{m-1} \left| \sum_{i=0}^{2n} c_i [y(s_{j+1} - t_i) - y(s_j - t_i)] \right| \leq \sum_{j=0}^{m-1} \sum_{i=0}^{2n} | c_i | \cdot | y(s_{j+1} - t_i) - y(s_j - t_i) | \leq \sum_{i=0}^{2n} | c_i | \cdot | V_{c-t_i}^d(y) \leq | T | \cdot V_{c-b}^{d-a}(y). \]

The lemma now follows at once.

**Theorem 3.3.** If \( y \) is quasi-continuous, then \( Ty \) is quasi-continuous; if \( y \) is of bounded variation, then \( Ty \) is of bounded variation; if \( y \) is continuous, then \( Ty \) is continuous.

**Proof.** Suppose first that \( y \) is quasi-continuous. Let \( y_1, y_2, y_3, \ldots \) denote a sequence of step-functions which converges uniformly to \( y \). By (v) of Definition 1.1, \( Ty_1, Ty_2, Ty_3, \ldots \) converges uniformly to \( Ty \). Now if \([c, d]\) is an interval and \( n \) is a positive integer, then by Lemma 3.3c, \( Ty_n \) is of bounded variation in \([c, d]\). Hence each of the functions \( Ty_1 \) is quasi-continuous; by Lemma 2.3, \( Ty \) is quasi-continuous.

Suppose now that \( y \) is of bounded variation, and that \([c, d]\) is an interval, and \( V \geq V_{c-b}^d(y) \). Let \( y_1, y_2, y_3, \ldots \) denote a sequence of step-functions converging uniformly to \( y \) such that \( V_{c-q}^d(y_n) \leq V, \quad n = 1, 2, 3, \ldots \). Then \( Ty_1, Ty_2, Ty_3, \ldots \) is a sequence of functions converging to \( Ty \) uniformly, and \( V_{c-b}(Ty_n) \leq | T | V \) for \( n = 1, 2, 3, \ldots \). By Lemma 2.4, \( V_{c-b}(Ty) \leq | T | V \); hence \( Ty \) is of bounded variation.

Suppose, finally, that \( y \) is continuous and that \([c, d]\) is an interval. Let \( \varepsilon \) denote a positive number, and let \( \delta \) denote a positive number such that if \( c - b \leq p < q \leq d - a \) and \( q - p < \delta \), then \( | y(p) - y(q) | < \varepsilon \). Now suppose that \( c \leq s_1 < s_2 < d \) and \( s_2 - s_1 < \delta \). If \( a \leq t \leq b \), then \( | y(s_1 - t) - y(s_2 - t) | < \varepsilon \); so \( | Ty(s_1) - Ty(s_2) | < \varepsilon | T | \). Hence \( Ty \) is continuous. This completes the proof.

**Corollary 3.3a.** If \( T_1 \) is a Q operator over the interval \([a_1, b_1]\) and \( T_2 \) is a Q operator over the interval \([a_2, b_2]\), and \( T_3y = T_1(T_2y) \) for each quasi-continuous function \( y \), then \( T_3 \) is a Q operator over the interval \([a_1 + a_2, b_1 + b_2]\), and \( | T_3 | \leq | T_1 | \cdot | T_2 | \).

**Proof.** By Theorem 3.3, if \( y \) is quasi-continuous, then so is \( T_2y \); therefore \( T_3y \) is a function. It can readily be verified that \( T_3 \) has the properties listed as (ii), (iii), and (iv) in Definition 1.1. Let \( s \) denote a real number, suppose that \( y \) is quasi-continuous, and that \( M \geq | y(s - t) | \) if \( a_1 + a_2 \leq t \leq b_1 + b_2 - a_2 \). Now if \( a_1 \leq a_2 \leq b_2 \), then \( a_1 + a_2 \leq s_1 + t \leq b_1 + b_2 \); hence \( | y(s - s_1 - t) | \leq M \) if \( a_1 \leq s_1 \leq b_1 \) and \( a_2 \leq t \leq b_2 \), so that \( | T_3y(s - s_1) | \leq | T_2 | M \) if \( a_1 \leq s_1 \leq b_1 \), and therefore \( | T_3y(s) | \leq | T_1 | \cdot | T_2 | M \). Hence \( T_3 \) is a Q operator over \([a_1 + a_2, b_1 + b_2]\).
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Example 3.1. We give an example of Corollary 3.3a for which $|T_3| < |T_1| \cdot |T_2|$. Suppose that if $y$ is quasi-continuous and $s$ is a real number, then $T_1y(s) = [y(s^-) + y(s^+)]/2$ and $T_2y(s) = [y(s) - y(s^-)]/2$. Then $T_3y(s) = [y(s^-) - y(s^+)])/4$. Hence $|T_1| = 1$, $|T_2| = 1$, and $|T_3| = 1/2$.

4. Integral representation of $Q$ operators. In this section we suppose that there are given an interval $[a, b]$ and a $Q$ operator over $[a, b]$. We introduce the following notation:

$J(t) = 0$ if $t < 0$ and $J(t) = 1$ if $t \geq 0$,

$L(s) = TJ_L(s) \quad$ and $R(s) = TJ_R(s) = TJ(s)$ for each real number $s$,

$x_1(t) = 2L(t) - L(t^-)$ and $x_2(t) = 2R(t) - R(t^+)$ for each real number $t$.

We note in passing that $L(t) = [x_1(t^-) + x_1(t)]/2$ and $R(t) = [x_2(t) + x_2(t^+)])/2$.

Lemma 4.1a. $R(s) = L(s) = L(a) = 0$ if $s < a$; and $R(s) = L(s) = R(b)$ if $s > b$. Moreover, $R(s^-) = L(s^-)$ and $R(s^+) = L(s^+)$ for each real number $s$.

Proof. If $s < a$ and $a \leq t \leq b$, then $s - t < 0$, so that $J_L(s-t) = J_R(s-t) = 0$, and therefore $L(s) = R(s) = L(a) = 0$. If $s > b$ and $a \leq t \leq b$, then $s - t > 0$, so that $J_L(s-t) = J_R(s-t) = J_R(b-t) = 1$, and therefore $L(s) = R(s) = R(b)$.

Suppose now that $s$ is a real number. Let $s_1, s_2, s_3, \cdots$ denote a decreasing sequence which converges to $s$. Let $a_1, a_2, a_3, \cdots$ denote a number sequence such that if $p$ is a positive integer then $|a_p| = 1$ and $a_p[R(s_p) - L(s_p)] \geq 0$. For each positive integer $n$, let $f_n$ denote the step-function such that if $t$ is a real number then $f_n(t) = \sum_{p=1}^{a_p}[J_L(s_p + t) - J_L(s_p + t)]$. Then $Tf_n(0) = \sum_{p=1}^{a_p} |R(s_p) - L(s_p)|$. But if $a \leq t \leq b$, then $|f_n(-t)| \leq 1$, whence $|Tf_n(0)| \leq |T|$. Hence $\sum_{p=1}^{a_p} |R(s_p) - L(s_p)|$ converges, and therefore $|R(s_p) - L(s_p)| \to 0$ as $p \to +\infty$. It follows that $R(s) = L(s^+)$. By a similar argument, if $s$ is a real number, then $R(s^-) = L(s^-)$. This completes the proof.

Lemma 4.1b. If $s$ is a real number, then

$L(s) = \int_a^{b+} J_L(s - t)dx_1(t)$

and

$R(s) = \int_a^{b} J_R(s - t)dx_2(t)$.

Proof. By Lemma 4.1a, $x_1(t) = x_2(t) = x_1(a) = 0$ if $t < a$; and $x_1(t) = x_2(t) = x_2(b)$ if $t > b$. Moreover, by Theorem 3.3, $R$ and $L$ are of bounded variation, and therefore $x_1$ and $x_2$ are of bounded variation. Suppose that $s$ is a real number and $p$ is a positive number such that $p > s - b$ and $p > a - s$. Then $\int_a^{b+} J_L(s - t)dx_1(t) = \int_a^{b+} J_L(s - t)dx_1(t)$. By the integration by parts formula,
\[ \int_{a-p}^{b+p} J_L(s - t)dx_1(t) = J_L(s - b - p)x_1(b + p) - J_L(s - a + p)x_1(a - p) \]
\[ - \int_{a-p}^{b+p} x_1(t)dJ_L(s - t), \]
or
\[ \int_{a}^{b+p} J_L(s - t)dx_1(t) = [x_1(s-) + x_1(s)]/2 = L(s). \]

By a similar argument,
\[ \int_{a-p}^{b} J_R(s - t)dx_2(t) = [x_2(s) + x_2(s+)]/2 = R(s). \]

**Theorem 4.1.** Suppose that \( y \) is quasi-continuous and \( s \) is a real number. Let \( g \) and \( h \) denote quasi-continuous functions such that \( g_L = g, h_R = h, \) and \( g + h = y. \) Then

\[ T^y(s) = \int_{a}^{b+p} g(s - t)dx_1(t) + \int_{a-p}^{b} h(s - t)dx_2(t). \]

**Proof.** If \( g \) is a step-function, it follows from Lemma 4.1b that \( Tg(s) = \int_{a}^{b+p} g(s - t)dx_1(t); \) if \( g \) is not a step-function, it follows from Lemmas 2.3 and 4.1b of the present paper and Lemma 4.1a of \([1]\) that \( Tg(s) = \int_{a}^{b+p} g(s - t)dx_1(t). \) Similarly, \( Th(s) = \int_{a-p}^{b} h(s - t)dx_2(t). \) The theorem now follows from (ii) of Definition 1.1.

**Remark 4.1.** Upon comparing Theorem 4.1 with (v) of Definition 1.1, one might suppose that \( x_1(b+) = x_1(b) \) and \( x_2(a-) = x_2(a). \) To show that this need not be so, we give the following example. For each quasi-continuous function \( y \) and real number \( s, \) let \( Ty(s) = [y(s) + y(s-1)]/2, \) so that \( T \) is a \( Q \) operator over \([0, 1].\) For this example,

\[ x_1(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1/2 & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1. \end{cases} \]
\[ x_2(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1/2 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t \geq 1. \end{cases} \]

**Theorem 4.2.** Suppose that \([c, d]\) is an interval and \( u_1 \) and \( u_2 \) are functions of bounded variation such that

(i) \( u_1(t) = u_2(t) = u_1(c) = 0 \) if \( t < c, \)
(ii) \( u_1(t) = u_2(t) = u_2(d) \) if \( t > d, \) and
(iii) \( u_1(t-) = u_2(t-) \) and \( u_1(t+) = u_2(t+) \) for each real number \( t. \)

If \( s \) is a real number and \( g \) and \( h \) are quasi-continuous functions such that \( g_L = g \) and \( h_R = h, \) let \( Ug(s), Uh(s), \) and \( Uy(s), \) where \( y = g + h, \) denote the numbers \( \int_c^{d+} g(s - t)du_1(t), \int_c^{d-} h(s - t)du_2(t), \) and \( Ug(s) + Uh(s), \) respectively. Then \( U \) is a
Proof. We show first that if \( y \) is quasi-continuous, then there is just one function which is \( Uy \). Suppose that \( y \) is quasi-continuous and that \( g, h, p, q \) is a sequence of quasi-continuous functions such that \( g_L = g, h_R = h, p_L = p, q_R = q \), \( g + h = y \), and \( p + q = y \). Suppose that \( f = g - p \); then \( f_L = f \). But \( f = q - h \), and therefore \( f_R = f \); so \( f \) is continuous. Now if \( s \) is a real number, then

\[
\int_{s-\infty}^{s+\infty} f(s-t)du_1(t) - \int_{s-\infty}^{s+\infty} f(s-t)du_2(t) = \int_{s-\infty}^{s+\infty} f(s-t)d[u_1(t) - u_2(t)].
\]

Since \( f \) is continuous, \( u_1 - u_2 \) is of bounded variation, and \( u_1 - u_2 \) is zero except for a countable subset of \( [c, d] \), it follows that \( \int_{s-\infty}^{s+\infty} f(s-t)d[u_1(t) - u_2(t)] = 0 \), so that \( \int_{s-\infty}^{s+\infty} f(s-t)du_1(t) = \int_{s-\infty}^{s+\infty} f(s-t)du_2(t) \), or \( \int_{s-\infty}^{s+\infty} g(s-t)du_1(t) - \int_{s-\infty}^{s+\infty} p(s-t)du_1(t) = \int_{s-\infty}^{s+\infty} h(s-t)du_2(t) - \int_{s-\infty}^{s+\infty} q(s-t)du_2(t) \), or \( Ug(s) - Up(s) = Uq(s) - Uh(s) \). Hence \( Ug(s) + Uh(s) = Up(s) + Uq(s) \); i.e., there is just one function which is \( Uy \). Moreover, if \( |y(s-t)| \leq M < M_1 \) for each number \( t \) in \( [c, d] \), then by Lemma 2.5 the functions \( p \) and \( q \) may be chosen so that \( |p(s-t)| \leq 1.5M_1 \) and \( |q(s-t)| \leq 1.5M_1 \) for each number \( t \) in \( [c, d] \); so by Theorem 2.1 of [1], \( |Uy(s)| \leq 1.5M_1 [V_{d+}^e(u_1) + V_{d-}^e(u_2)] \). The theorem now follows from Definition 1.1 and the properties of integrals (cf. Theorem 2.1 of [1]).

Remark 4.2. Compare Theorem 4.1 with the representation given in [2]; for an expression for the norm \( |T| \), see [6].

5. \( Q_1 \) operators. In this section, we show that if \( T \) is a \( Q_1 \) operator and \( y \) is a quasi-continuous function, then \( Ty \) is an integral; and we obtain a condition sufficient for the product of two \( Q_1 \) operators to be a \( Q_1 \) operator.

Theorem 5.1. For \( T \) to be a \( Q_1 \) operator over the interval \( [a, b] \) it is necessary and sufficient that there is a function \( x \) such that

(i) \( x \) is of bounded variation, \( x(t) = 0 \) if \( t \leq a \), \( x(t) = x(b) \) if \( t > b \), and

(ii) if \( y \) is quasi-continuous and \( s \) is a real number, then

\[
Ty(s) = \int_{a}^{b} y(s-t)dx(t).
\]

Proof. A. Suppose that \( T \) is a \( Q_1 \) operator over \( [a, b] \), and let \( R, L, x_1, \) and \( x_2 \) be defined as in §4. By Definition 1.2, \( R(s+) - R(s-) = 2[R(s) - L(s)] \) for each real number \( s \); and by Lemma 1.4a, \( R(s-) = L(s-) \), so that \( 2R(s) - R(s+) = 2L(s) - L(s-) \), or \( x_2(s) = x_1(s) \) for each real number \( s \). It follows from Theorem 3.3 that \( x_1 \) is of bounded variation, from Lemma 4.1a that \( x_1(t) = 0 \) if \( t \leq a \) and \( x_1(t) = x_1(b) \) if \( t > b \), and from Theorem 4.1 that \( Ty(s) = \int_{a}^{b} y(s-t)dx(t) \) for each quasi-continuous function \( y \) and real number \( s \).

B. Suppose that \( x \) is a function such that (i) and (ii) of the theorem are true. By Theorem 4.2, \( T \) is a \( Q \) operator over \( [a, b] \). Let \( L \) and \( R \) be defined as in §4. Now if \( s \) is a real number, then \( R(s) = [x(s) + x(s+)]/2 \) and \( L(s) = [x(s-) + x(s)]/2 \), whence \( 2R(s) - R(s+) = 2L(s) - L(s-) = x(s) \); since \( R(s+) = L(s+) \) and \( R(s-) = L(s-) \), it follows that \( R(s+) - R(s-) = 2[R(s) - L(s)] \) and \( L(s+) - L(s-) = 2[R(s) - L(s)] \). Now if \( y \) is a step-function and \( s \) is a real number, then there are a number sequence \( a_1, a_2, \ldots, a_{2n} \) and a real-number sequence \( t_1, t_2, \ldots, t_n \) such that
\[ y(s - t) = \sum_{p=1}^{n} \left[ a_{2p-1} J_R(s - t_p - t) + a_{2p} J_L(s - t_p - t) \right] \text{ if } a \leq t \leq b; \]

hence \( T y(s) = \sum_{p=1}^{n} \left[ a_{2p-1} R(s - t_p) + a_{2p} L(s - t_p) \right], \) so that \( T y(s) = 2 \left[ T y_R(s) - T y_L(s) \right]. \) It now follows from Lemmas 2.1, 3.3c, and 2.3 that if \( y \) is quasi-continuous and \( s \) is a real number, then \( T y(s+) - T y(s-) = 2 \left[ T y_R(s) - T y_L(s) \right]. \) Hence \( T \) is a \( Q_1 \) operator over \([a, b]\). This completes the proof.

**Remark 5.1.** If \( T \) is a \( Q_1 \) operator over an interval \([a, b]\) then there is just one function \( x \) such that

(i) \( x(a) = 0 \) and

(ii) if \( y \) is quasi-continuous and \( s \) is a real number, then \( T y(s) = \int_{a}^{s} y(t) dt \). Moreover, \( x \) is of bounded variation.

**Theorem 5.2.** Suppose that \( U \) is a \( Q_1 \) operator over the interval \([a, b]\), \( V \) is a \( Q_1 \) operator over the interval \([c, d]\), and \( T y = U(V y) \) for each quasi-continuous function \( y \). Let \( u \) and \( v \) denote the functions such that \( u(a) = v(c) = 0 \) and \( U y(s) = \int_{a}^{s} y(t) dt \) and \( V y(s) = \int_{c}^{s} y(t) dt \) if \( y \) is quasi-continuous and \( s \) is a real number. If \( u_L = u_R \) or \( v_L = v_R \), then \( T \) is a \( Q_1 \) operator over \([a+c, b+d]\).

**Proof.** By Corollary 3.3a, \( T \) is a \( Q \) operator over \([a+c, b+d]\). Suppose that \( u_L = u_R \) or \( v_L = v_R \). Let \( R \) and \( L \) denote the functions \( T J_R \) and \( T J_L \), respectively. Now \( V J_L = (v_L + v)/2 \) and \( V J_R = (v + v_R)/2 \); hence if \( s \) is a real number, then \( 2L(s) = \int_{a}^{s} [v_L(s-t) + v(s-t)] du(t) \) and \( 2R(s) = \int_{a}^{s} [v(s-t) + v_R(s-t)] du(t) \), so that \( 2[R(s) - L(s)] = \int_{a}^{s} [v_R(s-t) - v_L(s-t)] du(t) \). Since \( v \) is of bounded variation, so is \( v_R - v_L \); and if \( v_R \neq v_L \), then there is a countable real-number set \( K \) such that \( v_R(t) \neq v_L(t) \) if and only if \( t \) is in \( K \). Since \( u \) is of bounded variation, it follows from Theorem 3.1 of [1] and Lemma 4.2a of [1] that

\[ 4[R(s) - L(s)] = \sum_{(i)} [v_R(s - t) - v_L(s - t)] [u(t) - u(t-)] \]

\[ + \sum_{(i)} [v_R(s - t) - v_L(s - t)] [u(t+) - u(t)] \]

\[ = \sum_{(i)} [v_R(s - t) - v_L(s - t)] [u_R(t) - u_L(t)]. \]

But by hypothesis, \( v_R = v_L \) or \( u_R = u_L \); hence \( R = L \).

Now \( 2L = U(v_L + v) \), where \( U \) is a \( Q_1 \) operator; hence if \( s \) is a real number, then

\[ 2[L(s+) - L(s-)] = 4[U v_R(s) - U v_L(s)]. \]

But \( U v_R - U v_L = U(v + v_R) - U(v_L + v) = 2(R - L) = 0 \). Hence if \( s \) is a real number then \( L(s+) - L(s-) = 0 \); and by Lemma 4.1a, \( R(s+) = L(s+) \) and \( R(s-) = L(s-) \); hence \( R(s+) - R(s-) = 2[R(s) - L(s)] = 0 \). By the argument in part B of the proof of Theorem 5.1, it follows that \( T \) is a \( Q_1 \) operator over \([a+c, b+d]\). This completes the proof.
6. \(Q_1\) operators having specified properties. In this section we suppose that \(T\) is a \(Q_1\) operator over an interval \([a, b]\), and that \(x\) is the function of bounded variation such that \(x(t) = 0\) if \(t \leq a\), \(x(t) = x(b)\) if \(t > b\), and \(Ty(s) = \int_0^s y(s-t)dx(t)\) if \(y\) is quasi-continuous and \(s\) is a real number. We shall obtain conditions on \(x\) which are sufficient to assure that \(T\) has various ones of the properties described in the following definition.

**Definition 6.1.** (i) The statement that \(T\) is symmetric means that if \(y\) is quasi-continuous and \(z(t) = y(-t)\) for each real number \(t\), then \(Tz(s) = Ty(-s)\) for each real number \(s\).

(ii) The statement that \(T\) has property B means that if \(y\) is quasi-continuous, then \(Ty\) is of bounded variation.

(iii) The statement that \(T\) has property C means that if \(y\) is quasi-continuous, then \(Ty\) is continuous.

(iv) The statement that \(T\) has property D means that (a) if \(y\) is quasi-continuous, then \(Ty\) has a derivative, and (b) there is a positive number \(N\) such that if \(y\) is quasi-continuous, \(s\) is a real number, and \(M \geq |y(s-t)|\) for each number \(t\) in \([a, b]\), and \(g = Ty\), then \(|g'(s)| \leq MN\).

(v) If \(K\) is a set of quasi-continuous functions, then the statement that the members of \(K\) are invariant under \(T\) means that if \(y\) is in \(K\) then \(Ty = y\).

**Theorem 6.1.** For \(T\) to be symmetric, it is necessary and sufficient that \(x(-t) - x(0) = x(0) - x(t)\) for each real number \(t\).

**Proof.** A. Suppose that \(T\) is symmetric. If \(y\) is quasi-continuous, and \(z(t) = y(-t)\) for each real number \(t\), then \(Tz(0) = Ty(0)\), or \(\int_{-\infty}^\infty z(-t)dx(t) = \int_{-\infty}^\infty y(-t)dx(t)\), or \(\int_{-\infty}^\infty [y(t) - y(-t)]dx(t) = 0\). Hence if \(f\) is an odd quasi-continuous function, then \(\int_{-\infty}^\infty f(t)dx(t) = 0\). Suppose that if \(p > 0\) then

\[
 f_p(t) = \begin{cases} 
 0 & \text{if } t < -p, \text{ or } t = 0 \text{ or } t > p, \\
 -1 & \text{if } -p \leq t < 0, \\
 1 & \text{if } 0 < t \leq p.
\end{cases}
\]

Since \(f_p\) is an odd function, it follows that \(\int_{-\infty}^\infty f_p(t)dx(t) = 0\). Integration by parts gives the equation \(\int_{-\infty}^\infty x(t)df_p(t) = 0\). By Theorem 3.1 of [1],

\[
 [x(-p-) - x(0)] + [x(-p) - x(0)] - [x(0-) - x(0)] - [x(0+) - x(0)]
 + [x(p) - x(0)] + [x(p+) - x(0)] = 0
\]

for each positive number \(p\). Consequently, upon considering a sequence of positive numbers \(p_1, p_2, p_3, \ldots\) which converges to zero, we conclude that \(x(0-) - x(0) = x(0) - x(0+)\). By a similar argument, if \(p > 0\), then \(x(-p-) - x(0) = x(0) - x(p+)\); hence \(x(-p) - x(0) = x(0) - x(p)\).

B. Suppose that \(x(-t) - x(0) = x(0) - x(t)\) for each real number \(t\). Suppose that \(y\) is quasi-continuous, \(z(t) = y(-t)\) for each real number \(t\), and \(s\) is a real number. Then
\[ Tz(s) = \int_{-\infty}^{+\infty} z(s - t)dx(t) = \int_{-\infty}^{+\infty} z(s + t)dx(-t) \]
\[ = \int_{+\infty}^{\infty} y(-s - t)d[x(-t) - x(0)] \]
\[ = -\int_{+\infty}^{\infty} y(-s - t)d[x(t) - x(0)] = \int_{-\infty}^{+\infty} y(-s - t)dx(t) = Ty(-s). \]

This completes the proof.

**Theorem 6.2.** Suppose that if \([c, d]\) is an interval then there is a positive number \(N\) such that if \(s_0, s_1, \cdots, s_m\) is a subdivision of \([c, d]\) and \(t_0, t_1, \cdots, t_n\) is a subdivision of \([a - d, b - c]\), then
\[ m-1 n-1 \sum \sum |x(s_{i+1} + t_{j+1}) - x(s_{i+1} + t_j) - x(s_i + t_{j+1}) + x(s_i + t_j)| \leq N. \]

**Then** \(T\) **has property B.**

**Proof.** Suppose that \(y\) is quasi-continuous, \([c, d]\) is an interval, and \(M \geq |y(s - t)|\) if \(s\) is in \([c, d]\) and \(t\) is in \([a, b]\). Let \(s_0, s_1, \cdots, s_m\) denote a subdivision of \([c, d]\), and suppose that \(\epsilon > 0\). Now if \(s\) is a real number, then \(Ty(s) = \int_{-\infty}^{+\infty} y(s - t)dx(t) = \int_{-\infty}^{+\infty} y(-t)dx(s + t)\); hence if \(I\) denotes the sum \(\sum_{i=0}^{m-1} Ty(s_{i+1}) - Ty(s_i)\), then \(I = \sum_{i=0}^{m-1} \int_{-\infty}^{+\infty} y(-t)d[x(s_{i+1} + t) - x(s_i + t)]\).

Let \(t_0, t_1, \cdots, t_n\) denote a subdivision of \([a - d, b - c]\) such that if \(i\) is one of the integers 0, 1, \cdots, \(m-1\), then
\[ \int_{-\infty}^{+\infty} y(-t)d[x(s_{i+1} + t) - x(s_i + t)] \]
\[ - 2^{-1} \sum_{j=0}^{n-1} [y(-t_j) + y(-t_{j+1})][x(s_{i+1} + t_{j+1}) - x(s_i + t_{j+1})] \]
\[ - x(s_{i+1} + t_j) + x(s_i + t_j)] < \epsilon/m. \]

Now \(|y(-t)| \leq M\) if \(a - d \leq t \leq b - c\); hence \(I < MN + \epsilon\), and consequently \(V^* (Ty) \leq MN\). This completes the proof.

**Corollary 6.2a.** If \(x\) has a derivative which is of bounded variation, then \(T\) **has property B.**

**Proof.** Suppose that \([c, d]\) is an interval. If \(D\) is a subdivision \(s_0, s_1, \cdots, s_m\) of \([c, d]\) and \(E\) is a subdivision \(t_0, t_1, \cdots, t_n\) of \([a - d, b - c]\), let \(S(D, E)\) denote the sum \(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x(s_{i+1} + t_{j+1}) - x(s_{i+1} + t_j) - x(s_i + t_{j+1}) + x(s_i + t_j)\). Suppose that \(D\) and \(E\) are such subdivisions and that \(\epsilon > 0\). By hypothesis, \(x'\) is of bounded variation and (by Lemma 2.7) is therefore continuous; moreover, \(x'(t) = 0\) if \(t < a\) or \(t > b\). Hence there is a positive number \(\delta\) such that if
\(p\) and \(q\) are real numbers and \(|p - q| < \delta\), then \(|x'(p) - x'(q)| < \epsilon\). Let \(u_0, u_1, \ldots, u_r\) denote a refinement \(E'\) of \(E\) such that \(u_{j+1} - u_j < \delta, j = 0, 1, \ldots, r - 1\). Now

\[
S(D, E') = \sum_{i=0}^{m-1} \sum_{j=0}^{r-1} \left| \frac{x(s_{i+1} + u_{j+1}) - x(s_{i+1} + u_j)}{u_{j+1} - u_j} \right| \left| \frac{x(s_i + u_{j+1}) - x(s_i + u_j)}{u_{j+1} - u_j} \right| (u_{j+1} - u_j);
\]

and by Lemma 2.6,

\[
S(D, E') < 2\left[ (b - c) - (a - d) \right] \epsilon (2)^{1/2} + \sum_{i=0}^{m-1} \sum_{j=0}^{r-1} \left| x'(s_{i+1} + u_j) - x'(s_i + u_j) \right| (u_{j+1} - u_j)
\]

\[
< 2\left[ (b - c) - (a - d) \right] \epsilon (2)^{1/2} + \sum_{j=0}^{r-1} (u_{j+1} - u_j) V^+_{\infty}(x')
\]

\[
< \left[ (b - c) - (a - d) \right] [2\epsilon (2)^{1/2} + V^+_{\infty}(x')].
\]

But since \(E'\) is a refinement of \(E\), it follows that \(S(D, E) \leq S(D, E')\). Consequently, \(S(D, E) \leq \left[ (b - c) - (a - d) \right] V^+_{\infty}(x')\). By Theorem 6.2, \(T\) has property B. This completes the proof.

**Theorem 6.3.** For \(T\) to have property C, it is necessary and sufficient that \(x\) is continuous.

**Proof.** Suppose that \(T\) has property C; then \(TJL\) is continuous and \(TJR\) is continuous. Since \(TJL = (x_L + x)/2\) and \(TJR = (x + x_R)/2\), it follows that \(x\) is continuous. Suppose now that \(x\) is continuous and that \(\epsilon > 0\). Since \(x(t) = 0\) if \(t \leq a\) and \(x(t) = x(b)\) if \(t > b\), there is a positive number \(\delta\) such that if \(p\) and \(q\) are real numbers and \(|p - q| < \delta\), then \(|x(p) - x(q)| < \epsilon\). Now if \(s_1\) and \(s_2\) are two real numbers and \(|s_1 - s_2| < \delta\), and \(g(t) = x(s_1 + t) - x(s_2 + t)\) for each real number \(t\), then \(|g(t)| < \epsilon\) for each real number \(t\), and \(V^+_{\infty}(g) \leq 2V^+_{\infty}(x)\). Suppose that \(y\) is quasi-continuous, \(s_1\) is a real number, and \([c, d]\) is an interval containing \(s_1\). If \(s_2\) is in \([c, d]\), then \(|T_y(s_1) - T_y(s_2)| = \int_{s_1}^{s_2} \frac{d}{dt} \int_{x(s_1 + t)}^{x(s_2 + t)} dx'(t)|\); so by Lemma 4.2a of [1], \(T_y(s_2) - T_y(s_1)\) as \(s_2 \to s_1\). That is to say, \(T\) is continuous, or \(T\) has property C. This completes the proof.

**Theorem 6.4.** For \(T\) to have property D, it is necessary and sufficient that \(x\) has a derivative which is of bounded variation. Moreover, if \(T\) has property D, \(y\) is quasi-continuous, and \(g = Ty\), then \(g'\) is continuous, and \(g'(s) = \int_{s-t}^{s} y(s-t) dx'(t)\) for each real number \(s\).

**Proof.** A. Suppose that \(T\) has property D. If \(y\) is quasi-continuous and \(g = Ty\), let \(Uy\) denote the function \(g'\). It follows from Definition 1.1 and (iv) of
Definition 6.1 that \( U \) is a \( Q \) operator over \([a, b]\). By Theorem 3.3, if \( y \) is quasi-
continuous, then \( Uy \) is quasi-continuous; and by Lemma 2.7, \( Uy \) is continu-
iuous; i.e., if \( y \) is quasi-continuous and \( g = Ty \), then \( g' \) is continuous. In par-

ticular, \( UJ_R \) and \( UJ_L \) are continuous; so by Lemma 4.1a, \( UJ_R = UJ_L \), and it
follows that \( U \) is a \( Q_1 \) operator over \([a, b]\). Let \( u \) denote the function \( UJ \).

If \( y \) is quasi-continuous and \( s \) is a real number, then \( Uy(s) = \int_a^s y(s-t)du(t) \). But
if \( R = TJ_R \), then \( u = R' \); and since \( R \) has a derivative, \( R \) is continuous
so that \( x = R \), whence \( x' = R' = u \). Consequently, if \( y \) is quasi-continuous and
\( s \) is a real number, then \( Uy(s) = \int_a^s y(s-t)dx'(t) \). Moreover, since \( U \) is a \( Q_1 \)
operator, \( x' \) is of bounded variation.

B. Suppose that \( x \) has a derivative which is of bounded variation. By
Lemma 2.7, \( x' \) is continuous; and we observe that \( x'(t) = 0 \) if \( t \leq a \) or \( t \geq b \).
Suppose that \( \epsilon > 0 \), and let \( h \) denote a real number other than zero such that
if \( p < q \) and \( q - p \leq |h| \) then \( |x'(p) - x'(q)| < \epsilon \). For each real number \( t \), let
\( g_h(t) \) denote the number \( \frac{x(t+h) - x(t)}{h} \). By Lemma 2.6, \( |g_h(t) - x'(t)| < 2\epsilon \)
for each real number \( t \); and by Lemma 2.9, the total variation of \( g_h \) is less
than or equal to the total variation of \( x' \). Now if \( y \) is quasi-continuous and
\( s \) is a real number, then \( \left[ Ty(s+h) - Ty(s) \right]/h = \int_{-\infty}^{+\infty} y(-t)dx'(s+t) \); and by
Lemma 4.2a of [1], \( \left[ Ty(s+h) - Ty(s) \right]/h = \int_{-\infty}^{+\infty} y(-t)dx'(s+t) = \int_a^s y(s-t)dx'(t) \)
as \( h \to 0 \). If \( y \) is quasi-continuous and \( s \) is a real number, let \( T'y(s) \) denote the
number \( \int_a^s y(s-t)dx'(t) \). Then \( T' \) is a \( Q_1 \) operator over \([a, b]\), such that if \( y \) is
quasi-continuous and \( g = Ty \), then \( g' = T'y \). Hence \( T \) has property D. This
completes the proof.

Theorem 6.5. Suppose that \( n \) is a positive integer. For all polynomials of
degree \( n \) or less to be invariant under \( T \), it is necessary and sufficient that \( x(b) = 1 \),
and \( \int_a^b px(t)dx(t) = 0 \) for \( p = 1, 2, \ldots, n \). If \( T \) is symmetric, and \( n \) is an even
integer, and all polynomials of degree \( n \) or less are invariant under \( T \), then all
polynomials of degree \( n+1 \) or less are invariant under \( T \).

Proof is omitted, since the theorem readily follows if for each polynomial
\( y \) and real number \( s \) we consider the Maclaurin expansion of \( y(s-t) \) in powers
of \( t \).

7. A family of smoothing operators and differentiating operators. In this
section we suppose that \( n \) is a positive integer and \([a, b]\) is an interval of
unit length. We shall exhibit a \( Q_1 \) operator \( T \) on \([a, b]\) such that
(i) all polynomials of degree 2 or less are invariant under \( T \), and
(ii) if \( y \) is quasi-continuous, then \( Ty \) has an \( n \)th derivative.

Let polynomials \( u, v, \) and \( w \) be defined as follows. If \( t \) is a real number, then
\[
2u(t) = 1 + \sum_{p=0}^{n} \binom{2p}{p} (t-a)^p (b-t)^p,
\]
\[
v(t) = (n+2) \binom{2n+3}{n+1} (t-a)^{n+1} (b-t)^{n+1},
\]
\[ w(t) = (n + 2) \left( \frac{2n + 3}{n + 1} \right) (t - a)^{n+2} (b - t)^{n+1}. \]

Let \( A = (2n+5) [2ab+(n+1)/(2n+3)] \) and \( B = (a+b+A)/2 \). Let \( x \) denote the function such that \( x(t) = 0 \) if \( t < a \), \( x(t) = 1 \) if \( t > b \), and \( x(t) = u(t) + Bv(t) - Aw(t) \) if \( a \leq t \leq b \). If \( y \) is quasi-continuous and \( s \) is a real number, let \( T_y(s) \) denote the number \( \int_a^b s^2 y(s-t) dx(t) \), and let \( T'_y(s) = \int_a^b s^2 y(s-t) dx'(t) \).

By some rather tedious manipulation, it can be seen that \( x \) has a continuous \( n \)th derivative which is of bounded variation, and that \( x(a) = 0 \), \( x(b) = 1 \), \( \int_a^b dx(t) = 0 \), and \( \int_a^b t^2 dx(t) = 0 \), so that polynomials of degree 2 or less are invariant under \( T \). Moreover, if \( a = -1/2 \), then \( T \) is symmetric, so that polynomials of degree 3 or less are invariant under \( T \). If \( y \) is quasi-continuous and \( g = Ty \), then \( g' = T'_y \).

In particular, if \( n = 3 \) and \( a \leq t \leq b \), then
\[
x(t) = 0.5 + 0.5(2t - a - b) \left[ 1 + 2(1 - a)(1 - b) + 6(1 - a)^2 (1 - b)^2 \right.
+ 20(1 - a)^3 (1 - b)^3 - 140(22 + 99ab)(1 - a)^4 (1 - b)^4 \]
\[ + 315(a + b)(1 - a)^4 (1 - b)^4, \]

and
\[
x'(t) = 140 (1 - a)^3 (1 - b)^3 \left[ (45 + 198ab) + 9(a + b)(a + b - 2t) \right.
- (198 + 891ab)(1 - a)(1 - b)].
\]

In particular, if \( n = 3 \), \( a = -1/2 \), and \( -1/2 \leq t \leq 1/2 \), then
\[
x(t) = 0.5 + t \left[ 1 + 2(.25 - t^2) + 6(.25 - t^2)^2 + 20(.25 - t^2)^3 \right.
+ 385(.25 - t^2)^4],
\]

and
\[
x'(t) = 315(.25 - t^2)^3 \left[ 11(.25 - t^2) - 2 \right],
\]

and \( T \) is a limit of the "most powerful" smoothing operators (i.e., operators with minimum smoothing coefficients) described, e.g., in [4]. For this instance the operators \( T \) and \( T' \) have been used with quite satisfactory results on a digital computer with experimental data, the integrals being approximated by an approximating sum as in [1], with the subdivision \( -.50, -.45, -.40, \ldots, .45, .50 \) of the interval \([-1/2, 1/2]\).

If \( n = 3 \), \( a = 0 \), and \( 0 \leq t \leq 1 \), then
\[
x(t) = 0.5 + 0.5(2t - 1) \left[ 1 + 2t(1 - t) + 6t^2 (1 - t)^2 + 20t^3 (1 - t)^3 \right.
- 3080t^4 (1 - t)^4 \] + 315t^4 (1 - t)^4,
\]

and
\[ x'(t) = \frac{2520 t^3 (1 - t)^3 (3 - 12t + 11t^2)}{3}; \]

in this instance \( T \) is not symmetric, and the application of the operators \( T \) and \( T' \) to experimental data does not give results as satisfactory as those obtained with the symmetric operator previously described.

References


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