PRIMARY INTERSECTIONS FOR TWO SIDED IDEALS OF A NOETHERIAN MATRIX RING

BY

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Introduction. The purpose of this paper is to obtain certain primary intersections as described in [1] for all two sided ideals in the matrix ring D_n where D is a Noetherian ring. We refer to such a ring as a Noetherian matrix ring. The primary intersections will depend only upon the Noetherian ring D. The following discussion will show that if the primary intersections of the ideals in D are known one can immediately write primary intersections for all two sided ideals in D_n.

1. **Reformulation of theorems.** The following two theorems are reformulations of the author's Theorems 2.5 and 2.7 of [1]. The proofs are very similar to those of Noether [5] and Krull [4] and therefore were not included in [1] and are not included here. (See [2, pp. 172–181]). The definitions of [1] are used here.

THEOREM 1.1. *Let* $N = N_1 \cap \cdots \cap N_s = N_1^\# \cap \cdots \cap N_s^\#$ *be irredundant intersections where* N_i, $N_i^\#$ *are irreducible R submodules,* $i = 1, 2, \cdots, s$. *Let H be a subring of* $\bigcap_{i=1}^{s} [V^*(N_i) \cap V^*(N_i^\#)]$ *containing the identity element, then the set of distinct H radicals of* $N_1, \cdots, N_s$ *is identical with the set of distinct H radicals of* $N_1^\#, \cdots, N_s^\#$ *in H.*

From this theorem and Theorem 2.6 of [1] we have

THEOREM 1.2. *Let N be an R submodule of the A-R module M which satisfies the A.C.C. for R submodules*(1). *Let* α *be an index that ranges over a possibly infinite set G whose cardinal number is* ψ *and let* $N = \bigcap_{i=1}^{t} N_{i\alpha}$ *be a set of* ψ *irredundant representations of N as the intersection of irreducible R submodules* $N_{i\alpha}$ *of M. Let* $H(G)$ *be a subring with identity of* $\bigcap V^*(N_{i\alpha})$ *where i ranges from* 1 *to t and* α *ranges over G. Then for the* α*th intersection there exist H primary R submodules* $N'_{1\alpha}, \cdots, N'_{s\alpha}$ *with distinct H radicals* $p_1, \cdots, p_s$ *such that* $N = N'_{1\alpha} \cap \cdots \cap N'_{s\alpha}$. *If* $N = N'_{1\beta} \cap \cdots \cap N'_{r\beta}$ *is another such intersection where* β *is an index of G then* $r = s$ *and for a suitable rearrangement of the subscripts the corresponding H radicals are equal.*

2. **Primary intersections for two sided ideals in** D_n. A ring with identity as a A-R module if one takes as A the ring of left multiplications and as R the ring of right multiplications. Thus theorems of this paper and [1] apply to rings with identity that satisfy the A.C.C. for right ideals.

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(1) An A-R module is defined as a right A, right R module in [3, page 17].

Let D be a Noetherian ring. We shall consider the application of these theorems to all two sided ideals of D_n, the ring of n by n matrices with elements in D.

If I is an ideal of D then the set of all matrices (a_{ij}) with $a_{ij} \in D$ for $i \neq k$ and $a_{kj} \in I$ is a right ideal of D_n which we shall denote by (I, k).

STATEMENT 2.1. If I is an ideal of D and (I, k) is contained in a right ideal H of D_n then H is of the form (I', k) where I' is an ideal of D which contains I.

Proof. Suppose as in the statement that $(I, k) \subseteq H$. The set of elements I' that appear in the kth row of H is an ideal of D. For suppose $a, b \in I'$ and say a appears in the matrix A of H in the (k, i) position and b appears in the matrix B of H in the (k, j) position. Let E_{ij} denote the matrix with 1 in the (i, j) position and zero elsewhere. Then $AE_{i1}+BE_{j1}$ is a matrix which contains $a+b$ in the kth row. If $c \in D$ then $AE_{i1}c$ is a matrix which contains ac in the kth row. Next we shall show that if c is an element of I' then H contains a matrix with c in position $(k, 1)$ and zero elsewhere. Since $c \in I'$ there exist a matrix with c in the kth row and by proper multiplication by the elements of D_n on the right H must contain a matrix (a_{ij}), $a_{i1} \in D$, $a_{k1}=c$, $a_{ij}=0$ for $j>1$. Since $(I, k) \subseteq H$, H contains a matrix (b_{ij}), $b_{i1}=a_{i1}$, $b_{k1}=0$, $b_{ij}=0$ for $j>1$. Hence $(a_{ij})-(b_{ij})=(c_{ij})$ where $c_{1k}=c$, $c_{ij}=0$ for $i \neq 1$ and $j \neq k$. Consequently, since the right ideal $(0, k) \subseteq (I, k) \subseteq H$, we have $(c_{ij})+(0, k) \subseteq H$, i.e., the element c appears in the kth row of the first column and hence in every column with all combinations of the elements of D in the $j \neq k$ rows. Since this is true for all elements $c \in I'$, we have $H=(I', k)$.

STATEMENT 2.2. If I is an irreducible(2) ideal of D, then (I,k), $k=1, 2, \cdots, n$, is an irreducible(2) right ideal of D_n.

Proof. We shall prove that (I, k) is irreducible. Suppose $(I, k)=H_1 \cap H_2$ where H_1 and H_2 are right ideals of D_n which properly include (I, k). Then by Statement 2.1, H_1 and H_2 are of the form (I_1, k) and (I_2, k), where I_1 and I_2 are ideals of D. Hence $I=I_1 \cap I_2$ where I_1 and I_2 properly include I—contradiction.

For an irreducible ideal I of D we have from [1, Theorem 2.1] and the preceding statement that (I, k) is $V^*[(I, k)]$ primary, where $V^*[(I, k)]$ is the set of elements A in D_n such that $A(I, k) \subseteq (I, k)$, in the sense that if $AB \in (I, k)$, $B \notin (I, k)$, $A \in V^*[(I, k)]$, then $A^t \in (I, k)$ for some positive integer t.

Let (I_1, I_2, k) denote the set of all matrices (a_{ij}) where $a_{ij} \in D$, $i \neq k$, $a_{kj} \in I_1$, $j \neq k$, and $a_{kk} \in I_2$ where I_1 and I_2 are ideals of D.

STATEMENT 2.3. $V^*[(I, k)]=(I, D, k)$.

Proof. Certainly $(I, D, k) \subseteq V^*[(I, k)]$. Suppose $A=(a_{ij}) \notin (I, D, k)$, say $a_{kj} \notin I$ for $j \neq k$. Let E_{jk} denote the matrix with 1 in the (j, k) position and zero

(2) An ideal here is irreducible in the sense that it is not the intersection of two right ideals which properly contain it.

elsewhere. Then for E_{jk} contained in (I, k) we have $AE_{jk} \notin (I, k)$ since AE_{jk} contains a_{kj} in the (k, k) position. This proves the statement.

The V^* radical of (I, k) in (I, D, k) is the set of matrices A in (I, D, k) such that $A^t \in (I, k)$ for some positive integer t.

STATEMENT 2.4. The V^* radical of (I, k) is (I, P, k) where P is the radical of the ideal I in D. (I, P, k) is a completely prime two sided ideal of V^*.

Proof. If $(a_{ij}) \in (I, D, k)$, then $(a_{ij})^t = (b_{ij})$ where $b_{ij} \in D$, $i \neq k$, $b_{kj} \in I$, $j \neq k$, $b_{kk} = c + a_{kk}^t$ for $c \in I$. Hence $(a_{ij})^t$ is contained in (I, k) if and only if $a_{kk}^t \in I$ for some positive integer t, i.e., a_{kk} is contained in the radical of I in D. The second part follows from Theorem 2.2 of [1].

From [3, p. 40] the ideals of D_n are of the form I_n where I is an ideal of D. From the irreducible intersections for I in D we can write irreducible intersections for I_n in D_n. This is displayed in the next theorem the proof of which is most direct and is therefore omitted.

THEOREM 2.1. *Let I_n be a two sided ideal in D_n where D is a Noetherian ring. Let α be an index that ranges over a possibly infinite set E whose cardinal number is σ and let $I = I_{1\alpha} \cap I_{2\alpha} \cap \cdots \cap I_{s\alpha}$ be a set of σ irredundant representations of I as an intersection of irreducible ideals $I_{i\alpha}$ in D. Then the equation*

$$I_n = \bigcap_{i=1}^{s} \bigcap_{k=1}^{n} (I_{i\alpha_k}, k) \tag{Z}$$

in which, for each value of k, α_k is an arbitrary index from the set E, defines σ^n representations of I_n as an irredundant intersection of irreducible right ideals of D_n.

For an intersection of the form (Z), since the V^* radical of $(I_{i\alpha_k}, k)$ is $(I_{i\alpha_k}, P, k)$ where P is the radical of $I_{i\alpha_k}$, these radicals will all be different. In addition if different ideals of I are used in two intersections for I_n of the form (Z) none of the V^* radicals will be equal.

Let us now apply Theorems 2.6 of [1], 1.1, and 1.2 of this paper. For the σ intersections of Theorem 2.1, consider as in Theorem 2.1 the set S of σ^n intersections for I_n which can be formed from this set. Then $H = \bigcap V^*(I_{i\alpha}, k)$ where this intersection is taken over all i, k, and α, which of course could be an infinite intersection. However for any one intersection of the form (Z) the intersection of the n $V^*(I_{i\alpha}, k)$'s involved in this intersection is the set of all matrices (a_{ij}) where $a_{ij} \in I$ for $i \neq j$, $a_{ii} \in D$ which we shall denote by $(I \backslash D)$. Since this is true for all intersections of the form (Z) we have $H = (I \backslash D)$ which is a finite intersection. In general if I_1 and I_2 are ideals of D we shall denote by $(I_1 \backslash I_2)$ the set of all matrices (a_{ij}) with $a_{ij} \in I_1$ for $i \neq j$, $a_{ii} \in I_2$. Then the radical of $(I_{i\alpha}, k)$ in $(I \backslash D)$ will be the set of all matrices (a_{ij}) where $a_{ij} \in I$, $i \neq j$, $a_{ii} \in D$ for $i \neq k$, $a_{kk} \in P$ where P is the radical of $I_{i\alpha}$ in D. We shall denote such an ideal by $(I\backslash D, P, k)$(3). Consequently in $(I\backslash D)$ two radicals $(I\backslash D, P_1, k_1)$ and $(I\backslash D, P_2, k_2)$ will be equal if and only if $P_1 = P_2$ and $k_1 = k_2$. Thus we can apply Theorem 2.6 of [1] and combine the right ideals of (Z) which have the same radicals in $(I\backslash D)$. This will result in an intersection of primary $(I\backslash D)$ right ideals of D_n which is equal to I_n.

Since two radicals $(I\backslash D, P_1, k)$ and $(I\backslash D, P_2, m)$ will be equal if and only if $P_1 = P_2$ and $k = m$ then in (Z) we have $(I_{i\alpha_k}, k) \cap (I_{j\alpha_m}, m)$ will be $(I\backslash D)$ primary if and only if $k = m$ and $I_{i\alpha_k} \cap I_{j\alpha_m}$ is a primary ideal of D. Thus in applying Theorem 2.6 of [1] we combine the right ideals of (Z) to write $(I\backslash D)$ primary intersections for I_n with distinct $(I\backslash D)$ radicals.

From the previous discussion and Theorem 1.2 we have

THEOREM 2.2. *Let I_n be a two sided ideal of D_n where D is a Noetherian ring. Let α be an index that ranges over a possibly infinite set F whose cardinal number if ρ and let $I = J_{1\alpha} \cap \cdots \cap J_{r\alpha}$ be a set of ρ intersections for I where $J_{i\alpha}$ are primary ideals of D. Then the equation*

$$I_n = \bigcap_{i=1}^{r} \bigcap_{k=1}^{n} (J_{i\alpha_k}, k) \tag{Y}$$

in which, for each value of k, α_k is an arbitrary index from the set F, defines ρ^n representations of I_n as the intersection of $(I\backslash D)$ primary right ideals where the $(I\backslash D)$ radicals are distinct for each representation. For any two of these representations the $(I\backslash D)$ radicals are for some ordering equal.

One may now ask: (1) Are the primary intersections for I_n discussed here the only such intersections? (2) How can one write primary intersections for all right ideals of D_n?

BIBLIOGRAPHY

1. E. H. Feller, *The lattice of submodules of a module over a noncommutative ring*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 342–357.

2. N. Jacobson, *Lectures in abstract algebra*, Van Nostrand Co., 1951.

3. ———, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.

4. W. Krull, *Ein neuer Beweis für die Haupsätze der allgemeinen Idealtheorie*, Math. Ann. vol. 90 (1923) p. 55.

5. E. Noether, *Idealtheorie in Ringbereichen*, Math. Ann. vol. 83 (1921) pp. 24–66.

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(3) The radical of I in $(I\backslash D)$ will be equal to $(I\backslash P)$ where P is the radical of I in D.