CONNECTED ORDERED TOPOLOGICAL SEMIGROUPS WITH IDEMPOTENT ENDPOINTS II(1)

BY

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This paper is a continuation of an earlier paper(2) of the same title. The semigroups described in the title are called *threads*. Part I determined the structure of all possible threads with a zero element. The present paper will determine all threads without zero. The numbering of the sections, lemmas, and theorems will continue that of Part I. References to the literature in square brackets will be to the bibliography of Part I.

Let S be a thread. Since S is connected and has endpoints, it is compact. By a theorem of Numakura(3) and Wallace [11, Lemma 4], S contains a kernel K, i.e. an ideal contained in every ideal of S. K is closed and connected (Wallace [11, Lemma 4]), and so must be a closed subinterval $[\alpha, \omega]$ of S. K degenerates to a single point ($\alpha=\omega$) if and only if S has a zero; since this case was handled in Part I, we assume throughout Part II that $\alpha<\omega$.

The algebraic structure of K is one of the two trivial types: either (1) $\kappa\lambda=\lambda$ for all $\kappa, \lambda\in K$, or (2) $\kappa\lambda=\kappa$ for all $\kappa, \lambda\in K$. This is an immediate consequence of Faucett's Theorem 1.3 in [6]; see Lemma 11 below. By passing to the product-dual of S if necessary, we can and shall assume (1). Then K is *right-simple*, i.e. contains no proper right ideal. Conversely, any right-simple thread has the multiplication (1); indeed, any simple thread coincides with its kernel, and so has this structure or its product-dual.

If J is an ideal of a thread S, and is also a closed interval in S, then we can order the Rees [10] factor semigroup $T=S/J$ in the obvious way, and T becomes thereby a thread with zero. We call S a *linear extension of J by T*. In particular, any thread S with nondegenerate kernel K is a linear extension of K by $T=S/K$. Part II is thus concerned with the problem of finding all possible linear extensions S of a right-simple thread K by a thread T with zero.

There is a natural division in the results according to whether T is commutative or not. Assume T is commutative. If the zero element of T is at one end, then an extension S of K by T always exists, and all such are given by Theorem 8 below. Let $T=[f, e]$ with $f<0<e$. If $ef(=fe)=0$, then an

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(2) A. H. Clifford, *Connected ordered topological semigroups with idempotent endpoints*, I. Trans. Amer. Math. Soc. vol. 88 (1958) pp. 80–98.

(3) K. Numakura, *On bicompact semigroups*, Math. J. Okayama Univ. vol. 1 (1952) pp. 99–108. The reference is to Theorem 2, p. 104.

extension of K by T always exists, and all such are given by Theorem 10. If $ef \neq 0$, we can assume that $ef < 0$ by passing to the order-dual of T if necessary. Then a necessary and sufficient condition for the existence of an extension of K by T is that there exist an idempotent $g > 0$ in T such that (1) $fg = 0$, and (2) K is order-anti-isomorphic with some homomorphic image of the subthread $[0, g]$ of T. (See §11 for discussion of this second condition.) A description of how to obtain all extensions of K by T when these conditions are satisfied is given by Theorems 9 and 10.

Assume T is noncommutative. Then, by Theorem 7 of Part I, T is a contact extension of a january thread J, which may be either left or right-handed. If K is a right simple thread, then an extension of K by T exists if and only if J is right-handed. All possible extensions are described in Theorems 11 and 12.

In order to keep the paper from becoming unduly long, proofs of the converse halves of all the theorems have been omitted. These will be made available privately in mimeographed form.

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11. **Preliminary remarks.** In all of the theorems of this paper we meet a situation which is best formulated in advance. Let T be a standard thread $[0, e]$, i.e. a thread with lower endpoint acting as zero, and upper endpoint as identity element (see introduction to Part I). Let K be a right simple thread. A continuous, monotone nonincreasing mapping θ of T into K such that if $\theta(x_1) = \theta(x_2)$ for some x_1, x_2 in T, then $\theta(x_1 x) = \theta(x_2 x)$ for all x in T, will be called a *closed, convex, order-antihomomorphism* (abbreviation: c.c.o-a-h.) for the following reason. If we define the relation ρ in T by $x_1 \rho x_2$ if and only if $\theta(x_1) = \theta(x_2)$, then ρ is a congruence relation in T. Since the congruence classes are closed intervals of T, ρ is by definition (§5) a closed, convex, congruence relation. By Lemma 9 (1), T/ρ is a standard thread. $\theta(T)$ is a closed interval in K which is *order*-anti-isomorphic with T/ρ. Conversely, if some closed interval I in K is order-anti-isomorphic with a homomorphic image $\overline{T}$ of T, then there exists a c.c.o-a-h. θ of T onto I. In fact, $\theta = \phi\psi$ where ψ is a homomorphism (with respect to both order and product) of T onto $\overline{T}$, and ϕ is an order-anti-isomorphism of $\overline{T}$ onto I. (This is, of course, a much stronger statement than to say merely that I is an order-anti-homomorphic image of T.) Condition (2) of the result mentioned in the fourth from last paragraph of the introduction is to be considered in the light of these remarks, taking $I = K$.

We shall adhere throughout the paper to the notation used in the introduction. S will denote a thread with proper and nondegenerate kernel $K = [\alpha, \omega]$, $\alpha < \omega$. The upper and lower endpoints of S will be denoted by e

and f respectively. T will denote the Rees factor semigroup S/K. The zero element of T will be denoted by 0. We shall identify the complement $S\backslash K$ of K in S with $T^0 = T\backslash\{0\}$. If $f=\alpha$ ($e=\omega$) we shall say that K is "at the lower (upper) end of S."

The following lemma is an immediate consequence of Faucett's Theorem 1.3 in [6]. We give a proof, however, since it can be made much shorter than his in the case under consideration.

LEMMA 11. *Let S be a thread without zero, and K its kernel. Either* (1) *K is a minimal right ideal of S, and $x\kappa=\kappa$ for all x in S and κ in K*; *or* (2) *K is a minimal left ideal of S, and $\kappa x=\kappa$ for all x in S and κ in K.*

Proof. As shown by Numakura (loc. cit., Theorem 6) K is the union of disjoint minimal left (right) ideals of S, and these are closed sets. Moreover, each minimal left (right) ideal has the form $K\lambda$ (λK) with λ an element of K, and so (being a continuous image of the connected set K) is connected. Hence the minimal left (right) ideals of S are closed intervals in K. Moreover, they are isomorphic (and homeomorphic), so that if one is degenerate, they all are. Suppose first that there exists a nondegenerate minimal right ideal R in K.

Let λ be an interior point of R. λ belongs to the minimal left ideal $L=S\lambda$. $H=L\cap R$ is a closed and connected topological subgroup of K, hence a closed interval, and hence $H=\{\lambda\}$ since a nondegenerate closed interval is not homogeneous. But this entails $L=\{\lambda\}$. For the interval L could not meet the interval R only at an interior point λ of R, unless it is degenerate. Thus every minimal left ideal of K is degenerate. If $\kappa\in K$, then $\{\kappa\}$ is a left ideal of S, and so $x\kappa=\kappa$ for all x in S. Since R meets every left ideal of S, it must coincide with K. Thus (1) holds. If there exists a nondegenerate minimal left ideal in K, then the dual argument shows that (2)holds. Were all minimal left ideals in K and all minimal right ideals in K degenerate, K itself would be degenerate, contrary to hypothesis.

Because of Lemma 11, K is either right-simple or left-simple. By passing to the product-dual of S if necessary, we can assume that the former is the case. Throughout this paper, K will be a right-simple thread. If K is the kernel of S then $x\kappa=\kappa$ for all x in S and κ in K.

12. **Threads with kernel at one end.** Throughout this section, S will denote a thread with a proper nondegenerate kernel $K=[\alpha, \omega]$ at one end. By passing to the order-dual of S if necessary, we can and shall assume that K is at the lower end of S. Thus $S=[\alpha, e]$, and $T=S/K$ is a thread $[0, e]$ with zero at the lower end. By the corollary to Lemma 1, e is the identity element of T, and T is a standard thread.

LEMMA 12. *The closed interval $[\omega, e]$ is a subthread of S isomorphic with T.*

Proof. Let $x, y \in [\omega, e]$, and assume that $xy \leqq \omega$. From $xy \leqq \omega \leqq y = ey$ and Lemma 1, there exists t in $[x, e]$ such that $\omega = ty$. From $\omega t \leqq x \leqq et$, there exists s in $[\omega, e]$ such that $x = st$. Hence $xy = sty = s\omega = \omega$. It follows that $[\omega, e]$ is a subthread of S. The mapping ϕ of T onto $[\omega, e]$ defined by

$$\phi(x) = \begin{cases} x & \text{if } x \neq 0 \text{ in } T, \\ \omega & \text{if } x = 0 \text{ in } T, \end{cases}$$

is clearly an isomorphism.

LEMMA 13. *For each κ in K, the set of all elements x of S such that $\kappa x = \omega$ is a closed interval contained in $[\omega, e]$.*

Proof. Since $\kappa\lambda = \lambda$ for every λ in K, $\kappa x = \omega$ implies $x \geqq \omega$. The set of all such x is evidently closed. To show that it is connected, suppose that $\kappa x_2 = \omega$ and $\omega < x_1 < x_2$. From $x_1 < x_2$ we have $x_1 = x_2 x_3$ for some x_3 in $S \backslash K$. Since ω is the zero element of the thread $[\omega, e]$, $\kappa x_1 = \kappa x_2 x_3 = \omega x_3 = \omega$.

LEMMA 14. *Ke is a closed interval $[\beta, \omega]$ in K, and if $\omega < x_1 < x_2$ then $\beta x_1 \geqq \beta x_2$.*

Proof. Ke is a continuous image of the connected, compact space K, and so is likewise connected and compact. Hence Ke is a closed interval in K, and it contains ω since $\omega e = \omega$ by Lemma 12.

Suppose that $\omega < x_1 < x_2$ and $\beta x_1 \leqq \beta x_2$. Let s be an element of $S \backslash K$ such that $x_1 = x_2 s$ (Lemma 4). From $(\beta x_2)s = \beta x_1 \leqq \beta x_2 \leqq \omega = \omega s$ and Lemma 1, we have $[\beta x_2, \omega] \subseteq [\beta x_2, \omega]s$. Since S is compact, we may use Theorem 3.4 in Wallace [12] to obtain $[\beta x_2, \omega] = [\beta x_2, \omega]s$. Hence $\beta x_1 = \beta x_2 s \geqq \beta x_2$, and hence $\beta x_1 = \beta x_2$.

THEOREM 8. *Let S be a thread with nondegenerate and proper kernel K at one end. By passing to the order dual of S if necessary, we can and shall assume that K is at the lower end of S. Let $K = [\alpha, \omega]$, $S = [\alpha, e]$, $\alpha < \omega < e$. The Rees quotient $T = S/K$, ordered in the obvious way, is a standard thread $[0, e]$, and therefore has the structure described in Theorem 3. Let $T^0 = T \backslash \{0\} = S \backslash K$. $[\omega, e]$ is a subthread of S isomorphic with T.*

The mapping ψ of K into K defined by $\psi(\kappa) = \kappa e$ is a retraction of K onto a subinterval $[\beta, \omega]$ of K. The mapping θ of T into K defined by

$$\theta(t) = \begin{cases} \beta t & \text{if } t \in T^0, \\ \omega & \text{if } t = 0, \end{cases}$$

is a closed, convex, order-anti-homomorphism of T onto $\psi(K) = [\beta, \omega]$. The binary operation in S, which we now denote by $\circ$, is completely determined by that in T, which we denote by juxtaposition, and the mappings ψ and θ as follows (wherein $x, y \in T^0$; $t \in T$; $\kappa, \lambda \in K$):

(N1) $$x \circ y = \begin{cases} xy & \textit{if } xy \neq 0 \textit{ in } T, \\ \omega & \textit{if } xy = 0 \textit{ in } T; \end{cases}$$

(N2) $$\kappa \circ x = \theta(tx) \qquad \textit{if } \psi(\kappa) = \theta(t);$$

(N3) $$x \circ \kappa = \kappa;$$

(N4) $$\lambda \circ \kappa = \kappa.$$

Conversely, let $T = [0, e]$ be a nondegenerate standard thread, and let K be a nondegenerate right simple thread $[\alpha, \omega]$. Let θ be a closed, convex, order-anti-homomorphism of T into K. Let ψ be a retraction of K onto $\theta(T)$. Let $T^0 = T\backslash\{0\}$, $S = K \cup T^0$. Define product $\circ$ in S by (N1–4). *Introduce an ordering $<$ into S preserving those in K and T^0 and such that $\kappa < x$ for every κ in K and x in T^0. Then S is a thread $[\alpha, e]$ with kernel $K = [\alpha, e]$ at the lower end, and $S/K \cong T$.*

Proof. All of the first paragraph of the theorem was noted in the introduction, except the last sentence, and this is just Lemma 12. ψ is evidently a retraction of K onto Ke, and Ke is a closed subinterval $[\beta, \omega]$ of K by Lemma 14. Because of the isomorphism between T and $[\omega, e]$, we can identify T with $[\omega, e]$, i.e. 0 with ω, and then θ is simply the mapping $t \rightarrow \beta t$ for all t in $[\omega, e]$. By continuity of product in S, θ is continuous. By Lemma 14, θ is monotone nonincreasing. For any t in $[\omega, e]$, $\beta te = \beta t$, and hence $\theta(t) = \beta t \in Ke = \psi(K)$; conversely,

$$\psi(K) = [\beta, \omega] = [\beta e, \beta\omega] \subseteq \beta[\omega, e] = \theta([\omega, e]).$$

Thus θ maps $[\omega, e]$ onto $\psi(K)$. If $\theta(x_1) = \theta(x_2)$, i.e. $\beta x_1 = \beta x_2$, then $\theta(x_1 x) = \beta x_1 x = \beta x_2 x = \theta(x_2 x)$, for all x in $[\omega, e]$. Hence θ is a c.c.o-a-h. of T onto $\psi(K)$, as defined in §11.

(N1) follows from Lemma 12. (N3) and (N4) follow from Lemma 11. To prove (N2), let $\psi(\kappa) = \theta(t)$, i.e. $\kappa e = \beta t$. Dropping the notation $\circ$ (which is introduced primarily for the converse half of the theorem), we have $\kappa \circ x = \kappa x = \kappa e x = \beta t x = \theta(tx)$.

As promised in the introduction, we omit the proof of the converse, and proceed to a discussion of the construction of all linear extensions S of a given nondegenerate right simple thread $K = [\alpha, \omega]$ by a given nondegenerate standard thread $T = [0, e]$. There always exists a linear extension S of K by T, since we can take $\psi(\kappa) = \omega$ for all κ in K, and $\theta(t) = \omega$ for all t in T. This is, moreover, the only possible extension when we take $\beta = \omega$. Now let β be a fixed element of K such that $\beta < \omega$. Then there will exist a linear extension S of K by T with the property $Ke = [\beta, \omega]$ if and only if there exists a c.c.o-a-h. θ of T onto $[\beta, \omega]$. ψ can be any retraction of K onto $[\beta, \omega]$, e.g.

$$\psi(\kappa) = \begin{cases} \beta & \text{if } \kappa \in [\alpha, \beta], \\ \kappa & \text{if } \kappa \in [\beta, \omega]. \end{cases}$$

It is now clear that we will obtain all possible threads S with proper nondegenerate kernel at the lower end if we perform in all possible ways the following sequence of constructions.

(1) Construct a standard thread $T=[0, e]$ as prescribed by Theorem 3.

(2) Construct a homomorphic image $\overline{T}=[\overline{0}, \overline{e}]$ of T as prescribed by Lemma 9(2).

(3) Let $K=[\alpha, \omega]$ be any compact, connected, ordered set containing a subinterval $[\beta, \omega]$ order-anti-isomorphic with $\overline{T}$. (In other words, we effectively invert $\overline{T}$ as an ordered set, and tack onto it at the lower end any compact, connected, ordered set $[\alpha, \beta]$.) Let θ be the composition of the homomorphism $T\rightarrow\overline{T}$ and the anti-isomorphism $\overline{T}\rightarrow[\beta, \omega]$. Define $\kappa\lambda=\lambda$ for all κ, λ in K.

(4) Let ψ be any retraction of K onto $[\beta, \omega]$.

(5) Let S be the linear extension of K by T determined by the mappings θ and ψ as described in Theorem 8.

13. **Threads S with interior kernel K such that S/K is commutative.** In this section we shall consider threads $S=[f, e]$ having nondegenerate kernel $K=[\alpha, \omega]$ such that $f<\alpha<\omega<e$, and such that $T=S/K$ is commutative. At first we shall consider the case where e is the identity element of T. The structure of T is given by Theorem 4: T is the linear extension of $T''=[f, 0]$ by $T'=[0, e]$ defined by $\phi(x')=fx'(=x'f)$, $x'\in T'$.

Since T' and T'' are subthreads of T, and K is an ideal of S, it is clear that the intervals $[\alpha, e]$ and $[f, \omega]$ are subthreads of S of the type discussed in the preceding section. Since $eS\cap Se$ is a connected set containing e and f, it must coincide with S, and we conclude that e is the identity element of S. Thus $[\alpha, e]$ has the structure described in Theorem 8 with ψ the identity mapping of K onto itself ($\beta=\alpha$). As a corollary of the following lemma, we shall see that $[f, \omega]$ is at the other extreme: the analogous ψ maps K onto the single element α (which of course plays the rôle of ω in Theorem 8 since K is at the upper end of $[f, \omega]$). We also note from Theorem 8 that $[\omega, e]$ is a subthread of S isomorphic with T', and $[f, \alpha]$ is a subthread of S isomorphic with T''.

Lemma 15. (1) *There exists an idempotent element $g>\omega$ in S such that $fg=gf=\alpha$.*

(2) *If $x\in[f, \alpha]$ and if $y\in[\alpha, g]$ then $yx=\alpha$ and $xy=\alpha y$.*

(3) $\alpha[g, e]=\alpha$.

Proof. (1) Since $\alpha\in[f, \omega]=[fe, f\omega]\subseteq f[\omega, e]$, $\alpha=fc$ for some c in (ω, e). Thus $\alpha c=(\alpha f)c=\alpha(fc)=\alpha^2=\alpha$. Since $[\omega, e]$ is isomorphic with T' and thus it is a standard thread, $g=\lim c^n$ is an idempotent element, $\omega\leqq g\leqq c$, and g is a zero for $[g, e]$. From $\alpha c=\alpha$ we conclude $\alpha c^n=\alpha$ for every positive integer n. Hence $\alpha g=\alpha \lim c^n=\lim \alpha c^n=\alpha$. Moreover, since $cg=g$, $fg=f(cg)=(fc)g=\alpha g=\alpha$. Were $g=\omega$ we would have $\alpha=\alpha g=\alpha\omega=\omega$. Hence $g>\omega$.

Since $\alpha g=g\alpha=\alpha$ and since Sg and gS are connected, g must be an identity for $[\alpha, g]$. Furthermore, $fg=\alpha\in K$ implies $gf=fg=0$ in T. Hence $gf\in K$, and $gf=f(gf)=(fg)f=\alpha f=\alpha$.

(2) Let $x\in[f, \alpha]$ and $y\in[\alpha, g]$. Then $xy=(xf)(gy)=x(fg)y=x\alpha y=\alpha y$ and $yx=(yg)(fx)=y(gf)x=y\alpha x=\alpha x=\alpha$.

(3) Finally, since g is a zero for $[g, e]$, $\alpha[g, e]=(\alpha g)[g, e]=\alpha g=\alpha$.

THEOREM 9. *Let S be a thread $[f, e]$ with nondegenerate interior kernel $K=[\alpha, \omega]$ such that the Rees quotient $T=S/K$ is commutative, and such that e is the identity element of T. K is right simple or left simple, and by passing to the product dual of S if necessary, we can and shall assume the former.*

(1) *There exists an idempotent element $g>0$ in T such that $fg=0$.*

(2) *e is the identity element of S.*

(3) *$[\alpha, e]$ is a subthread of S having the structure described in Theorem 8, with ψ the identity mapping of K onto itself, and θ the mapping $\theta(t)=\alpha t$ of $[\omega, e]$ onto K. θ is a closed, convex, order-anti-homomorphism of $[\omega, e]$ onto K; in fact, θ maps $[\omega, g]$ onto K, and maps $[g, e]$ onto the single element α.*

(4) *The binary operation in S, which we now denote by $\circ$, is completely determined by that in T, which we now denote by juxtaposition, and the mapping θ, as follows (wherein $x, y\in T^0=T\backslash\{0\}$; $t\in T$; $\kappa, \lambda\in K$):*

$$\text{(M1)}\qquad x\circ y=\begin{cases} xy & \text{if } xy\neq 0 \text{ in } T,\\ \omega & \text{if } xy=0 \text{ in } T,\ x>0, \text{ and } y>0,\\ \theta(y) & \text{if } xy=0 \text{ in } T,\ x<0, \text{ and } y>0,\\ \alpha & \text{if } xy=0 \text{ in } T, \text{ and } y<0;\end{cases}$$

$$\text{(M2)}\qquad \kappa\circ x=\begin{cases} \theta(tx) & \text{if } \kappa=\theta(t) \text{ and } x>0,\\ \alpha & \text{if } x<0;\end{cases}$$

$$\text{(M3)}\qquad x\circ\kappa=\kappa;$$

$$\text{(M4)}\qquad \lambda\circ\kappa=\kappa.$$

Conversely, let T be a thread $[f, e]$ with identity element e and interior zero element 0, and let T satisfy condition (1) above. Let K be a nondegenerate right simple thread $[\alpha, \omega]$. Let $T^0=T\backslash\{0\}$ and $S=K\cup T^0$. Let θ be a closed, convex, order-anti-homomorphism of $T'=[0, e]$ onto K such that $\theta(g)=\alpha$, where g is defined in (1); thus $\theta(0)=\omega$ and $\theta(x)=\alpha$ for all x in $[g, e]$. Define product $\circ$ in S by (M1–4). Define order in S preserving that in K and T^0, and such that $x<\kappa$ or $x>\kappa$ (x in T^0 and κ in K) according to whether $x<0$ or $x>0$ in T. Then S is a thread with kernel K such that $S/K\cong T$.

Proof. (1) follows from part (1) of Lemma 15. (2) and all but the last assertion in (3) follows from Theorem 8 and the remarks made prior to Lemma 15. The last assertion in (3) follows from part (3) of Lemma 15.

In proving (M1–4) we suppress the notation $\circ$ (which is for the benefit of the converse). Let x and y be elements of $S\backslash K$ such that $xy\in K$. (M1) will be established when we show that

$$xy = \begin{cases} \omega & \text{if } x > \omega \text{ and } y > \omega, \\ \theta(y) & \text{if } x < \alpha \text{ and } y > \omega, \\ \alpha & \text{if } y < \alpha. \end{cases}$$

The first of these follows from Lemma 12. If $x<\alpha$ and $\omega<y$ then, since $xy\in K$ and α is a zero for x, $xy=\alpha(xy)=(\alpha x)y=\alpha y=\theta(y)$. As for the third, let $y<\alpha$. If $x<\alpha$, $xy=\alpha$ by Lemma 12. If $x\geqq g$ then, by part (3) of Lemma 15, $\alpha x=\alpha$; hence $xy=\alpha(xy)=\alpha y=\alpha$. And if $0<x\leqq g$ then $xy=\alpha$ by part (2) of Lemma 15 (with x and y interchanged).

To show (M2), first let $\kappa=\theta(t)=\alpha t$ and let $x>\omega$. Then $\kappa x=\alpha tx=\theta(tx)$. Next let $x<\alpha$. Then $\kappa x=\alpha$ by part (2) of Lemma 15, with y replaced by κ. (M3) and (M4) follow from Lemma 11. We omit the proof of the converse.

Corollary. *Let T be a (commutative) thread $[f, e]$ with interior zero element 0 such that e is the identity element of T.*

(1) *Let K be a nondegenerate right simple thread $[\alpha, \omega]$. Then there will exist a linear extension S of K by T if and only if T contains an idempotent element $g>0$ such that $fg=0$ and such that K is order-anti-isomorphic with a homomorphic image of the standard subthread $[0, g]$ of T. S is completely determined by a closed, convex, order-anti-homomorphism of $[0, g]$ onto K.*

(2) *There exist a nondegenerate right simple thread K and a linear extension S of K by T if and only if T contains an idempotent element $g>0$ such that $fg=0$.*

Remark. It will be clear from Theorem 10 below that this corollary holds for any (commutative) thread $T=[f, e]$ with interior zero 0 such that $ef=fe<0$.

Proof. (1) Let S be an extension of K by T. By part (1) of Theorem 9, T contains an idempotent $g>0$ such that $fg=0$. By part (3) of Theorem 9, $\theta(t)=\alpha t$ is a c.c.o-a-h. of $[\omega, e]$ onto K. By part (3) of Lemma 15, $\theta(g)=\alpha g=\alpha$, and so θ induces a c.c.o-a-h. of $[\omega, g]$ onto K. Conversely, if g is an idempotent element >0 of T such that $fg=0$ and such that there exists c.c.o-a-h. θ of $[0, g]$ onto K, then we can extend θ to a c.c.o-a-h. of $[0, e]$ onto K by defining $\theta(x)=\alpha$ for all $x>g$. Thus θ satisfies the requirements of the converse half of Theorem 9, and so defines a linear extension S of K by T. That S is uniquely determined by θ follows from the direct half of Theorem 9.

(2) The necessity follows from Theorem 9. For the sufficiency, we may take K to be an order-anti-isomorphic image of $[0, g]$.

It is clear that we shall obtain all possible threads $S=[f, e]$ with nondegenerate right-simple interior kernel K and such that $T=S/K$ has e as identity element (and so is commutative), by the following procedure:

(1) Construct a thread $T=[f, e]$ with interior zero and such that e is the identity element; the method for doing this is given by Theorem 4. T will

possess a largest idempotent g such that $fg=0$. If $g=0$, discard T. If $g>0$, proceed to step (2).

(2) Construct a nontrivial closed, convex congruence relation ρ on the standard subthread $[0, g]$ of T; see Lemma 9(2) for method. Take K to be order-anti-isomorphic with $[0, g]/\rho$. Let θ be the resulting c.c.o-a-h. of $[0, g]$ onto K.

(3) Extend θ to $[0, e]$ by defining $\theta(x)=\theta(g)$ for all $x>g$, and let S be the linear extension of K by T described in Theorem 9.

THEOREM 10. *Let S be a thread $[f, e]$ with nondegenerate interior kernel $K=[\alpha, \omega]$ such that $T=S/K$ is commutative. By passing to the order-dual of S if necessary, we can assume that $ef=fe\leqq 0$ in T. If $ef<0$, then* (*by Theorem* 5) *T is the contact extension of $T_1=[ef, e]_T$ by the standard thread $[f, ef]$. In this case, S is the contact extension of $S_1=[ef, e]_S$ by $[f, ef]$. S_1 is a linear extension of K by T_1, and since e is the identity element of T_1, the structure of S_1 is given by Theorem* 9. *For the remainder of this theorem, we assume that $ef=fe=0$ in T. Let $T'=[0, e]$, $T'^0=(0, e]$, $T''=[f, 0]$, $T''^0=[f, 0)$.*

(1) *There exist closed, convex, order-anti-homomorphisms θ and η of T' and T'', respectively, into K, such that $\theta(0)=\omega$ and $\eta(0)=\alpha$.*

(2) *There exist retractions ψ and χ of K onto $\theta(T')$ and $\eta(T'')$, respectively, such that $\psi\chi(\kappa)=\psi(\alpha)$ and $\chi\psi(\kappa)=\chi(\omega)$ for every κ in K.*

(3) *The binary operation in S, which we now denote by $\circ$ is completely determined by that in T, which we denote by juxtaposition, and the mappings θ, η, ψ, χ as follows* (*wherein $x, x'\in T'^0$; $t\in T'$; $y, y'\in T''^0$; $s\in T''$; $\kappa, \lambda\in K$*):

(L1)
$$x\circ x' = \begin{cases} xx' & \text{if } xx'\neq 0 \text{ in } T', \\ \omega & \text{if } xx' = 0 \text{ in } T'; \end{cases}$$

$$y\circ y' = \begin{cases} yy' & \text{if } yy'\neq 0 \text{ in } T'', \\ \alpha & \text{if } yy' = 0 \text{ in } T''; \end{cases}$$

$$x\circ y = \eta(sy) \quad \text{if } \chi(\omega) = \eta(s);$$

$$y\circ x = \theta(tx) \quad \text{if } \psi(\alpha) = \theta(t);$$

(L2)
$$\kappa\circ x = \theta(tx) \quad \text{if } \psi(\kappa) = \theta(t);$$
$$\kappa\circ y = \eta(sy) \quad \text{if } \chi(\kappa) = \eta(s);$$

(L3)
$$x\circ\kappa = y\circ\kappa = \kappa;$$

(L4)
$$\lambda\circ\kappa = \kappa.$$

Conversely, let $T=[f, e]$ be a thread with interior zero 0 *such that*([4]) *$ef=fe=0$. Define T', T'^0, T'', T''^0 as above. Let K be a nondegenerate right simple thread $[\alpha, \omega]$. Assume that there exist mappings θ, η, ψ, χ satisfying* (1) *and* (2). *Let $T^0=T\backslash\{0\}$ and $S=K\cup T^0$. Define a binary operation $\circ$ in S by* (L1–4), *and*

([4]) The converse of the case in which $ef=fe\neq 0$ is too trivial to mention.

order S in the same way as in Theorem 9. *Then S is a thread with interior kernel K, and $S/K \cong T$.*

Proof. First assume that $ef=fe<0$ in T. Let $x \in [f, ef]$, $y \in [ef, e]$. If $xy \notin K$ then $xy=(ef)y$, and if $yx \notin K$ then $yx=y(ef)$, since these products are the same in T as in S, and T is the contact extension of $[ef, e]$ by $[f, ef]$. If $xy \in K$, then $xy=(ef)xy=(ef)y$ since $(ef)x=ef$. Now e is the identity element of $[ef, e]$ by part (2) of Theorem 9. If $ex \in K$, then $ef=(ef)x=(fe)x=f(ex) \in K$, which is contrary to the assumption $ef<0$ in T. Hence $ex \notin K$ so that $ex=e(ef)=ef$, and $yx=(ye)x=y(ex)=y(ef)$. Hence S is the contact extension of $S_1=[ef, e]$ by $[f, ef]$.

Assume now that $ef=fe=0$ in T. This means that ef and fe belong to K; unlike the former case, we may now have $ef \neq fe$. By the corollary to Theorem 5, $T'T''=T''T'=\{0\}$. $[\alpha, e]$ and $[f, \omega]$ are subthreads of S having the structure given by Theorem 8. Let $Ke=[\beta, \omega]$, $\psi(\kappa)=\kappa e$, $\theta(x)=\beta x$ for x in $[\omega, e]$. Similarly, let $Kf=[\alpha, \gamma]$, $\chi(\kappa)=\kappa f$, $\eta(y)=\gamma y$ for y in $[f, \alpha]$. By Theorem 8, θ and η satisfy (1), while ψ and χ are retractions of K onto $\theta(T')$ and $\eta(T'')$, respectively. Likewise all of (L1–4) follow from Theorem 8 except for the last two parts of (L1), which, because of (L2), can be expressed $x \circ y=\omega \circ y$ and $y \circ x=\alpha \circ x$. We shall prove these first, dropping the $\circ$ notation.

Let $x \in (\omega, e]$ and let $y \in [f, \alpha)$. Then $xy=(xe)y=x(ey)=ey$, since $ey \in K$. Thus xy is independent of x; and, letting $x \to \omega$, we get $\omega y=ey$. Hence $xy=ey=\omega y$. The proof that $yx=\alpha x$ is dual.

We proceed to prove (2). Let $\kappa \in K$. Since $\chi(K)=\eta(T'')$, there exists y in T'' such that $\chi(\kappa)=\eta(y)=\gamma y$. Then $\psi\chi(\kappa)=\psi(\gamma y)=\gamma ye=ye$, since $ye \in K$. But, by the foregoing with $x=e$, $ye=\alpha e=\psi(\alpha)$. Hence $\psi\chi(\kappa)=ye=\psi(\alpha)$. The proof that $\chi\psi(\kappa)=\chi(\omega)$ is dual.

We omit proof of the converse.

We proceed to discuss the construction of all possible linear extensions S of a given nondegenerate right simple thread $K=[\alpha, \omega]$ by a given commutative thread $T=[f, e]$ with interior zero element 0. We can assume $ef(=fe) \leqq 0$. If $ef<0$ then T is the contact extension of $T_1=[ef, e]$ by the standard thread $[f, ef]$, and we saw in Theorem 10 that there exists a linear extension S of K by T if and only if there exists a linear extension S_1 of K by T_1, and that we can obtain every S by a contact extension of an S_1 by $[f, ef]$. Consequently the problem is reduced to the consideration of T_1, and this is the substance of the corollary to Theorem 9. Moreover we get every thread $S=[f, e]$ such that $ef=fe \notin K$ by adding a fourth step to the three steps given after this corollary, namely that of taking a contact extension.

We pass on to the case $ef=fe=0$. In this case an extension S of K by T always exists. In the terminology of Theorem 10, we may take $\theta(x)=\psi(\kappa)=\omega$ and $\eta(y)=\chi(\kappa)=\alpha$ for all κ in K, x in $T'=[0, e]$, and y in $T''=[f, 0]$. In general, let θ, η, ψ, χ be mappings satisfying conditions (1) and (2) of Theorem 10, and let $\psi(K)=[\beta, \omega]$, $\chi(K)=[\alpha, \gamma]$. By (2), ψ maps all of $\chi(K)$ onto the single element $\psi(\alpha)$, and is the identity mapping on $\psi(K)$. Hence $\psi(K)$ and $\chi(K)$ can have at most one element in common, and so $\alpha \leqq \gamma \leqq \beta \leqq \omega$.

Starting with given K and T, let β and γ be elements of K with $\gamma \leqq \beta$. A linear extension of K by T such that $Ke=[\beta, \omega]$ and $Kf=[\alpha, \gamma]$ will exist if and only if there exist linear extensions of $[\beta, \omega]$ by $[0, e]$ and of $[\alpha, \gamma]$ by $[f, 0]$, such that $\beta e=\beta$ and $\gamma f=\gamma$. Moreover, these can be entirely arbitrary. Let θ and η be the corresponding c.c.o-a-h.'s. Define ψ in any way such that it maps all of $[\alpha, \gamma]$ into a single element of $[\beta, \omega]$, is the identity map on $[\beta, \omega]$, and maps $[\gamma, \beta]$ continuously into $[\beta, \omega]$. Likewise define χ in any way such that it maps all of $[\beta, \omega]$ into a single element of $[\alpha, \gamma]$, is the identity map on $[\alpha, \gamma]$, and maps $[\gamma, \beta]$ continuously into $[\alpha, \gamma]$. (Evidently such ψ and χ always exist.) Then define product $\circ$ in $S=T^0 \cup K$ by (L1–4) of Theorem 10. Moreover, the foregoing procedure will produce every possible thread $S=[f, e]$ with interior kernel K, such that ef and fe both belong to K.

14. **Threads S with interior kernel K such that S/K is noncommutative.** The notion of a "january thread" was defined in the introduction to Part I. By Theorem 6, a thread $T=[f, e]$ with zero 0 is a right-handed (left-handed) january thread if and only if $ef=f$ and $fe=e$ ($ef=e$ and $fe=f$). If T is right-handed, e and f are left identity elements of T, while the mappings $x \rightarrow xf$ and $y \rightarrow ye$ are mutually inverse, order-reversing, isomorphisms of $T'=[0, e]$ onto $T''=[f, 0]$ and vice-versa; moreover, $T'T''=T''$ and $T''T'=T'$. The product-dual assertions hold if T is left-handed.

THEOREM 11. *Let S be a thread $[f, e]$ with nondegenerate interior kernel $K=[\alpha, \omega]$ such that $T=S/K$ is a january thread. K is either right or left simple, and as usual we can and shall assume the former. Then T must be right-handed.*

(1) *There exist elements β and γ of K, $\beta \geqq \gamma$, and retractions ψ and ψ' of K onto $[\beta, \omega]$ and $[\alpha, \gamma]$ respectively, such that the restriction of ψ to $[\alpha, \gamma]$ and that of ψ' to $[\beta, \omega]$ are mutually inverse order-anti-isomorphisms.*

(2) *There exists a closed, convex, order-anti-homomorphism θ of $T'=[0, e]$ onto $\psi(K)=[\beta, \omega]$.*

(3) *The binary operation S, which we now denote by $\circ$, is completely determined by that in T, which we denote by juxtaposition, and the mappings ψ, ψ', θ as follows (wherein $x, y \in T^0=T\backslash\{0\}$; $t \in T'$; $\kappa, \lambda \in K$):*

$$\text{(J1)} \qquad x \circ y = \begin{cases} xy & \text{if } xy \neq 0 \text{ in } T, \\ \omega & \text{if } xy = 0 \text{ in } T \text{ and } y > 0, \\ \alpha & \text{if } xy = 0 \text{ in } T \text{ and } y < 0; \end{cases}$$

$$\text{(J2)} \qquad \kappa \circ x = \begin{cases} \theta(tx) & \text{if } \psi(\kappa) = \theta(t) \text{ and } x > 0, \\ \psi'\theta(txe) & \text{if } \psi(\kappa) = \theta(t) \text{ and } x < 0; \end{cases}$$

$$\text{(J3)} \qquad x \circ \kappa = \kappa;$$

$$\text{(J4)} \qquad \lambda \circ \kappa = \kappa.$$

Conversely, let $T=[f, e]$ *be a right-handed january thread, and let* $K=[\alpha, \omega]$ *be a nondegenerate right simple thread. Suppose furthermore that there exist elements* β *and* γ *of* K *and mappings* ψ, ψ', *and* θ *satisfying conditions* (1) *and* (2) *above. Let* $T^0=T\setminus\{0\}$ *and* $S=K\cup T^0$. *Let a binary operation* $\circ$ *be defined in* S *by* (J1–4). *Order* S *as usual* (*see e.g. Theorem* 9). *Then* S *is a thread with interior kernel* K, *and* $S/K\cong T$.

Proof. Let $T'=[0, e]$, $T''=[f, 0]$. Suppose T were left-handed. Then $T'T''=T'$ and $T''T'=T''$; moreover e and f are right identities of T. Consider in S the product $f\omega=\omega$. By continuity of multiplication, there exists an open interval $V(\omega)$ about ω such that $fV(\omega)\subseteq(\alpha, e]$. $V(\omega)$ must have some element x in common with $(\omega, e]$. But $fx<0$ in T, and hence $fx<\alpha$ in S, contrary to $fV(\omega)\subseteq(\alpha, e]$. We conclude that T must be right-handed.

$[\alpha, e]$ is a subthread of S with kernel $[\alpha, \omega]=K$ at the lower end, and $[\alpha, e]/K\cong T'$. By Theorem 8, $\kappa\to\psi(\kappa)=\kappa e$ is a retraction of K onto a subinterval $[\beta, \omega]$ of K. Dually, $\lambda\to\psi'(\lambda)=\lambda f$ is a retraction of K onto a subinterval $[\alpha, \gamma]$ of K.

If $x<\alpha$, $xe\in T''T'=T'$, and $xe\neq 0$ in T, so that $xe>\omega$ in S. The set of all x in $[f, \alpha]$ such that $xe\geqq\omega$ is closed and contains $[f, \alpha)$, hence contains α. Therefore $\psi(\alpha)=\alpha e=\omega$. Also, $\psi(\omega)=\omega e=\omega$, by Corollary 2 to Lemma 13. Dually, $\psi'(\omega)=\psi'(\alpha)=\alpha$. For κ in $[\alpha, \gamma]$, $\psi(\kappa)\in[\beta, \omega]$, and $\psi'\psi(\kappa)=\kappa ef=\kappa f=\kappa$. For λ in $[\beta, \omega]$, $\psi'(\lambda)\in[\alpha, \gamma]$, and $\psi\psi'(\lambda)=\lambda fe=\lambda e=\lambda$. Hence ψ restricted to $[\alpha, \gamma]$ and ψ' restricted to $[\beta, \omega]$ are mutually inverse mappings. Since they are continuous, and $\psi(\alpha)=\omega$, $\psi'(\omega)=\alpha$, it follows that they are order-reversing; in particular, $\psi(\gamma)=\beta$ and $\psi'(\beta)=\gamma$. We now see that $\gamma\leqq\beta$. For, were $\gamma>\beta$, ψ would be the identity mapping in $[\beta, \gamma]$ and at the same time order-reversing, evidently absurd. Hence (1) is established.

We define a mapping θ of T' into K by $\theta(x)=\beta x$ $(x>0)$ and $\theta(0)=\omega$. (2) then follows from Theorem 8.

We proceed to the proof of (J1–4), dropping the notation $\circ$. To establish (J1), we must show that if x and y are elements of $S\setminus K$ such that $xy\in K$, then $xy=\omega$ if $y>\omega$ and $xy=\alpha$ if $y<\alpha$. This follows from Lemma 12. For if $y>\omega$, then $xy=x(ey)=(xe)y$, and $xe>\omega$ also; and if $y<\alpha$, then $xy=x(fy)=(xf)y$, and $xf<\alpha$ also. To show (J2), let $\psi(\kappa)=\theta(t)$, i.e. $\kappa e=\beta t$. If $x>\omega$, then $\kappa x=\kappa ex=\beta tx=\theta(tx)$. If $x<\alpha$, then $\kappa x=\kappa xf=\kappa exef=\beta txef=\psi'(\beta txe)=\psi'\theta(txe)$.

THEOREM 12. *Let* $S=[f, e]$ *be a thread with nondegenerate interior kernel* $K=[\alpha, \omega]$ *such that* $T=S/K$ *is noncommutative. As usual, we may assume* K *is right simple. Then* $ef<\alpha<\omega<fe$, *and* $[ef, fe]/K$ *is a right-handed january thread. Let* $T^0=T\setminus\{0\}$, $T'=[0, e]$, $T'^0=(0, e]$, $T''=[f, 0]$, $T''^0=[f, 0)$. *Let* $f_1=ef$ *and* $e_1=fe$.

(1) *There exist elements* β, β_1, γ, γ_1 *of* K *such that* $Ke=[\beta, \omega]$, $Ke_1=[\beta_1, \omega]$, $Kf=[\alpha, \gamma]$, $Kf_1=[\alpha, \gamma_1]$, *and* $\gamma_1\leqq\gamma\leqq\beta\leqq\beta_1$.

(2) *The mappings* ψ *and* χ *of* K *into itself defined respectively by* $\psi(\kappa)=\kappa e$ *and* $\chi(\kappa)=\kappa f$ *have the following properties*:

(a) *ψ and χ are retractions of K onto $[\beta, \omega]$ and $[\alpha, \gamma]$, respectively.*

(b) *ψ maps $[\alpha, \gamma_1]$ in a one-to-one order-reversing fashion onto $[\beta_1, \omega]$, and the restriction of χ to $[\beta_1, \omega]$ is the inverse thereof.*

(c) *ψ maps $[\gamma_1, \gamma]$ onto β_1; χ maps $[\beta, \beta_1]$ onto γ_1.*

(d) *$\psi(\kappa) \in (\beta_1, \omega]$ if and only if $\chi(\kappa) \in [\alpha, \gamma_1)$, and then $\psi\chi(\kappa) = \psi(\kappa)$ and $\chi\psi(\kappa) = \chi(\kappa)$.*

(3) *The mappings θ and η of T' and T'', respectively, into K, defined by*

$$\theta(x) = \beta x \quad \text{for } x \text{ in } T'^0, \quad \theta(0) = \omega,$$
$$\eta(x) = \gamma x \quad \text{for } x \text{ in } T''^0, \quad \eta(0) = \alpha,$$

have the following properties:

(a) *θ and η are closed, convex, order-anti-homomorphisms of T' onto $[\beta, \omega]$ and T'' onto $[\alpha, \gamma]$ respectively.*

(b) *$\theta(e_1) = \beta_1$ and $\eta(f_1) = \gamma_1$; hence θ maps $[0, e_1]$ onto $[\beta_1, \omega]$, and η maps $[f_1, 0]$ onto $[\alpha, \gamma_1]$.*

(c) *$\chi\theta(xe_1) = \eta(xf_1)$ and $\psi\eta(xf_1) = \theta(xe_1)$ for all x in T.*

(4) *The binary operation in S, which we now denote by $\circ$, is completely determined by that in T, which we denote by juxtaposition, and the mappings ψ, χ, θ, η as follows (wherein $x, y \in T^0$; $t \in T'$; $s \in T''$; $\kappa, \lambda \in K$):*

$$\text{(H1)} \qquad x \circ y = \begin{cases} xy & \text{if } xy \neq 0 \text{ in } T, \\ \omega & \text{if } xy = 0 \text{ in } T \text{ and } y > 0, \\ \alpha & \text{if } xy = 0 \text{ in } T \text{ and } y < 0; \end{cases}$$

$$\text{(H2)} \qquad \kappa \circ x = \begin{cases} \theta(tx) & \text{if } x > 0 \text{ and } \psi(\kappa) = \theta(t), \\ \eta(sx) & \text{if } x < 0 \text{ and } \chi(\kappa) = \eta(s); \end{cases}$$

$$\text{(H3)} \qquad x \circ \kappa = \kappa;$$

$$\text{(H4)} \qquad \lambda \circ \kappa = \kappa.$$

Conversely, let $T = [f, e]$ be a noncommutative thread with zero element 0 such that $ef < fe$, and hence such that $[ef, fe]$ is a right-handed january thread. Let $f_1 = ef$ and $e_1 = fe$. Define T^0, T', T'^0, T'', T''^0 as above. Let K be a right simple thread $[\alpha, \omega]$. Assume furthermore that there exist four elements $\gamma_1 \leqq \gamma \leqq \beta \leqq \beta_1$ of K, and four mappings ψ, χ, θ, η with all the properties stated in (2) *and* (3) *above. Let $S = K \cup T^0$. Define a binary operation $\circ$ in S by* (H1–4). *Order S as usual. Then S is a thread with kernel K such that $S/K \cong T$.*

Proof. By Faucett's Theorem 4 of [5], or by Theorem 5 of Part I, the noncommutativity of T implies $ef \neq fe$. By Theorem 7, the zero element 0 of T lies between ef and fe; if $ef < fe$, then $[ef, fe]_T$ is a right january thread; if $fe < ef$, then $[fe, ef]_T$ is a left january thread. By Theorem 11, applied to $[ef, fe]_S$ or $[fe, ef]_S$, and the assumption that K is right simple, we must have $ef < fe$.

Let $e_1=fe$, $f_1=ef$, and $T_1=[f_1, e_1]_T$. By Theorem 7, T is a two-ended contact extension of the right january thread T_1 by the standard thread $[f, f_1]$ on the left and the standard thread $[e_1, e]$ on the right. We shall conduct the proof of the present theorem as though $f<f_1$ and $e_1<e$. The trifling modifications required if $f=f_1$ or $e_1=e$ are evident. From the meaning of "contact extension," and the fact (Theorem 6) that e_1 and f_1 are left identities in T_1, it follows that if

$$w \in [f, f_1], \qquad x \in [f, \alpha), \qquad y \in (\omega, e_1], \qquad z \in [e_1, e],$$

then some of the products of these elements with each other are given in the accompanying table. We shall use these without comment.

	w	x	y	z
w		x	y	e_1
x	x			xe_1
y	yf_1			y
z	f_1	x	y	

The intervals $[\alpha, e]$ and $[f, \omega]$ are subthreads of S to which we may apply Theorem 8 or its order-dual. From this theorem we infer the following. There exist elements β and γ of K such that $Ke=[\beta, \omega]$ and $Kf=[\alpha, \gamma]$. The mappings ψ and χ defined in (2) have the property (2a). The mappings θ and η defined in (3) have the property (3a).

The intervals $[\alpha, e_1]$ and $[f_1, \omega]$ are also subthreads of S, and we infer from Theorem 8 that there exist elements β_1 and γ_1 of K such that $Ke_1 =[\beta_1, \omega]$ and $Kf_1=[\alpha, \gamma_1]$. Since $\beta_1 e=(\beta_1 e_1)e=\beta_1(e_1 e)=\beta_1 e_1=\beta_1$, we infer that $\beta_1 \in Ke=[\beta, \omega]$, and so $\beta \leqq \beta_1$. Similarly, $\gamma_1 \leqq \gamma$. We postpone the proof of $\gamma \leqq \beta$.

Now $\beta e_1 \in Ke_1=[\beta_1, \omega]$, so that $\beta \leqq \beta_1 \leqq \beta e_1$. Hence $\beta_1 \in [\beta, \beta e_1]=[\beta e, \beta e_1] \subseteq \beta[e_1, e]$ by Lemma 1, so that $\beta_1=\beta x$ for some x in $[e_1, e]$. Hence $\beta_1=\beta_1 e_1 =\beta x e_1=\beta e_1=\theta(e_1)$. Similarly, $\gamma_1=\gamma f_1=\eta(f_1)$. This proves (3b), the second part being immediate from (3a).

To show (2b), let $\kappa \in [\alpha, \gamma_1]$. Then $\kappa f_1 = \kappa$, so that $\psi(\kappa) = \kappa e = \kappa f_1 e =\kappa e_1 \in [\beta_1, \omega]$, and $\chi\psi(\kappa)=\chi(\kappa e_1)=\kappa e_1 f=\kappa f_1=\kappa$. Similarly, $\psi\chi(\lambda)=\lambda$ for every λ in $[\beta_1, \omega]$. Now $xe>\omega$ for all x in $[f, \alpha)$, and the set of all x in $[f, \alpha]$ for which $xe \geqq \omega$ is closed. Hence $\alpha e \geqq \omega$, and so $\psi(\alpha)=\alpha e=\omega$. Since ψ is a one-to-one continuous mapping of $[\alpha, \gamma_1]$ onto $[\beta_1, \omega]$, and $\psi(\alpha)=\omega$, we conclude that ψ is order-reversing, and that $\psi(\gamma_1)=\beta_1$.

To show (2c), let $\kappa \in [\gamma_1, \gamma]$. Since $[\gamma_1, \gamma]=[\gamma f_1, \gamma f] \subseteq \gamma[f, f_1]$, $\kappa=\gamma w$ for some w in $[f, f_1]$. Then $\psi(\kappa)=\kappa e=\gamma w e=\gamma e_1=\gamma f_1 e=\gamma_1 e$ by (3b), and $\gamma_1 e =\psi(\gamma_1)=\beta_1$ by (2b). The proof that $\chi(\lambda)=\gamma_1$ for every λ in $[\beta, \beta_1]$ is similar.

At this point we may complete the proof of (1) by showing that $\gamma \leqq \beta$. Were $\beta<\gamma_1$, we would have $\beta=\psi(\beta)>\psi(\gamma_1)=\beta_1$ by (2b), contrary to $\beta \leqq \beta_1$, shown above. Hence $\beta \geqq \gamma_1$, and similarly we can show $\gamma \leqq \beta_1$. Suppose now that $\beta \leqq \gamma$. From $\beta \in [\gamma_1, \gamma]$ and (2c), $\beta = \psi(\beta) = \beta_1 \geqq \gamma$, and so $\beta = \gamma$. We conclude that $\beta \geqq \gamma$.

To show (2d), let $\psi(\kappa) \in (\beta_1, \omega]$. Now $\chi(\kappa) \in [\alpha, \gamma]$; assume by way of contradiction that $\chi(\kappa) \in [\gamma_1, \gamma]$. By (2c), $\psi\chi(\kappa) = \beta_1$, that is, $\kappa fe = \kappa e_1 = \beta_1$. But from $\beta_1 < \psi(\kappa) = \kappa e$, we have $(\kappa e)e_1 = \kappa e$, hence $\beta_1 = \kappa e_1 = \kappa e > \beta_1$. Hence $\chi(\kappa) \in [\alpha, \gamma_1]$. The proof that $\chi(\kappa) \in [\alpha, \gamma_1)$ implies $\psi(\kappa) \in (\beta_1, \omega]$ is similar. Assume now that either, and hence both, of these hold. From $\kappa e = \psi(\kappa) \in (\beta_1, \omega]$ we have $\kappa e_1 = \kappa(ee_1) = (\kappa e)e_1 = \kappa e$, and hence $\psi\chi(\kappa) = \kappa fe = \kappa e_1 = \kappa e = \psi(\kappa)$. The proof that $\chi\psi(\kappa) = \chi(\kappa)$ is similar.

We are left with (3c). We first note that if $x_1 \in (0, e_1]$ then $\theta(x_1) = \beta x_1 = \beta e_1 x_1 = \beta_1 x_1$, and similarly, if $y_1 \in [f_1, 0)$ then $\eta(y_1) = \gamma_1 y_1$. Now let x be any element of T^0. Since $xe_1 \in (0, e_1]$, $\theta(xe_1) = \beta_1 xe_1$, and $\chi\theta(xe_1) = \beta_1 xe_1 f = \beta_1 xf_1$. Since $xf_1 \in [f_1, 0)$, $\eta(xf_1) = \gamma xf_1$. Since $\beta_1 f_1 = \beta_1 e_1 f = \beta_1 f = \chi(\beta_1) = \gamma_1$, we have $\beta_1 xf_1 = \beta_1 f_1 xf_1 = \gamma_1 xf_1$, and so $\chi\theta(xe_1) = \eta(xf_1)$. By (2b), the restriction of $\psi\chi$ to $[\beta_1, \omega]$ is the identity mapping. Since $\theta(xe_1) \in [\beta_1, \omega]$ we conclude that $\psi\eta(xf_1) = \psi\chi\theta(xe_1) = \theta(xe_1)$. (3c) holds also for $x = 0$ since $\chi(\omega) = \alpha$ and $\psi(\alpha) = \omega$.

The relations (H2–4) follow directly from Theorem 8 applied to the subthreads $[\alpha, e]$ and $[f, \omega]$ of S, as well as those cases in (H1) in which x and y are on the same side of 0 in T. What is left to prove is (dropping the notation $\circ$) that

$$x < \alpha, \qquad y > \omega, \qquad xy \in K \text{ imply } xy = \omega;$$
$$x < \alpha, \qquad y > \omega, \qquad yx \in K \text{ imply } yx = \alpha.$$

But $xy = x(ey) = (xe)y$, and $xe > \omega$; $yx = y(fx) = (yf)x$, and $yf < \alpha$. The desired conclusions then follow from those cases of (H1) which follow from Theorem 8.

We omit proof of the converse, and proceed to a discussion of the construction of all possible linear extensions S of a given nondegenerate right simple thread $K = [\alpha, \omega]$ by a given noncommutative thread $T = [f, e]$ with 0.

By Theorem 12, no such S can exist unless $ef < fe$, i.e. $[ef, fe]$ is a right-handed january thread. In other words, T and K must be similarly oriented. But this condition is sufficient for the existence of at least one linear extension S. In the notation of Theorem 12, we can take $\beta = \beta_1 = \omega$, $\gamma = \gamma_1 = \alpha$; $\psi(\kappa) = \omega$ and $\chi(\kappa) = \alpha$, for all κ in K; $\theta(x) = \omega$ for all x in $T' = [0, e]$; $\eta(x) = \alpha$ for all x in $T'' = [t, 0]$. We get all possible linear extensions S of K by T by the following procedure.

(1) Select elements β_1 and γ_1 of K such that (a) $\gamma_1 \leqq \beta_1$, (b) the intervals $[\beta_1, \omega]$ and $[\alpha, \gamma_1]$ are order-anti-isomorphic, and (c) $[\beta_1, \omega]$ is order-anti-isomorphic with a homomorphic image of $[0, e_1]$. Let σ be an order-anti-isomorphism of $[\beta_1, \omega]$ onto $[\alpha, \gamma_1]$, and let τ be its inverse. Let θ_1 be a c.c.o-a-h. of $[0, e_1]$ onto $[\beta_1, \omega]$. Then the mapping η_1 defined by $\eta_1(x) = \sigma\theta_1(xe_1)$ is a c.c.o-a-h. of $[f_1, 0]$ onto $[\alpha, \gamma_1]$.

(2) Extend θ_1 to a c.c.o-a-h. θ of $[0, e]$ into K, and η_1 to a c.c.o-a-h. η of $[f, 0]$ into K, such that $\beta=\theta(e)\geqq\gamma=\eta(f)$. Let ψ be any continuous mapping of K into itself satisfying the following conditions:

$$\psi(\kappa)\begin{cases}=\tau(\kappa) & \text{if } \kappa\in[\alpha, \gamma_1],\\ =\beta_1 & \text{if } \kappa\in[\gamma_1, \gamma],\\ \in[\beta, \omega] & \text{if } \kappa\in[\gamma, \beta],\\ =\kappa & \text{if } \kappa\in[\beta, \omega].\end{cases}$$

Let χ be any continuous mapping of K into itself satisfying the following conditions:

$$\chi(\kappa)\begin{cases}=\kappa & \text{if } \kappa\in[\alpha, \gamma],\\ =\sigma\psi(\kappa) & \text{if } \kappa\in[\gamma, \beta] \text{ and } \psi(\kappa)\in[\beta_1, \omega],\\ \in[\gamma_1, \gamma] & \text{if } \kappa\in[\gamma, \beta] \text{ and } \psi(\kappa)\in[\beta, \beta_1],\\ =\gamma_1 & \text{if } \kappa\in[\beta, \beta_1],\\ =\sigma(\kappa) & \text{if } \kappa\in[\beta_1, \omega].\end{cases}$$

(The foregoing is always possible, since we may take, for example,

$$\begin{aligned}\theta(x) &= \beta_1 && \text{for all } x \text{ in } [e_1, e],\\ \eta(x) &= \gamma_1 && \text{for all } x \text{ in } [f, f_1],\\ \psi(\kappa) &= \beta_1 && \text{for all } \kappa \text{ in } [\gamma_1, \beta_1],\\ \chi(\kappa) &= \gamma_1 && \text{for all } \kappa \text{ in } [\gamma_1, \beta_1].\end{aligned}$$

This, of course, entails $\beta=\beta_1$ and $\gamma=\gamma_1$.)

(3) The conditions for the converse half of Theorem 12 now hold, and we obtain thereby a linear extension S of K by T.

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