WEIGHTED QUADRATIC NORMS AND ULTRASPERICAL POLYNOMIALS, I(1)

BY

RICHARD ASKEY AND ISIDORE HIRSCHMAN, JR.

1. Introduction. Let \( \nu \geq 0 \) be fixed and let

\[
2^n \left( \nu + \frac{1}{2} \right)_n W_{\nu}(n, x) = (-1)^n (1 - x^2)^{-\nu + 1/2} \left( \frac{d}{dx} \right)^n [(1 - x^2)^{n+\nu-1/2}]
\]

be the ultraspherical polynomial of degree \( n \) and index \( \nu \) normalized by the condition

\[
W_{\nu}(n, 1) = 1 \quad (n = 0, 1, \ldots).
\]

If we define

\[
d\Omega_{\nu}(x) = (1 - x^2)^{-\nu + 1/2} dx,
\]

\[
\omega_{\nu}(n) = \frac{(2\nu)_n(n + \nu)\Gamma(\nu)}{n!\Gamma(\nu + 1/2)},
\]

then

\[
\int_{-1}^{1} W_{\nu}(n, x)W_{\nu}(m, x)d\Omega_{\nu}(x) = \delta(n, m)/\omega_{\nu}(n),
\]

where \( \delta(n, m) \) is 1 if \( n = m \) and is 0 if \( n \neq m \). For all such formulas see [2, vol. 2, Chapter X]. The harmonic analysis of ultraspherical polynomials rests upon the dual convolution structure due to Lewitan [7] and Bochner [1] and described below. For a detailed discussion of the implications of the following formulas, as well as a general survey of the present subject, see [6]. Let \( f(x) \) be a measurable function on \([-1 \leq x \leq 1]\) and let us write \( f(x) \in P^n \) if

\[
\|f\|_1 = \int_{-1}^{1} |f(x)|d\Omega_{\nu}(x)
\]

is finite. For \( f \in B \), we define the transform \( \hat{f}(n) \) of \( f(x) \) by

Presented to the Society September 1, 1955; received by the editors June 28, 1957.

(1) This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF18(600)-568. Reproduction in whole or in part is permitted for any purpose of the United States Government.

(2) \((a)_r = \Gamma(a+r)/\Gamma(r)\).
QUADRATIC NORMS AND ULTRASPHERICAL POLYNOMIALS

\[ f^{(n)} = \int_{-1}^{1} f(x) W_r(n, x) d\Omega_r(x) \quad (n = 0, 1, \cdots). \]

We then have the (formal) inversion formula

\[ f(x) = \sum_{n=0}^{\infty} f^{(n)} W_r(n, x) \omega_r(n). \]

Let us set

\[ C_r(x, y, z) = 2^{1-2r} \Gamma(2\nu) \Gamma(\nu)^{-2} (1 - x^2 - y^2 - z^2 + 2xyz)^{-1} [(1 - x^2)(1 - y^2)(1 - z^2)]^{(1/2) - r} \]

if \((1 - x^2 - y^2 - z^2 + 2xyz) > 0\), otherwise let \(C_r(x, y, z) = 0\). By a formula of Gegenbauer \([1, \text{vol. 2, p. 177}]\)

\[ (1) \quad \int_{-1}^{1} C_r(x, y, z) W_r(n, z) d\Omega_r(z) = W_r(n, x) W_r(n, y). \]

Starting from this result it is possible to show that if \(f_1(x), f_2(x) \in B_r\) and if

\[ (2) \quad f_1 * f_2 \cdot (x) = \int_{-1}^{1} \int_{-1}^{1} f_1(y) f_2(z) C_r(x, y, z) d\Omega_r(y) d\Omega_r(z), \]

then \(f_1 * f_2 \cdot (x) \in B_r\) and \((f_1 * f_2) \sim (n) = f_1 \sim (n) f_2 \sim (n)\). Let \(F(n)\) be defined for \([n = 0, 1, \cdots]\) and let us write \(F(n) \in b_r\) if

\[ \|F\|_1 = \sum_{n=0}^{\infty} |F(n)| \omega_r(n) \]

is finite. For \(F \in b_r\), we define the transform \(F(\sim)(x)\) by

\[ F(\sim)(x) = \sum_{m=0}^{\infty} F(n) W_r(n, x) \omega_r(n) \quad [-1 \leq x \leq 1]. \]

The inversion formula is then

\[ F(n) = \int_{-1}^{1} F(\sim)(x) W_r(n, x) d\Omega_r(x). \]

Let

\[ c_r(k, j, n) = \frac{\pi^{2-2r}(\nu)_{\sigma-k}(\nu)_{\sigma-j}(\nu)_{\sigma-n}}{\Gamma(\nu)^2 (\sigma-k)! (\sigma-j)! (\sigma-n)!} \frac{k! \quad j! \quad n!}{(2\nu)_{k} (2\nu)_{j} (2\nu)_{n}} \frac{(2\nu)_{\sigma}}{(\nu)^{\sigma + \nu}} \]

if \(k+j+n\) is even and if \(\max(k, j, n) \leq \sigma\) where \(2\sigma = k+j+n\); otherwise let \(c_r(k, j, n) = 0\). A formula of Dougall \([8]\) asserts that

\[ (3) \quad \sum_{n=0}^{\infty} c_r(k, j, n) W_r(n, x) \omega_r(n) = W_r(k, x) W_r(j, x), \]
and from this it can be shown that if \( F_1(n), F_2(n) \in \mathfrak{b} \), and if

\[
F_1 \ast F_2 \cdot (n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} F_1(k)F_2(j)c_\nu(k, j, n)\omega_\nu(k)\omega_\nu(j),
\]

then \( F_1 \ast F_2 \cdot (n) \in \mathfrak{b} \) and \((F_1 \ast F_2)^\sim(x) = F_1^\sim(x)F_2^\sim(x)\).

We shall be concerned in the present paper with those linear transformations \( T \) of functions on \([-1, 1]\) into functions on \([-1, 1]\) which commute with the convolution operation (2); that is

\[
(Tf_1) \ast f_2 = f_1 \ast (Tf_2).
\]

It is easily seen that to every such transformation there corresponds a function \( t(n) \) defined on \( n = 0, 1, \cdots \) such that

\[
(Tf)^\sim(n) = f^\sim(n)t(n).
\]

An equivalent formulation is that if

\[
f(x) = \sum_{n=0}^{\infty} f^\sim(n)W_\nu(n, x)\omega_\nu(n)
\]

then (formally)

\[
Tf(x) = \sum_{n=0}^{\infty} f^\sim(n)t(n)W_\nu(n, x)\omega_\nu(n).
\]

Such transformations are called "multiplier" transformations.

For \( f(x) \) a real measurable function on \([-1, 1]\) we set

\[
\mathfrak{R}_{\alpha,\beta}[f] = \left[ \int_{-1}^{1} f(x)^2(1 + x)^\alpha(1 - x)^\beta d\Omega_\nu(x) \right]^{1/2}, \left( -\frac{1}{2} < \alpha, \beta < \frac{1}{2} \right).
\]

We shall also use \( \mathfrak{R}_{\alpha,\beta} \) to denote the space of functions \( f(x) \) for which \( \mathfrak{R}_{\alpha,\beta}[f] \) is finite.

Our objective in the present paper is to find rather general sufficient conditions which will insure that \( T = \{ t(n) \}_0^\infty \) be a bounded linear transformation of \( \mathfrak{R}_{\alpha,\beta} \) into itself. The dual theory, in which the roles of \( F \) and \( f \) are interchanged, will be dealt with in the subsequent paper. We shall there be concerned with those linear transformations \( t \) of functions on \([n = 0, 1, \cdots]\) into functions on \([n = 0, 1, \cdots]\) which commute with the convolution operation (4); that is

\[
tF_1 \ast F_2 = F_1 \ast tF_2.
\]

To every such transformation there corresponds a measurable function \( t(x) \) defined on \(-1 \leq x \leq 1\) such that

\[
[tF]^\sim(x) = F^\sim(x)t(x).
\]

A (formally) equivalent definition is that if
\[ F(n) = \int_{-1}^{1} F^{-}(x) W_{\nu}(n, x) d\Omega_{\nu}(x) \]

then

\[ (tF)(n) = \int_{-1}^{1} F^{-}(x)t(x) W_{\nu}(n, x) d\Omega_{\nu}(x). \]

We set

\[ \mathcal{M}_{\alpha}[F] = \left\{ \sum_{n=0}^{\infty} F(n)^{2} \omega_{\nu}(n)(n+1)^{2\alpha} \right\}^{1/2} \left( -\frac{1}{2} < \alpha < \frac{1}{2} \right). \]

We also use \( \mathcal{M}_{\alpha} \) to denote the space of functions \( F(n) \) for which \( \mathcal{M}_{\alpha}[F] \) is finite. In the succeeding paper we shall find rather general sufficient conditions on \( t = t(x) \) which will insure that \( t \) is a bounded transformation of \( \mathcal{M}_{\alpha} \) into itself. These papers thus complete investigations initiated in [4].

2. Weighted quadratic norms. Let \( \nu > 0 \) be fixed and let \( r(k) \) be a non-negative function defined for \( k = 1, 2, \ldots \) such that \( \sum_{k}^{\infty} r(k) < \infty \). We define

\[ s(x) = \sum_{k=1}^{\infty} [1 - W_{\nu}(k, x)] r(k), \]

\[ S(m, n) = \sum_{k=1}^{\infty} c_{\nu}(m, n, k) r(k). \]

We write \( f(x) \in L^1 \) if \( \int_{-1}^{1} |f(x)| d\Omega_{\nu}(x) < \infty \).

**Theorem 2a.** If \( s(x) \) and \( S(m, n) \) are defined as above and if \( f(x) \in L^1 \) then

\[ \int_{-1}^{1} f(x)^2 s(x) d\Omega_{\nu}(x) = \frac{1}{2} \sum_{m, n=0}^{\infty} [f^{-}(n) - f^{-}(m)]^2 S(n, m) \omega_{\nu}(n) \omega_{\nu}(m) \]

provided that the left hand side is finite.

The assumption \( f \in L^1 \) is, of course, necessary to insure that \( f^{-}(n) \) is defined. Let us suppose first that

\[ \int_{-1}^{1} f(x)^2 d\Omega_{\nu}(x) < \infty, \]

a restriction that will be removed at the end of the proof. We expand \( [f^{-}(n) - f^{-}(m)]^2 \) so that the right hand side of (1) splits into three terms,

\[ I_1 = \frac{1}{2} \sum_{m, n=0}^{\infty} f^{-}(n)^2 S(m, n) \omega_{\nu}(n) \omega_{\nu}(m), \]

\[ I_2 = \frac{1}{2} \sum_{m, n=0}^{\infty} f^{-}(m)^2 S(m, n) \omega_{\nu}(n) \omega_{\nu}(m), \]

\[ I_3 = \frac{1}{2} \sum_{m, n=0}^{\infty} f^{-}(n) f^{-}(m) [S(n, m) - S(n, n) - S(m, m) + S(n, m) + S(m, n)], \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
\[ I_3 = - \sum_{n, m=0}^{\infty} f(n) f(m) S(n, m) \omega_v(n) \omega_v(m). \]

Because the summand in \( I_1 \) is non-negative we have
\[ I_1 = \frac{1}{2} \sum_{n=0}^{\infty} f(n)^2 \omega_v(n) \sum_{m=0}^{\infty} S(n, m) \omega_v(m). \]

Now
\[ \sum_{m=0}^{\infty} S(n, m) \omega_v(m) = \sum_{m=0}^{\infty} \omega_v(m) \sum_{k=1}^{\infty} c_v(n, m, k) r(k) \]
\[ = \sum_{k=1}^{\infty} r(k) \sum_{m=0}^{\infty} c_v(n, m, k) \omega_v(m). \]

By (3) of §1 with \( x = 1 \)
\[ \sum_{m=0}^{\infty} c_v(n, m, k) \omega_v(m) = 1 \]
and thus
\[ I_1 = \frac{1}{2} \left[ \sum_{n=1}^{\infty} f(n)^2 \omega_v(n) \right] \left[ \sum_{k=1}^{\infty} r(k) \right]. \]

Using Parseval's equality we obtain
\[ I_1 = \frac{1}{2} \left[ \int_{-1}^{1} f(x)^2 d\Omega_v(x) \right] \left[ \sum_{k=1}^{\infty} r(k) \right]. \]

Similarly
\[ I_2 = \frac{1}{2} \left[ \int_{-1}^{1} f(x)^2 d\Omega_v(x) \right] \left[ \sum_{k=1}^{\infty} r(k) \right]. \]

As we have seen the infinite sums defining \( I_1 \) and \( I_2 \) converge absolutely. Now we have
\[ (3) \quad I_3 = - \sum_{n, m=0}^{\infty} f(n) f(m) \omega_v(n) \omega_v(m) \sum_{k=1}^{\infty} c_v(n, m, k) r(k), \]
and the inequality
\[ |f(n) f(m)| \leq \frac{1}{2} f(n)^2 + \frac{1}{2} f(m)^2 \]
shows that the sum in (3) also converges absolutely; thus
We have

\[
\sum_{m=0}^{\infty} f^{-}(m)c_{v}(n, m, k)\omega_{v}(m) = \sum_{m=0}^{\infty} \int_{-1}^{1} f(x)c_{v}(n, m, k)\omega_{v}(m)W_{v}(m, x)d\Omega_{v}(x).
\]

The formal infinite sum is actually finite here and thus the order of summation and integration can be interchanged. Using (3) of §1 we obtain

\[
\sum_{m=0}^{\infty} f^{-}(m)c_{v}(n, m, k)\omega_{v}(m) = \int_{-1}^{1} f(x)W_{v}(n, x)W_{v}(k, x)d\Omega_{v}(x).
\]

Since \(| W_{v}(k, x)| \leq 1, k = 0, 1, \ldots, -1 \leq x \leq 1\) it is easy to see that

\[
\sum_{k=1}^{\infty} r(k) \sum_{m=0}^{\infty} f^{-}(m)c_{v}(n, m, k)\omega_{v}(m) = \int_{-1}^{1} f(x)\left\{ \sum_{k=1}^{\infty} W_{v}(k, x)r(k) \right\} W_{v}(n, x)d\Omega_{v}(x),
\]

\[
= \int_{-1}^{1} f(x)\left\{ \sum_{k=1}^{\infty} r(k) - s(x) \right\} W_{v}(n, x)d\Omega_{v}(x).
\]

Making use of this and of the definition of \(f^{-}(n)\) and employing the general form of Parseval's equality in (4) we find that

\[
I_{3} = -\int_{-1}^{1} f(x)2i \sum_{k=1}^{\infty} r(k) - s(x)d\Omega_{v}(x).
\]

Combining our evaluations of \(I_{1}, I_{2},\) and \(I_{3}\) our theorem is proved, under the additional assumption (2).

Suppose that (2) is not satisfied. We set

\[
f_{j}(x) = \begin{cases} j & f(x) > j, \\ f(x) & -j \leq f(x) \leq j, \\ -j & f(x) < -j. \end{cases}
\]

Then

\[
\int_{-1}^{1} f_{j}(x)^{2}d\Omega_{v}(x) < \infty \quad j = 1, 2, \ldots,
\]

and thus by what we have already proved

\[
\int_{-1}^{1} f_{j}(x)^{2}s(x)d\Omega_{v}(x) = \frac{1}{2} \sum_{m, n=0}^{\infty} [f_{j}^{-}(n) - f_{j}^{-}(m)]^{2}S(m, n)\omega_{v}(m)\omega_{v}(n).
\]

Now
\[ \lim_{j \to \infty} \int_{-1}^{1} f_j(x)^2 s(x) d\Omega_\nu(x) = \int_{-1}^{1} f(x)^2 s(x) d\Omega_\nu(x) \]

while, since \( f_j(n) \to f(n) \) as \( j \to \infty \) \((n = 0, 1, \cdots)\), we have

\[ \liminf_{j \to \infty} \sum_{m,n=0}^{\infty} [f_j(n) - f(j)]^2 S(m, n) \omega_s(m) \omega_s(n) \]

\[ \geq \sum_{m,n=0}^{\infty} [f(n) - f(j)]^2 S(m, n) \omega_s(m) \omega_s(n). \]

Thus

\[ (5) \quad \int_{-1}^{1} f(x)^2 s(x) d\Omega_\nu(x) \geq \frac{1}{2} \sum_{m,n=0}^{\infty} [f(n) - f(j)]^2 S(m, n) \omega_s(m) \omega_s(n). \]

If

\[ \int_{-1}^{1} f(x)^2 s(x) d\Omega_\nu(x) < \infty \]

then given \( \epsilon > 0 \) we can choose functions \( g(x) \) and \( h(x) \) such that

\[ f(x) = g(x) + h(x), \]

\[ \int_{-1}^{1} g(x)^2 d\Omega_\nu(x) < \infty, \]

\[ \int_{-1}^{1} g(x)^2 s(x) d\Omega_\nu(x) \geq \int_{-1}^{1} f(x)^2 s(x) d\Omega_\nu(x) - \epsilon, \]

\[ \int_{-1}^{1} h(x)^2 s(x) d\Omega_\nu(x) < \epsilon. \]

From the inequality

\[ a^2 \leq (1 + \epsilon^\rho)(a + b)^2 + (1 + \epsilon^\rho)b^2, \]

valid for any value of \( \rho \), and the relation \( f(n) = g(n) + h(n) \), we obtain

\[ (1 + \epsilon^\rho) \sum_{m,n=0}^{\infty} [f(n) - f(m)]^2 S(m, n) \omega_s(m) \omega_s(n) \]

\[ \geq \sum_{m,n=0}^{\infty} [g(n) - g(m)]^2 S(m, n) \omega_s(m) \omega_s(n) \]

\[ - (1 + \epsilon^\rho) \sum_{m,n=0}^{\infty} [h(n) - h(m)]^2 S(m, n) \omega_s(m) \omega_s(n). \]

Using (1) for \( g(x) \) and (5) for \( h(x) \) we have
\[ \frac{1}{2} \sum_{m, n=0}^{\infty} \left[ f(m) - f(n) \right]^2 S(m, n) \omega_s(m) \omega_s(n) \geq \int_{-1}^{1} f(x)^2 s(x) d\Omega_s(x) - (2 + e^\rho) \epsilon. \]

Taking first \( \epsilon \) arbitrarily small and then \( \rho \) arbitrarily large we see that

\[ \frac{1}{2} \sum_{m, n=0; m \neq n}^{\infty} \left[ f(m) - f(n) \right]^2 S(m, n) \omega_s(m) \omega_s(n) \geq \int_{-1}^{1} f(x)^2 s(x) d\Omega_s(x). \]

The inequalities (5) and (6) together imply our desired result.

**Theorem 2b.** If \( f(x) \in L^1, g(x) \in L^1 \) and if

\[ \int_{-1}^{1} f(x)^2 s(x) d\Omega_s(x) < \infty, \quad \int_{-1}^{1} g(x)^2 s(x) d\Omega_s(x) < \infty, \]

then

\[ \int_{-1}^{1} f(x) g(x) s(x) d\Omega_s(x) = \frac{1}{2} \sum_{m, n=0; m \neq n}^{\infty} \left[ f(n) - f(m) \right] \left[ g(n) - g(m) \right] S(m, n) \omega_s(m) \omega_s(n). \]

This follows from Theorem 2a using the standard device of writing out (1) for \( f(x) + g(x) \) and for \( f(x) - g(x) \) and then subtracting the results.

3. **Approximations.** Let \( \nu > 0 \) be fixed. We set

\[ sa(x) = \sum_{n=1}^{\infty} \left[ 1 - W_s(n, x) \right] n^{-2a-1}, \]

\[ Sa(m, n) = \sum_{k=1}^{\infty} c_s(k, m, n) k^{-2a-1}. \]

Let us write \( A(y) \approx B(y) \) for \( y \in Y \) if there exist finite positive constants \( C_1 \) and \( C_2 \) such that \( A(y) \leq C_1 B(y) \) and \( B(y) \leq C_2 A(y) \) for \( y \in Y \).

**Lemma 3a.** If \( 0 < \alpha < 1/2 \) and if \( sa(x) \) is defined by (1) then

\[ sa(x) \approx (1 - x)^\alpha \quad (-1 \leq x \leq 1). \]

We have, see [2, vol. 2, p. 175],

\[ W_s(n, \cos \theta) = \frac{n!}{(2\nu)_n} \sum_{m=0}^{n} \frac{(\nu)_m (n-m)}{m!(n-m)!} \cos (n - 2m)\theta. \]

\(^{(1)}\) The case \( \nu = 0 \) requires certain small changes and is left to the reader.
Setting $\theta = 0$ we see that
\[
1 = \frac{n!}{(2\nu)_n} \sum_{m=0}^{n} \frac{(\nu)_m (\nu)_{n-m}}{m!(n-m)!}
\]
and thus
\[
1 - W_r(n, \cos \theta) = \frac{n!}{(2\nu)_n} \sum_{m=0}^{n} \frac{(\nu)_m (\nu)_{n-m}}{m!(n-m)!} [1 - \cos (n - 2m)\theta].
\]
Summing separately over $n$ even and $n$ odd we have
\[
\sum_{n=1}^{\infty} [1 - W_r(2n, \cos \theta)] (2n)^{-1-2\alpha} = 2 \sum_{j=1}^{\infty} [1 - \cos 2j\theta] \sum_{n \geq j} \frac{(\nu)_{n-j}(\nu)_{n+j}}{(n-j)!(n+j)!} \frac{(2n)!}{(2\nu)_{2n}} (2n)^{-1-2\alpha},
\]
\[
\sum_{n=0}^{\infty} [1 - W_r(2n+1, \cos \theta)] (2n+1)^{-1-2\alpha} = 2 \sum_{j=0}^{\infty} [1 - \cos (2j + 1)\theta] \sum_{n \geq j} \frac{(\nu)_{n-j}(\nu)_{n+j+1}(2n + 1)!}{(n-j)!(n+j+1)!(2\nu)_{2n+1}} (2n+1)^{-1-2\alpha}.
\]
Since $(\nu)_r/r! \approx r^{-1}$ we have that
\[
\sum_{n \geq j} \frac{(\nu)_{n-j}(\nu)_{n+j+1}(2n)!}{(n-j)!(n+j)!(2\nu)_{2n}} (2n)^{-1-2\alpha} \approx \sum_{n \geq j} n^{-2\gamma-2\alpha}(n-j)^{-1}(n+j)^{-1}
\]
\[
\approx \Sigma_1 + \Sigma_2
\]
where $\Sigma_1$ corresponds to the range $j \leq n \leq 2j$ and $\Sigma_2$ to the range $2j < n$. Now
\[
\Sigma_1 \approx j^{-\alpha-1} \sum_{n=j}^{2j} (n-j)^{-1} \approx j^{-\alpha-1},
\]
\[
\Sigma_2 \approx \sum_{n>2j} n^{-2\gamma-2} \approx j^{-\alpha-1},
\]
and thus
\[
\sum_{n=1}^{\infty} [1 - W_r(2n, \cos \theta)] (2n)^{-1-2\alpha} \approx \sum_{j=1}^{\infty} [1 - \cos 2j\theta] (2j)^{-2\alpha-1}.
\]
Similarly
\[
\sum_{n=0}^{\infty} [1 - W_r(2n+1, \cos \theta)] (2n+1)^{-1-2\alpha} \approx \sum_{j=0}^{\infty} [1 - \cos (2j + 1)\theta] (2j+1)^{-2\alpha-1}.
\]
Combining these results we see that

$$s_a(\cos \theta) \approx \sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2a-1}.$$

Now

$$\frac{d}{d\theta} \sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2a-1} = \sum_{j=1}^{\infty} (\sin j\theta) j^{-2a},$$

$$\frac{d}{d\theta} \sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2a-1} \sim \theta^{-1+2a} \quad (\theta \to 0^+).$$

For this last step see Zygmund [11, p. 114]. Integrating, we find that

$$\sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2a-1} \approx (1 - \cos \theta)^{a} \quad (0 \leq \theta \leq \pi);$$

that is

$$s_a(x) \approx (1 - x)^a \quad (-1 \leq x \leq 1).$$

**Lemma 3b.** If $0 < a < 1/2$ and if $S_a(m, n)$ is defined by (2) then

$$S_a(m, n) \approx (n + 1)^{-2r} (n - m)^{-1-2a} \quad (n > m).$$

If $n > m$ then

$$S_a(m, n) = \sum_{j=0}^{m} c_r(n - m + 2j, m)(n - m + 2j)^{-1-2a}.$$ 

It is easily verified from this, using the relation $(\alpha)_{r/r!} \approx (r+1)^{\alpha-1}$ that

$$S_a(m, n) \approx (m + 1)^{-2r}(n + 1)^{-r} \sum_{j=0}^{m} (m - j + 1)^{r-1}(n - m + j)^{-1}(j + 1)^{-1}(n - m + 2j)^{-2r-2a}.$$ 

We must distinguish between two cases, $n \geq 3m/2$ and $n < 3m/2$. Suppose $n \geq 3m/2$. Then $(n - m + j + 1) \approx n + 1$, $(n - m + 2j) \approx n + 1$, and

$$S_a(m, n) \approx (m + 1)^{1-2r}(n + 1)_{-1-2r-2a} \sum_{j=0}^{m} (m - j + 1)^{r-1}(j + 1)^{-r-1}.$$ 

Now

$$\sum_{j=0}^{m} (m - j + 1)^{r-1}(j + 1)^{-r-1} \approx (m + 1)^{2r-1},$$

and thus, since $n + 1 \approx n - m$ if $n \geq 3m/2$,

$$S_a(m, n) \approx (n + 1)^{-2r}(n - m)^{-1-2a} \quad (n \geq 3m/2).$$
If $n < 3m/2$ then we have

$$S_a(m, n) \approx \Sigma_1 + \Sigma_2 + \Sigma_3$$

where $\Sigma_1$ corresponds to the range $0 \leq j < n - m$, $\Sigma_2$ to the range $n - m \leq j < m/2$, and $\Sigma_3$ to the range $m/2 \leq j \leq m$. If $0 \leq j < n - m$ (and if $m < n < 3m/2$) then $(m - j + 1) \approx (n + 1)$, $(n - m + j) \approx (n - m)$, $(n - m + 2j) \approx (n - m)$, and thus

$$\Sigma_1 \approx (n + 1)^{-2\nu} (n - m)^{-2\alpha - 2} \sum_{0 \leq j < n - m} (j + 1)^{-r - 1},$$

$$\approx (n + 1)^{-2\nu} (n - m)^{-2\alpha}.$$

If $n - m \leq j < m/2$ then $(m - j + 1) \approx (n + 1)$, $(n - m + j) \approx (j + 1)$, $(n - m + 2j) \approx (j + 1)$, $(m + 1) \approx (n + 1)$, and thus

$$\Sigma_2 \approx (n + 1)^{-2\nu} \sum_{n - m \leq j < m/2} (j + 1)^{-2\alpha - 2},$$

$$\approx (n + 1)^{-2\nu} (n - m)^{-2\alpha}.$$  

If $m/2 \leq j \leq m$ then $(n - m + j) \approx (n + 1)$, $(j + 1) \approx (n + 1)$, $(n - m + 2j) \approx (n + 1)$, $(m + 1) \approx (n + 1)$ and

$$\Sigma_3 \approx (n + 1)^{-1 - 3\nu - 2\alpha} \sum_{m/2 \leq j \leq m} (m - j + 1)^{-1},$$

$$< \approx (n + 1)^{-2\nu} (n - m)^{-1 - 2\alpha}.$$  

Combining these estimates we see that

$$S_a(m, n) \approx (n + 1)^{-2\nu} (n - m)^{-1 - 2\alpha}$$

$(n < 3m/2)$.

Our demonstration is now complete.

Lemmas 3a and 3b and Theorem 2a together imply the following result.

**Theorem 3c.** If $0 < \alpha < 1/2$, and if $f \in \mathcal{R}_{0, \alpha}^*$ then

$$\mathcal{R}_{0, \alpha}^*[f]^2 \approx \sum_{m, n = 0; m \neq n}^{\infty} [f^r(n) - f^r(m)]^2 S_a(m, n) \omega_r(m) \omega_r(n).$$

4. **Some inequalities.** We begin by noting that the functions

$$W_r(n, \cos \theta)(\sin \theta)^{\nu \frac{1}{2}}(n)$$

$(n = 0, 1, \ldots)$, are orthonormal and uniformly bounded on $0 \leq \theta \leq \pi$, see [2, vol. 2, p. 174 and p. 206]. If $\phi_n(\theta)$, $n = 0, 1, \ldots$, are a uniformly bounded orthonormal set of functions on $[0, b]$ and if

$$a(n) = \int_0^b \psi(\theta) \phi_n(\theta) d\theta,$$
then if $0 \leq \alpha < 1/2$

$$\sum_{n=0}^{\infty} a(n)^2 (R(n) + 1)^{-2\alpha} \leq A(\alpha) \int_{0}^{b} \psi(\theta)^2 \theta^{2\alpha} d\theta,$$

see [5]. Here $R(0), R(1), R(2), \cdots$ is any rearrangement of $0, 1, 2, \cdots$. We have

$$f(n) = \int_{-1}^{1} f(s) W_r(n, x) ds.$$

Setting $x = \cos \theta$ we find that

$$\omega_r(n)^{1/2} f(n) = \int_{0}^{\pi} \{f(\cos \theta)(\sin \theta)^r\} \{\omega_r(n)^{1/2} W_r(n, \cos \theta)(\sin \theta)^r\} d\theta,$$

and thus

$$\sum_{n=0}^{\infty} f(n)^2 \omega_r(n) [R(n) + 1]^{-2\alpha} \leq A(\alpha) \int_{0}^{\pi} \{f(\cos \theta)(\sin \theta)^r\}^2 \theta^{2\alpha} d\theta,$$

$$\leq A'(\alpha) \int_{-1}^{1} f(x)^2 (1 - x)^{\alpha} d\Omega_r(x).$$

We have proved the following result.

**Theorem 4a.** If $0 \leq \alpha < 1/2$ and if $R(0), R(1), R(2), \cdots$ is any rearrangement of $0, 1, 2, \cdots$ then

$$\sum_{n=0}^{\infty} f(n)^2 \omega_r(n) [R(n) + 1]^{-2\alpha} \leq A(\alpha) \eta_{0, \alpha}^r f_1.$$

Let $S_N$ be the multiplier transformation which carries

$$f(x) \sim \sum_{n=0}^{\infty} f(n) \omega_r(n) W_r(n, x)$$

into

$$S_N f(x) \sim \sum_{n=0}^{N} f(n) \omega_r(n) W_r(n, x).$$

As a first application of our ideas we prove

**Theorem 4b.** If $0 \leq \alpha < 1/2$ then

$$\eta_{0, \alpha}[S_N f] \leq A(\alpha) \eta_{0, \alpha}[f].$$

We may suppose $\alpha > 0$ since the case $\alpha = 0$ follows from Parseval's equal-
ity. By Theorem 3c we have
\[ \mathfrak{M}^*_{0,\alpha}[S_N f]^2 \leq A \sum_{m,n \leq N} [f^-(n) - f^-(m)]^2 S_\alpha(m, n) \omega_\nu(m) \omega_\nu(n) \]
\[ + A \sum_{m \leq N; n > N} f^-(m)^2 S_\alpha(m, n) \omega_\nu(m) \omega_\nu(n). \]

A second application of this same result shows that
\[ (1) \sum_{m,n \leq N} [f^-(n) - f^-(m)]^2 S_\alpha(m, n) \omega_\nu(m) \omega_\nu(n) \leq A \mathfrak{M}^*_{0,\alpha}[f]^2. \]

Further, Lemma 3b implies that if \( m \leq N \) then
\[ \sum_{n > N} S_\alpha(m, n) \omega_\nu(n) \approx \sum_{n > N} (n - m)^{-1-2\alpha} \leq (N + 1 - m)^{-2\alpha}. \]
Thus
\[ \sum_{m \leq N; n > N} f^-(m)^2 S_\alpha(m, n) \omega_\nu(m) \omega_\nu(n) \approx \sum_{m=0}^N f^-(m)^2 \omega_\nu(m)(N + 1 - m)^{-2\alpha}, \]
\[ (2) < A \mathfrak{M}^*_{0,\alpha}[f]^2, \]
by Theorem 4a. The inequalities (1) and (2) together imply our desired result.

5. Bounded multiplier transformations. Let \( b_\mu = 3 \cdot 2^{\mu-2}, r_\mu = 2^{\mu-1} \), let \( \sigma_\mu \) be the set of integers \( b_\mu - r_\mu \leq k < b_\mu + r_\mu \), and let
\[ \rho_\mu(x) = [1 - r_\mu^{-2}(x - b_\mu)^2]. \]

If
\[ f(x) \sim \sum_{n=0}^\infty f^-(n) \omega_\nu(n) W_\nu(n, x) \]
then we set
\[ E_\mu(x) = \sum_{n \in \sigma_\mu} f^-(n) \rho_\mu(n) \omega_\nu(n) W_\nu(n, x). \]

**Lemma 5a.** If \( 0 \leq \alpha < 1/2 \) then
\[ \sum_{\mu=2}^\infty \mathfrak{M}^*_{0,\alpha}[E_\mu]^2 \leq A(\alpha) \mathfrak{M}^*_{0,\alpha}[f]^2. \]

Evidently we may suppose \( \alpha > 0 \). By Theorem 3c we have
\[ \mathfrak{M}^*_{0,\alpha}[E_\mu]^2 \approx \Sigma_1 + \Sigma_2 + \Sigma_3 \]
where
\[
\begin{align*}
\Sigma_1 &= \sum_{m, n \in \sigma, n > m} [\rho_\mu(n)f(n) - \rho_\mu(m)f(m)]^2S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n), \\
\Sigma_2 &= \sum_{m, n \in \sigma, n > m} \rho_\mu(m)^2f(m)^2S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n), \\
\Sigma_3 &= \sum_{n \in \sigma, m < n} \rho_\mu(n)^2f(n)^2S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n).
\end{align*}
\]

Let us begin with \(\Sigma_2\). We have
\[
\rho_\mu(m)^2 \leq 4(b_\mu + r_\mu - m)^2r_\mu^{-2} \quad (m \in \sigma_\mu),
\]
and
\[
\sum_{n > \sigma_\mu} S_\alpha(m, n)\omega_\nu(n) \leq A \sum_{n > \sigma_\mu} (n - m)^{-1-2\alpha} \leq A(b_\mu + r_\mu - m)^{-2\alpha}.
\]

Making use of the inequalities
\[
(b_\mu + r_\mu - m)^{2-2\alpha} \leq A(m + 1)^{2-2\alpha} \quad (m \in \sigma_\mu),
\]
\[
(m + 1) \leq Ar_\mu \quad (m \in \sigma_\mu),
\]
we obtain
\[
\Sigma_2 \leq A \sum_{m \in \sigma_\mu} f(n)^2(m + 1)^{-2\alpha}\omega_\nu(m).
\]

Exactly the same argument shows that
\[
\Sigma_3 \leq A \sum_{n \in \sigma_\mu} f(n)^2(n + 1)^{-2\alpha}\omega_\nu(n).
\]

It remains to treat \(\Sigma_1\). Since
\[
\rho_\mu(n)f(n) - \rho_\mu(m)f(m) = [f(n) - f(m)]\rho_\mu(n) + f(m)[\rho_\mu(n) - \rho_\mu(m)]
\]
and since \(0 \leq \rho_\mu(n) \leq 1\), we have
\[
\Sigma_1 < 2 \sum_{n, m \in \sigma_\mu, n > m} [f(n) - f(m)]^2S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n) \\
+ 2 \sum_{n, m \in \sigma_\mu, n > m} f(n)^2[\rho_\mu(n) - \rho_\mu(m)]^2S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n).
\]

We assert that
\[
\sum_{n=m+1}^{b_\mu+r_\mu} [\rho_\mu(n) - \rho_\mu(m)]^2S_\alpha(m, n)\omega_\nu(n) \leq A(m + 1)^{-2\alpha} \quad (m \in \sigma_\mu).
\]

To verify this note that
\[
\rho_\mu(n) - \rho_\mu(m) = -(n - m)(n + m - 2b_\mu)r_\mu^{-2},
\]
\[
|\rho_\mu(n) - \rho_\mu(m)| \leq A(n - m)r_\mu^{-1} \quad (n, m \in \sigma_\mu).
\]
It follows that
\[
\sum_{n=m+1}^{b_\mu+r_\mu} \left[ \rho_\mu(n) - \rho_\mu(m) \right]^2 S_\alpha(m, n) \psi_r(n) \leq A r_\mu^{2} \sum_{n=m+1}^{b_\mu+r_\mu} (n-m)^{1-2\alpha},
\]
\[
\leq A r_\mu^{2} (b_\mu + r_\mu - m)^{2-2\alpha},
\]
\[
\leq A (m+1)^{-2\alpha},
\]
as desired. Thus
\[
\sum_{n, m \in \sigma_\mu; n > m} f^-(m)^2 \left[ \rho_\mu(n) - \rho_\mu(m) \right]^2 S_\alpha(m, n) \psi_r(n) \psi_r(m)
\]
\[
\leq A \sum_{m \in \sigma_\mu} f^-(m)^2 (m+1)^{-2\alpha} \psi_r(m).
\]

Combining our results we have shown that
\[
\mathcal{R}^{\sigma_\mu}_{\alpha} \left[ E_\mu \right]^2 \leq \sum_{n, m \in \sigma_\mu; n > m} \left[ f^-(n) - f^-(m) \right]^2 S_\alpha(m, n) \psi_r(n) \psi_r(m)
\]
\[
+ A \sum_{m \in \sigma_\mu} f^-(m)^2 (m+1)^{-2\alpha} \psi_r(m).
\]

Since no integer belongs to more than three sets \( \sigma_\mu \) we see that
\[
\sum_{\mu=2}^{\infty} \mathcal{R}^{\sigma_\mu}_{\alpha} \left[ E_\mu \right]^2 \leq A \sum_{n > m} \left[ f^-(n) - f^-(m) \right]^2 S_\alpha(m, n) \psi_r(n) \psi_r(m)
\]
\[
+ A \sum_{m} f^-(m)^2 (m+1)^{-2\alpha} \psi_r(m).
\]

Applying Theorems 3c and 4a we have proved our desired result.

Let \( S_\mu \) be the set of integers \( 2^{\mu-1} \leq k < 2^\mu, \mu = 2, 3, \ldots \).

**Lemma 5b.** If \( 0 \leq \alpha < 1/2 \) and if \( n_\mu \in S_\mu \) then
\[
\sum_{\mu=2}^{\infty} \sum_{m \in S_\mu} f^-(m)^2 \left[ | m - n_\mu | + 1 \right]^{-2\alpha} \psi_r(m) \leq A (\alpha) \mathcal{R}^{\sigma_\mu}_{\alpha}[f]^2.
\]

By Theorem 4a
\[
\sum_{m \in \sigma_\mu} \rho_\mu(m)^2 f^-(m)^2 \left[ | m - n_\mu | + 1 \right]^{-2\alpha} \psi_r(m) \leq A \mathcal{R}^{\sigma_\mu}_{\alpha} \left[ E_\mu \right]^2.
\]

For \( m \in S_\mu \) \( \rho_\mu(m) \geq A \) and thus
\[
\sum_{m \in S_\mu} f^-(m)^2 \left[ | m - n_\mu | + 1 \right]^{-2\alpha} \psi_r(m)
\]
\[
\leq A \sum_{m \in \sigma_\mu} \rho_\mu(m)^2 f^-(m)^2 \left[ | m - n_\mu | + 1 \right]^{-2\alpha} \psi_r(m).
\]

These two inequalities together imply our desired result.
Definition. \( T = \{ t(n) \} \) \((0 \leq n \leq \infty)\) is said to belong to class \( M(C) \) if
\[
|t(n)| \leq C \quad (n = 0, 1, \ldots);
\]
\[
\sum_{2n}^{2n+1} |t(k) - t(k - 1)| \leq C \quad (n = 0, 1, \ldots).
\]

Theorem 5c. If \( 0 \leq \alpha < 1/2 \) and if
1. \( f(x) \sim \sum_{0}^{\infty} f(x) \omega(x) W(x, x) \quad f \in \mathcal{R}_0^\alpha, \)
2. \( T = \{ t(n) \} \in M(C), \)
3. \( Tf(x) \sim \sum_{0}^{\infty} f(x) t(n) \omega(x) W(x, x), \)
then
\[
\mathcal{R}_0^\alpha[Tf] \leq A(\alpha) C \mathcal{R}_0^\alpha[f].
\]

We set
\[
\delta_\mu(x) = \sum_{n \in s_\mu} f(x) t(n) \omega(x) W(x, x).
\]

Let us begin by supposing that \( t(0) = t(1) = 0 \); this restriction is unimportant and is made only for the sake of convenience. Let
\[
F_M(x) = \sum_{\mu = 2}^{M} \delta_\mu(x).
\]

It will be sufficient to show that
\[
\mathcal{R}_0^\alpha[F_M] \leq A C \mathcal{R}_0^\alpha[f]
\]
where \( A \) is independent of \( M \). We have
\[
\mathcal{R}_0^\alpha[F_M]^2 = \int_{-1}^{1} F_M(x)^2 s_\alpha(x) d\Omega_\alpha(x),
\]
and since
\[
\int_{-1}^{1} F_M(x)^2 s_\alpha(x) d\Omega_\alpha(x) = \sum_{\mu = 2}^{M} \int_{-1}^{1} \delta_\mu(x)^2 s_\alpha(x) d\Omega_\alpha(x)
\]
\[+ \sum_{\lambda, \mu = 2; \lambda \neq \mu}^{M} \int_{-1}^{1} \delta_\mu(x) \delta_\nu(x) s_\alpha(x) d\Omega_\alpha(x),
\]
it is sufficient to show that
(1) \[ \sum_{\mu, \lambda=2; \mu \neq \lambda}^{\infty} \left| \int_{-1}^{1} \delta_{\mu}(x) \delta_{\lambda}(x) s_{\alpha}(x) d\Omega_{\nu}(x) \right| \leq AC^{2} \mathfrak{R}_{0, \alpha}^{\nu} [f]^{2}, \]

and

(2) \[ \sum_{\mu=2}^{\infty} \int_{-1}^{1} \delta_{\mu}(x)^{2} s_{\alpha}(x) d\Omega_{\nu}(x) \leq AC^{2} \mathfrak{R}_{0, \alpha}^{\nu} [f]^{2}. \]

By Theorem 2b and the inequality \(|ab| \leq (a^{2} + b^{2})/2\) we have

\[ I_{\mu, \lambda} = \int_{-1}^{1} \delta_{\mu}(x) \delta_{\lambda}(x) s_{\alpha}(x) d\Omega_{\nu}(x) \]

\[ = \sum_{n \in S_{\mu}, m \in S_{\lambda}} \left[ t(n) f^{-}(n) - t(m) f^{-}(m) \right]^{2} S_{\alpha}(m, n) \omega_{\nu}(n) \omega_{\nu}(m), \]

\[ \leq 2C^{2} \sum_{n \in S_{\mu}} f^{-}(n)^{2} \omega_{\nu}(n) \sum_{m \in S_{\lambda}} S_{\alpha}(m, n) \omega_{\nu}(m) + 2C^{2} \sum_{m \in S_{\lambda}} f^{-}(m)^{2} \omega_{\nu}(m) \sum_{n \in S_{\mu}} S_{\alpha}(m, n) \omega_{\nu}(n). \]

Thus

\[ \sum_{\mu, \lambda=2; \mu \neq \lambda}^{\infty} \left| I_{\mu, \lambda} \right| \leq AC^{2} \sum_{\mu=2}^{\infty} \sum_{n \in S_{\mu}} f^{-}(n)^{2} \omega_{\nu}(n) \sum_{\lambda=2; \lambda \neq \mu}^{\infty} \sum_{m \in S_{\lambda}} S_{\alpha}(m, n) \omega_{\nu}(n). \]

Now, as is easily verified,

\[ \sum_{\lambda=2; \lambda \neq \mu}^{\infty} \sum_{m \in S_{\lambda}} S_{\alpha}(m, n) \omega_{\nu}(m) \leq A \left[ |n - 2^{n-1}| + 1 \right]^{-2\alpha} + A \left[ |n - 2^{n}| + 1 \right]^{-2\alpha} \]

so that

\[ \sum_{\mu, \lambda=2; \mu \neq \lambda}^{\infty} \left| I_{\mu, \lambda} \right| \leq AC^{2} \sum_{\mu=2}^{\infty} \sum_{n \in S_{\mu}} f^{-}(n)^{2} \left[ |n - 2^{n-1} + 1|^{-2\alpha} + |2^{n} + 1 - n|^{-2\alpha} \right] \]

so that using Lemma 5b (1) is seen to be valid.

Let us next consider

\[ \int_{-1}^{1} \delta_{\mu}(x)^{2} s_{\alpha}(x) d\Omega_{\nu}(x) = \mathfrak{R}_{0, \alpha}^{\nu} [\delta_{\mu}]^{2}. \]

For \(\mu\) fixed we set

\[ p(n, x) = \sum_{\nu=-r_{\mu}}^{n} \rho_{\mu}(n) f^{-}(n) \omega_{\nu}(n) W_{\nu}(n, x) \quad (n \in \sigma_{\mu}). \]

It follows from Theorem 4b that

\[ \mathfrak{R}_{0, \alpha}^{\nu} [p(n, x)] \leq A \mathfrak{R}_{0, \alpha}^{\nu} [E_{\mu}]. \]

If \(u(n) = t(n)/\rho_{\mu}(n)\), then

\[ \delta_{\mu}(x) = \sum_{n \in S_{\mu}} u(n) [p(n, x) - p(n - 1, x)]. \]
Summing by parts this becomes
\[ \delta_{\mu}(x) = \sum_{n \in \delta_{\mu}} p(n, x)[u(n) - u(n + 1)] + u(2^{\mu})p(2^{\mu} - 1, x) \
\quad - u(2^{\mu-1})p(2^{\mu-1} - 1, x), \]
from which using Theorem 4b it follows that
\[ \mathcal{N}_{0,a}[\delta_{\mu}] \leq A \mathcal{N}_{0,a}[E_{\mu}] \left\{ \sum_{n \in \delta_{\mu}} |u(n) - u(n + 1)| + |u(2^{\mu})| + |u(2^{\mu-1})| \right\}. \]

Now it is easily verified that
\[ \sum_{n \in \delta_{\mu}} |u(n) - u(n + 1)| + |u(2^{\mu})| + |u(2^{\mu-1})| \leq AC \]
and thus
\[ \mathcal{N}_{0,a}[\delta_{\mu}] \leq AC \mathcal{N}_{0,a}[E_{\mu}]. \]

Squaring and summing over \( \mu \) we see, using Lemma 5a, that (2) holds.

6. **Multiplier transformations continued.** Let \( p(\beta, \alpha) \) stand for the proposition that if \( T \in M(C) \) then \( \mathcal{N}_{0,a}[Tf] \leq AC \mathcal{N}_{0,a}[f] \) where \( A \) depends only upon \( \alpha, \beta \) and of course \( v \). Theorem 5c shows that \( p(0, \alpha) \) is valid for \( 0 \leq \alpha < 1/2 \). In this section we shall show that \( p(\beta, \alpha) \) is valid for \( (-1/2 < \beta, \alpha < 1/2) \). The following general principles are easily established, see \[4\].

i. If \( p(\beta, \alpha) \) is valid so is \( p(\alpha, \beta) \).

ii. If \( p(\beta, \alpha) \) is valid so is \( p(-\beta, -\alpha) \).

iii. If \( p(\beta_1, \alpha_1) \) and \( p(\beta_2, \alpha_2) \) are valid so is \( p(\beta, \alpha) \) where \( \beta = \min(\beta_1, \beta_2), \alpha = \min(\alpha_1, \alpha_2) \).

Using these it is easily shown that \( p(\alpha, \beta) \) is valid if \( -1/2 < \alpha, \beta < 1/2 \) and if in addition \( \alpha \beta \geq 0 \). To remove this restriction we require an additional argument.

**Lemma 6a.** If \(-1/2 < \beta \leq 0 \leq \alpha < 1/2 \), if \( T \in M(C) \), and if
\[ F(x) = (1 - x)T[f(x)] - T[(1 - x)f(x)], \]
then
\[ \mathcal{N}_{0,0}[F] \leq AC \mathcal{N}_{0,a}[f]. \]

The familiar recurrence formula for ultraspherical polynomials, see \[2, \text{vol. 2, p. 175}\], implies that
\[ (1 - x)W_{\nu}(n, x) = + [(2\nu + n)/2(n + \nu)][W_{\nu}(n, x) - W_{\nu}(n + 1, x)] \
\quad + [n/2(n + \nu)][W_{\nu}(n, x) - W_{\nu}(n - 1, x)]. \]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Supposing, as we may, that only finitely many $f^- (n)$ are not zero we find, after a short computation, that $F(x) = F_1(x) + F_2(x)$ where

$$F_1(x) = \sum_{n=0}^{\infty} \frac{n}{2(n + \nu)} f^-(n - 1) [t(n) - t(n - 1)] \omega_s(n) W_s(n, x),$$

$$F_2(x) = \sum_{n=0}^{\infty} \frac{2\nu + n}{2(n + \nu)} f^-(n + 1) [t(n) - t(n + 1)] \omega_s(n) W_s(n, x).$$

Let $g(x) \in \mathfrak{N}^r_{-\beta, 0}$ and let $g^- (n)$ be defined as usual. We have

$$\int_{-1}^{1} F_1(x) g(x) dx = \sum_{n=0}^{\infty} \frac{n}{2(n + \nu)} f^-(n - 1) g^-(n) [t(n) - t(n - 1)] \omega_s(n),$$

$$\left| \int_{-1}^{1} F_1(x) g(x) dx \right| \leq A \sum_{\mu=0}^{\infty} \sum_{n \in S_\mu} |f^-(n - 1)| |g^-(n)| |t(n) - t(n - 1)| (\omega_s(n))^{1/2} \omega_s(n - 1)^{1/2}.$$

If $f^*(\mu) = \text{l.u.b. } |f^-(n - 1)| \omega_s(n - 1)^{1/2}$ for $n \in S_\mu$, and

$$g^*(\mu) = \text{l.u.b. } |g^-(n)| \omega(n)^{1/2}$$

for $n \in S_\mu$, then

$$\left| \int_{-1}^{1} F_1(x) g(x) dx \right| \leq A \sum_{\mu=0}^{\infty} f^*(\mu) g^*(\mu) \sum_{n \in S_\mu} |t(n) - t(n - 1)|$$

$$\leq AC \sum_{\mu=0}^{\infty} f^*(\mu) g^*(\mu)$$

$$\leq AC \left[ \sum_{\mu=0}^{\infty} f^*(\mu)^2 \right]^{1/2} \left[ \sum_{\mu=0}^{\infty} g^*(\mu)^2 \right]^{1/2}$$

$$\leq AC \mathfrak{N}^*_{0, a} [f] \mathfrak{N}^*_{-\beta, 0} [g]$$

by Lemma 5b. Since this holds for every $g \in \mathfrak{N}^*_{-\beta, 0}$ it implies that $\mathfrak{N}^*_{\beta, 0} [F_1] \leq AC \mathfrak{N}^*_{0, a} [f]$. Similarly we can show that $\mathfrak{N}^*_{\beta, 0} [F_2] \leq AC \mathfrak{N}^*_{0, a} [f]$, and our lemma is established.

Using this we can now show that if $-1/2 < \beta \leq 0 \leq \alpha < 1/2$ then $p(\beta, \alpha)$ is valid. We have

$$\mathfrak{N}^*_{\beta, a} [Tf] \leq A \mathfrak{N}^*_{0, a} [Tf] + A \mathfrak{N}^*_{\beta, 0} [(1 - x)Tf].$$

Since $p(0, \alpha)$ is valid $\mathfrak{N}^*_{0, a} [Tf] \leq AC \mathfrak{N}^*_{0, a} [f] \leq AC \mathfrak{N}^*_{\beta, a} [f]$ if $F(x)$ is defined as above then
\[
\mathcal{N}_\beta,0[(1-x)Tf] = \mathcal{N}_\beta,0[T(1-x)f(x)] + F(x) \\
\leq \mathcal{N}_\beta,0[T(1-x)f(x)] + \mathcal{N}_\beta,0[F(x)].
\]

By \(p(\beta, 0)\),
\[
\mathcal{N}_\beta,0[T(1-x)f(x)] \leq A\mathcal{C}_\beta,0[1-x]f(x)] \\
\leq A\mathcal{C}_\beta,0[f(x)].
\]

Lemma 6a implies that
\[
\mathcal{N}_\beta,0[F(x)] \leq A\mathcal{C}_\beta,0[f] \leq A\mathcal{C}_\beta,0[f].
\]
Combining these results we have our desired result.

**Theorem 6a.** \(p(\alpha, \beta)\) is valid for \(-1/2 < \alpha, \beta < 1/2\).

This follows from the above.

The restriction \(-1/2 < \alpha, \beta < 1/2\) is essential in Theorem 6a and the result is not otherwise true. See in this connection the discussion at the end of §6 of [4].

An application of Theorem 6a to the theory of fractional integration is described in [6]. Proofs for the special case \(\nu=1/2\) are given in [4]. The modifications needed to adapt the proof to the case of general \(\nu\) are slight.

**Bibliography**


Washington University,
St. Louis, Mo.