INCLUSION THEOREMS FOR CONGRUENCE SUBGROUPS

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1. Introduction. We shall use the following notation throughout: $A^{(r)}$ denotes an $r \times r$ matrix $A$; $I^{(r)}$ denotes the $r$-rowed identity matrix; $0$ will be used for a zero matrix of appropriate size. Congruence of matrices will be interpreted as elementwise congruence. We write $a \mid b$ to indicate that $a$ divides $b$. Lower case italic letters will always denote integers.

Let $G_t$ be the proper unimodular group consisting of all $t \times t$ matrices with integral elements and determinant $+1$. For a fixed partition $t = r + s$ of $t$ into two positive integers $r$ and $s$, and for a fixed positive integer $n$, define the subgroup

\[ G_{r,s}(n) = \left\{ \begin{bmatrix} A^{(r)} & B \\ C & D^{(s)} \end{bmatrix} \in G_t : C \equiv 0 \pmod{n} \right\}. \]

We shall prove:

**Theorem 1.** Let $m, n$ be positive integers, and let $H$ be a group such that

\[ G_{r,s}(mn) \subseteq H \subseteq G_{r,s}(n). \]

Then there exists a divisor $d$ of $m$ such that

\[ H = G_{r,s}(dn). \]

Special cases of this have been proved in [1] and [3].

In the case where $t = 2r$, define

\[ G_{r}(m, n) = \left\{ \begin{bmatrix} A^{(r)} & B \\ C & D^{(r)} \end{bmatrix} \in G_{2r} : B \equiv 0 \pmod{m}, C \equiv 0 \pmod{n} \right\}. \]

Then we shall show:

**Theorem 2.** Let $H$ be a group satisfying

\[ G_{r}(m, n) \subseteq H \subseteq G_{2r}. \]

If $(m, n) = 1$, then there exist integers $m_1, n_1$ with $m_1 \mid m, n_1 \mid n$, and

\[ H = G_{r}(m_1, n_1). \]

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A special case of this (with \( r=1 \)) was proved in [2], where it was also shown that the hypothesis \((m, n)=1\) could not be dropped.

To generalize further, let \( n=(n_1, \ldots, n_{t-1}) \), and define

\[
G_t(n) = G_{1,t-1}(n_1) \cap G_{2,t-2}(n_2) \cap \cdots \cap G_{t-1,1}(n_{t-1}).
\]

Thus an element \( M \in G_t \) lies in \( G_t(n) \) if and only if for every partition \( t=r+s \) \((1 \leq r \leq t-1)\) we have

\[
M = \begin{bmatrix}
A^{(r)} & B \\
C & D^{(s)}
\end{bmatrix}, \quad C \equiv 0 \pmod{n_r}.
\]

We shall prove:

**Theorem 3.** Let \((m,n_1, m_j, n_j) = 1\) for \(1 \leq i, j \leq t-1, i \neq j\). Let \( H \) be a group such that

\[
G_t(mn) \subset H \subset G_t(n),
\]

where \( mn \) denotes \((m_1,n_1, \ldots, m_{t-1}n_{t-1})\). Then there exists a vector

\[
d = (d_1, \ldots, d_{t-1}),
\]

with \( d_i \mid m_i, \ldots, d_{t-1} \mid m_{t-1} \), such that

\[
H = G_t(bn).
\]

Finally, we shall prove analogues of Theorems 1 and 2 for the symplectic modular group \( \Gamma_t \) of order \( t \), which consists of all integral matrices

\[
\begin{bmatrix}
A^{(t)} & B \\
C & D^{(t)}
\end{bmatrix}
\]

satisfying

\[
AB' = B'A, \quad CD' = D'C, \quad AD' - DC' = I.
\]

2. We begin the proof of Theorem 1 with two lemmas.

**Lemma 1.** Let \( t=r+s \), and let \( n \) be a fixed positive integer. For each

\[
M = \begin{bmatrix}
A^{(r)} & B \\
C & D^{(s)}
\end{bmatrix} \in G_t,
\]

there exists an integral \( r \times s \) matrix \( X \) such that \(|A+XC|, n) = 1\).

**Proof.** It is sufficient to show that for every prime \( p \) there exists an integral matrix \( X_p \) such that \( p \mid |A+X_pC|\). For we may then find an integral matrix \( X \) satisfying \( X \equiv X_p \pmod{p} \) for each \( p \mid n \). Since \(|A+XC| \equiv |A+X_pC| \pmod{p} \), it then follows that \(|A+XC|, n) = 1\).

Now let \( p \) be a fixed prime, and let \( \alpha_1, \ldots, \alpha_s \) denote the rows of \( A \), and
\( \gamma_1, \ldots, \gamma_s \) those of \( C \). Since the rows of \( X_pC \) are linear combinations of those of \( C \), we need only show that there exist linear combinations

\[
\beta_i = \sum_{j=1}^{s} x_{ij} \gamma_j \quad (1 \leq i \leq r, \ x_{ij} \text{ integers})
\]

such that \( p \mid \det (\alpha_i + \beta_i) \). Thus, we seek integers \( x_{ij} \) for which the vectors \( \alpha_i + \beta_i \ (1 \leq i \leq r) \) are linearly independent modulo \( p \).

Since \( M \) is unimodular, the set \( \{ \alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_s \} \) contains exactly \( r \) linearly independent vectors modulo \( p \). Suppose that \( r' \) of the \( \alpha \)'s are linearly independent modulo \( p \) \( (r' \leq r) \); for simplicity of notation, suppose that these are \( \alpha_1, \ldots, \alpha_{r'} \). Then each \( \alpha_k \ (r' < k \leq r) \) is a linear combination modulo \( p \) of \( \alpha_1, \ldots, \alpha_{r'} \). Further, there exist \( r-r' \) vectors \( \gamma_1^*, \ldots, \gamma_{r-r'}^* \) among \( \gamma_1, \ldots, \gamma_s \) such that the set \( \{ \alpha_1, \ldots, \alpha_{r'}, \gamma_1^*, \ldots, \gamma_{r-r'}^* \} \) is linearly independent modulo \( p \). Then we need only choose \( \beta_1 = \cdots = \beta_{r'} = 0, \beta_{r'+1} = \gamma_1^*, \ldots, \beta_r = \gamma_{r-r'}^* \) to achieve the desired result.

**Lemma 2.** Let \( M \in G_{r,s}(n) \), and let \( m \) be a fixed positive integer. Then there exists an integral \( r \times s \) matrix \( X \) and an integral \( s \times r \) matrix \( Y \) such that

\[
W(nY)S(X)M \in G_{r,s}(mn),
\]

where

\[
W(nY) = \begin{bmatrix} I^{(r)} & 0 \\ nY & I^{(s)} \end{bmatrix}, \quad S(X) = \begin{bmatrix} I^{(r)} & X \\ 0 & I^{(s)} \end{bmatrix}.
\]

The entries of \( X \) and \( Y \) are integers determined only modulo \( m \). Therefore the set of products \( W(nY)S(X) \), as the entries of \( X \) and \( Y \) range over all residues modulo \( m \), contains a full set of left coset representatives of \( G_{r,s}(n) \) modulo \( G_{r,s}(mn) \). Consequently \( G_{r,s}(mn) \) is of finite index in \( G_{r,s}(n) \).

**Proof.** Set

\[
M = \begin{bmatrix} A^{(r)} & B \\ nC & D^{(s)} \end{bmatrix} \in G_{r,s}(n).
\]

By Lemma 1, we can determine \( X \) modulo \( m \) such that \( (|A+nXC|, m) = 1 \). Set \( A_0 = A+nXC \). Then

\[
S(X)M = \begin{bmatrix} A_0 & * \\ nC & * \end{bmatrix},
\]

and

\[
W(nY)S(X)M = \begin{bmatrix} * & * \\ nYA_0 + C & * \end{bmatrix}.
\]
In order for (10) to hold, we need only show that \( Y \) modulo \( m \) can be determined so that \( YA_0 + C \equiv 0 \) (mod \( m \)).

Now \( (|A_0|, m) = 1 \), so that we may find an integer \( a \) with \( a |A_0| = 1 \) (mod \( m \)). Letting \( A_0^{adj} \) denote the adjoint of \( A_0 \), we set

\[
(11) \quad Y \equiv -aCA_0^{adj} \pmod{m}.
\]

Using \( A_0^{adj}A_0 = |A_0| I \), we obtain

\[
YA_0 \equiv - C \pmod{m},
\]
as desired.

The remainder of the lemma follows at once from (10).

We now proceed with the proof of Theorem 1. Let \( H \) be a group such that

\[
Gr_{r,s}(mn) \subset H \subset Gr_{r,s}(n).
\]

Using the argument in [1], we find by induction on the total number of prime factors of \( m \) that the conclusion of Theorem 1 is valid unless for every \( d \) dividing \( m \), \( d \neq 1 \), we have

\[
H \cap Gr_{r,s}(dn) = Gr_{r,s}(mn).
\]

Suppose now that \( H \neq Gr_{r,s}(mn) \). The above then shows that there exists a matrix

\[
M = \begin{bmatrix} A^{(r)} & B \\ nC & D^{(s)} \end{bmatrix} \in H
\]
such that \( C \equiv 0 \) (mod \( d \)) for any divisor \( d \) of \( m \), \( d \neq 1 \). Choose \( X, Y \) as in Lemma 2, and use the fact that \( S(X) \subseteq H \). Then we see that \( W(nY) \subseteq H \), where \( Y \) is chosen by use of (11). Hence also \( Y \equiv 0 \) (mod \( d \)) for any divisor \( d \) of \( m \), \( d \neq 1 \).

Call an \( s \times r \) matrix \( T \) permissible if \( W(nT) \subseteq H \). We have shown the existence of a permissible matrix \( Y \) such that \( Y \equiv 0 \) (mod \( d \)) for any divisor \( d \) of \( m \), \( d \neq 1 \). We shall use this to deduce that every matrix is permissible. Since already \( S(X) \subseteq H \) for all \( X \), it will then follow from Lemma 2 that \( H = Gr_{r,s}(n) \), and the theorem will be proved.

Now we have

\[
W(nT_1) \cdot W(nT_2) = W(n(T_1 + T_2)),
\]
and

\[
\begin{bmatrix} V^{-1} & 0 \\ 0 & U \end{bmatrix} W(nT) \begin{bmatrix} V & 0 \\ 0 & U^{-1} \end{bmatrix} = W(nUTV), \quad U \in G_s, \ V \in G_r.
\]

Therefore if \( T_1 \) and \( T_2 \) are permissible, so is \( T_1 + T_2 \). If \( T \) is permissible, then
so is \(-T\); and if \(U \in G_s, V \in G_r\), then \(UTV\) is also permissible.

Starting with the permissible \(Y\) above, set \(Y_1 = UYV\), with \(U \in G_s, V \in G_r\). Then \(Y_1\) is also permissible, and with proper choice of \(U\) and \(V\), we may take \(Y_1\) in Smith normal form:

\[
Y_1 = \begin{bmatrix}
    h_1 & & \\
    & h_2 & \\
    & & \ddots \\
    & & & h_\mu
\end{bmatrix}, \quad \mu = \min (r, s),
\]

where \(h_1 | h_2 | \cdots | h_\mu\). If \((h_1, m) > 1\), then there is a prime \(p | m\) such that \(Y_1 \equiv 0 \pmod{p}\). Then also \(Y \equiv 0 \pmod{p}\), which is impossible. Hence \((h_1, m) = 1\). Let us choose \(a\) so that \(ah_1 \equiv 1 \pmod{m}\). Then \(Y_2 = aY_1\) is also permissible. Since a permissible matrix remains permissible when multiples of \(m\) are added to its entries, we therefore have the permissible matrix

\[
Y_3 = \begin{bmatrix}
    1 & k_2 & & \\
    & & \ddots & \\
    & & & k_\mu
\end{bmatrix}.
\]

Hence also

\[
Y_4 = \begin{bmatrix}
    0 & -k_2 & & \\
    1 & 0 & & \\
    & k_3 & \ddots & \\
    & & \ddots & k_\mu
\end{bmatrix}
\]

and

\[
Y_5 = Y_3 - Y_4 = \begin{bmatrix}
    1 & k_2 & & \\
    -1 & -k_2 & & \\
    & 0 & \ddots & \\
    & & \ddots & 0
\end{bmatrix}
\]

are permissible. In \(Y_5\) add the second row to the first row, and then subtract the matrix so obtained from \(Y_5\), obtaining the permissible matrix which has 1 in the \((1, 1)\) place, \(-k_2\) in the \((1, 2)\) place, and 0 elsewhere. In this matrix add \(k_2\) times the first column to the second column, thereby obtaining the permissible matrix.
Since also $UY^V$ is permissible for all $U \in G_s, V \in G_r$, we find that every matrix whose entries are all zeros except for a single 1, must be permissible. Therefore all matrices are permissible, and Theorem 1 is proved.

3. We now prove Theorem 2. Let $H$ be a group satisfying

$$ G_r(m, n) \subset H \subset G_{2r}, $$

where $G_r(m, n)$ is defined by (4), and where $(m, n) = 1$. Choose integers $a, b$ satisfying $am - bn = 1$, and set

$$ K = \begin{bmatrix} amI^{(r)} & I \\ bnI & I^{(r)} \end{bmatrix} \in G_{2r}. $$

Then as in [2] we find that $K^{-1}G_r(m, n)K = G_{r,r}(mn)$, and the remainder of the proof of Theorem 2 follows from Theorem 1 just as in [2].

Theorem 2 is false for $(m, n) > 1$, as is shown in [2].

4. To prove Theorem 3, we begin with several lemmas.

**Lemma 3.** Let $n_1, \cdots, n_{t-1}$ be pairwise coprime, and let $M \in G_t$. Then there exists an upper triangular matrix $S \in G_t$ such that for each $r$ $(1 \leq r \leq t-1)$ we have

$$ M = \begin{bmatrix} A^{(r)} & B \\ C & D^{(t-r)} \end{bmatrix}, \quad S = \begin{bmatrix} I^{(r)} & X_r \\ 0 & I^{(t-r)} \end{bmatrix} \pmod{n_r}, $$

and

$$ (\lfloor A^{(r)} + X_rC \rfloor, n_r) = 1. $$

**Proof.** Let $M$ be fixed. For each $r$, write $M$ in the form (12). By Lemma 1, we may then choose $X_r$ such that (13) holds. We then use the Chinese remainder theorem to determine an upper triangular matrix $S$ satisfying

$$ S = \begin{bmatrix} I^{(r)} & X_r \\ 0 & I^{(t-r)} \end{bmatrix} \pmod{n_r}, \quad 1 \leq r \leq t-1. $$

This completes the proof of the lemma.

**Lemma 4.** Let $S$ be an integral $t \times t$ matrix such that $|S| \equiv 1 \pmod{n}$. Then there exists a matrix $T \in G_t$ such that $T \equiv S \pmod{n}$.

**Proof.** (Although this lemma is known, references are hard to come by, and so we insert a proof.)
Set $T = S + nY$; we need only choose $Y$ so that $|S + nY| = 1$. Let $U$, $V \in G_t$ be chosen so that $USV = D$ is diagonal, and set $X = UYV$. Then

$$|S + nY| = |D + nX|,$$

so it suffices to show that we can find $X$ such that $|D + nX| = 1$, where $D$ is diagonal and $|D| = 1 \pmod{n}$.

Let $D = \text{diag}(d_1, \ldots, d_t)$, and set $|D| = 1 + nd$. Choose $X$ so that

$$D + nX = \begin{bmatrix}
    d_1 + nx & 0 & 0 & \cdots & 0 & ny \\
    n & d_2 & 0 & \cdots & 0 & 0 \\
    0 & n & d_3 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & n & d_t
\end{bmatrix}.$$

Then

$$|D + nX| = 1 + n(d + xd_2 \cdots d_t \pm n^{t-1}y).$$

Since $(d_2 \cdots d_t, n) = 1$, we may choose integers $x, y$ such that

$$d + xd_2 \cdots d_t \pm n^{t-1}y = 0,$$

which completes the proof.

**Lemma 5.** Let $m = (m_1, \ldots, m_{t-1})$, $n = (n_1, \ldots, n_{t-1})$, where $(m_i, n_i) = 1$ for $1 \leq i \leq t-1$, $(m_im_i, m_jn_j) = 1$ for $1 \leq i, j \leq t-1, i \neq j$, and let $M \in G_t(n)$. Then there is an upper triangular matrix $S \in G_t$ and a lower triangular matrix $W \in G_t$ such that $WSM \in G_t(mn)$. The entries of $W$ and $S$ are determined only modulo $m_1 \cdots m_{t-1}$, and hence $G(mn)$ is of finite index in $G(n)$.

**Proof.** This lemma follows readily from Lemma 3 in the same way that Lemma 2 follows from Lemma 1.

We now proceed with the proof of Theorem 3. Let $m, n$ be chosen as in the above lemma, and let $H$ be a group such that

$$G_t(mn) \subset H \subset G_t(n).$$

As in the proof of Theorem 1, by using induction on the total number of prime factors of $m_1m_2 \cdots m_{t-1}$, we see that the theorem holds unless for every vector $a = (a_1, \ldots, a_{t-1})$ such that $a_1|m_1, \ldots, a_{t-1}|m_{t-1}$, except

$$a = (1, \ldots, 1),$$

we have

$$H \cap G_t(an) = G_t(mn).$$

Suppose that $H \not\supseteq G_t(mn)$; then $H$ must contain an element $M$ such that for each $r$ ($1 \leq r \leq t-1$) we have
Now choose an upper triangular matrix \( S \) and a lower triangular matrix \( W \) as in Lemma 5, such that \( WSM \in G_t(\text{mn}) \subseteq H \). Since also \( S \in H \), this shows that \( W \in H \). Further, for each \( r \) we have

\[
W = \begin{bmatrix}
I^{(r)} & 0 \\
0 & I^{(t-r)}
\end{bmatrix} \pmod{m_r},
\]

where \( Y_r \neq 0 \pmod{a_r} \) for any \( a_r \) dividing \( m_r \), \( a_r \neq 1 \).

Call a lower triangular matrix in \( G_t \) permissible if it is an element of \( H \). The above-constructed \( W \) is permissible. If we can show that all lower triangular matrices in \( G_t(n) \) are permissible, then using Lemma 5 we will deduce that \( H = G_t(n) \), and Theorem 3 will be established.

Define the non-negative integer \( k \) by \( m_1 = \cdots = m_{k-1} = 1 \), \( m_k > 1 \). (If \( m_1 > 1 \), then choose \( k = 1 \).) We shall show that also \( m_{k+1} = \cdots = m_{t-1} = 1 \). For let \( m_0 = m_{k+1} \cdots m_{t-1} \); then \( (m_0, m_k) = 1 \).

Now we remark that the matrix \( Y_r \) was determined only modulo \( m_r \), and hence since \( (m_r, n_r) = 1 \), we could have chosen the permissible matrix \( W \) so that instead of (15) we have (for each \( r \))

\[
W = \begin{bmatrix}
I^{(r)} & 0 \\
Y_r n_r & I^{(t-r)}
\end{bmatrix} \pmod{m_r n_r}.
\]

Then \( W \in H \), so also \( W^{m_0} \in H \). Now for each \( r \) (\( 1 \leq r \leq t-1 \)) we have

\[
W^{m_0} = \begin{bmatrix}
I^{(r)} & 0 \\
Y_r n_r m_0 & I^{(t-r)}
\end{bmatrix} \pmod{m_r n_r},
\]

whence

\[
W^{m_0} \in G_t(n_1, \cdots, n_k, m_{k+1} n_{k+1}, \cdots, m_{t-1} n_{t-1}).
\]

Unless \( (1, \cdots, 1, m_{k+1}, \cdots, m_{t-1}) = (1, \cdots, 1) \), we deduce from (15) that \( W^{m_0} \in G_t(\text{mn}) \), which is impossible because \( W^{m_0} \in G_{k-1, t-k+1}(m_k n_k) \). We thus have shown that \( m = (1, \cdots, 1, m_k, 1, \cdots, 1) \).

We are now supposing that

\[
G_t(\text{mn}) \subset H \subset G_t(n),
\]

where \( m = (1, \cdots, 1, m_k, 1, \cdots, 1) \), \( m_k > 1 \), that (14) holds, and that \( H \neq G_t(\text{mn}) \). We have shown the existence of a lower triangular matrix \( W \in H \) such that (16) holds, with \( Y_k \neq 0 \pmod{a_k} \) for any \( a_k \) dividing \( m_k \), \( a_k \neq 1 \). We are trying to prove that every lower triangular matrix in \( G_t(n) \) is permissible (that is, lies in \( H \)), and consequently that \( H = G_t(n) \).
Let \( U \in G_k, V \in G_{t-k} \) be arbitrary. By Lemma 4, there exists a matrix \( R \in G_t \) such that

\[
R = I \quad \pmod{n_r}, \quad 1 \leq r \leq t-1, r \neq k,
\]

\[
R = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \quad \pmod{m_k n_k}.
\]

Then \( R \in G_t(\text{mn}) \subseteq H \), and hence also \( W_1 = R W R^{-1} \in H \). But we have

\[
W_1 = \begin{bmatrix} I^{(k)} & 0 \\ n_k V Y_k U^{-1} & I^{(t-k)} \end{bmatrix} \quad \pmod{m_k n_k},
\]

and

\[
W_1 = \begin{bmatrix} I^{(r)} & 0 \\ n_r Y_r & I^{(t-r)} \end{bmatrix} \quad \pmod{n_r}
\]

for \( 1 \leq r \leq t-1, r \neq k \). The same reasoning as in the proof of Theorem 1 then shows that all lower triangular matrices in \( G_t(\text{n}) \) lie in \( H \), whence \( H = G_t(\text{n}) \) and Theorem 3 is proved.

5. We conclude with an examination of the symplectic modular group \( \Gamma_t \) of order \( t \) (see [4]). Let

\[
\Gamma_t(\text{m}, \text{n}) = \begin{cases} A^{(t)} & B \\ C & D^{(t)} \end{cases} \in \Gamma_t: \quad B \equiv 0 \pmod{m}, \quad C \equiv 0 \pmod{n},
\]

and set \( \Gamma_t(\text{n}) = \Gamma_t(1, \text{n}) \). We shall prove analogues of Theorems 1 and 2. We begin with

**Lemma 6.** Let \( n \) be a fixed positive integer, and let

\[
M = \begin{bmatrix} A^{(t)} & B \\ C & D^{(t)} \end{bmatrix} \in \Gamma_t.
\]

Then there exists a symmetric \( t \times t \) matrix \( X \) such that \( \det(A + X C) \equiv 1 \pmod{n} \).

**Proof.** As in the proof of Lemma 1, it suffices to show for each prime \( p \) that there exists a symmetric matrix \( X_p \) for which \( p \nmid \det(A + X_p C) \). For \( U, V \in G_t \) we have

\[
\begin{bmatrix} U & 0 \\ 0 & U'^{-1} \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & V'^{-1} \end{bmatrix} = \begin{bmatrix} A_1^{(t)} & B_1 \\ C_1 & D_1^{(t)} \end{bmatrix} \in \Gamma_t,
\]

with \( A_1 = U A V, C_1 = U'^{-1} C V \). Set \( Y_p = U X_p U' \); then

\[
A_1 + Y_p C_1 = U(A + X_p C)V.
\]

Hence we need only find a symmetric matrix \( Y_p \) such that \( p \nmid \det(A_1 + Y_p C_1) \).
By proper choice of $U$, $V \in G_i$, we may assume that $A_1$ is diagonal. Let

$$A_1 = \begin{bmatrix} E^{(k)} & 0 \\ 0 & 0 \end{bmatrix} \pmod{p},$$

where $E$ is diagonal and nonsingular modulo $p$. (The case where $A \equiv 0 \pmod{p}$ is easily disposed of separately.) Setting

$$C_1 = \begin{bmatrix} C^{(k)}_{11} & C_{12} \\ C_{21} & C^{(l-k)}_{22} \end{bmatrix},$$

the symmetry of $A_1 C_1$ shows that $C_{12} \equiv 0 \pmod{p}$. Hence

$$\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} \equiv \begin{bmatrix} E & 0 \\ 0 & 0 \\ C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \pmod{p},$$

whence $p \mid \det C_{22}$. Then set

$$Y_p = \begin{bmatrix} 0 & 0 \\ 0 & I^{(l-k)} \end{bmatrix},$$

and obtain

$$A_1 + Y_p C_1 \equiv \begin{bmatrix} E & 0 \\ 0 & C_{21} \\ C_{21} & C_{22} \end{bmatrix} \pmod{p};$$

which shows that $p \mid \det (A_1 + Y_p C_1)$. This completes the proof of the lemma.

**Lemma 7.** Let $M \in \Gamma_t(n)$, and let $m$ be a fixed positive integer. Then there exist symmetric integral $t \times t$ matrices $X$, $Y$, whose entries are determined only modulo $m$, such that

$$W(nY) S(X) M \in \Gamma_t(mn),$$

where

$$W(nY) = \begin{bmatrix} I^{(t)} \\ nY & I^{(t)} \end{bmatrix}, \quad S(X) = \begin{bmatrix} I^{(t)} \\ 0 & X \\ 0 & I^{(t)} \end{bmatrix}.$$  

**Proof.** The proof follows that of Lemma 2. The only additional fact needed is that the matrix $Y$ determined by Equation (11) can be chosen to be symmetric, since the symmetry of $A_0' C$ implies that of $CA_0^{\text{adj}}$.

We now have

**Theorem 4.** Let $m$, $n$ be positive integers, and let $H$ be a group such that

$$\Gamma_t(mn) \subset H \subset \Gamma_t(n).$$
Then there exists a divisor $d$ of $m$ such that $H = \Gamma_t(dn)$.

**Proof.** This theorem follows from Lemmas 6 and 7 in the same manner that Theorem 1 follows from Lemmas 1 and 2. We omit the details.

**Theorem 5.** Let $m$, $n$ be positive coprime integers, and let $H$ be a group satisfying

$$\Gamma_t(m, n) \subset H \subset \Gamma_t.$$

Then there exist integers $m_1$, $n_1$ with $m_1 | m$, $n_1 | n$, and $H = \Gamma_t(m_1, n_1)$.

**Proof.** The proof of Theorem 2 carries over to this case with minor modifications. We omit the details.

**References**


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