ON A HOMOMORPHISM PROPERTY OF CERTAIN JORDAN ALGEBRAS(1)

BY

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1. **Introduction.** The purpose of this paper is to provide a proof of the following basic result.

THEOREM. *Let $\mathfrak{D}$ be any (nonassociative) algebra with a unity element e over a field $\mathfrak{F}$ of characteristic not two and possessing an involution T over $\mathfrak{F}$. Let $\mathfrak{H}_0$ be the algebra of all three rowed T-hermitian matrices over $\mathfrak{D}$ relative to Jordan multiplication. Then, if $\mathfrak{H}_0$ is a homomorphic image of a special Jordan algebra $\mathfrak{H}$, the algebra $\mathfrak{D}$ is associative.*

This result has a number of important consequences. It may be seen to imply that no simple exceptional finite dimensional Jordan algebra of characteristic not two is a homomorphic image of a special(2) Jordan algebra. But there is an exceptional simple Jordan algebra $\mathfrak{H}$ (of dimension 27) over a field $\mathfrak{F}$ of characteristic not two which is generated by three of its elements. Then $\mathfrak{H}$ is a homomorphic image of the free Jordan algebra $\mathfrak{J}_3$ on three generators. It follows that $\mathfrak{J}_3$ *is not special*, and we also know that $\mathfrak{J}_3$ *is not isomorphic to the free special Jordan algebra* $\mathfrak{J}[x, y, z, 1]$ consisting of all *reversible* polynomials(3) in the free associative algebra $\mathfrak{F}[x, y, z, 1]$ of all polynomials on the three generators x, y, z.

2. **Elementary properties of $\mathfrak{H}_0$ and certain subalgebras.** We begin with a brief description of the algebra $\mathfrak{H}_0$ of our theorem. Let $\mathfrak{D}$ be an algebra with a unity element e over a field $\mathfrak{F}$ of characteristic not two. Suppose that $\mathfrak{D}$ has an involution T over $\mathfrak{F}$; that is, an antiautomorphism

$$x \longrightarrow x^T = \bar{x}$$

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(2) The case where $\mathfrak{H}$ is finite-dimensional was considered in an earlier note entitled *A property of special Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. vol. 42 (1956) pp. 624–625.

(3) The identification of the free special Jordan algebra on three generators x, y, z with the Jordan algebra of all reversible polynomials in x, y, z was derived by P. M. Cohn, *On homomorphic images of special Jordan algebras*, Canadian Journal of Mathematics vol. 6 (1954) pp. 253–264.

of period 2 over $\mathfrak{F}$. The set of all self-adjoint elements $x=\bar{x}$ of $\mathfrak{D}$ contains the field $e\mathfrak{F}$.

Now consider the set $\mathfrak{D}_3$ of all three rowed matrices with elements in $\mathfrak{D}$ and its subspace $\mathfrak{H}_0$ of all three rowed T-hermitian matrices. Then $\mathfrak{H}_0$ consists of all matrices

$$h = \begin{pmatrix} \alpha & u & v \\ \bar{u} & \beta & w \\ \bar{v} & \bar{w} & \gamma \end{pmatrix}, \tag{1}$$

where α, β, γ are in the subspace of self-adjoint elements of $\mathfrak{D}$ and u, v, w are in $\mathfrak{D}$. Let hk be the ordinary matrix product in $\mathfrak{D}_3$ and define a product $h \cdot k$ by

$$2h \cdot k = hk + kh. \tag{2}$$

Then $\mathfrak{H}_0$ is an algebra relative to the product $h \cdot k$ and it is known(4) that if $\mathfrak{D}$ is not associative the algebra $\mathfrak{H}_0$ is not a special Jordan algebra. In our case, since $\mathfrak{H}_0$ is to be the homomorphic image of a Jordan algebra $\mathfrak{H}$, $\mathfrak{H}_0$ will be an exceptional Jordan algebra if $\mathfrak{D}$ is not associative.

The restriction of $\mathfrak{H}_0$ to T-hermitian or self-adjoint elements of $\mathfrak{D}_3$ under a standard involution is unnecessary and the present discussion can be modified if $\mathfrak{H}_0$ is defined as the self-adjoint elements of $\mathfrak{D}_3$ under a canonical involution of $\mathfrak{D}_3$. In this case, we obtain as a corollary to our theorem the result noted in the previous paragraph. However, the simplicity afforded by assuming that $\mathfrak{H}_0$ consists of T-hermitian elements of $\mathfrak{D}_3$ will be retained for the remainder of the paper.

We shall be interested in the Jordan subalgebra $\mathfrak{J}_0$ of $\mathfrak{H}_0$ generated by the elements

$$x = \begin{pmatrix} e & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & e & e \\ e & 0 & e \\ e & e & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & u & v \\ \bar{u} & 0 & w \\ \bar{v} & \bar{w} & 0 \end{pmatrix} \tag{3}$$

of $\mathfrak{H}_0$ where u, v and w are arbitrary elements of $\mathfrak{D}$. The unity element of $\mathfrak{J}_0$ and of $\mathfrak{H}_0$ is the identity matrix

$$f = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{4}$$

and f is the sum of the three pairwise orthogonal idempotents

(4) See N. Jacobson, *Structure of alternative and Jordan bimodules*, Osaka Math. J. vol. 6 (1954) pp. 1–71.

$$(5) \qquad e_1 = \frac{x^2+x}{2}, \qquad e_2 = \frac{x^2-x}{2}, \qquad e_3 = f - (e_1+e_2) = f - x^2,$$

represented in the representation (1) by

$$e_1 = \begin{bmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{bmatrix}.$$

Then $\mathfrak{J}_0$ is the vector space direct sum

$$(6) \qquad \mathfrak{J}_0 = \mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3 + \mathfrak{J}_{12} + \mathfrak{J}_{23} + \mathfrak{J}_{13},$$

where $\mathfrak{J}_i$ is the set of all elements h_i of $\mathfrak{J}_0$ such that

$$(7) \qquad h_i \cdot e_i = h_i \qquad (i = 1, 2, 3)$$

while $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ for $i \neq j$ and is the set of all elements h_{ij} of $\mathfrak{J}_0$ such that

$$(8) \qquad 2e_i \cdot h_{ij} = 2e_j \cdot h_{ij} = h_{ij} \qquad (i \neq j;\ i, j = 1, 2, 3).$$

Since $f = e_i + e_j + e_k$ we see that (8) implies that

$$(9) \qquad e_k \cdot h_{ij} = 0 \qquad (i \neq j;\ k \neq i, j;\ i, j, k = 1, 2, 3).$$

We record our assumption that

$$(10) \qquad e_i \cdot e_i = e_i^2 = e_i, \qquad e_i \cdot e_j = 0 \qquad (i \neq j;\ i, j = 1, 2, 3).$$

We shall now characterize the spaces $\mathfrak{J}_i$ and $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ in terms of the *Jordan triple product.* This product $\{ghk\}$ can be defined in terms of elements g, h, k of $\mathfrak{J}_0$ by

$$(11) \qquad \{ghk\} = (g \cdot h) \cdot k + (h \cdot k) \cdot g - (k \cdot g) \cdot h.$$

Write, for an arbitrary element h of $\mathfrak{J}_0$,

$$(12) \qquad h = h_i + h_j + h_k + h_{ij} + h_{jk} + h_{ik},$$

where

$$(13) \qquad h_i \cdot e_i = h_i, \qquad h_i \cdot e_j = 0 \qquad (i \neq j;\ i, j = 1, 2\ 3)$$

and (7) and (8) both hold. Then, we see that

$$\begin{aligned} \{e_i h e_i\} &= (e_i \cdot h) \cdot e_i + (e_i \cdot h) \cdot e_i - (e_i^2 \cdot h) \\ &= 2h_i + \frac{1}{2}(h_{ij} + h_{ik}) - \left\{ h_i + \frac{1}{2}(h_{ij} + h_{ik}) \right\} = h_i. \end{aligned}$$

and that

$$\begin{aligned} 2\{e_i h e_j\} = 2\{e_j h e_i\} &= (e_j \cdot h) \cdot e_i + 2(e_i h) e_j - (e_i \cdot e_j) \cdot h \\ &= 2(e_i \cdot h) \cdot e_j + 2(e_j \cdot h) \cdot e_i = h_{ij} \end{aligned}$$

for ($i \neq j$; $i, j = 1, 2, 3$). Thus we have derived the following result.

LEMMA 1. *Let $\mathfrak{J}_i$ be the set of all elements h_i of $\mathfrak{J}_0$ such that $h_i \cdot e_i = h_i$, and $\mathfrak{J}_{ij} = \mathfrak{J}_{ji}$ be the set of all elements $h_{ij} = h_{ji}$ in $\mathfrak{J}_0$ such that $2e_i \cdot h_{ij} = 2e_j \cdot h_{ij} = h_{ij}$. Then every element h of $\mathfrak{J}_0$ is uniquely expressible in the form $h = h_i + h_j + h_k + h_{ij} + h_{jk} + h_{ik}$, where*

$$h_i = \{e_i h e_i\}, \qquad h_{ij} = 2\{e_i h e_j\} = 2\{e_j h e_i\}. \tag{14}$$

If we apply Lemma 1 to the element h of (1) and let

$$u_{12} = \begin{pmatrix} 0 & u & 0 \\ \bar{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad w_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & \bar{w} & 0 \end{pmatrix}, \qquad v_{23} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ \bar{v} & 0 & 0 \end{pmatrix} \tag{15}$$

it is easy to see that

$$h_1 = \alpha e_1, \quad h_2 = \beta e_2, \quad h_3 = \gamma e_3, \quad h_{12} = u_{12}, \quad h_{23} = w_{23}, \quad h_{13} = v_{13}.$$

Moreover, for the Jordan product (2), we complete our product relations (7), (8), (9), and (10) by the consequences

$$2u_{12}w_{23} = (uw)_{13}, \qquad 2u_{12} \cdot v_{13} = (\bar{u}v)_{23}, \qquad 2v_{13} \cdot w_{23} = (v\bar{w})_{12} \tag{16}$$

of (2) and (15), where u, v, w are elements of $\mathfrak{D}$ and the indicated products uw, $\bar{u}v$, $v\bar{w}$ are the products in the algebra $\mathfrak{D}$.

Observe that for the generator y of $\mathfrak{J}_0$, the elements

$$2\{e_i y e_j\} = e_{ij} \qquad (i \neq j;\ i, j = 1, 2, 3), \tag{17}$$

where

$$e_{12} = \begin{pmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix}, \qquad e_{13} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}. \tag{18}$$

Also we see that for the generator z of $\mathfrak{J}_0$,

$$2\{e_1 z e_2\} = u_{12}, \qquad 2\{e_2 z e_3\} = w_{23}, \qquad 2\{e_3 z e_1\} = v_{13}. \tag{19}$$

Now note that (16) and (18) imply

$$2a_{12} \cdot e_{23} = a_{13}, \qquad 2b_{12} \cdot e_{13} = \bar{b}_{23}, \tag{20}$$

so that we have the basic formula

$$8(a_{12} \cdot e_{23}) \cdot (b_{12} \cdot e_{13}) = (ab)_{12}. \tag{21}$$

This formula enables us to identify the set of elements d of $\mathfrak{D}$ occurring in elements h_{12} of $\mathfrak{J}_{12}$ as the smallest self-adjoint subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$ containing e, u, v and w. Hence $\mathfrak{J}_0$ can be considered as an algebra $\mathfrak{H}_0$ with $\mathfrak{D}_0$ as the basic algebra. Consequently the mapping

$$d_{12} \to d \tag{22}$$

for elements of $\mathfrak{J}_{12}$ upon $\mathfrak{D}_0$ can be considered as an algebra isomorphism of $\mathfrak{J}_{12}$ upon $\mathfrak{D}_0$ relative to the multiplication defined by (21).

3. **The Jordan algebra $\mathfrak{H}$ and induced homomorphisms.** Consider a special Jordan algebra $\mathfrak{H}$ over a field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{H}$ is a subspace of an associative algebra $\mathfrak{A}$ over $\mathfrak{F}$. Let HK be the associative product for $\mathfrak{A}$ so that $\mathfrak{A}$ is a Jordan algebra $\mathfrak{A}^{(+)}$ relative to the product operation $H \cdot K$ defined by

$$2H \cdot K = HK + KH \tag{23}$$

and $\mathfrak{H}$ is a subalgebra over $\mathfrak{F}$ of $\mathfrak{A}^{(+)}$. Assume that $\mathfrak{N}$ is an ideal of $\mathfrak{H}$ such that $\mathfrak{H}-\mathfrak{N}$ is isomorphic to $\mathfrak{H}_0$. The elements of $\mathfrak{H}-\mathfrak{N}$ are then the cosets

$$H + \mathfrak{N} \qquad (H \text{ in } \mathfrak{H}), \tag{24}$$

and we can then say that every element h in $\mathfrak{H}_0$ is a coset

$$h = H + \mathfrak{N} = H' \tag{25}$$

defined uniquely for every H of $\mathfrak{H}$. Since the mapping $H \to H'$ is a homomorphism and we have both (23) and (2), the property

$$(H \cdot K)' = H' \cdot K' \tag{26}$$

is reflected as

$$(HK + KH)' = H'K' + K'H'. \tag{27}$$

We now derive the following result.

LEMMA 2. *The algebras $\mathfrak{A}$ and $\mathfrak{H}$ may be selected so that $\mathfrak{A}$ has a unity I, I is in $\mathfrak{H}$, and $I+\mathfrak{N}=I'=f$ is given by* (4).

We adjoin I to $\mathfrak{A}$ and obtain an associative algebra $\mathfrak{A}_1=\mathfrak{A}+I\mathfrak{F}$ which is the vector space direct sum of $\mathfrak{A}$ and the one-dimensional space $I\mathfrak{F}$. We also adjoin I to $\mathfrak{H}$ to obtain a Jordan algebra $\mathfrak{H}_1=\mathfrak{H}+I\mathfrak{F}$ which is again a vector space direct sum of $\mathfrak{H}$ and $I\mathfrak{F}$. Evidently $\mathfrak{H}_1$ is special and I is in $\mathfrak{H}_1$. Also $\mathfrak{N}$ is an ideal of $\mathfrak{H}_1$ and $\mathfrak{H}_1-\mathfrak{N}$ is clearly isomorphic to the direct sum $\mathfrak{H}_0+I'\mathfrak{F}$. But $\mathfrak{H}_0$ has a unity element $f=F'$, where F is in $\mathfrak{H}$ and $I'=F'+G'$ for an idempotent class G' orthogonal to F'. It follows that G' is actually orthogonal to $\mathfrak{H}_0$, $\mathfrak{H}_1-\mathfrak{N}=\mathfrak{H}_0\oplus G'\mathfrak{F}$ and $G'\mathfrak{F}$ is an ideal of $\mathfrak{H}_1-\mathfrak{N}$. Then the elements of the coset $(I-F)+\mathfrak{N}=G+\mathfrak{N}$ form an ideal $\mathfrak{N}_1$ of $\mathfrak{H}_1$ and $\mathfrak{H}_1-\mathfrak{N}_1$ is isomorphic to $\mathfrak{H}_0$. Also $I=F+G$, $I'=F'=f$ as desired. This completes the proof.

Henceforth we shall assume that $\mathfrak{A}$ and $\mathfrak{H}$ are selected as in Lemma 2.

Suppose that X, Y, and Z are the elements of $\mathfrak{H}$ whose images are the elements x, y, and z of $\mathfrak{H}_0$, and let $\mathfrak{J}$ be the special Jordan subalgebra of $\mathfrak{H}$ generated by I, X, Y and Z. The homomorphism of $\mathfrak{H}$ upon $\mathfrak{H}_0$ induces a homomorphism of $\mathfrak{J}$ upon $\mathfrak{J}_0$ under which

$$X' = x, \qquad Y' = y, \qquad Z' = z, \qquad I' = f, \tag{28}$$

and there is an ideal $\mathfrak{M}$ of $\mathfrak{J}$ such that $\mathfrak{J}-\mathfrak{M}\cong\mathfrak{J}_0$. The relations (24), (25), (26) and (27) are quite valid for this induced homomorphism if $\mathfrak{H}$ is replaced by $\mathfrak{J}$ and $\mathfrak{N}$ by $\mathfrak{M}$.

We wish to give another description of $\mathfrak{J}$. Let

$$\mathfrak{P} = \mathfrak{F}[X, Y, Z, I] \tag{29}$$

be the subset of $\mathfrak{A}$ of all polynomials

$$\phi = \phi[X, Y, Z] \tag{30}$$

in X, Y, Z including the constant polynomials. Every monomial $\phi=\alpha A_1 \cdots A_n$ in $\mathfrak{P}$ is a product of $\alpha\neq 0$ in $\mathfrak{F}$ and factors $A_t=I, X, Y$ or Z. Define the reverse ϕ^* of ϕ to be $\alpha A_n \cdots A_1$. Every element ϕ of $\mathfrak{P}$ is a sum $\phi=\phi_1+ \cdots +\phi_r$ of monomials ϕ_i, and we define its reverse to be $\phi^*=\phi_1^*+ \cdots +\phi_r^*$. The mapping

$$\phi \to \phi^* \tag{31}$$

is then an *involution over* $\mathfrak{F}$ of the algebra $\mathfrak{P}$. The set $\mathfrak{R}$ of all *reversible* polynomials $\phi=\phi^*$ of $\mathfrak{P}$ is a Jordan algebra. The elements I, X, Y, Z generate the Jordan subalgebra $\mathfrak{J}$ and it is actually known [3] that $\mathfrak{J}=\mathfrak{R}$.

4. **Proof of the theorem.** We shall write $A\equiv B$ for any two elements A and B of $\mathfrak{J}$ if $A-B$ is in $\mathfrak{M}$. Then $A\equiv 0$ if and only if A is in $\mathfrak{M}$ so that $A'=0$ in $\mathfrak{J}_0$. We now propose to characterize certain subspaces $\mathfrak{R}_i$ and $\mathfrak{R}_{ij}$ of $\mathfrak{J}$. We define $\mathfrak{R}_i$ to be the set of all elements H_i of $\mathfrak{J}$ such that

$$H_i' = h_i \qquad (h_i \text{ in } \mathfrak{J}_i \text{ of } \mathfrak{J}_0), \tag{32}$$

and we define $\mathfrak{R}_{ij}=\mathfrak{R}_{ji}$ for $i\neq j$ to be the set of all elements $H_{ij}=H_{ji}$ of $\mathfrak{J}$ such that

$$H_{ij}' = h_{ij} \qquad (h_{ij} \text{ in } \mathfrak{J}_{ij} \text{ of } \mathfrak{J}_0). \tag{33}$$

However in a special Jordan algebra

$$\begin{aligned}4\{GHK\} &= (GH+HG)K+K(GH+HG)+(HK+KH)G+G(HK+KH)\\ &\quad -(KG+GK)H-H(GK+KH) = 2(GHK+KHG)\end{aligned}$$

so that we have the formula

$$2\{GHK\} = GHK + KHG. \tag{34}$$

We now use Lemma 1 to obtain the following result as an immediate consequence of (14) and (34).

LEMMA 3. *Let F_1, F_2, F_3 be any element of $\mathfrak{J}$ such that $F_i'=e_i$ for $i=1, 2, 3$. Then every element H of $\mathfrak{J}$ is congruent to an element of the form*

$$H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13}, \tag{35}$$

where the elements

$$H_i = F_i H F_i \tag{36}$$

are in $\mathfrak{R}_i$, *and the elements*

$$H_{ij} = F_i H F_j + F_j H F_i \tag{37}$$

are in $\mathfrak{R}_{ij}$.

We use the fact that (6) is a vector space direct sum to obtain the following result.

LEMMA 4. *If* $H \equiv H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13}$, *then* H *is in* $\mathfrak{R}_{ij}$ *if and only if* $H_1 \equiv H_2 \equiv H_3 \equiv H_{ik} \equiv H_{jk} \equiv 0$.

In $\mathfrak{J}_0$, $x^3 - x = 0$ so that $X^3 - X \equiv 0$ and (5) implies that, if

$$E_1 = \frac{X^2 + X}{2}, \qquad E_2 = \frac{X^2 - X}{2}, \qquad E_3 = I - (E_2 + E_3), \qquad F_i = E_i^2, \tag{38}$$

then

$$E_i' = F_i' = e_i \qquad (i = 1, 2, 3). \tag{39}$$

We then use Lemma 3 to obtain the following result.

LEMMA 5. *If* H *is any* (*reversible*) *polynomial of* $\mathfrak{J}$ *the elements*

$$H_i = E_i H E_i, \qquad H_i^{(0)} = F_i H F_i, \tag{40}$$

are in $\mathfrak{R}_i$, *and the elements*

$$H_{ij} = E_i H E_j + E_j H E_i, \qquad H_{ij}^{(0)} = F_i H F_j + F_j H F_i \tag{41}$$

are in $\mathfrak{R}_{ij}$. *Every element* H *is congruent to each of the sums* $H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{13}$ *and* $H_1^{(0)} + H_2^{(0)} + H_3^{(0)} + H_{12}^{(0)} + H_{23}^{(0)} + H_{13}^{(0)}$, *so that* $H_i \equiv H_i^{(0)}$, $H_{ij} \equiv H_{ij}^{(0)}$.

We now derive the following auxiliary vital characterization.

LEMMA 6. *If* G *is any polynomial of* $\mathfrak{P}$ *the element* $F_i G F_j + F_j G^* F_i$ *is in* $\mathfrak{R}_{ij}$. *Every element of* $\mathfrak{R}_{ij}$ *has a representative of the form* $F_i G F_j + F_j G^* F_i$ *for* G *in* $\mathfrak{P}$.

For if G is in $\mathfrak{P}$ the element $H = E_i G E_j + E_j G^* E_i$ is in $\mathfrak{R}$. By Lemma 5 the element $E_i H E_j + E_j H E_i = E_i^2 G E_j^2 + E_j^2 G^* E_i + E_i E_j (G + G^*) E_i E_j$ is in $\mathfrak{R}_{ij}$. But $E_i^2 = F_i$, $M = E_i E_j$ is in $\mathfrak{M}$, $\{M(G+G^*)M\} = M(G+G^*)M \equiv 0$ and so $E_i H E_j + E_j H E_i \equiv F_i G F_j + F_j G^* F_i$ as desired. Lemma 5 implies that every element of $\mathfrak{R}_{ij}$ is congruent to an element $F_i G F_j + F_j G^* F_i$ for $G = G^*$ in $\mathfrak{R}$ and our proof is complete.

Lemma 6 states that the mapping

$$G \to (F_1 G F_2 + F_2 G^* F_1)' \tag{42}$$

maps the space $\mathfrak{P}$ of all polynomials in I, X, Y, Z onto $\mathfrak{J}_{12}$. We already have a mapping (22) of $\mathfrak{J}_{12}$ to $\mathfrak{D}_0$ and so we have the induced mapping

$$(43) \qquad G \to G'' = g, \quad \text{where} \quad g_{12} = (F_1GF_2 + F_2G^*F_1)',$$

of $\mathfrak{P}$ to the subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$.

We use the result of (40) with $H = Y$, $Y' = y$ of (17) to see that

$$(44) \qquad E_{ij} = F_iYF_j + F_jYF_i, \qquad E'_{ij} = e_{ij},$$

and then use (20) to see that

$$(45) \quad 2H_{12}\cdot E_{23} = H_{13}, \qquad 2K_{12}\cdot E_{13} = K_{23}, \qquad 8(H_{12}\cdot E_{23})\cdot(K_{12}\cdot E_{13}) = L_{12},$$

for H_{ij}, K_{ij}, L_{ij} in $\mathfrak{R}_{ij}$. The first expression in (45) yields the computation

$$\begin{aligned}(F_1GF_2 &+ F_2G^*F_1)(F_2YF_3 + F_3YF_2) + (F_2YF_3 + F_3YF_2)(F_1GF_2 + F_2G^*F_1) \\ &= [F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1] + [F_1(GF_2F_3Y)F_2 + F_2(YF_3F_2G^*)F_1] \\ &\quad + [F_2(G^*F_1F_2Y)F_3 + F_3(YF_2F_1G)F_2] + F_2[(G^*F_1F_3Y) + YF_3F_1G)]F_2.\end{aligned}$$

The second bracketed expression is in $\mathfrak{R}_{12}$, the third in $\mathfrak{R}_{23}$ and the last in $\mathfrak{R}_2$ by Lemma 5. Since the first and the sum is in $\mathfrak{R}_{13}$ by (45), we use Lemma 4 to see that

$$(46) \qquad 2(F_1GF_2 + F_2G^*F_1)\cdot E_{23} \equiv F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1.$$

We similarly compute

$$\begin{aligned}(F_1HF_2 &+ F_2H^*F_1)(F_1YF_3 + F_3YF_1) + (F_1YF_3 + F_3YF_1)(F_1HF_2 + F_2H^*F_1) \\ &= [F_2(H^*F_1^2Y)F_3 + F_3(YF_1^2H)F_2] + [F_2(H^*F_1F_3Y)F_1 + F_1(YF_3F_1H)F_2] \\ &\quad + [F_1(HF_2F_1Y)F_3 + F_3(YF_1F_2H^*)F_1] + F_1[(HF_2F_3Y) + (YF_3F_2H^*)]F_1,\end{aligned}$$

and obtain

$$(47) \qquad 2(F_1HF_2 + F_2H^*F_1)\cdot E_{13} \equiv F_2(H^*F_1^2Y)F_3 + F_3(YF_1^2H)F_2.$$

We finally compute

$$\begin{aligned}[F_1(GF_2^2Y)F_3 &+ F_3(YF_2^2G^*)F_1][F_3(YF_1^2H)F_2 + F_2(H^*F_1^2Y)F_3] \\ &+ [F_3(YF_1^2H)F_2 + F_2(H^*F_1^2Y)F_3][F_1(GF_2^2Y)F_3 + F_3(YF_2^2G^*)F_1] \\ &\equiv F_1G(F_2^2YF_3^2YF_1^2)HF_2 + F_2H^*(F_1^2YF_3^2YF_2^2)G^*F_1\end{aligned}$$

since the remaining components are in $\mathfrak{R}_{13}$, $\mathfrak{R}_{23}$ and $\mathfrak{R}_3$ and hence in $\mathfrak{M}$.

We have now shown that, if

$$(48) \qquad S = F_2^2YF_3^2YF_1^2,$$

then

$$(49) \quad 8[(F_1GF_2 + F_2G^*F_1)\cdot E_{23}]\cdot[(F_1HF_2 + F_2H^*F_1)\ E_{13}] \equiv F_1KF_2 + F_2K^*F_1,$$

where

$$K = [G, H] = GSH. \tag{50}$$

The polynomial space $\mathfrak{P}$ is clearly an algebra $\mathfrak{P}_0$ with respect to the product defined by (50) and it follows from (21) that the mapping $G \to G''$ of (43) is an algebra homomorphism of $\mathfrak{P}_0$ onto the subalgebra $\mathfrak{D}_0$ of $\mathfrak{D}$. But

$$[[G, H], A] = (GSH)SA = GS(HSA) = [G, [H, A]],$$

and so $\mathfrak{P}_0$ is associative. Hence $\mathfrak{D}_0$ must be associative and so $u(vw) = (uv)w$. Since u, v and w were any elements of $\mathfrak{D}$ the algebra $\mathfrak{D}$ is associative and the proof of our main theorem is complete.

5. **Consequences.** It is well known(5) that the only exceptional simple Jordan algebra over an algebraically closed field Ω of characteristic not two is the algebra $\mathfrak{H}_0$ defined by selecting $\mathfrak{D}$ to be the unique eight dimensional split Cayley algebra $\mathfrak{C}$ over Ω. Thus the following corollary is an immediate consequence of our theorem when $\mathfrak{F}$ is algebraically closed.

COROLLARY 1. *Let $\mathfrak{H}$ be a simple exceptional Jordan algebra of finite dimension (necessarily* 27) *over any field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{H}$ is not a homomorphic image of any special Jordan algebra over $\mathfrak{F}$.*

To complete our proof we let Ω be the algebraic closure of $\mathfrak{F}$ and $\mathfrak{H}_1 = \mathfrak{H}_\Omega$ and $\mathfrak{H}_0$ be defined as above for $\mathfrak{D}$ the split Cayley algebra over Ω. Then $\mathfrak{H}_1$ and $\mathfrak{H}_0$ are isomorphic. If $\mathfrak{H}$ were the homomorphic image of a special Jordan algebra $\mathfrak{J}$ it should be clear that $\mathfrak{H}_\Omega$ is such an image whereas $\mathfrak{H}_0$ is not such an image and our proof is complete.

We also have the following result.

COROLLARY 2. *The free Jordan algebra $\mathfrak{J}_3$ on three generators is not special.*

To derive this result it is only necessary to note that the algebra $\mathfrak{H}_0$ of all three rowed hermitian matrices with elements in a Cayley algebra $\mathfrak{C}$ is generated by three of its matrices. This is clearly a consequence of the fact that the matrices x, y, z of (3) generate $\mathfrak{H}_0$ if we select u, v, w as generators of $\mathfrak{C}$. It is well known(6) that $\mathfrak{C}$ has three generators and so $\mathfrak{H}_0$ has three generators and is a homomorphic image of the algebra $\mathfrak{J}_3$. Our theorem then implies that $\mathfrak{J}_3$ cannot be special.

Thus we learn also that $\mathfrak{J}_3$ is not isomorphic to the free special Jordan algebra $\mathfrak{J}[x, y, z, 1]$ of all reversible elements in the free associative algebra

(5) See A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 446–447.

(6) See, for example, the discussion of composition algebras in the paper of A. A. Albert and N. Jacobson, *On reduced exceptional Jordan algebras*, Ann. of Math. vol. 66 (1957) pp. 400–417.

$\mathfrak{F}[x, y, z, 1]$ of all polynomials on three generators. This is, of course, a quite different situation to that which exists in the case of the free Jordan algebra $\mathfrak{J}_2$ on two generators where $\mathfrak{J}_2$ is isomorphic to $\mathfrak{J}[x, y, 1]$. It would then be of great interest to derive the identities which must exist and which are satisfied by all special Jordan algebras but not by $\mathfrak{J}_3$ or by $\mathfrak{H}_0$.

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