EMBEDDING ANY SEMIGROUP IN A
\(\mathcal{D}\)-SIMPLE SEMIGROUP\(^{(1)}\)

BY

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1. Introduction. A semigroup is said to be \textit{simple} if it has no proper two-sided ideals. Let \(S\) be a semigroup and let \(\mathcal{L}\) and \(\mathcal{R}\) be the equivalences:

\[
\mathcal{L} = \{(a, b) : a, b \in S \text{ and } Sa \cup a = Sb \cup b\}; \\
\mathcal{R} = \{(a, b) : a, b \in S \text{ and } aS \cup a = bS \cup b\}.
\]

Denote by \(\circ\) the operation of composition, so that if \(A \subseteq X \times Y\) and \(B \subseteq Y \times Z\) then \(A \circ B = \{(x, z) : (x, y) \in A \text{ and } (y, z) \in B \text{ for some } y\}\). Then the minimal equivalence on \(S\) containing both \(\mathcal{L}\) and \(\mathcal{R}\) is \(\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}\) (J. A. Green \cite{1}). A semigroup is said to be \(\mathcal{D}\)-simple if it consists of a single \(\mathcal{D}\)-class. A \(\mathcal{D}\)-simple semigroup is necessarily simple, a completely simple semigroup is \(\mathcal{D}\)-simple, but in general a simple semigroup is not \(\mathcal{D}\)-simple (see \cite{1} and also \S 2 below). A semigroup is said to be \textit{regular} if \(a \in aSa\) for each \(a\) in \(S\). If \(S\) is \(\mathcal{D}\)-simple then, as shown by D. D. Miller and A. H. Clifford \cite{2}, \(S\) is regular if and only if \(S\) contains an idempotent. In particular, therefore, a \(\mathcal{D}\)-simple semigroup with an identity is necessarily regular. A regular simple semigroup is not necessarily \(\mathcal{D}\)-simple (see \S 2).

A recent result of R. H. Bruck \cite[II, Theorem 8.3, p. 48]{3} shows that any semigroup \(S\) can be embedded in a simple semigroup \(T\), say, with identity. I show below that the simple semigroup \(T\) constructed by Bruck is \(\mathcal{D}\)-simple if and only if \(S\) both has an identity and is \(\mathcal{D}\)-simple. The main result of this paper is the following theorem: \textit{any semigroup can be embedded in a (necessarily regular) \(\mathcal{D}\)-simple semigroup with an identity}. As a preliminary to the proof of this theorem we obtain (\S 3) a characterization of the \(\mathcal{D}\)-classes of the semigroup of all mappings of a set into itself.

2. The construction of R. H. Bruck. Let \(S\) be a semigroup. If \(S\) has an identity element \(e\), say, write \(S = S^1\). If \(S\) has no identity element then, by the adjunction of a single element \(e\), say, to \(S\), we can embed \(S\) in a semigroup \(S^1\) with identity. In either case \(S^1\) is a semigroup with \(e\) as identity. Let \(N\) denote the set of non-negative integers. Let \(T\) be the set product \(N \times S^1 \times N\) and define a product in \(T\) by the rule:

\[
(m, s, n)(m', s', n') = (m + [m' - n], f(n - m'; s, s'), n' + [n - m']),
\]

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where for any integer \(x\),

\[
[x] = \begin{cases} 
  x & \text{if } x \geq 0; \\
  0 & \text{if } x < 0; 
\end{cases}
\]

and

\[
f(x; s, s') = s, ss' \text{ or } s' \text{ according as } x > 0, x = 0 \text{ or } x < 0.
\]

Then Bruck shows that with this product \(T\) becomes a simple semigroup, with \((0, e, 0)\) as its identity, in which \(S\) is embedded.

We now prove the result claimed in the introduction: \(T\) is \(\mathcal{D}\)-simple if and only if \(S\) has an identity and is \(\mathcal{D}\)-simple.

Firstly suppose that \(S\) has an identity and is \(\mathcal{D}\)-simple so that \(S^1\) is \(\mathcal{D}\)-simple. Let \((m, s, n)\) and \((m', s', n')\) be any two elements of \(T\). Since \(S^1\) is \(\mathcal{D}\)-simple there exists \(s''\) in \(S^1\) such that \(sLs''Rs'\). Hence, since \(S^1\) has an identity, there exist \(x, y, u, v\) in \(S^1\) such that \(xs = s'', ys'' = s, s''u = s'\) and \(s'v = s''\). Then it may be verified that \((m', x, m)(m, s, n) = (m', s'', n)\) and \((m, y, m')(m', s'', n) = (m, s, n)\) so that \((m, s, n)L(m', s'', n)\). Similarly we have that \((m', s'', n)(u, u, v') = (m', s', n')\) and \((m', s', n')(v, n) = (m', s'', n)\) so that \((m', s'', n)R(m', s', n')\). Thus \((m, s, n)L(m', s'', n)R(m', s', n')\) and so \((m, s, n)D(m', s', n')\); and this proves that \(T\) is \(\mathcal{D}\)-simple.

Conversely suppose that \(T\) is \(\mathcal{D}\)-simple. For any \(s, s'\) in \(S^1\) it follows, in particular, that \((0, s, 0)D(0, s', 0)\). Thus there exists an element \((m, s'', n)\) in \(T\) such that \((0, s, 0)L(m, s'', n)R(0, s', 0)\). We will show that this implies that \(sLs''Rs'\).

Since \((0, s, 0)L(m, s'', n)\) there exist \((p, x, q)\) and \((p', y, q')\) in \(T\) such that \((p, x, q)(0, s, 0) = (m, s'', n)\) and \((p', y, q')(m, s'', n) = (0, s, 0)\). Thus

\[
(p + [-q], f(q; x, s), [q]) = (m, s'', n)
\]

and

\[
(p' + [m - q'], f(q' - m; y, s''), n + [q' - m]) = (0, s, 0).
\]

The second equation implies that \(n = 0\) and it then follows from the first equation that \(q = 0\). Hence \(f(q; x, s) = xs\) and so from the first equation we have \(xs = s''\). Again the second equation gives \([q' - m] = 0\) and \([m - q'] = 0\) and these together imply that \(q' = m\). Hence \(f(q' - m; y, s'') = ys''\) and we have \(ys'' = s\). Thus \(xs = s''\) and \(ys'' = s\) i.e. \(sLs''\). By a similar argument we deduce that also \(s''Rs'\). Hence we have \(sDs'\); and this shows that \(S^1\) is \(\mathcal{D}\)-simple.

It now follows that \(S = S^1\). For if \(S \neq S^1\) then the identity element \(e\) of \(S^1\) is not \(\mathcal{D}\)-equivalent to any element of \(S\) and so \(S^1\) could not be \(\mathcal{D}\)-simple. This completes the proof of our assertion.

Let \(S\) be regular but not \(\mathcal{D}\)-simple. Then \(T\) is regular. For let \((m, a, n) \in T\). The regularity of \(S\) implies that \(S^1\) is regular and hence there exists an \(x\) in \(S^1\) such that \(axa = a\). Then \((m, a, n)(n, x, m)(m, a, n) = (m, a, n)\) which shows that \(T\) is regular. Thus \(T\) is a simple regular semigroup which is not \(\mathcal{D}\)-simple: which proves an assertion made earlier.
3. Determination of the \( \mathcal{D} \)-classes of the semigroup of all mappings of a set into itself. Let \( \Sigma(=\Sigma(A)) \) be the semigroup of all single-valued mappings of \( A \) into \( A \), combined under composition. The composition of the mapping \( \alpha \) with the mapping \( \beta \) is the mapping obtained by following \( \alpha \) by \( \beta \) (we write operators or mappings on the right). Regarding a mapping \( \alpha \) of \( A \) into \( A \) as a subset of \( A \times A \), namely the subset \( \{ (a, \alpha a) : a \in A \} \), then this operation of composition is the same as that defined earlier in the introduction. It will be convenient in what follows to write sometimes \( \alpha \beta \) and sometimes \( \alpha \circ \beta \) for the composition of the mapping \( \alpha \) with the mapping \( \beta \). If \( \alpha \subset A \times B \) then \( \alpha^{-1} \) denotes the set \( \{ (x, y) : (y, x) \in \alpha \} \); if \( C \subset B \) then \( C^{-1} \) denotes the set \( \{ x : (x, y) \in \alpha, y \in C \} \).

Since \( \Sigma \) has an identity element, namely the identical mapping of \( A \) onto \( A \), two elements \( \alpha, \beta \) in \( \Sigma \) are \( \mathcal{D} \)-equivalent if and only if there exist \( \gamma, \delta \) in \( \Sigma \) such that \( \gamma \alpha = \beta \) and \( \delta \beta = \alpha \). A similar comment applies to \( \mathcal{R} \)-equivalent elements. We now give two lemmas which determine the \( \mathcal{L} \)-classes and the \( \mathcal{R} \)-classes of \( \Sigma \).

Lemma 1. If \( \alpha, \beta \in \Sigma \), then \( (\alpha, \beta) \in \mathcal{E} \) if and only if \( A\alpha = A\beta \).

Proof. If \( (\alpha, \beta) \in \mathcal{E} \) then there exist \( \gamma, \delta \) in \( \Sigma \) such that \( \gamma \alpha = \beta \) and \( \delta \beta = \alpha \). Hence \( A\beta = A\gamma \alpha \subseteq A\alpha \) and \( A\alpha = A\delta \beta \subseteq A\beta \). Thus if \( (\alpha, \beta) \in \mathcal{E} \) then \( A\alpha = A\beta \).

Conversely suppose that \( A\alpha = A\beta \). Define the mapping \( \gamma \) of \( A \) into \( A \) as follows: for each element \( b \) in \( A\beta \) let \( \gamma \) map the elements of the set \( b\beta^{-1} \) onto a single element in \( b\alpha^{-1} \). Then \( \gamma \alpha = \beta \). Similarly there exists \( \delta \) in \( \Sigma \) such that \( \delta \beta = \alpha \). Thus \( (\alpha, \beta) \in \mathcal{E} \).

Lemma 2. If \( \alpha, \beta \in \Sigma \) then \( (\alpha, \beta) \in \mathcal{R} \) if and only if \( \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \).

Proof. If \( (\alpha, \beta) \in \mathcal{R} \) then there exist \( \gamma, \delta \) in \( \Sigma \) such that \( \alpha \gamma = \beta \) and \( \beta \delta = \alpha \). Hence \( \alpha \circ \alpha^{-1} = (\beta \delta) \circ (\delta \beta^{-1}) = \beta \circ (\delta \circ \delta^{-1}) \circ \beta^{-1} \subseteq \beta \circ \beta^{-1} \); similarly, \( \beta \circ \beta^{-1} \subseteq \alpha \circ \alpha^{-1} \). Hence \( \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \).

Conversely suppose that \( \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \). Then we may define a mapping \( \gamma \) as follows. Let \( \gamma \) map \( A \setminus A\alpha \) identically and for \( b \) in \( A\alpha \) let \( \gamma \) map \( b \) onto \( (b\alpha^{-1})\beta \). The condition \( \alpha \circ \alpha^{-1} = \beta \circ \beta^{-1} \) implies that \( (b\alpha^{-1})\beta \) is a single element, for \( \alpha \circ \alpha^{-1} \) is the equivalence relation on \( A \) determined canonically by \( \alpha \): \( (x, y) \in \alpha \circ \alpha^{-1} \) if and only if \( x\alpha = y\alpha \). Thus \( \alpha \gamma = \beta \). Similarly there exists \( \delta \) in \( \Sigma \) such that \( \delta \beta = \alpha \). Thus \( (\alpha, \beta) \in \mathcal{R} \).

Using these lemmas we now easily have the following determination of the \( \mathcal{D} \)-classes of \( \Sigma \). Denote by \( |X| \) the cardinal of a set \( X \).

Theorem 1. Let \( \Sigma \) be the semigroup of all mappings of the set \( A \) into \( A \) combined under composition. Then \( \alpha, \beta \) in \( \Sigma \) are \( \mathcal{D} \)-equivalent if and only if \( |A\alpha| = |A\beta| \).

Proof. We know, since \( \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} \), that \( (\alpha, \beta) \in \mathcal{D} \) if and only if there exists \( \gamma \) in \( \Sigma \) such that \( (\alpha, \gamma) \in \mathcal{L} \) and \( (\gamma, \beta) \in \mathcal{R} \). By Lemmas 1 and 2
this is equivalent to the existence of a mapping \( \gamma \) in \( \Sigma \) such that \( A\alpha = A\gamma \) and \( \gamma \circ \gamma^{-1} = \beta \circ \beta^{-1} \).

Consequently, denoting by \( A/\rho \) the quotient set determined by the equivalence \( \rho \) on \( A \), if \( \alpha \) is \( \mathfrak{D} \)-equivalent to \( \beta \), then \( |A\alpha| = |A\gamma| = |A/(\gamma \circ \gamma^{-1})| = |A/(\beta \circ \beta^{-1})| = |A\beta| \), so that \( |A\alpha| = |A\beta| \).

Conversely, suppose that \( |A\alpha| = |A\beta| \). Denote by \( \rho \) the equivalence \( \beta \circ \beta^{-1} \) on \( A \). Since \( |A\beta| = |A/\rho| \), we have \( |A/\rho| = |A\alpha| \). Let \( \delta \) be any \((1,1)\)-mapping of \( A/\rho \) onto \( A\alpha \). Then let \( \gamma \) be the mapping of \( A \) into \( A \) which maps the elements in each \( \rho \)-class onto the image of the \( \rho \)-class under \( \delta \). Then \( \gamma \circ \gamma^{-1} = \rho = \beta \circ \beta^{-1} \) and \( A\gamma = A\alpha \). Thus \((\alpha, \beta) \in \mathfrak{D} \); and this completes the proof of the theorem.

4. The embedding theorem. If \( S \) is a semigroup with an identity then it is easily verified that \( S \) is \( \mathfrak{D} \)-simple if and only if for any two elements \( a, b \) in \( S \) there exist elements \( s, t, u, v \) in \( S \) such that \( as = ub, ast = a \) and \( vub = b \).

To embed an arbitrary semigroup \( S \) in a \( \mathfrak{D} \)-simple semigroup we can clearly suppose, without loss of generality, that \( S \) contains an identity element. The first stage in our construction is to embed \( S \) in a semigroup \( S(1) \), say, with the same identity element as \( S \) and such that for each pair of elements \( a, b \) in \( S \) there exist elements \( s, t, u, v \) in \( S(1) \) such that \( as = ub, ast = a \) and \( vwb = b \).

Let \( B \) be a set of elements disjoint from \( S \) and such that if \( |S| \) is finite then \( |B| \) is countably infinite, whilst if \( |S| \) is infinite then \( |B| = |S| \). Let \( A = B \cup S \) and let \( \Sigma = \Sigma(A) \), the semigroup of all mappings of \( A \) into \( A \). Each element \( s \) in \( S \) then determines an element \( \rho_s \) in \( \Sigma \) defined thus:

\[
x_{\rho_s} = \begin{cases} x, & \text{if } x \in B, \\ x_{s} & \text{if } x \in S, \end{cases}
\]

We easily verify, since by assumption \( S \) contains an identity element, that the mapping \( s \mapsto \rho_s \) embeds \( S \) isomorphically into \( \Sigma \) and that the identity element of \( S \) is mapped onto the identity element of \( \Sigma \).

Now, for each \( s \) in \( S \), \( |A\rho_s| \geq |B| = |A| \), and hence \( |A\rho_s| = |A| \). Hence, by Theorem 1, all the elements \( \rho_s \) of \( \Sigma \) belong to the same \( \mathfrak{D} \)-class in \( \Sigma \). Thus for each pair of elements \( s, t \) in \( S \) we may select a set of four elements \( \alpha, \beta, \xi, \eta \) in \( \Sigma \) such that \( \rho_s \alpha = \xi \rho_t, \rho_s \alpha \beta = \rho_s \), and \( \eta \xi \rho_t = \rho_t \). For each pair of elements \( s, t \) in \( S \) select a definite set of four such elements and let \( P \) denote the set of all such elements so selected. Let \( S(1) \) be the subsemigroup of \( \Sigma \) generated by \( P \cup \{ \rho_s : s \in S \} \). Then, regarding \( S \) as identified with its image in \( S(1) \) under the mapping \( s \mapsto \rho_s \), we have embedded \( S \) in a semigroup \( S(1) \) with the properties we required, and the first stage of the construction is completed.

Now construct \( S(2) \) from \( S(1) \) in exactly the same way as \( S(1) \) was constructed from \( S \). Similarly we construct \( S(n+1) \) from \( S(n) \) for any integer \( n \geq 1 \). Let \( T = \bigcup_{n=1}^{\infty} S(n) \).

Then \( T \) contains an identity element, viz. the common identity of all the
Further for any \( a, b \) in \( T \) there exists an integer \( n \) such that \( a, b \in S(n) \) and then there necessarily exist \( s, t, u, v \) in \( S(n+1) \), and hence in \( T \), such that \( as = ub, ast = a \) and \( vub = b \). Thus, in view of a remark made earlier, \( T \) is \( \mathcal{D} \)-simple and we have proved the following theorem.

**Theorem 2.** Any semigroup can be embedded in a (necessarily regular) \( \mathcal{D} \)-simple semigroup with identity.

We note finally that our construction is such that if \( S \) is infinite then \( |T| = |S| \) and that if \( S \) is finite then \( T \) is at most countably infinite.

**Acknowledgment.** In the author's original proof of Theorem 2 \( S(1) \) was constructed as a free semigroup subject to certain relations. In a letter to the author (February, 1958) Dr. M. P. Schützenberger suggested that it could probably be shown that when \( A \) is an infinite set then the subset \( \Omega(A) \) of \( \Sigma(A) \) consisting of all those mappings \( \alpha \) such that (i) \( |A| = |A\alpha| \) and (ii) for all \( b \) in \( A\alpha \), \( |b\alpha^{-1}| < |A| \), formed a \( \mathcal{D} \)-simple subsemigroup of \( \Sigma \). In fact \( \Omega \) is a semigroup if and only if \( |A| \) is a regular cardinal and when it is a semigroup it is \( \mathcal{D} \)-simple. Then, as Dr. Schützenberger suggested, the proof of Theorem 2 can be completed in one step by the mapping \( s \to \rho_s \) of the previous section which now embeds \( S \) in \( \Omega(A) \) if we choose \( B \) such that \( |B| > |S| \). The referee suggested yet a further proof of Theorem 2. The proof given in the paper results from a combination of the referee’s proof with the author’s proof of Dr. Schützenberger’s conjecture.

**References**


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