INFINITE CARTESIAN PRODUCTS AND A PROBLEM CONCERNING HOMOLOGY LOCAL CONNECTEDNESS

BY

SIBE MARDEŠIĆ

If $M \subset N$ are subsets of a topological space X, we denote by $H_p(M)$ and $H_p(N)$ the singular homology groups (with integer coefficients) of M and N respectively; the image of $H_p(M)$ in $H_p(N)$ (under the homomorphism induced by inclusion $M \subset N$) will be denoted by $H_p(M|N)$. The space X is said to be p-lc_s (i.e. p-locally connected in the sense of singular homology) at the point $x \in X$ if for every neighborhood U of x there is a neighborhood V of x, $V \subset U$, such that $H_p(V|U)=0$; if $p=0$ augmented homology is used. X is lc_s^q at x if it is p-lc_s at x, for all $0 \leqq p \leqq q$. X is lc_s^q if it is lc_s^q at all $x \in X$. Replacing singular homology by Čech homology (arbitrary open coverings and integer coefficients) and by homotopy, one obtains the definition of properties lc_c^q and LC^q respectively.

These notions are well-known and have been studied by various authors. In a recent paper [9], the present author has shown that for Hausdorff locally paracompact spaces the property lc_s^q implies lc_c^q(1). The implication $\mathrm{lc}_s^q \Rightarrow \mathrm{lc}_c^q$ can not be reversed (not even in the category of metrizable compacta) as has been shown by H. B. Griffiths [5, p. 477]. Griffiths has also proved [7] that for locally compact metrizable spaces $\mathrm{LC}^q \Rightarrow \mathrm{lc}_s^q$. However, the question of the possibility of reversing this last implication has remained open and has been pointed out by Griffiths in [5, p. 479] and in [6, 3, p. xi,]. The corresponding question with Čech homology has been settled previously (see [1, p. 573]) by the well-known example of an "infinite bouquet" of Poincaré spaces, which is lc_c^1 but fails to be LC^1 at the base point of the bouquet. Griffiths has shown [5, p. 477] that an infinite bouquet of LC^1 spaces can never provide an example of an lc_s^1 space which would not be LC^1 at the same time. This different behavior is due to the fact that singular homology is not continuous with respect to inverse limits.

In this paper we describe a whole category of 2-dimensional metrizable compacta which are lc_s^1 but fail to be LC^1 in certain points(2), proving thus that the implication $\mathrm{LC}^1 \Rightarrow \mathrm{lc}_s^1$ can not be reversed (Theorem 7). If one admits examples of infinite dimension, then the problem is easily settled by an in-

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(1) The same result has also been obtained by H. B. Griffiths in an unpublished paper.

(2) The case $q=1$ is easier to handle because of the simple relation between the fundamental group and H_1 given by the Poincaré theorem. This case deserves special attention due to the fact that for locally compact metrizable spaces (lc_c^q and LC^1)$\Rightarrow \mathrm{LC}^q$ (see [10]).

finite Cartesian product of Poincaré spaces (Theorem 8). The main part of the paper is concerned with a construction giving a 2-dimensional subset of the infinite Cartesian product which, roughly speaking, in the neighborhood of some points has the fundamental group of the entire infinite product (see §3,1). We hope that the main Theorem 6 might prove useful in other connections too.

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1. Preliminaries. 1. The following four propositions will often be referred to in the sequel. The proofs can be easily supplied and are omitted.

1.1. If M is a metric space with metric ρ and $N\subset M$, then $U(N, \epsilon)$ will denote the ϵ-neighborhood around N, i.e. the set $\{x \mid x\in M, \rho(N, x)<\epsilon\}$.

Let $C_0, C_1, \cdots$ be a sequence of compact subsets of a metric space M. If there is a sequence of reals $\epsilon_n>0$, $\lim \epsilon_n=0$, such that $C_n\subset U(C_0, \epsilon_n)$, then $\bigcup_0^\infty C_n$ is compact.

1.2. Let I be the unit interval and let $f^p: I\to M$, $p=1, 2, \cdots$, be a sequence of loops in a metric space M, based at a point $o\in M$. Let $F^{p\ p+1}$ be homotopies in M, connecting f^p and f^{p+1}, such that $\operatorname{diam} F^{p\ p+1} = \max_x \operatorname{diam} F^{p\ p+1}(x, I)\leqq c_p$, where $\sum_1^\infty c_p$ is a convergent series. Then $f(x)=\lim_p f^p(x)$ exists and is a loop homotopic to all f^p. One can choose the homotopy F, connecting f^1 and f, so as to take place in the union of images of all $F^{p\ p+1}$.

F can be obtained by considering $F^{p\ p+1}$ as defined over $I\times I_p$, where $I_p=[(p-1)/p, p/(p+1)]$ and setting

$$F(x, t) = F^{p\ p+1}(x, t), \text{ for } x\in I, t\in I_p, p\in\{1, 2, \cdots\}, \tag{1}$$

and $F(x, 1)=f(x)$.

Whenever we speak of homotopies of loops and paths we mean homotopies with fixed end-points.

1.3. Let M^* be a metric space obtained from its closed subset M by attaching an n-cell e^n, $n>1$. Every loop f in M^* with base point in M can be deformed (inside M^*) into a path g in M in such a way that the deformation $F(x, t)=f(x)$, whenever $f(x)\in M$ and $F(x, t)\in(e^n)^-$, whenever $f(x)\in e^n$.

1.4. Let M be a metric space with a base point o and $f: I\to M$ a path. Furthermore, let U be an open set of I such that $f(\overline{U}\setminus U)=o$. U is obviously the union of at most countably many disjoint open intervals $V\subset I$, which are components of U; $f\mid\overline{V}$ are loops in M, based on o.

If for every V, $F_V: \overline{V}\times I\to M$ is a deformation of the loop $f\mid\overline{V}$ and for every $\epsilon>0$ there is a $\delta(\epsilon)>0$ with the property that $\operatorname{diam}\overline{V}<\delta$ implies $\operatorname{diam} F_V<\epsilon$, then the following relations define a deformation F of the path f:

$$F(x, t) = F_V(x, t), \qquad \text{for } x\in\overline{V}, \tag{2}$$

$$F(x, t) = o, \qquad \text{for } x\in\overline{U}\setminus U \tag{3}$$

and

$$F(x, t) = f(x), \qquad \text{for } x \in I \backslash U. \tag{4}$$

2. By a finite cell complex K we mean in this paper a finite cell complex which admits a simplicial subdivision (see e.g. [2, p. 152]). We use the same letter to denote the complex and the underlying polyhedron. There is no loss of generality in assuming that K is provided with a metric $d \leqq 1$ and that every point x of K has arbitrarily small δ-neighborhoods $U(x)$ admitting a cell-preserving contraction into x (with respect to K)(3). Moreover, if dim $K = n$ and K^p denotes the p-skeleton of K, we can assume that this contraction is composed first of a cell-preserving deformation retraction of U onto $U \cap K^{n-1}$, then of a cell-preserving deformation retraction of $U \cap K^{n-1}$ onto $U \cap K^{n-2}$, etc. Clearly, $U(x)$ has to be contained in the open star $\mathrm{St}_K(x)$. We shall often have the additional assumption that K has a single vertex o; closed 1-cells will therefore be 1-spheres and thus never contained entirely in such a neighborhood $U(x)$.

3. Let M be a metric space with a given metric $d \leqq 1$. The infinite Cartesian product of a sequence $M_1, M_2, \cdots$ of copies of M will be denoted by $\prod M$. If $x \in M$, we shall usually denote the nth coordinate of x by x_n. We shall consider M as metrized by the metric

$$\rho(x, y) = \sum_1^\infty d(x_n, y_n) 2^{-n}. \tag{5}$$

If $a = (a_1, \cdots, a_n) = a_1 \times \cdots \times a_n$ is a point of the n-fold Cartesian product $M \times \cdots \times M$ and $b = (b_1, \cdots) = b_1 \times \cdots$ is a point of the infinite product $\prod M$, we shall often denote the point $(a_1, \cdots, a_n, b_1, \cdots) \in \prod M$ simply by $a \times b$. If $A \subset M \times \cdots \times M$ and $B \subset \prod M$, the meaning of the notation $A \times B \subset \prod M$ is clear.

2. **Infinite Cartesian products of cell complexes.** 1. Let K be a finite cell complex(4) having a single vertex o. We can assume that dim $K \leqq 2$ (otherwise we should replace K by the 2-skeleton K^2 in (4)). The infinite Cartesian product $\prod K$ will be denoted hereafter by P_0. All sets encountered throughout §§2–4 will be subsets of P_0. The cellular structure of K induces a decomposition of P_0 into disjoint "cells"

$$\sigma = \sigma_1 \times \sigma_2 \times \cdots, \tag{1}$$

where σ_n are (open) cells of K. We define

$$\dim \sigma = \sum_1^\infty \dim \sigma_n \leqq \infty. \tag{2}$$

Let $X_0(Y_0)$ denote the "2-skeleton" ("1-skeleton") of this decomposition of

(3) A deformation is said to be cell-preserving if, during the deformation, no point can leave the closure of the cell containing that point at $t=0$.

(4) See §1,2 and §1,3.

P_0. The "0-skeleton" consists of a single point $O=(o, o, \cdots)$. Denoting by L the 1-skeleton of K and by o^n the point $(o, o, \cdots, o)$ of the n-fold product $K\times\cdots\times K$ (o^0 meaning the "empty symbol"), we have

$$(3)\quad Y_0=\bigcup_{n=0}^{\infty} o^n\times L\times O=(L\times O)\cup(o\times L\times O)\cup(o\times o\times L\times O)\cup\cdots,$$

$$(4)\quad X_0=\left(\bigcup_{n=0}^{\infty} o^n\times K\times O\right)\cup\left(\bigcup_{n=0}^{\infty} o^n\times L\times Y_0\right).$$

Observing that[4] diam $(o^n\times K\times O)\leqq 2^{-n-1}$ and diam $(o^n\times L\times Y_0)\leqq 2^{-n}$ we conclude readily (by 1.1.1) that Y_0 and X_0 are compacta. Notice also that a point of $Y_0(X_0)$ can have at most one (two) coordinates different from o.

Although the described decomposition of P_0 is not a complex, we shall prove in this section

THEOREM 1. *The inclusion $X_0\subset P_0$ induces an isomorphism of $\pi_1(X_0)$ onto $\pi_1(P_0)$.*

2. DEFINITION 1. A loop $f: I\to P_0$ (based at O) is said to be a *standard loop* if $f((n-1)/n)=O$, for all $n=1, 2, \cdots$ and if $f(I_n)\subset o^{n-1}\times L\times O$ (recall that $I_n=[(n-1)/n, n/(n+1)]$).

LEMMA 1. *If f and g are standard loops, homotopic in P_0, then they are homotopic already in X_0.*

Proof. Let F be a homotopy in P_0 connecting f and g and let F_n, f_n and g_n be maps obtained from F, f and g respectively by composition with the natural projection $P_0=\prod K\to o^{n-1}\times K\times O$. F_n is obviously a homotopy connecting f_n and g_n. However, $f_n(x)=f(x)$, $g_n(x)=g(x)$, for $x\in I_n$, otherwise $f_n(x)=g_n(x)=O$, hence, the loops $f|I_n$ and $g|I_n$ are homotopic in $o^{n-1}\times K\times O\subset X_0$; let G^n be a connecting homotopy. Defining G by $G(x, t)=G^n(x, t)$, for $(x, t)\in I_n\times I$, $n=1, 2, \cdots$, and by $G(1, t)=O$, we obtain a homotopy in X_0 connecting f and g.

If f_n and g_n both lie in a subset of $o^{n-1}\times K\times O$, which is contractible to O (O fixed during contraction), then we can take for G^n a connecting homotopy contained in that subset. Using this remark we can prove

LEMMA 2. *For every $\epsilon>0$ there is a $\delta(\epsilon)>0$ such that any two standard loops f and g, homotopic in P_0 and lying in $U(O, \delta)$, can be connected by a homotopy in $X_0\cap U(O, \epsilon)$.*

Indeed, choose p so large that $2^{-p}<\epsilon$ and $0<\eta<\epsilon$ such that $U(o, \eta)$ is contractible to o in K. Let $\delta(\epsilon)=\eta 2^{-p}$. If $f, g\subset U(O, \delta)$, then $f_n, g_n\in o^{n-1}\times U(o, \eta)\times O$, for $n=1, \cdots, p$. Choose now G^n, $n=1, \cdots, p$, in $o^{n-1}\times U(o, \eta)\times O$ (no requirements on $G^{p+1}, \cdots$). Clearly, $G(x, t)\in U(O, \delta)\subset U(O, \epsilon)$, for $x\in I_1\cup\cdots\cup I_p$. For $x\in I_n$, $n>p$, we obtain $G(x, t)\in U(O, \epsilon)$

as a consequence of diam $o^{n-1}\times K\times O\leqq 2^{-n}$ and of the choice of p. Lemma 2 will be used in §3.

LEMMA 3. *Every loop f in P_0 (based at O) can be deformed (in P_0) into a standard loop.*

Proof. The nth coordinate f_n of f, being a loop in K, admits a deformation F_n (in K) into a loop g_n of the 1-skeleton L of K. One can easily achieve that $g_n(I\backslash I_n)=o$. $F=(F_1, F_2, \cdots)$ is then obviously a deformation of f into a standard loop g.

3. The main part of the proof of Theorem 1 is contained in the following

LEMMA 4. *Every loop f in X_0 (based at O) can be deformed, in X_0, into a standard loop.*

Proof. Observe first that cell-preserving deformations of coordinates f_n of f give a deformation of f in P_0 which actually takes place in the "2-skeleton" X_0 of P_0. Since the deformations occurring in the cell-approximation theorem are cell-preserving, we can assume that f_n are loops in the 1-skeleton L; and, consequently, that f is contained in the second summand of (4). Moreover, we can achieve (say, by simplicial approximations with respect to some simplicial subdivisions of L) that, for $n=1, 2, \cdots$, the open set $U_n=\{x|f_n(x)\neq o\}\subset I$ is the union of finitely many disjoint open intervals.

Given a point $a\in U_n$, it is clear that the particular open interval of U_n which contains a is mapped by f_n entirely into a 1-cell of L. Therefore, it is easy to define a cell-preserving deformation, affecting only that particular interval (without changing the total number of components of U_n) and yielding a new loop f_n with $f_n(a)=o$. In view of this remark we can assume from now on that for every $n=1, 2, \cdots$, $f_n(I)\subset L$, that U_n consists of a finite number of disjoint open intervals and that $f_r(\overline{U}_n\backslash U_n)=o$, for $r\geqq n$; a loop having the last two properties will be referred to as a "normal" loop.

Consider now the sets

$$S_p = Y_0\cup\left(\bigcup_{n=p-1}^{\infty} o^n\times L\times Y_0\right). \tag{5}$$

All S_p are compact (by 1.1.1) and $X_0\supset S_1\supset S_2\supset\cdots\supset\bigcap S_p=Y_0$. In view of the above remarks, $f\subset S_1$.

We shall now define, by induction, a sequence of loops $f=f^1, f^2, \cdots, f^p, \cdots$, with $f^p\subset S_p$, and a sequence of homotopies $F^{p\ p+1}: I\times I\to S_p$, connecting f^p and f^{p+1} and satisfying

$$\text{diam } F^{p\ p+1}\leqq 2^{-p+1}. \tag{6}$$

1.1.2 will then provide a limit loop $f=\lim f^p$, obviously contained in Y_0 and homotopic to f in $\bigcup_{p-1}^{\infty} S_p\subset X_0$.

Suppose that $f=f^1, \cdots, f^p$ and $F^{1\ 2}, \cdots, F^{p-1\ p}$ have already been de-

fined and satisfy the conditions of above; in order to carry through the induction, we assume in addition that $f^1, \cdots, f^p$ are "normal" loops. For $p=1$, these conditions are verified as established in the preceding remarks concerning f. Consider now $U_p^p = \{x \mid f_p^p(x) \neq o\}$, f_p^p denoting the pth coordinate of f^p. Since $f^p \subset S_p$, it follows immediately from (5) that

$$f^p(\overline{U}_p^p) \subset o^{p-1} \times L \times Y_0. \tag{7}$$

Now let (a, b) be one of the finitely many components of U_p^p. In order to define $F^{p\ p+1}$, choose a point c, $a<c<b$, and put

$$c_t = c + (1-t)(b-c),\ d_t = a + t(c-a), \qquad t \in I. \tag{8}$$

Furthermore, let $\alpha_t(x)$ be the transformation mapping $[a, c_t]$ linearly onto $[a, b]$ and sending $[c_t, b]$ into b; let β_t be the transformation mapping $[a, d_t]$ into a and mapping $[d_t, b]$ linearly onto $[a, b]$. $\alpha_t(x)$ and $\beta_t(x)$ are mappings of $[a, b] \times I$ into $[a, b]$, leaving end-points a and b fixed.

Define now $F^{p\ p+1}$: $[a, b] \times I \to o^{p-1} \times L \times Y_0 \subset S_p$ by

$$F^{p\ p+1}(x, t) = o^{p-1} \times f_p^p \alpha_t(x) \times (f_{p+1}^p \times f_{p+2}^p \times \cdots)\beta_t(x). \tag{9}$$

Clearly, $F^{p\ p+1}(x, 0) = f^p(x)$. As to $f^{p+1}(x) = F^{p\ p+1}(x, 1)$, observe first that $f_r^p(a) = f_r^p(b) = o$, for $r \geqq p$ (f^p is "normal"). It now follows, from (9), that

$$f^{p+1}(x) = o^{p-1} \times f_p^p \alpha_1(x) \times o \times o \times \cdots, \qquad \text{for } x \in [a, c], \tag{10}$$

$$f^{p+1}(x) = o^p \times (f_{p+1}^p \times f_{p+2}^p \times \cdots)\beta_1(x), \qquad \text{for } x \in [c, b], \tag{11}$$

showing that $f^{p+1}([a, b]) \subset Y_0 \subset S_{p+1}$.

We define $F^{p\ p+1}$ on other components of U_p^p in exactly the same way (they are in a finite number) and complete the definition by

$$F^{p\ p+1}(x, t) = f^p(x), \qquad \text{for } x \in I \setminus U_p^p. \tag{12}$$

$F^{p\ p+1}$ is continuous on $I \times I$, because (10), (11) and (12) give $F^{p\ p+1}(a, t) = F^{p\ p+1}(b, t) = O$. Moreover, for $x \in I \setminus U_p^p$, $f_p^p(x) = o$ and thus $f^{p+1}(x) = f^p(x)$ belongs actually to $S_{p+1} \subset S_p$. (6) follows from diam $o^{p-1} \times L \times Y_0 \leqq 2^{-p+1}$. Finally, it is readily checked that f^{p+1} is "normal." This completes the argument showing that every loop of X_0 (based on O) can be deformed, in X_0, into a loop of Y_0.

To complete the proof of Lemma 4, we have to show now that every loop f of Y_0 (based at O) can be deformed, in X_0, into a standard loop. For that purpose we shall define by induction a sequence of loops $f = f^0, f^1, \cdots, f^p, \cdots$ in Y_0 and a sequence of homotopies $F^{p\ p+1}$: $I \times I \to X_0$, connecting f^p and f^{p+1} and having diam $F^{p\ p+1} \leqq 2^{-p}$. For $p>0$, we require in addition

$$f^p(I_q) \subset o^{q-1} \times L \times O, \qquad q \leqq p, \tag{13}$$

$$f^p\left(\frac{q}{q+1}\right) = o, \qquad q \leqq p, \tag{14}$$

$$f^p\left(\left[\frac{p}{p+1}, 1\right]\right) \subset o^p \times Y_0. \tag{15}$$

Once such a sequence is defined, 1.1.2 will yield a limit loop $f = \lim f^p$, homotopic to f in X_0, and actually a standard loop (due to (13) and (14)).

Assume that $f^1, \cdots, f^p$ and $F^{0\ 1}, \cdots, F^{p-1\ p}$ have already been defined in accordance with the above requirements. Denote $p/(p+1)$, $(p+1)/(p+2)$ and 1 by a, c and b respectively and let c_t, d_t, $\alpha_t(x)$ and $\beta_t(x)$, for $x \in [a, b]$, be defined as in the preceding argument; moreover, let $\alpha_t(x) = \beta_t(x) = x$, for $x \in I \backslash (a, b)$. We define $F^{p\ p+1}$ by

$$F^{p\ p+1}(x, t) = f^p(x), \qquad \text{for } x \in \left[0, \frac{p}{p+1}\right], \qquad t \in I, \tag{16}$$

$$F^{p\ p+1}(x, t) = o^p \times f^p_{p+1}\alpha_t(x) \times (f^p_{p+2} \times f^p_{p+3} \times \cdots)\beta_t(x), \qquad \text{for } x \in \left[\frac{p}{p+1}, 1\right], \qquad t \in I, \tag{17}$$

$$f^{p+1}(x) = F^{p\ p+1}(x, 1). \tag{18}$$

All the required properties are readily checked (notice that $f^p\beta_t(x) \in Y_0$ implies $(f^p_{p+2} \times f^p_{p+3} \times \cdots)\beta_t(x) \in Y_0$).

4. The following lemma will be needed in §3.

LEMMA 5. *For every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that every loop f, lying in $U(O, \delta) \subset P_0$ (in $X_0 \cap U(O, \delta)$), can be deformed into a standard loop by a deformation lying in $U(O, \epsilon)$ (in $X_0 \cap U(O, \epsilon)$).*

Proof. Choose p, η and δ as in the proof of Lemma 2 (with the additional requirement that $U(o, \eta)$ admits a cell-preserving contraction to o). If $f \subset U(O, \delta)$, then $f_1, \cdots, f_p \subset U(o, \eta) \subset K$. Composing these coordinates with a (cell-preserving) contraction of $U(o, \eta)$ to o, while leaving $f_{p+1}, \cdots$ unchanged, one obtains a deformation F of f, in P_0 (in X_0), into a loop $g \subset o^p \times P_0 (g \subset o^p \times X_0)$. Since $\rho(F(x, t), F(x, 0)) < \eta + \cdots + \eta 2^{-p+1} = 2\eta(1 - 2^{-p})$ and $F(x, 0) = f(x) \in U(O, \delta)$, it follows that $\rho(O, F(x, t)) < 2\eta < 2\epsilon$. Consequently, $F \subset U(O, 2\epsilon)$. Applying now Lemma 3 (Lemma 4) to g and $o^p \times P_0 (o^p \times X_0)$ we deform g further into a standard loop by a deformation of diameter lesser than diam $(o^p \times P_0) \leqq 2^{-p} < \epsilon$. The total deformation is thus contained in $U(O, 2\epsilon)$.

5. **Proof of Theorem** 1. Lemma 3 proves that the homomorphism $i: \pi_1(X_0) \to \pi_1(P_0)$, induced by $X_0 \subset P_0$, is an epimorphism. Combining Lemmas 4 and 1, we conclude that i is a monomorphism (the constant loop $g(x) = O$ is a standard loop).

REMARK. Theorem 1 holds also in the case of an infinite product of different complexes $K_1, K_2, \cdots$ (5).

3. Continuous curve $\overline{X}$ and its fundamental group.

1. *Description of the basic construction.* Let K be a finite cell complex having one single vertex o and at least one 1-cell. Choose a sequence of finite (nonempty) disjoint subsets $A_1, \cdots, A_k, \cdots$ of the 1-skeleton L of K in such a way that $o \in A_1$ and that

$$A = \bigcup_{k=1}^{\infty} A_k \tag{1}$$

is dense in L; these sets will be considered as fixed throughout this section. We define next, by induction on n, a finite subset B_n of the n-fold product $K \times \cdots \times K$, by

$$B_n = \bigcup_{k=1}^{n} B_{n-k} \times A_k \times o^{k-1}. \tag{2}$$

B_0, as well as A_0, o^0 and o^{-1} are considered to represent "empty symbols"; e.g. $B_1 = A_1$, $B_2 = A_1 \times A_1 \cup A_2 \times o$. Notice that $o^n \in B_n$, for all $n \geqq 1$. Let X_0 and Y_0 be as in §2. Consider the following subsets of P_0

$$X = \bigcup_{n=0}^{\infty} B_n \times X_0 \text{ and} \tag{3}$$

$$Y = \bigcup_{n=0}^{\infty} B_n \times Y_0. \tag{4}$$

Let $\overline{Y}$ and $\overline{X}$ be the closures of Y and X taken with respect to P_0.

In this section, and the following section, we are concerned with a proof of the basic

THEOREM 2. *$\overline{X}$ and $\overline{Y}$ are continuous curves*(6) *with* $\dim \overline{X} = 2$, $\dim \overline{Y} = 1$. *$\overline{Y} \subset \overline{X}$ and points of $\overline{X} \setminus \overline{Y}$ have 2-dimensional Euclidean neighborhoods (with respect to $\overline{X}$). The inclusion $\overline{X} \subset P_0$ induces an isomorphism $\pi_1(\overline{X}) \approx \pi_1(P_0)$. Every $x \in \overline{Y}$ has a basis of connected (open) neighborhoods (with respect to $\overline{X}$) $U(x)$, such that $U(x) \subset \overline{X}$ induces a monomorphism of $\pi_1(U)$ into $\pi_1(\overline{X})$ with an image isomorphic to $\pi_1(P_0)$.*

2. For purposes of proof we introduce certain subsets of P_0 approximating $\overline{X}$ and $\overline{Y}$. Let

$$X_1 = \bigcup_{k=0}^{\infty} A_k \times o^{k-1} \times X_0, \qquad Y_1 = \bigcup_{k=0}^{\infty} A_k \times o^{k-1} \times Y_0. \tag{5}$$

(5) It seems likely that the restriction to complexes having a single vertex (imposed in view of applications in forthcoming sections) should not be essential for the validity of Theorem 1.

(6) I.e. metrizable compact connected and locally connected spaces.

X_1 and Y_1 are compact (1.1.1) and connected. The same is true for

$$X_{p+1} = X_p \cup B_p \times X_1 = \bigcup_0^p B_n \times X_1 \tag{6}$$

and

$$Y_{p+1} = Y_p \cup B_p \times X_1 = \bigcup_0^p B_n \times Y_1, \qquad p = 1, 2, \cdots. \tag{7}$$

Denote by $B_{n\ p}$ the union of the last p terms in the expression (2), $p \leqq n$,

$$B_{n\ p} = \bigcup_{k=n-p+1}^{n} B_{n-k} \times A_k \times o^{k-1}. \tag{8}$$

Notice that $B_{n\ n} = B_n$. One obtains new expressions for X_p and X

$$X_p = \left(\bigcup_{n=0}^{p-1} B_n \times X_0\right) \cup \left(\bigcup_{n=p}^{\infty} B_{n\ p} \times X_0\right), \quad p = 1, 2, \cdots, \tag{9}$$

$$X = \bigcup_0^\infty X_p. \tag{10}$$

Analogous formulae hold for Y_p and Y. Notice that $X_p \subset X_{p+1}$, $Y_p \subset Y_{p+1}$, $Y_{p+1} \subset X_p$. We conclude from (10) that connectedness of X_n and Y_n implies that of X and Y as well as $\overline{X}$ and $\overline{Y}$.

In order to obtain suitable approximations of $\overline{X}$ and $\overline{Y}$ "from outside" we introduce

$$P_p = \left(\bigcup_{n=0}^{p-1} B_n \times X_0\right) \cup \left(\bigcup_{n=p}^{\infty} B_{n\ p} \times P_0\right) \tag{11}$$

and

$$Q_p = \left(\bigcup_{n=0}^{p-1} B_n \times Y_0\right) \cup \left(\bigcup_{n=p}^{\infty} B_{n\ p} \times P_0\right). \tag{12}$$

Notice that

$$X_p \subset P_p, \qquad Y_p \subset Q_p \subset P_p. \tag{13}$$

In order to prove

$$P_{p+1} \subset P_p, \qquad Q_{p+1} \subset Q_p, \tag{14}$$

it suffices to show that $B_{n\ p+1} \times P_0 \subset (B_p \times P_0) \cup (B_{n\ p} \times P_0)$, $n \geqq p+1$. All but the first term of $B_{n\ p+1} \times P_0$ are contained in $B_{n\ p} \times P_0$; however, this term is $B_p \times A_{n-p} \times o^{n-p-1} \times P_0 = B_p \times B_{n-p\ 1} \times P_0 \subset B_p \times P_0$.

A consequence of (13) and (14) is

$$X_p \subset P_q, \qquad Y_p \subset Q_q, \tag{15}$$

for arbitrary p, q.

Observe now that $O \in X_0$ and $B_{n\ p} \times O, \subset B_{n\ p} \times X_0 \subset X_p$, $n \geqq p$; therefore, diam $(b \times P_0) \leqq 2^{-n}$, $b \in B_{n\ p}$, implies $P_p \subset U(X_p, 2^{-p+1}) \subset U(\overline{X}, 2^{-p+1})$. Firstly, we conclude (1.1.1) that P_p is compact because X_p is compact. Secondly, since $\overline{X} \subset \cap P_p$ (by (15)),

$$\overline{X} = \bigcap_1^\infty P_p. \tag{16}$$

Analogous arguments show that Q_p is compact and

$$\overline{Y} = \bigcap_1^\infty Q_p. \tag{17}$$

3. We list here several simple propositions needed in the sequel.

3.1. $x = (x_1, \cdots, x_n) \in B_n$ implies $x_k \in A_1 \cup \cdots \cup A_{n-k+1}$, $k = 1, \cdots, n$. Proof immediate by induction on n.

3.2. $x = (x_1, \cdots, x_n) \in B_n$ and $x_q \neq o$, $2 \leqq q \leqq n$, implies $(x_1, \cdots, x_{q-1}) \in B_{q-1}$.

Proof of induction on $n \geqq q$ (q fixed). x can not belong to the last $q-1$ terms of (2) because the qth coordinate would be o. Hence, $x \in B_{n-k} \times A_k \times o^{k-1}$, $k \in \{1, \cdots, n-q+1\}$. If $n-k < q$, then actually $n-k = q-1$ (otherwise we would have $x_q = o$). However, in this case $x \in B_{q-1} \times A_{n-q+1} \times o^{n-q}$ and $(x_1, \cdots, x_{q-1}) \in B_{q-1}$. In the remaining cases $q \leqq n-k$ and $(x_1, \cdots, x_q, \cdots, x_{n-k}) \in B_{n-k}$ so that the hypothesis of induction is applicable.

3.3. For arbitrary q, n, $B_q \times B_n \subset B_{q+n}$. Proof by induction on n. Substitute (2) for B_n, apply the inductive hypothesis and notice that the resulting expression gives the first n terms of (2) for B_{q+n}.

3.4. If $\varnothing$ denotes the empty set, then $(B_p \times P_0) \cap (B_{n\ p} \times P_0) = \varnothing$, for $n > p$, and $(B_{n\ p} \times P_0) \cap (B_{m\ p} \times P_0) = \varnothing$, for $n > m \geqq p$.

It suffices to prove the first assertion, because of $B_{n\ p} \subset B_{n\ m}$, $B_{m\ p} \subset B_m$. Assume that $x \in B_{n\ p} \times P_0$; there exists then an $s \in \{n-p+1, \cdots, n\}$ (by (8)) such that $x \in B_{n-s} \times A_s \times o^{s-1} \times P_0$, hence $x_{n-s+1} \in A_s$, $n-s+1 \leqq p$. If at the same time $x \in B_p \times P_0$, then 3.1 would imply $x_{n-s+1} \in A_1 \cup \cdots \cup A_{s-(n-p)}$. However, this set is disjoint with A_s (because of $n > p$ and the definition of sets A_k), which presents a contradiction.

3.5. If $q > p$, we have

$$\left(\bigcup_{n=0}^{p-1} B_n \times Y_0\right) \cap (B_{q\ p} \times P_0) = B_{q\ p} \times O. \tag{18}$$

Indeed, if $b \in B_{q\ p}$, it follows immediately (by (8)), that $b \times O \in B_n \times Y_0$, for an $n \in \{0, \cdots, p-1\}$. On the other hand, for $n \leqq p-1$, $(B_n \times Y_0) \cap (B_{q\ n} \times P_0)$

$=\emptyset$ by 3.4, so that $x\in(B_n\times Y_0)\cap(B_{q\ p}\times P_0)$ implies $x\in B_{q-k}\times A_k\times o^{k-1}\times P_0$, with $2\leqq q-p+1\leqq k\leqq q-n$. Since $o\in A_1$ and $x_{q-k+1}\in A_k$ $(A_k\cap A_1=\emptyset)$, we have $x_{q-k+1}\neq o$, showing that at least one of the coordinates $x_{n+1},\cdots,x_p$ is $\neq o$. However, $x\in B_n\times Y_0$ implies $(x_{n+1},\cdots,x_p,x_{p+1},\cdots)\in Y_0$ and thus $(x_{p+1},\cdots)=O$ (see 2.1); a fortiori $(x_{q+1},\cdots)=O$.

3.6.

$$Y_{p+1}\cap(B_p\times P_0)=B_p\times Y_1, \tag{19}$$

$$Q_{p+1}\cap(B_p\times P_0)=B_p\times Q_1,\qquad P_{p+1}\cap(B_p\times P_0)=B_p\times P_1. \tag{20}$$

Notice first that $x\in Y_1$ implies $(x_2,x_3,\cdots)\in Y_0\subset Y_1$. Therefore, $x\in Y_{p+1}$ implies $(x_{p+1},\cdots)\in Y_1$ (see (7)); this proves (19). In order to prove the first relation in (20) (proof of the second relation is analogous), notice first that, for $n\geqq p+1$, $B_{n\ p+1}\times P_0=(B_{n\ p}\times P_0)\cup(B_p\times A_{n-p}\times o^{n-p-1}\times P_0)$. Using 3.4, we conclude that $x\in(B_{n\ p+1}\times P_0)\cap(B_p\times P_0)$ implies $x\in B_p\times A_{n-p}\times o^{n-p-1}\times P_0\subset B_p\times Q_1$. If on the other hand $x\in(B_n\times Y_0)\cap(B_p\times P_0)$, $n\leqq p$, then $(x_{p+1},\cdots)\in Y_0$ and thus $x\in B_p\times Y_0\subset B_p\times Q_1$; this proves $\subset$ in (20). The other inclusion follows from the fact that, for $n\geqq 1$, $B_p\times B_{n\ 1}\times P_0=B_p\times A_n\times o^{n-1}\times P_0$ is the first term of $B_{p+n\ p+1}\times P_0\subset Q_{p+1}$.

3.7.

$$Y_{p+1}=(Y_p\backslash(B_p\times P_0))\cup(B_p\times Y_1), \tag{21}$$

$$\begin{aligned} Q_{p+1}&=(Q_p\backslash(B_p\times P_0))\cup(B_p\times Q_1),\\ P_{p+1}&=(P_p\backslash(B_p\times P_0))\cup(B_p\times P_1). \end{aligned} \tag{22}$$

(21) is an immediate consequence of (7) and (19). To prove the first relation of (22) (the second is proved analogously) notice that the first summand in (12) is also contained in the expansion for Q_{p+1}. Furthermore, for $n\geqq p+1$, $B_{n\ p}\times P_0\subset B_{n\ p+1}\times P_0\subset Q_{p+1}$. Since the only remaining term in (12) is $B_p\times P_0$, we conclude that $Q_p\backslash(B_p\times P_0)\subset Q_{p+1}$. This and (20) prove $\supset$ in (22). The other inclusion follows from (14) and (20).

3.8. The following sets (23) and (24) are compact, $q\geqq p$,

$$(Q_p\backslash(B_{q\ p}\times P_0))\cup(B_{q\ p}\times O), \tag{23}$$

$$(Y_p\backslash(B_{q\ p}\times P_0))\cup(B_{q\ p}\times O). \tag{24}$$

It suffices to prove that (23) is compact, the assertion for (24) will then follow (using the fact that Y_p is compact and $Y_p\subset Q_p$).

Given a sequence $x^1,\cdots,x^k,\cdots$ of points of $(Q_p\backslash(B_{q\ p}\times P_0))$ we can assume that it converges towards a limit $x\in Q_p$ (because Q_p is compact); we have to show that x belongs to the set (23). This is certainly the case if x is not in $B_{q\ p}\times P_0$. Assume therefore that $x\in Q_p\cap(B_{q\ p}\times P_0)$. If $x^k\in b\times P_0$, $b\in B_{m\ p}$, $m\geqq p, m\neq q$, replace x^k (in the sequence) by $y^k=b\times O\in\bigcup_{n=0}^{p-1}(B_n\times Y_0)$ (see (8)); notice also that $b\times O\in B_{m\ p}\times P_0$ and thus does not belong to $B_{q\ p}\times P_0$ (see 3.4). There can only be finitely many terms x^k in a given

$B_{m\ p}\times P_0$, $m\geqq p$, $m\neq q$, otherwise we would have $x\in B_{m\ p}\times P_0$ contradicting the assumption $x\in B_{q\ p}\times P_0$ (see 3.4). Since $\rho(x^k,\ b\times O)\leqq 2^{-m}$, the new sequence y^k, obtained from x^k in the described way, converges to the same x and is contained in $(\bigcup_{n=0}^{p-1}(B_n\times Y_0))\backslash(B_{q\ p}\times P_0)$; the first term of this expression being compact, we get

$$x\in\left(\bigcup_{n=0}^{p-1}(B_n\times Y_0)\right)\cap(B_{q\ p}\times P_0),\qquad q\geqq p. \tag{25}$$

If $q>p$, our assertion follows immediately from (25) and (18). In the case $q=p$, we have to prove that $x_{p+1}=x_{p+2}=\cdots=o$. Suppose on the contrary that there is an $r\geqq 1$ with $x_{p+r}\neq o$. Let k be so large that $y^k_{p+r}\neq o$, too. Since $y^k\in B_n\times Y_0$, for some $0\leqq n\leqq p-1$, it follows that $(y^k_{n+1},\cdots,y^k_{p+r},\cdots)\in Y_0$ and thus $y^k_{n+1}=\cdots=y^k_{p+r-1}=o$. Hence, $y^k\in B_n\times o^{p-n}\times o^r\times P_0\subset B_p\times P_0$, contradicting the fact that y^k does not belong to $B_p\times P_0$.

3.9.

$$(B_p\times P_0)\cap\overline{Y}=B_p\times\overline{Y}. \tag{26}$$

Let $x\in(B_p\times P_0)\cap\overline{Y}$. Since $\overline{Y}\subset Q_{p+1}$ we conclude (from (20)) that $x\in B_p\times Q_1$. Hence, x is either in $B_p\times Y_0\subset B_p\times\overline{Y}$ or in $(B_p\times B_{n_1 1}\times P_0)\cap\overline{Y}$, for an $n_1\geqq 1$. Since also $\overline{Y}\subset Q_{p+n_1+1}$, we see that, in the second case, $x\in B_p\times B_{n_1 1}\times Q_1$ (notice that, by 3.3, $B_p\times B_{n_1 1}\subset B_p\times B_{n_1}\subset B_{p+n_1}$) and thus either $x\in B_p\times B_{n_1 1}\times Y_0\subset B_p\times\overline{Y}$ or $x\in(B_p\times B_{n_1 1}\times B_{n_2 1}\times P_0)\cap\overline{Y}$, for an $n_2\geqq 1$. Continuing this argument we conclude that either $x\in B_p\cap\overline{Y}$ or there is a sequence $n_1, n_2,\cdots\geqq 1$, such that $(x_{p+1},\cdots,x_{n_k})\in B_{n_1+\cdots+n_k}$. However, in this last case, points $(x_{p+1},\cdots,x_{n_k},o,o,\cdots)\in B_{n_1+\cdots+n_k}\times O\in\overline{Y}$ converge to $(x_{p+1},\cdots)$, proving again that $x\in B_p\times\overline{Y}$. In order to prove the other inclusion in (26) it suffices to observe that $B_p\times Y\subset(B_p\times P_0)\cap Y$ is an immediate consequence of (4) and 3.3.

3.10.

$$\overline{Y}=\left(\bigcup_{n=0}^{p-1}B_n\times Y_0\right)\cup\left(\bigcup_{n=p}^{\infty}B_{n\ p}\times\overline{Y}\right),\qquad \text{for } p\geqq 1. \tag{27}$$

Immediate consequence of 3.9 and the fact that $\overline{Y}=\overline{Y}\cap Q_p$.

4. Lemma 6. *Every loop f in $\overline{Y}$ (based at O) can be deformed in $\overline{X}$ into a standard loop (contained in $Y_0\subset X_0$).*

4.1. According to (17), f can be considered as a loop of Q_p, for every $p=0, 1, 2,\cdots$ $(Q_0=P_0)$. We shall define now deformations F^p of f (in Q_p) such that

(i)$_p$ $f(x)\in b\times P_0$, $b\in B_{n\ p}$, $n\geqq p\geqq 1$, implies $F^p(x,\ t)\in b\times P_0$ and $f^p(x)=F^p(x,\ 1)\in b\times Y_0\subset Y_p$,

(ii)$_p$ $f(x)\in Q_p\backslash(\bigcup_{n\geqq p}B_{n\ p}\times P_0)$, $p\geqq 1$, implies $F^p(x,\ t)=f(x)\in Y_p$, requiring in addition that f^0 be standard. (i)$_p$ and (ii)$_p$ imply diam $F^p\leqq 2^{-p}$ and thus $\lim f^p=f$. The next step will consist in defining homotopies $G^{p\ p+1}$, connecting

f^p and f^{p+1}, in X_p, and satisfying diam $G^{p\ p+1} \leqq 2^{-p}$. An application of 1.1.2 will then prove that f is homotopic, in $\overline{X}$, to $f^0 \subset Y_0$.

4.2. F^0 exists by Lemma 3. For $p \geqq 1$, let $R_p = (Q_p \backslash (B_p \times P_0)) \cup (B_p \times O)$ and let $b \in B_p$. Obviously $Q_p = [R_p \cup ((B_p \backslash \{b\}) \times P_0)] \cup (b \times P_0)$; both summands are compact (see 3.8) and their intersection is the single point $b \times O$. Since $f \subset Q_p$, the set $U = \{x \mid f(x) \in (b \times P_0) \backslash \{b \times O\}\} \subset I$ is open and $f(\overline{U} \backslash U) = b \times O$. If V is any one of the components of U, then $\overline{V} \backslash V \subset \overline{U} \backslash U$, so that $f \mid \overline{V}$ is a loop in $b \times P_0$, based at $b \times O$. We can apply now Lemma 5 (the part concerning P_0) to obtain homotopies deforming loops $f \mid \overline{V}$ into loops of $b \times Y_0$ in such a way that 1.1.4 is applicable and produces a deformation of f, defined over the entire interval I. Repeating the process with all b of the finite set B_p, we arrive at a deformation, satisfying $(\mathrm{i})_p$, for $n = p$, and having the following property ("approximating" property $(\mathrm{ii})_p$): for $f(x) \in Q_p \backslash (B_p \times P_0)$, the deformation equals $f(x)$.

4.3. Now repeat the process described in 4.2, this time applied to the loop we obtained in 4.2 and to all $b \in B_{p+1\ p}$ (we consider $R_{p+1\ p} = (Q_p \backslash (B_{p+1\ p} \times P_0)) \cup (B_{p+1\ p} \times O)$ and the decomposition $Q_p = [R_{p+1\ p} \cup ((B_{p+1\ p} \backslash \{b\}) \times P_0)] \cup (b \times P_0)$). The resulting deformation affects only the set $B_{p+1\ p} \times P_0$ (disjoint to $B_p \times P_0$) and does not interfere with the gain (in the direction of obtaining $(\mathrm{i})_p$ and $(\mathrm{ii})_p$) achieved in the preceding step. Defining in this manner a sequence of deformations and passing finally to the limit (1.1.2), one arrives at a deformation F^p, satisfying $(\mathrm{i})_p$ and $(\mathrm{ii})_p$ (1.1.2 is applicable because the diameter of the deformation in the step involving $B_{n\ p} \times P_0$ is $\leqq 2^{-n}$).

4.4. We proceed now to define $G^{p\ p+1}$. Consider again $b \in B_p$ and the sets R_p and U, defined as above. Points of $\overline{U} \backslash U$ can be approached arbitrarily close from U as well as from $I \backslash U$. Since F^p maps U in $b \times P_0$ and $I \backslash U$ in $(Q_p \backslash (b \times P_0)) \cup (b \times O)$ (due to $(\mathrm{i})_p$ and $(\mathrm{ii})_p$), and these two sets are compact (see 3.8), we conclude that $F^p((\overline{U} \backslash U) \times I)$ is contained in their intersection, i.e.

$$F^p((\overline{U} \backslash U) \times I) = b \times O, \qquad b \in B_p. \tag{28}$$

In a similar way, using $(\mathrm{i})_{p+1}$ and $(\mathrm{ii})_{p+1}$, one can see that $F^{p+1}(U \times I) \subset b \times P_0$ and $F^{p+1}((I \backslash U) \times I) \subset (Q_p \backslash (b \times P_0)) \cup (b \times O)$ and therefore

$$F^{p+1}((\overline{U} \backslash U) \times I) = b \times O, \qquad b \in B_p. \tag{29}$$

Now let V be any one of the components of U. Then $F^p \mid \overline{V} \times I$ and $F^{p+1} \mid \overline{V} \times I$ are homotopies in $b \times P_0$, connecting the loop $f \mid \overline{V}$ with the loops $f^p \mid \overline{V}$ and $f^{p+1} \mid \overline{V}$ respectively; these loops are therefore homotopic in $b \times P_0$. Moreover, $f^p(\overline{V}) \subset b \times Y_0 \subset b \times X_0$, by $(\mathrm{i})_p$, while $(\mathrm{i})_{p+1}$ and $(\mathrm{ii})_{p+1}$ imply $f^{p+1}(\overline{V}) \subset Y_{p+1}$. Applying (19) we conclude that actually $f^{p+1}(\overline{V}) \subset b \times Y_1 \subset b \times X_0$. It follows (Theorem 1) that $f^p \mid \overline{V}$ and $f^{p+1} \mid \overline{V}$ are homotopic already in $b \times X_0$.

Notice now that f^p and f^{p+1} are uniformly continuous on I and therefore,

for every $\epsilon>0$, there is a $\delta>0$ such that diam $V\leqq\delta$ implies $f^p\subset(b\times X_0)\cap U(b\times O, \epsilon)$ and $f^{p+1}\subset(b\times X_0)\cap U(b\times O, \epsilon)$. Now take into account Lemma 5 (the part concerning X_0) and Lemma 2. It is clear that we can define homotopies F_V, connecting $f^p|\overline{V}$ and $f^{p+1}|\overline{V}$ in $b\times X_0$, for every V, in such a way that 1.1.4 is applicable (with $M=Y_p\cup(b\times X_0)$, base point $b\times O$, open set U, mapping $f^p: I\rightarrow M$ and homotopies F_V), producing a homotopy in $Y_p\cup(b\times X_0)\subset X_p$, defined over $I\times I$. Repeating the whole construction for every $b\in B_p$, we arrive at a homotopy contained in $Y_p\cup(B_p\times X_0)\subset X_p$ and equal to $f^p(x)$ on $\{x|f(x)\in Q_p\setminus(B_p\times P_0)\}$; f^p is deformed by this homotopy into a map which coincides with f^{p+1} on $\{x|f(x)\in(B_p\times P_0)\}$.

4.5. Repeat now the process described in 4.4 with all $b\in B_{p+1\ p}$ (R_p has to be replaced by $R_{p+1\ p}$, $U=\{x|f(x)\in(b\times P_0)\setminus\{b\times O\}\}$) and apply 1.1.4 to the loop obtained from f^p as the result of the deformation described in 4.4. Continue this process for $B_{p+2\ p}, \cdots$. The step involving $B_{k\ p}$, $k\geqq p$, affects only the set $\{x|f(x)\in(B_{k\ p}\times P_0)\}$ and has a diameter $\leqq 2^{-k}$; the resulting loop coincides with $f^{p+1}(x)$ on $\{x|f(x)\in\bigcup_{n=p}^{k}(B_{n\ p}\times P_0)\}$. Applying 1.1.2 (and $(\mathrm{i})_p$, $(\mathrm{ii})_p$, $(\mathrm{i})_{p+1}$, $(\mathrm{ii})_{p+1}$) we conclude, finally, that there is a homotopy $G^{p\ p+1}$ contained in X_p and connecting f^p with f^{p+1}; if $f(x)\in b\times P_0$, $b\in B_{n\ p}\times P_0$, $n\geqq p$, then $G^{p\ p+1}(x, t)\in b\times X_0\subset X_p$, otherwise $G^{p\ p+1}(x, t)=f^p(x)=f^{p+1}(x)$. Consequently, diam $G^{p\ p+1}\leqq 2^{-p}$, so that 1.1.2 is applicable.

Notice that the deformation G^p, that one obtains applying 1.1.2 to the sequence $G^{p\ p+1}$, $G^{p+1\ p+2}, \cdots$, has some special properties that we state here (for future usage):

LEMMA 7. *Given any loop f in $\overline{Y}$ (based at O) and any integer $p\geqq 0$, there is a loop $f^p\subset Y_p$ and a homotopy $G^p\subset\overline{X}$, connecting f^p and f, and having the property that, for $f(x)\in b\times P_0$, $b\in B_{n\ p}$, $n\geqq p$, we have $G^p(x, t)\in b\times P_0$, while otherwise $G^p(x, t)=f^p(x)=f(x)$.*

5. If a sequence of (Euclidean) cells in a metric space has the property that the diameters of the cells tend to zero, we shall speak of a 0-sequence of cells.

LEMMA 8. *$\overline{X}$ can be obtained from $\overline{Y}$ by attaching a 0-sequence of disjoint 2-dimensional cells.*

We precede the proof by some consequences.

LEMMA 9. *Every loop f in $\overline{X}$ (based at O) can be deformed in $\overline{X}$ into a loop of $\overline{Y}$.*

A proof follows from Lemma 8 and Propositions 1.1.3 and 1.1.2.

THEOREM 3. *$\overline{X}$ is an arcwise connected subset of P_0. The inclusions $X_0\subset\overline{X}\subset P_0$ induce isomorphisms of the corresponding fundamental groups.*

Proof follows from Lemmas 9 and 6 and Theorem 1.

Proof of Lemma 8. If σ is a 2-cell (open) of K, then $\sigma\times O$ is a 2-cell imbedded in P_0 and contained in $K\times O\subset X_0$. Let L_n be the subdivision of L obtained by considering all points of $A_1\cup\cdots\cup A_n$ as vertexes of L_n, $n\geqq 1$. If σ is a 1-cell (open) of L_n and τ a 1-cell (open) of L, then $\sigma\times o^{n-1}\times\tau\times O$ is a 2-cell imbedded in P_0 and contained in $L\times o^{n-1}\times L\times O\subset X_0$. The described 2-cells will be referred to in the sequel as 2-cells of the first and of the second kind respectively. It is not difficult to see that these cells are disjoint one from each other and from Q_1, while their boundaries lie in $Y_1\subset Q_1$, e.g. in the case of cells of the second kind, the boundary is lying in $\bigcup_{k\leqq n}(A_k\times o^{n-1}\times L\times O)\cup(L\times O)\subset Y_1$. Moreover, it is easy to see that all the described 2-cells can be ordered in a sequence $e_1, e_2, \cdots$ with lim diam $e_n=0$. (Observe that the set A from §3, (1) is dense in L and that there are only finitely many cells of the first kind.) Finally,

$$P_1 = Q_1\cup\bigcup_{n=1}^{\infty} e_n, \tag{30}$$

showing that P_1 is obtained from Q_1 by attaching the described 0-sequence of cells. We prove next

$$P_{p+1} = Q_{p+1}\cup(\cup e_n)\cup(\cup B_1\times e_n)\cup\cdots\cup(\cup B_p\times e_n). \tag{31}$$

The inclusion $\supset$ is immediate because of $e_n\subset X_0$. The inclusion $\subset$ can be proved by induction on p, using (30) and both relations in (22). Finally,

$$\overline{X} = \overline{Y}\cup(\cup e_n)\cup(\cup B_1\times e_n)\cup\cdots\cup(\cup B_p\times e_n)\cup\cdots. \tag{32}$$

Recall the relations (16) and (17). If $x\in\overline{X}\backslash\overline{Y}$, let $p+1$ be the smallest integer such that x does not belong to Q_{p+1}. Since $x\in\overline{X}\subset P_{p+1}$, it follows from (31) that x belongs to the set on the right side of (32). The other inclusion is obvious, since $e_n\subset X_0$.

Observe now that $Q_1\cap e_n=\emptyset$ implies (by (20)) that $\overline{Y}\cap(B_p\times e_n)\subset Q_{p+1}\cap(B_p\times e_n)=\emptyset$. It implies also $(B_p\times e_n)\cap(B_q\times e_m)=\emptyset$, for $p>q$. Indeed, if $x\in(B_q\times e_m)$, then $(x_{q+1},\cdots)\in e_m$ and thus obviously $(x_{q+2},\cdots)\in Y_0$. Furthermore, $p>q$ implies $(x_{p+1},\cdots)\in Y_0\subset Q_1$, while $x\in(B_p\times e_n)$ would imply $(x_{p+1},\cdots)\in e_n$. The boundary of e_n lies in Y_1, therefore, the boundary of $B_p\times e_n$ lies in $B_p\times Y_1\subset Y_p\subset\overline{Y}$. Finally, since diam e_n tends towards zero, the cells appearing in (32) can be ordered into a 0-sequence.

Notice that Lemma 8 proves also that points of $\overline{X}\backslash\overline{Y}$ have Euclidean 2-neighborhoods.

4. Local properties of $\overline{X}$. 4.1. We shall now consider particular open sets of $\overline{X}$, referred to in the sequel as standard open sets. A standard open set of $\overline{X}$ is the intersection of $\overline{X}$ and an open set U of P_0 of the form $U=U_1\times\cdots\times U_q\times P_0$, where U_n are open in K, provided that one can find a point $b\times O$, $b\in B_p$, $p\leqq q$, contained in U. Moreover, if b_n denotes the nth coordinate of $b\times O$, $U_n\cap K$ should admit a cell-preserving (with respect to K) deforma-

tion retraction to $U_n \cap L$ and $U_n \cap L$ should be contractible to b_n; for $n<p$, this contraction should be cell-preserving with respect to the subdivision L_{p-n}. Notice that these requirements imply that, for $b_n \neq o$, U_n can not contain o and that, for $n<p$, U_n can not contain points of $A_1 \cup \cdots \cup A_{p-n}$ except b_n, which may belong to that set.

4.2. LEMMA 10. *Standard open sets of $\overline{X}$ form a basis of neighborhoods at every point x belonging to $\overline{Y}$.*

If $x \in \overline{Y}$ and W is an open set of P_0, $x \in W$, we have to find a standard open set $U \cap \overline{X}$ such that $x \in U \subset W$. Clearly, we can find $V = V_1 \times \cdots \times V_q \times P_0$, $x \in V \subset W$, such that V_n is open and admits a cell-preserving deformation retraction of $V_n \cap K$ to $V_n \cap L$ and a contraction of $V_n \cap L$ into x_n, $n \leqq q$ (see 1.2).

Assume now first that $x \in Y$ or more precisely that $x \in B_p \times o^m \times L \times O$ (see § 3, (4) and §2, (3)). We can also assume that $q \geqq p+m+1$. Let r be such an integer that $(A_1 \cup \cdots \cup A_r) \cap V_{p+m+1} \neq \emptyset$ (A is dense in L). If x_{p+m+1} does not belong to $A_1 \cup \cdots \cup A_r$, it belongs to a 1-cell of L_r and one of the end-points of that 1-cell has to be in V_{p+m+1}; denote that end-point by a. If $x_{p+m+1} \in A_1 \cup \cdots \cup A_r$, put $a = x_{p+m+1}$, thus, in all cases $a \in A_k$, $k \leqq r$. It is now possible to choose a new neighborhood $U_{p+m+1} \subset V_{p+m+1}$ around a, containing x_{p+m+1} and satisfying the requirements concerning retraction and contraction with respect to L_k. Let $b = (x_1, \cdots, x_{p+m}) \times a \times o^{k-1} \subset B_{p+m} \times A_k \times o^{k-1} \subset B_{p+m+k}$. Replace $V_1, \cdots, V_{p+m}, V_{p+m+2}, \cdots, V_q$ by smaller neighborhoods $U_1, \cdots, U_{p+m}, U_{p+m+2}, \cdots, U_q$ around $x_1 = b_1, \cdots, x_{p+m} = b_{p+m}$, $x_{p+m+2} = b_{p+m+2}, \cdots, x_q = b_q$; these neighborhoods should be chosen so as to fulfil the requirements in the definition of a standard open set. If necessary, one can replace a few terms K in V by similar neighborhoods in order to achieve that $U = U_1 \times \cdots \times U_{q'} \times P_0$ and $q' \geqq p+m+k$.

Assume now that $x \in \overline{Y} \setminus Y$. Since $\overline{Y} \subset Q_q$, it follows (by (12)) that $x \in b \times P_0$, where $b \in B_{n\ q} \subset B_n$, $n \geqq q$. x and $b \times O$ coincide in the first n coordinates, it is therefore easy to replace $V_1, \cdots, V_q, K, \cdots, K$ by smaller neighborhoods $U_1, \cdots, U_q, U_{q+1}, \cdots, U_n$ (containing $x_1, \cdots, x_n$ respectively) in such a way that $\overline{X} \cap (U_1 \times \cdots \times U_n \times P_0)$ is a standard neighborhood centered at $b \times O$ and containing x.

Lemma 10 and the following Theorem 4 prove the assertions of Theorem 2 concerning neighborhoods of points in $\overline{Y}$:

THEOREM 4. *A standard open set $U \cap \overline{X}$ is connected. The inclusion $U \cap \overline{X} \subset \overline{X}$ induces a monomorphism of corresponding fundamental groups. The image of $\pi_1(U \cap \overline{X})$ in $\pi_1(\overline{X})$ under this monomorphism is isomorphic to $\pi_1(P_0) \approx \pi_1(\overline{X})$.*

The proof is based on two lemmas.

4.3. LEMMA 11. *Consider all the cells*(7) $b\times e_n$, $b\in B_r$, $r=0, 1, \cdots$, *which have points in common with* $U\cap\overline{X}$, *but* $U\cap\overline{X}$ *does not contain their entire closure* $b\times(e^n)^-$. *The set obtained from* $U\cap\overline{X}$ *by removing exactly these cells is a deformation retract of* $U\cap\overline{X}$.

The fact that these cells are disjoint and can be ordered in a sequence with diameters tending to zero, makes it sufficient (see 1.1.2) to prove the corresponding proposition involving the removal of only one such cell, denoted henceforth by $c\times e$, $c\in B_r$ (U and $b\in B_p$ as in 4.1).

We assume that $r<q$, the other case being trivial. If $e=\sigma\times O$, i.e. of the first kind, we have $(c\times e)\cap U=c\times(\sigma\cap U_{r+1})\times O$. It suffices now to subject U_{p+1} to a (cell-preserving) deformation retraction into $(\dot{\sigma}\cap U_{r+1})$, $\dot{\sigma}$ being the boundary of σ.

If $e=\sigma\times o^{n-1}\times\tau\times O$, i.e. of the second kind, we have either:

$$(1)\quad r+n+1\leqq q,\ (c\times e)\cap U=c\times(\sigma\cap U_{r+1})\times o^{n-1}\times(\tau\cap U_{r+n+1})\times O$$

or

$$(2)\quad r+n+1>q,\ (c\times e)\cap U=c\times(\sigma\cap U_{r+1})\times o^{n-1}\times\tau\times O.$$

Observe that $\bar{\sigma}\cap U_{r+1}$ and $\bar{\tau}\cap U_{r+n+1}$ are simple arcs, while $\bar{\tau}$ is a simple closed curve. Therefore, it is an elementary task to verify that if in the case (1) $o\in U_{r+n+1}$, then $(c\times e)\cap U$ admits a deformation retraction to $(c\times\dot{e})\cap U$, where $\dot{e}$ is the boundary of e. Similarly, if U_{r+1} contains exactly one end-point of $\bar{\sigma}$, then (for (1) as well as for (2)), $(c\times e)\cap U$ admits a deformation retraction into $(c\times\dot{e})\cap U$. We shall show now that at least one of the two cases described is always present.

Assume first (1). Let b_i denote the ith coordinate of $b\times O\in U$, $b\in B_p$. If $b_{r+n+1}=o$, then $o\in U_{r+n+1}$, because $b_{r+n+1}\in U_{r+n+1}$. Suppose now that $b_{r+n+1}\neq o$ and thus $r+n+1\leqq p$. By 3.3.2 we conclude that $(b_1, \cdots, b_{r+n})\in B_{r+n}$, so that 3.3.1 gives $b_{r+1}\in A_1\cup\cdots\cup A_n$, showing that b_{r+1} is a vertex of L_n. However, σ is by supposition a 1-cell of L_n, so that b_{r+1} does not belong to σ. Since $b_{r+1}\in U_{r+1}$ and $U_{r+1}\cap\sigma\neq\emptyset$, U_{r+1} contains at least one end-point of $\bar{\sigma}$. On the other hand, U_{r+1} can contain at most one point of the set $A_1\cup\cdots\cup A_{p-r-1}$ (see 4.1), while both end-points of $\bar{\sigma}$ belong to its subset $A_1\cup\cdots\cup A_n$ ($n\leqq p-r-1$).

Assume now (2). If $p\leqq r$, then $b_{r+1}=o$ is disjoint with σ, hence, U_{r+1} contains at least one end-point of $\bar{\sigma}$. However, if U_{r+1} would contain both end-points, i.e. entire $\bar{\sigma}$, then U would contain entire $c\times\bar{e}$, contrary to our assumption. Suppose now that $p>r$. b_{r+1} is now the $(r+1)$st coordinate of $b\in B_p$ and thus 3.3.1 gives $b_{r+1}\in A_1\cup\cdots\cup A_{p-r}$. Since, in this case, $n>q-r-1\geqq p-r-1$ or $n\geqq p-r$, we see that b_{r+1} is a vertex of L_n and thus disjoint with σ. The rest of the argument is as above.

(7) For the definition of cells e_m, see the proof of Lemma 8.

4.4. LEMMA 12. *If $U\cap\overline{X}$ is a standard open set, then $U\cap\overline{Y}$ and $U\cap\overline{X}$ are connected. Every loop f in $\overline{X}\cap U$ can be deformed, inside $\overline{X}\cap U$, into a loop g of $b\times o^{q-p}\times Y_0$, such that $(g_{q+1}, \cdots)$ is a standard loop of Y_0.*

Proof. In view of Lemma 11, it suffices to prove that $\overline{Y}\cap U$ is connected and that every loop f of $\overline{Y}\cap U$ admits a deformation of the kind required by Lemma 12 (in order to "push" the loop out of the cells $c\times e$ whose closure is in $\overline{X}\cap U$, apply 1.1.3 and 1.1.2). Observe now that $\overline{Y}\subset Q_q$ and if f has points in $c\times P_0$, $c\in B_{n\ q}$, $n\geqq q$, then $(c_1, \cdots, c_q)\in U_1\times\cdots\times U_q$, hence, $(c\times P_0)\subset U$. By 3.3.9, we know that $(c\times P_0)\cap\overline{Y}=(c\times\overline{Y})\subset U\cap\overline{Y}$ is connected. Since $c\times O\in(\bigcup_{n\leqq q-1}B_n\times Y_0)\cap U$, the connectedness of $\overline{Y}\cap U$ will follow from that of $(\bigcup_{n\leqq q-1}B_n\times Y_0)\cap U$. As to the deformation of f, apply Lemma 7 to f and q; the resulting deformation G^q takes place in $\overline{X}\cap U$ (due to special properties of G^q listed in Lemma 7) and enables us to assume hereafter that $f\subset Y_q\cap U$.

Assume now that $b_q=o$ (qth coordinate of $b\times O$, $b\in B_p$). If $c\in B_{q-1}$ and $(c\times Y_0)\cap U\neq\emptyset$, then it follows easily that $(c\times Y_0)\cap U=(c\times(L\cap U_q)\times O)\cup(c\times o\times Y_0)$ (observe that $o=b_q\in U_q$ and $c\in U_1\times\cdots\times U_{q-1}$). Both of the terms are, obviously, connected and have in common the point $c\times O\in\bigcup_{n\leqq q-2}B_n\times Y_0$. The question is thus reduced to proving that $(\bigcup_{n\leqq q-2}B_n\times Y_0)\cap U$ is connected. We can continue this process one step further if $b_{q-1}=o$. We now distinguish two cases. Either we meet a coordinate $b_r\neq o$, $r\geqq 2$, and have to prove that $(\bigcup_{n\leqq r-1}B_n\times Y_0)\cap U$ is connected (obviously, $r\leqq p$), or we have to prove the obvious statement that $Y_0\cap U$ is connected (in the last case $b\times O=b_1\times O\in U$, $b_1\in L$).

In order to prove our assertion in the first case, let us prove that $b_r\neq o$, $2\leqq r\leqq p$, $b\in B_p$, implies

$$Y_n\cap U\subset(Y_{n-1}\cap U)\cup(b_1\times\cdots\times b_{r-1})\times(L\cap U_r)\times O, \qquad n=1,\cdots,r-1. \tag{3}$$

Indeed, let $x\in(Y_n\cap U)\backslash(Y_{n-1}\cap U)\subset(B_{n-1}\times Y_1)\cap U$ (see §3,(21)). Then $(x_1, \cdots, x_{n-1})\in B_{n-1}$ and thus (by 3.3.1) $x_s\in A_1\cup\cdots\cup A_{n-s}\subset A_1\cup\cdots\cup A_{p-s}$, $s\leqq n-1$. Since $x_s\in U_s$, it follows from 4.1 that $x_s=b_s$, $s=1, \cdots, n-1$. Assume now more precisely that

$$x\in B_{n-1}\times A_k\times o^{k-1}\times Y_0, \qquad k\in\{0, 1, \cdots\}. \tag{4}$$

Since $x_r\in U_r$ and $b_r\neq o$, 4.1 implies that $x_r\neq o$. We infer from (4) that this is possible only if $n+k\leqq r$ and that $(x_{n+1}, \cdots, x_r, \cdots)\in Y_0$. Therefore, $x_n\in A_k\subset A_1\cup\cdots\cup A_{r-n}\subset A_1\cup\cdots\cup A_{p-n}$. This fact, together with $x_n\in U_n$, proves that $x_n=b_n$ (by 4.1). Finally, $x_r\neq o$ implies that all other coordinates of $(x_{n+1}, \cdots)$ equal o. To $x_{n+1}, \cdots, x_{r-1}$, we apply again the argument involving 4.1 and obtain (3). It is easy to see that (3) remains valid for $n=0$ if we put $Y_{-1}=\emptyset$. Applying (3) subsequently with $n=r-1, \cdots, 1, 0$, we obtain

$$(5) \qquad Y_{r-1} \cap U \subset b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times O.$$

Notice now that $(b_1 \times \cdots \times b_{r-1}) \in B_{r-1}$ (see 3.3.2) so that the set on the right side of (5) is contained in $(\bigcup_{n \leqq r-1} B_n \times Y_0) \cap U \subset Y_{r-1} \cap U$ and we obtain

$$(6) \quad \left(\bigcup_{n \leqq r-1} B_n \times Y_0\right) \cap U = Y_{r-1} \cap U = b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times O;$$

the examined set is thus an arc and therefore is connected. This completes the proof of the connectedness of $\overline{Y} \cap U$ and $\overline{X} \cap U$.

Consider now the loop $f \subset (Y_q \cap U)$ and suppose that $b_q = o$. Observe that for $c \in B_{q-1}$, $(c \times Y_1) \cap U \subset c \times (L \cap U_q) \times Y_0$ and that $(c \times Y_1) \cap U \neq \emptyset$ implies $c \times (L \cap U_q) \times Y_0 \subset (c \times X_0) \cap U \subset \overline{X} \cap U$. Define now a deformation of the set $(Y_q \backslash (c \times P_0)) \cup (c \times (L \cap U_q) \times Y_0)$ by taking identity on the first summand, on the second summand we keep all the coordinates fixed except the qth which we subject to a contraction of $L \cap U_q$ to the point $b_q = o$, this point being kept fixed during the deformation ($c \times O$ is the only common point of the two summands). The described deformation induces a deformation of the loop f, which takes place in $\overline{X} \cap U$ and brings f into $(Y_q \backslash (c \times P_0)) \cup (c \times o \times Y_0)$. Repeating the process for all $c \in B_{q-1}$, we obtain a deformation of f in $\overline{X} \cap U$, giving a loop in $Y_{q-1} \cap U$ (see §3,(21)). We can continue this reducing process one step further if $b_{q-1} = o$ (by similar arguments), etc. If there is no $b_r \neq o$, $r \geqq 2$, then we have only to see that a loop $f \subset Y_0 \cap U$ can be brought to the required form. Suppose now that there is a $b_r \neq o$, $r \geqq 2$. Then we can assume that $f \subset (Y_r \cap U)$. Since

$$(7) \qquad Y_r \cap U = [(Y_{r-1} \backslash (B_{r-1} \times P_0)) \cap U] \cup [(B_{r-1} \times Y_1) \cap U]$$

and in this case $b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times O \subset (B_{r-1} \times Y_0) \subset (B_{r-1} \times P_0) \cap Y_{r-1}$, we infer from (6) that the first term in (7) is empty and thus $Y_r \cap U = (B_{r-1} \times Y_1) \cap U$. However, f being connected, it has to lie entirely in a set $c \times Y_1$, $c \in B_{r-1}$. Since $b \times O$ is the base point of f we conclude that $c = (b_1, \cdots, b_{r-1})$, hence, $f \subset (b_1 \times \cdots \times b_{r-1} \times Y_1) \cap U \subset b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times (Y_0 \cap (U_{r+1} \times \cdots \times U_q \times P_0)) \subset \overline{X} \cap U$. A deformation of this set, determined by a contraction of $L \cap U_r$ to b_q, induces a deformation of f, in $\overline{X} \cap U$, into a loop of

$$(b_1 \times \cdots \times b_r \times Y_0) \cap U \subset (b_1, \cdots, b_r) \times [((L \cap U_{r+1}) \times O)$$
$$\cup (o \times (L \cap U_{r+2}) \times O) \cup \cdots \cup (o^{q-r-1} \times (L \cap U_q) \times O) \cup (o^{q-r} \times Y_0)].$$

All the terms of this set, except the last one, can be contracted to their only common point $b_1 \times \cdots \times b_r \times O$. These contractions induce a deformation of f into a loop in $b_1 \times \cdots \times b_r \times o^{q-r} \times Y_0 = b \times o^{q-p} \times Y_0$. Since $b \times o^{q-p} \times X_0 \subset U \cap \overline{X}$, we can apply Lemma 4 to obtain, finally, a loop as required by Lemma 12.

4.5. **Proof of Theorem** 4. Let f be a loop of $U \cap \overline{X}$, which is homotopic to $b \times O$ in P_0. f can be deformed in $U \cap \overline{X}$ to a loop g as in Lemma 12. Clearly,

$(g_{q+1}, \cdots)$ is homotopic to O in P_0 and thus (by Theorem 1) homotopic to O already in X_0. Consequently, f is homotopic to $b\times O$ in $(b\times o^{q-p}\times X_0)\cap U\subset\overline{X}\cap U$. $\pi_1(\overline{X}\cap U)\to\pi_1(P_0)$ is thus a monomorphism.

Now associate with every loop of $\overline{X}\cap U$ a "standard" loop g of $b\times o^{q-p}\times P_0$. Two loops, homotopic in $\overline{X}\cap U$, give rise to loops which are homotopic in $b\times o^{q-p}\times P_0$. This defines a monomorphism $\pi_1(\overline{X}\cap U)\to\pi_1(b\times o^{q-p}\times P_0)$, which is clearly an epimorphism, because every loop of $b\times o^{q-p}\times P_0$ can be deformed, in $b\times o^{q-p}\times P_0$, into a "standard" loop g (see Lemma 3), which belongs to $\overline{X}\cap U$. Since $b\times o^{q-p}\times P_0$ is homeomorphic to P_0, we obtain

$$\pi_1(\overline{X}\cap U)\approx\pi_1(P_0). \tag{8}$$

4.6. *Dimension of $\overline{X}$ and $\overline{Y}$.* To complete the proof of Theorem 2, we now prove

$$\dim\overline{Y}=1, \tag{9}$$

$$\dim\overline{X}=2. \tag{10}$$

Since K has at least one 1-cell and $\overline{Y}\supset Y_0\supset L\times O$, we have $\dim\overline{Y}\geqq 1$. Similarly, $\dim\overline{X}\geqq 2$, because of $\overline{X}\supset X_0\supset L\times L\times O$. $\dim\overline{X}\leqq 2$ is an easy consequence of $\dim\overline{Y}\leqq 1$ and Lemma 8 (apply the sum theorem of dimension theory). To prove that $\dim\overline{Y}\leqq 1$, consider open sets $U=U_1\times\cdots\times U_q\times P_0$ of P_0, where U_n is open in K and $(\overline{U}_n\backslash U_n)$ intersect L in a finite set, which is disjoint with the countable set A. Sets $U\cap\overline{Y}$, obviously, form a basis of open sets for $\overline{Y}$. Since the boundary of $U\cap\overline{Y}$ (with respect to $\overline{Y}$) is contained in $(\overline{U}\backslash U)\cap\overline{Y}$, it suffices to show that $(\overline{U}\backslash U)\cap\overline{Y}$ is a finite set. Notice now that

$$\overline{U}\backslash U=\bigcup_{n=1}^{q}[\overline{U}_1\times\cdots\times\overline{U}_{n-1}\times(\overline{U}_n\backslash U_n)\times\overline{U}_{n+1}\times\cdots\times\overline{U}_q\times P_0]. \tag{11}$$

It is clear that our assertion will follow from this proposition: given a fixed point $a\in L\backslash A$ and an integer $p\geqq 1$, the set of all $x\in\overline{Y}$ with $x_p=a$ is a finite set. In order to prove this proposition, observe that the pth coordinate of a point from $B_{n\ p}\times\overline{Y}$, $n\geqq p$, belongs to A. Therefore, our set has to be contained in $\bigcup_{n=0}^{p-1}B_n\times Y_0$ (see §3, (27)). However, if $c\in B_n$, $n\leqq p-1$, and $x\in c\times Y_0$, then $(x_{n+1},\cdots,x_p,\cdots)\in Y_0$, and since $x_p=a$ is not in A, x_p must be different from o, hence, $(x_{n+1},\cdots,x_p,\cdots)=o^{p-n-1}\times a\times O$, showing that there is only one such x. This proves the assertion.

5. **First singular homology group of the infinite Cartesian product.** 5.1. The first singular homology group (with integer coefficients) $H_1(X)$ of an arcwise connected space X is the factor group of $\pi_1(X)$ by the commutator subgroup (Theorem of Poincaré). $H_1(X)$ is zero if and only if $\pi_1(X)$ is a perfect group(8). If $G_1, G_2, \cdots$ is a sequence of groups, let $\prod G_n$ denote their (complete) direct product(9); if $G_1=G_2=\cdots=G$, we use the notation $\prod G$. If

(8) I.e. a group coinciding with its commutator subgroup.

(9) Elements of the product are sequences $(g_1, g_2, \cdots)$, $g_n\in G_n$, all g_n can be different from the unit; $(g_1, g_2, \cdots)\cdot(h_1, h_2, \cdots)=(g_1h_1, g_2h_2, \cdots)$ (see [8, p. 122]).

$X_1, X_2, \cdots$ is a sequence of arcwise connected spaces, then $\prod X_n$ is arcwise connected and $\pi_1(\prod X_n) \approx \prod(\pi_1(X_n))$.

LEMMA 13. *If G is a (nontrivial) perfect finite group, then $\prod G$ is also (nontrivial) perfect.*

Since G is finite and perfect, there is an integer p, such that every element of G is a product of p commutators (some of which may be trivial, i.e. of type $eee^{-1}e^{-1}$, e being the unit of G). Let $g=(g_1, g_2, \cdots)\in G$ and let

$$g_n = a_{n1}b_{n1}a_{n1}^{-1}b_{n1}^{-1} \cdots a_{np}b_{np}a_{np}^{-1}b_{np}^{-1}, \qquad n = 1, 2, \cdots. \tag{1}$$

Furthermore, let

$$a_k = (a_{1k}, a_{2k}, \cdots), \qquad b_k = (b_{1k}, b_{2k}, \cdots), \qquad k = 1, 2, \cdots, p. \tag{2}$$

Then

$$a_k^{-1} = (a_{1k}^{-1}, a_{2k}^{-1}, \cdots), \qquad b_k^{-1} = (b_{1k}^{-1}, b_{2k}^{-1}, \cdots), \tag{3}$$

and it is readily verified that

$$g = a_1b_1a_1^{-1}b_1^{-1} \cdots a_pb_pa_p^{-1}b_p^{-1}; \tag{4}$$

every $g\in G$ is thus a product of p commutators.

Examples of nontrivial finite perfect groups are provided by the alternating group A_n of degree $n>4$ (see [3, p. 38]); another example is the "binary icosahedral group" (see [11, p. 218]) defined by two generators a, b and relations $a^5=b^3=(ab)^2$.

5.2. If G_n is a sequence of perfect groups (possibly $G_n=G$, for all n) and G_n has at least one element $h_n\in G_n$, which is not a product of fewer than n commutators, then $\prod G_n$ is not perfect. It suffices to see that the element $h=(h_1, h_2, \cdots)\in\prod G_n$ is not a product of finitely many commutators. The assumption that h is a product of, say, r commutators, would imply that h_n is a product of r commutators for all n. However, if $n>r$, this is in contradiction with the choice of h_n.

An example of such a situation is provided as follows. Let G be a perfect nontrivial group (possibly finite); let G_n be the n-fold free product $G_n = G * \cdots * G$ and let $h_n\in G_n$ be given by $h_n=g_1g_2 \cdots g_n$, where $g_k\in G$ and is different from the unit of G, $k=1, \cdots, n$. A theorem, due to H. B. Griffiths [4, p. 245], asserts that h_n is not a product of fewer than n commutators in G_n.

Here is a geometric consequence.

THEOREM 5. *There exists a sequence of (connected 2-dimensional) finite polyhedra P_n, $n=1, 2, \cdots$, with vanishing homology groups $H_q(P_n)=0$, $q=1, 2, \cdots$, and such that the first singular homology group of the infinite Cartesian product $H_1(\prod P_n) \neq 0$.*

Let P be the 2-skeleton of the well-known "Poincaré space" described in [11, p. 216]. It is known that $H_2(P)=0$ and that $\pi_1(P)$ is the "binary icosahedral group." Take now for P_n n copies of P attached at a single common point. Obviously, $\pi_1(P_n)=\pi_1(P)*\cdots*\pi_1(P)$; this group is perfect, because $\pi_1(P)$ is a perfect group. Moreover, $H_2(P_n)=0$, so that all the hypotheses of Theorem 5 are fulfilled. However, by the above remarks, $\pi_1(\prod P_n)=\prod(\pi_1(P_n))$ is not perfect and thus $H_1(\prod P_n)\neq 0$.

It is well-known that the singular homology groups of the Cartesian product of finitely many spaces are completely determined by the homology groups of these spaces. Theorem 5 shows that this is not the case for infinite products.

6. **Main theorem and lc_s^1 spaces which fail to be LC^1.** 6.1. Given any finitely presented([10]) group G, there exists a finite (2-dimensional) cell complex K, having a single vertex o and satisfying $\pi_1(K)=G$ (see [12]). Assigning to G such a K and to K the continuous curves $\overline{X}$ and $\overline{Y}$ described in 3.1, we derive from Theorem 2 our main result:

THEOREM 6. *Given any finitely presented group G, there exist a 2-dimensional continuous curve $C(G)$ and a 1-dimensional continuous curve $D(G)\subset C(G)$, having the following properties: $\pi_1(C(G))\approx\prod G$; every point $x\in C(G)\setminus D(G)$ has neighborhoods homeomorphic to the Euclidean plane and every point $x\in D(G)$ has a basis of connected (open) neighborhoods $U(x)$ in $C(G)$ such that $U(x)\subset C(G)$ induces a monomorphism of $\pi_1(U(x))$ into $\pi_1(C(G))$ with an image isomorphic to $\prod G$.*

6.2. Now take for G a nontrivial perfect finite group. Then $\prod G$ is nontrivial and perfect (see Lemma 13). Therefore, every $x\in C(G)$ has a basis of connected neighborhoods $U(x)$ with $H_1(U(x))=0$, showing that $C(G)$ is a 2-dimensional continuous curve, everywhere lc_s^1. On the other hand, if $x\in D(G)$, $\pi_1(U(x))\approx\prod G$ and thus nontrivial. Since $\pi_1(U(x))\to\pi_1(C(G))$ is a monomorphism it follows that the space is not semi-1-LC at the points of $D(G)$([11]); a fortiori it is not LC^1 in those points. This proves

THEOREM 7. *Every nontrivial perfect finite group gives rise to a 2-dimensional continuous curve which is lc_s^1, but fails to be LC^1 in a subset of dimension 1.*

CONJECTURE. *A continuous curve which is everywhere lc_s^1 can not fail to be LC^1 in exactly one point.*

This statement, if true, should explain why the examples exhibited in this paper are of a rather involved nature.

6.3. We now state (proof is easily supplied using Lemma 13).

([10]) I.e. group defined by a finite number of generators and relations.

([11]) A space X is semi-1-LC at $x\in X$ if there is a neighborhood V of x such that the image of $\pi_1(V)$ in $\pi_1(X)$ (under the homomorphism induced by $V\subset X$) is trivial, i.e. the unit subgroup of $\pi_1(X)$.

THEOREM 8. *If K is a finite complex with a single vertex o and $\pi_1(K)$ is a nontrivial finite perfect group, then $\prod K$ is an infinite dimensional continuous curve, everywhere* lc_s^1 *and nowhere* LC^1.

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INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.
UNIVERSITY OF ZAGREB,
ZAGREB, YUGOSLAVIA