INFINITE CARTESIAN PRODUCTS AND A PROBLEM CONCERNING HOMOLOGY LOCAL CONNECTEDNESS

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If \( M \subset N \) are subsets of a topological space \( X \), we denote by \( H_p(M) \) and \( H_p(N) \) the singular homology groups (with integer coefficients) of \( M \) and \( N \) respectively; the image of \( H_p(M) \) in \( H_p(N) \) (under the homomorphism induced by inclusion \( M \subset N \)) will be denoted by \( H_p(M \mid N) \). The space \( X \) is said to be \( p \)-lc\(_s\) (i.e. \( p \)-locally connected in the sense of singular homology) at the point \( x \in X \) if for every neighborhood \( U \) of \( x \) there is a neighborhood \( V \) of \( x \), \( V \subset U \), such that \( H_p(V \mid U) = 0 \); if \( p = 0 \) augmented homology is used. \( X \) is lc\(_s\) at \( x \) if it is \( p \)-lc\(_s\) at \( x \), for all \( 0 \leq p \leq q \). \( X \) is lc\(_s\) if it is lc\(_s\) at all \( x \in X \). Replacing singular homology by Čech homology (arbitrary open coverings and integer coefficients) and by homotopy, one obtains the definition of properties lc\(_s\) and LC\(_s\) respectively.

These notions are well-known and have been studied by various authors. In a recent paper [9], the present author has shown that for Hausdorff locally paracompact spaces the property lc\(_s\) implies LC\(_s\)(\(^1\)). The implication LC\(_s\) =⇒ lc\(_s\) can not be reversed (not even in the category of metrizable compacta) as has been shown by H. B. Griffiths [5, p. 477]. Griffiths has also proved [7] that for locally compact metrizable spaces LC\(_s\) =⇒ LC\(_s\). However, the question of the possibility of reversing this last implication has remained open and has been pointed out by Griffiths in [5, p. 479] and in [6, 3, p. xi,]. The corresponding question with Čech homology has been settled previously (see [1, p. 573]) by the well-known example of an “infinite bouquet” of Poincaré spaces, which is lc\(_s\) but fails to be LC\(_s\) at the base point of the bouquet. Griffiths has shown [5, p. 477] that an infinite bouquet of LC\(_s\) spaces can not provide an example of an lc\(_s\) space which would not be LC\(_s\) at the same time. This different behavior is due to the fact that singular homology is not continuous with respect to inverse limits.

In this paper we describe a whole category of 2-dimensional metrizable compacta which are lc\(_s\) but fail to be LC\(_s\) in certain points(\(^2\)), proving thus that the implication LC\(_s\) =⇒ lc\(_s\) can not be reversed (Theorem 7). If one admits examples of infinite dimension, then the problem is easily settled by an in-

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(\(^1\)) The same result has also been obtained by H. B. Griffiths in an unpublished paper.

(\(^2\)) The case \( q = 1 \) is easier to handle because of the simple relation between the fundamental group and \( H_1 \) given by the Poincaré theorem. This case deserves special attention due to the fact that for locally compact metrizable spaces (lc\(_s\) and LC\(_s\)) =⇒ LC\(_s\) (see [10]).

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finite Cartesian product of Poincaré spaces (Theorem 8). The main part of the paper is concerned with a construction giving a 2-dimensional subset of the infinite Cartesian product which, roughly speaking, in the neighborhood of some points has the fundamental group of the entire infinite product (see §3,1). We hope that the main Theorem 6 might prove useful in other connections too.

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1. Preliminaries. 1. The following four propositions will often be referred to in the sequel. The proofs can be easily supplied and are omitted.

1.1. If \( M \) is a metric space with metric \( \rho \) and \( N \subseteq M \), then \( U(N, \epsilon) \) will denote the \( \epsilon \)-neighborhood around \( N \), i.e. the set \( \{ x \mid x \in M, \rho(N, x) < \epsilon \} \).

Let \( C_0, C_1, \ldots \) be a sequence of compact subsets of a metric space \( M \). If there is a sequence of reals \( \epsilon_n > 0 \), \( \lim \epsilon_n = 0 \), such that \( C_n \cup U(C_0, \epsilon_n) \) is compact.

1.2. Let \( I \) be the unit interval and let \( f^p : I \to M, p = 1, 2, \ldots \), be a sequence of loops in a metric space \( M \), based at a point \( o \in M \). Let \( F^p \) and \( f^{p+1} \) be homotopies in \( M \), connecting \( f^p \) and \( f^{p+1} \), such that \( \text{diam } F^p \leq c_p \), \( \sum c_p \) is a convergent series. Then \( f(x) = \lim f^p(x) \) exists and is a loop homotopic to all \( f^p \). One can choose the homotopy \( F \), connecting \( f^1 \) and \( f \), so as to take place in the union of images of all \( F^p \).

\( F \) can be obtained by considering \( F^p \) as defined over \( I \times I_p \), where \( I_p = \left[ (p-1)/p, p/(p+1) \right] \) and setting

\( F(x, t) = F^p(x, t) \), for \( x \in I, t \in I_p, p \in \{1, 2, \ldots \} \),

and \( F(x, 1) = f(x) \).

Whenever we speak of homotopies of loops and paths we mean homotopies with fixed end-points.

1.3. Let \( M^* \) be a metric space obtained from its closed subset \( M \) by attaching an \( n \)-cell \( e^n \), \( n > 1 \). Every loop \( f \) in \( M^* \) with base point in \( M \) can be deformed (inside \( M^* \)) into a path \( g \) in \( M \) in such a way that the deformation \( F(x, t) = f(x) \) whenever \( f(x) \in M \) and \( F(x, t) \in (e^n)^- \), whenever \( f(x) \in e^n \).

1.4. Let \( M \) be a metric space with a base point \( o \) and \( f : I \to M \) a path. Furthermore, let \( U \) be an open set of \( I \) such that \( f(U \setminus U) = o \). \( U \) is obviously the union of at most countably many disjoint open intervals \( V \subseteq I \), which are components of \( U \); \( f|V \) are loops in \( M \), based on \( o \).

If for every \( V \), \( F_V : \overline{V} \times I \to M \) is a deformation of the loop \( f|V \) and for every \( \epsilon > 0 \) there is a \( \delta(\epsilon) > 0 \) with the property that \( \text{diam } \overline{V} < \delta \) implies \( \text{diam } F_V < \epsilon \), then the following relations define a deformation \( F \) of the path \( f \):

\( F(x, t) = F_V(x, t) \), for \( x \in \overline{V} \),

\( F(x, 1) = o \), for \( x \in U \setminus U \).
and

\[ F(x, t) = f(x), \quad \text{for } x \in \mathcal{I} \setminus U. \]

2. By a finite cell complex \( K \) we mean in this paper a finite cell complex which admits a simplicial subdivision (see e.g. [2, p. 152]). We use the same letter to denote the complex and the underlying polyhedron. There is no loss of generality in assuming that \( K \) is provided with a metric \( d \leq 1 \) and that every point \( x \) of \( K \) has arbitrarily small \( \delta \)-neighborhoods \( U(x) \) admitting a cell-preserving contraction into \( x \) (with respect to \( K \))(3). Moreover, if \( \dim K = n \) and \( K^p \) denotes the \( p \)-skeleton of \( K \), we can assume that this contraction is composed first of a cell-preserving deformation retraction of \( U \) onto \( U \cap K^{n-1} \), then of a cell-preserving deformation retraction of \( U \cap K^{n-1} \) onto \( U \cap K^{n-2} \), etc. Clearly, \( U(x) \) has to be contained in the open star \( \text{St}_K(x) \). We shall often have the additional assumption that \( K \) has a single vertex \( o \); closed 1-cells will therefore be 1-spheres and thus never contained entirely in such a neighborhood \( U(x) \).

3. Let \( M \) be a metric space with a given metric \( d \leq 1 \). The infinite Cartesian product of a sequence \( M_1, M_2, \ldots \) of copies of \( M \) will be denoted by \( \prod M \). If \( x \in M \), we shall usually denote the \( n \)th coordinate of \( x \) by \( x_n \). We shall consider \( M \) as metrized by the metric

\[ \rho(x, y) = \sum_1^\infty d(x_n, y_n)2^{-n}. \]

If \( a = (a_1, \ldots , a_n) = a_1 \times \cdots \times a_n \) is a point of the \( n \)-fold Cartesian product \( M \times \cdots \times M \) and \( b = (b_1, \ldots ) = b_1 \times \cdots \) is a point of the infinite product \( \prod M \), we shall often denote the point \( (a_1, \ldots , a_n, b_1, \ldots ) \in \prod M \) simply by \( a \times b \). If \( A \subset M \times \cdots \times M \) and \( B \subset \prod M \), the meaning of the notation \( A \times B \subset \prod M \) is clear.

2. Infinite Cartesian products of cell complexes. 1. Let \( K \) be a finite cell complex(4) having a single vertex \( o \). We can assume that \( \dim K \leq 2 \) (otherwise we should replace \( K \) by the 2-skeleton \( K^2 \) in (4)). The infinite Cartesian product \( \prod K \) will be denoted hereafter by \( P_0 \). All sets encountered throughout §§2-4 will be subsets of \( P_0 \). The cellular structure of \( K \) induces a decomposition of \( P_0 \) into disjoint "cells"

\[ \sigma = \sigma_1 \times \sigma_2 \times \cdots , \]

where \( \sigma_n \) are (open) cells of \( K \). We define

\[ \dim \sigma = \sum_1^\infty \dim \sigma_n \leq \infty. \]

Let \( X_0(Y_0) \) denote the "2-skeleton" ("1-skeleton") of this decomposition of

(4) A deformation is said to be cell-preserving if, during the deformation, no point can leave the closure of the cell containing that point at \( t = 0 \).

(4) See §1.2 and §1.3.
$P_0$. The “0-skeleton” consists of a single point $O = (o, o, \cdots)$. Denoting by $L$ the 1-skeleton of $K$ and by $o^n$ the point $(o, o, \cdots, o)$ of the $n$-fold product $K \times \cdots \times K$ ($o^0$ meaning the “empty symbol”), we have

$3. \ Y_0 = \bigcup_{n=0}^{\infty} o^n \times L \times O = (L \times O) \cup (o \times L \times O) \cup (o \times o \times L \times O) \cup \cdots,$

$4. \ X_0 = \left( \bigcup_{n=0}^{\infty} o^n \times K \times O \right) \cup \left( \bigcup_{n=0}^{\infty} o^n \times L \times Y_0 \right).$

Observing that

$$\text{diam } (o^n \times K \times O) \leq 2^{-n-1} \quad \text{and} \quad \text{diam } (o^n \times L \times Y_0) \leq 2^{-n},$$

we conclude readily (by 1.1.1) that $Y_0$ and $X_0$ are compacta. Notice also that a point of $Y_0(X_0)$ can have at most one (two) coordinates different from $o$.

Although the described decomposition of $P_0$ is not a complex, we shall prove in this section

**Theorem 1.** The inclusion $X_0 \subseteq P_0$ induces an isomorphism of $\pi_1(X_0)$ onto $\pi_1(P_0)$.

2. **Definition 1.** A loop $f: I \to P_0$ (based at $O$) is said to be a standard loop if $f((n-1)/n) = O$, for all $n=1, 2, \cdots$ and if $f(I_n) \subseteq o^{n-1} \times L \times O$ (recall that $I_n = [(n-1)/n, n/(n+1)]$).

**Lemma 1.** If $f$ and $g$ are standard loops, homotopic in $P_0$, then they are homotopic already in $X_0$.

**Proof.** Let $F$ be a homotopy in $P_0$ connecting $f$ and $g$ and let $F_n, f_n$ and $g_n$ be maps obtained from $F, f$ and $g$ respectively by composition with the natural projection $P_0 = \prod K \to o^{n-1} \times K \times O$. $F_n$ is obviously a homotopy connecting $f_n$ and $g_n$. However, $f_n(x) = f(x), g_n(x) = g(x)$, for $x \in I_n$, otherwise $f_n(x) = g_n(x) = O$, hence, the loops $f|I_n$ and $g|I_n$ are homotopic in $o^{n-1} \times K \times O \subseteq X_0$; let $G^n$ be a connecting homotopy. Defining $G$ by $G(x, t) = G^n(x, t)$, for $(x, t) \in I_n \times I$, $n=1, 2, \cdots$, and by $G(1, t) = O$, we obtain a homotopy in $X_0$ connecting $f$ and $g$.

If $f_n$ and $g_n$ both lie in a subset of $o^{n-1} \times K \times O$, which is contractible to $O$ (fixed during contraction), then we can take for $G^n$ a connecting homotopy contained in that subset. Using this remark we can prove

**Lemma 2.** For every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that any two standard loops $f$ and $g$, homotopic in $P_0$ and lying in $U(O, \delta)$, can be connected by a homotopy in $X_0 \cap U(O, \epsilon)$.

Indeed, choose $p$ so large that $2^{-p} < \epsilon$ and $0 < \eta < \epsilon$ such that $U(o, \eta)$ is contractible to $o$ in $K$. Let $\delta(\epsilon) = \eta 2^{-p}$. If $f, g \subseteq U(O, \delta)$, then $f_n, g_n \subseteq o^{n-1} \times U(o, \eta) \times O$, for $n=1, \cdots, p$. Choose now $G^n, n=1, \cdots, p$, in $o^{n-1} \times U(o, \eta) \times O$ (no requirements on $G^{p+1}, \cdots$). Clearly, $G(x, t) \subseteq U(O, \delta) \subseteq U(O, \epsilon)$, for $x \in I_1 \cup \cdots \cup I_p$. For $x \in I_n, n > p$, we obtain $G(x, t) \subseteq U(O, \epsilon)$.
as a consequence of \( \text{diam } o^{n-1} \times K \times O \leq 2^{-n} \) and of the choice of \( p \). Lemma 2 will be used in §3.

**Lemma 3.** Every loop \( f \) in \( P_0 \) (based at \( O \)) can be deformed (in \( P_0 \)) into a standard loop.

**Proof.** The \( n \)th coordinate \( f_n \) of \( f \), being a loop in \( K \), admits a deformation \( F_n \) (in \( K \)) into a loop \( g_n \) of the 1-skeleton \( L \) of \( K \). One can easily achieve that \( g_n(I \setminus I_n) = o \). \( F = (F_1, F_2, \cdots) \) is then obviously a deformation of \( f \) into a standard loop \( g \).

3. The main part of the proof of Theorem 1 is contained in the following

**Lemma 4.** Every loop \( f \) in \( X_0 \) (based at \( O \)) can be deformed, in \( X_0 \), into a standard loop.

**Proof.** Observe first that cell-preserving deformations of coordinates \( f_n \) of \( f \) give a deformation of \( f \) in \( P_0 \) which actually takes place in the “2-skeleton” \( X_0 \) of \( P_0 \). Since the deformations occurring in the cell-approximation theorem are cell-preserving, we can assume that \( f_n \) are loops in the 1-skeleton \( L \); and, consequently, that \( f \) is contained in the second summand of (4). Moreover, we can achieve (say, by simplicial approximations with respect to some simplicial subdivisions of \( L \)) that, for \( n = 1, 2, \cdots \), the open set \( U_n = \{ x | f_n(x) \not= o \} \subset I \) is the union of finitely many disjoint open intervals.

Given a point \( a \in U_n \), it is clear that the particular open interval of \( U_n \) which contains \( a \) is mapped by \( f_n \) entirely into a 1-cell of \( L \). Therefore, it is easy to define a cell-preserving deformation, affecting only that particular interval (without changing the total number of components of \( U_n \)) and yielding a new loop \( f_n \) with \( f_n(a) = o \). In view of this remark we can assume from now on that for every \( n = 1, 2, \cdots \), \( f_n(I) \subset L \), that \( U_n \) consists of a finite number of disjoint open intervals and that \( f_r(U_n \setminus U_n) = o \), for \( r \geq n \); a loop having the last two properties will be referred to as a “normal” loop.

Consider now the sets

\[
S_p = Y_0 \cup \left( \bigcup_{n=p+1}^\infty o^n \times L \times Y_0 \right).
\]

All \( S_p \) are compact (by 1.1.1) and \( X_0 \supset S_1 \supset S_2 \supset \cdots \supset \cap S_p = Y_0 \). In view of the above remarks, \( f \subset S_1 \).

We shall now define, by induction, a sequence of loops \( f = f^1, f^2, \cdots, f^p, \cdots \), with \( f^p \subset S_p \), and a sequence of homotopies \( F^p, p+1 : I \times I \to S_p \), connecting \( f^p \) and \( f^{p+1} \) and satisfying

\[
\text{diam } F^p, p+1 \leq 2^{-p+1}.
\]

1.1.2 will then provide a limit loop \( f = \lim f^p \), obviously contained in \( Y_0 \) and homotopic to \( f \) in \( \bigcup_{p=1}^\infty S_p \subset X_0 \).

Suppose that \( f = f^1, \cdots, f^p \) and \( F^1, 2, \cdots, F^{p-1} \) have already been de-
fined and satisfy the conditions of above; in order to carry through the induction, we assume in addition that \( f^1, \cdots, f^p \) are "normal" loops. For \( p = 1 \), these conditions are verified as established in the preceding remarks concerning \( f \). Consider now \( U_p^p = \{ x | f_p^p(x) \neq 0 \} \), \( f_p^p \) denoting the \( p \)th coordinate of \( f_p^p \). Since \( f_p^p \subset S_p \), it follows immediately from (5) that

\[
(7) \quad f_p^p(U_p^p) \subset o^{p-1} \times L \times Y_0.
\]

Now let \((a, b)\) be one of the finitely many components of \( U_p^p \). In order to define \( F_p^{p+1} \), choose a point \( c, a < c < b \), and put

\[
(8) \quad c_t = c + (1 - t)(b - c), \quad d_t = a + t(c - a), \quad t \in I.
\]

Furthermore, let \( \alpha_t(x) \) be the transformation mapping \([a, c]\) linearly onto \([a, b]\) and sending \([c_t, b]\) into \( b \); let \( \beta_t \) be the transformation mapping \([a, d_t]\) into \( a \) and mapping \([d_t, b]\) linearly onto \([a, b]\). \( \alpha_t(x) \) and \( \beta_t(x) \) are mappings of \([a, b] \times I\) into \([a, b] \), leaving end-points \( a \) and \( b \) fixed.

Define now \( F_p^{p+1} : [a, b] \times I \to o^{p-1} \times L \times Y_0 \subset S_p \) by

\[
(9) \quad F_p^{p+1}(x, t) = o^{p-1} \times f_p^p(x) \times (f_{p+1}^p \times f_{p+2}^p \times \cdots) \beta_t(x).
\]

Clearly, \( F_p^{p+1}(x, 0) = f_p^p(x) \). As to \( f_p^{p+1}(x) = F_p^{p+1}(x, 1) \), observe first that \( f_p^r(a) = f_p^r(b) = o \), for \( r \geq p \) (\( f_p^p \) is "normal"). It now follows, from (9), that

\[
(10) \quad f_p^{p+1}(x) = o^{p-1} \times f_p^p(x) \times o \times o \times \cdots, \quad \text{for } x \in [a, c],
\]

\[
(11) \quad f_p^{p+1}(x) = o^{p} \times (f_{p+1}^p \times f_{p+2}^p \times \cdots) \beta_t(x), \quad \text{for } x \in [c, b],
\]

showing that \( f_p^{p+1}([a, b]) \subset Y_0 \subset S_{p+1} \).

We define \( F_p^{p+1} \) on other components of \( U_p^p \) in exactly the same way (they are in a finite number) and complete the definition by

\[
(12) \quad F_p^{p+1}(x, t) = f_p^p(x), \quad \text{for } x \in I \setminus U_p^p.
\]

\( F_p^{p+1} \) is continuous on \( I \times I \), because (10), (11) and (12) give \( F_p^{p+1}(a, t) = F_p^{p+1}(b, t) = o \). Moreover, for \( x \in I \setminus U_p^p \), \( f_p^p(x) = o \) and thus \( f_p^{p+1}(x) = f_p^p(x) \) belongs actually to \( S_{p+1} \subset S_p \). (6) follows from \( \text{diam } o^{p-1} \times L \times Y_0 \leq 2^{-p+1} \).

Finally, it is readily checked that \( f_p^{p+1} \) is "normal." This completes the argument showing that every loop of \( X_0 \) (based on \( O \)) can be deformed, in \( X_0 \), into a loop of \( Y_0 \).

To complete the proof of Lemma 4, we have to show now that every loop \( f \) of \( Y_0 \) (based at \( O \)) can be deformed, in \( X_0 \), into a standard loop. For that purpose we shall define by induction a sequence of loops \( f = f^0, f^1, \cdots, f^p, \cdots \) in \( Y_0 \) and a sequence of homotopies \( F_p^{p+1} : I \times I \to X_0 \), connecting \( f^p \) and \( f^{p+1} \) and having diam \( F_p^{p+1} \leq 2^{-p} \). For \( p > 0 \), we require in addition

\[
(13) \quad f_p^q(I_q) \subset o^{q-1} \times L \times O, \quad q \leq p,
\]
(14) \[ f^p \left( \frac{q}{q+1} \right) = o, \quad q \leq p, \]

(15) \[ f^p \left( \left[ \frac{p}{p+1}, 1 \right] \right) \subset o^p \times Y_0. \]

Once such a sequence is defined, 1.1.2 will yield a limit loop \( f = \lim f^p \), homotopic to \( f \) in \( X_0 \), and actually a standard loop (due to (13) and (14)).

Assume that \( f^1, \ldots, f^p \) and \( F^0, \ldots, F^{p-1} \) have already been defined in accordance with the above requirements. Denote \( p/(p+1) \), \( (p+1)/(p+2) \) and 1 by \( a, c \) and \( b \) respectively and let \( c_t, d_t, \alpha_t(x) \) and \( \beta_t(x) \), for \( x \in [a, b] \), be defined as in the preceding argument; moreover, let \( \alpha_t(x) = \beta_t(x) = x \), for \( x \in I \setminus (a, b) \). We define \( F^{p+1} \) by

\[ F^{p+1}(x, t) = f^p(x), \quad \text{for} \ x \in \left[ 0, \frac{p}{p+1} \right], \quad t \in I, \]

(16) \[ F^{p+1}(x, t) = o^p \times f^p_{p+1} \alpha_t(x) \times (f^p_{p+2} \times f^p_{p+3} \times \cdots) \beta_t(x), \]

\[ \text{for} \ x \in \left[ \frac{p}{p+1}, 1 \right], \quad t \in I, \]

(17) \[ F^{p+1}(x) = F^{p+1}(x, 1). \]

All the required properties are readily checked (notice that \( f^p \beta_t(x) \in Y_0 \) implies \( (f^{p+1} \times f^{p+2} \times \cdots) \beta_t(x) \in Y_0 \)).

4. The following lemma will be needed in §3.

**Lemma 5.** For every \( \epsilon > 0 \) there is a \( \delta(\epsilon) > 0 \) such that every loop \( f \), lying in \( U(o, \delta) \subset P_0 \) (in \( X_0 \cap U(o, \delta) \)), can be deformed into a standard loop by a deformation lying in \( U(o, \epsilon) \) (in \( X_0 \cap U(o, \epsilon) \)).

**Proof.** Choose \( p, \eta \) and \( \delta \) as in the proof of Lemma 2 (with the additional requirement that \( U(o, \eta) \) admits a cell-preserving contraction to \( o \)). If \( f \subset U(O, \delta) \), then \( f_1, \ldots, f_p \subset U(o, \eta) \subset K \). Composing these coordinates with a (cell-preserving) contraction of \( U(o, \eta) \) to \( o \), while leaving \( f_{p+1}, \ldots \) unchanged, one obtains a deformation \( F \) of \( f \), in \( P_0 \) (in \( X_0 \)), into a loop \( g \subset o^p \times P_0 \subset o^p \times X_0 \). Since \( \rho(F(x, t) \neq F(x, 0)) < \eta + \cdots + \eta 2^{-p+1} = 2\eta(1 - 2^{-p}) \) and \( F(x, 0) = f(x) \in U(O, \delta) \), it follows that \( \rho(O, F(x, t)) \leq 2\eta < 2\epsilon \). Consequently, \( F \subset U(O, 2\epsilon) \). Applying now Lemma 3 (Lemma 4) to \( g \) and \( o^p \times P_0 \) (in \( X_0 \)) we deform \( g \) further into a standard loop by a deformation of diameter lesser than \( \text{diam} (o^p \times P_0) \leq 2^{-p} \leq \epsilon \). The total deformation is thus contained in \( U(O, 2\epsilon) \).

5. **Proof of Theorem 1.** Lemma 3 proves that the homomorphism \( i: \pi_1(X_0) \rightarrow \pi_1(P_0) \), induced by \( X_0 \subset P_0 \), is an epimorphism. Combining Lemmas 4 and 1, we conclude that \( i \) is a monomorphism (the constant loop \( g(x) = O \) is a standard loop).
Remark. Theorem 1 holds also in the case of an infinite product of different complexes $K_1, K_2, \cdots(6)$.

3. Continuous curve $\overline{X}$ and its fundamental group.

1. Description of the basic construction. Let $K$ be a finite cell complex having one single vertex $o$ and at least one 1-cell. Choose a sequence of finite (nonempty) disjoint subsets $A_1, \cdots, A_k, \cdots$ of the 1-skeleton $L$ of $K$ in such a way that $o \in A_1$ and that

$$(1) \quad A = \bigcup_{k=1}^{\infty} A_k$$

is dense in $L$; these sets will be considered as fixed throughout this section. We define next, by induction on $n$, a finite subset $B_n$ of the $n$-fold product $K \times \cdots \times K$, by

$$B_n = \bigcup_{k=1}^{n} B_{n-k} \times A_k \times o^{k-1}.$$  

$B_0$, as well as $A_0, o^0$ and $o^{-1}$ are considered to represent "empty symbols"; e.g. $B_1 = A_1, B_2 = A_1 \times A_1 \cup A_2 \times o$. Notice that $o^n \in B_n$, for all $n \geq 1$. Let $X_0$ and $Y_0$ be as in §2. Consider the following subsets of $P_0$

$$(3) \quad X = \bigcup_{n=0}^{\infty} B_n \times X_0 \quad \text{and} \quad (4) \quad Y = \bigcup_{n=0}^{\infty} B_n \times Y_0.$$  

Let $\overline{Y}$ and $\overline{X}$ be the closures of $Y$ and $X$ taken with respect to $P_0$.

In this section, and the following section, we are concerned with a proof of the basic

Theorem 2. $\overline{X}$ and $\overline{Y}$ are continuous curves(6) with $\dim \overline{X} = 2$, $\dim \overline{Y} = 1$. $\overline{Y} \subset \overline{X}$ and points of $\overline{X} \setminus \overline{Y}$ have 2-dimensional Euclidean neighborhoods (with respect to $\overline{X}$). The inclusion $\overline{X} \subset P_0$ induces an isomorphism $\pi_1(\overline{X}) = \pi_1(P_0)$. Every $x \in \overline{Y}$ has a basis of connected (open) neighborhoods (with respect to $\overline{X}$) $U(x)$, such that $U(x) \subset \overline{X}$ induces a monomorphism of $\pi_1(U)$ into $\pi_1(\overline{X})$ with an image isomorphic to $\pi_1(P_0)$.

2. For purposes of proof we introduce certain subsets of $P_0$ approximating $\overline{X}$ and $\overline{Y}$. Let

$$(5) \quad X_1 = \bigcup_{k=0}^{\infty} A_k \times o^{k-1} \times X_0, \quad Y_1 = \bigcup_{k=0}^{\infty} A_k \times o^{k-1} \times Y_0.$$  

(6) It seems likely that the restriction to complexes having a single vertex (imposed in view of applications in forthcoming sections) should not be essential for the validity of Theorem 1.

(6) I.e. metrizable compact connected and locally connected spaces.
$X_1$ and $Y_1$ are compact (1.1.1) and connected. The same is true for

$$X_{p+1} = X_p \cup B_p \times X_1 = \bigcup_{0}^{p} B_n \times X_1$$

and

$$Y_{p+1} = Y_p \cup B_p \times X_1 = \bigcup_{0}^{p} B_n \times Y_1, \quad p = 1, 2, \ldots .$$

Denote by $B_{n,p}$ the union of the last $p$ terms in the expression (2), $p \leq n$,

$$B_{n,p} = \bigcup_{k=n-p+1}^{n} B_{n-k} \times A_k \times o^{k-1}.$$ 

Notice that $B_{n,n} = B_n$. One obtains new expressions for $X_p$ and $X$,

$$X_p = \left( \bigcup_{n=0}^{p-1} B_n \times X_0 \right) \cup \left( \bigcup_{n=p}^{\infty} B_n \times X_0 \right), \quad p = 1, 2, \ldots ,$$

and

$$X = \bigcup_{p=0}^{\infty} X_p.$$ 

Analogous formulae hold for $Y_p$ and $Y$. Notice that $X_p \subset X_{p+1}$, $Y_p \subset Y_{p+1}$, $Y_{p+1} \subset X_p$. We conclude from (10) that connectedness of $X_n$ and $Y_n$ implies

that of $X$ and $Y$ as well as $\overline{X}$ and $\overline{Y}$.

In order to obtain suitable approximations of $\overline{X}$ and $\overline{Y}$ “from outside” we introduce

$$P_p = \left( \bigcup_{n=0}^{p-1} B_n \times X_0 \right) \cup \left( \bigcup_{n=p}^{\infty} B_n \times P_0 \right)$$

and

$$Q_p = \left( \bigcup_{n=0}^{p-1} B_n \times Y_0 \right) \cup \left( \bigcup_{n=p}^{\infty} B_n \times P_0 \right).$$

Notice that

$$X_p \subset P_p, \quad Y_p \subset Q_p \subset P_p.$$ 

In order to prove

$$P_{p+1} \subset P_p, \quad Q_{p+1} \subset Q_p,$$

it suffices to show that $B_{n,p+1} \times P_0 \subset (B_p \times P_0) \cup (B_n \times P_0)$, $n \geq p + 1$. All but the first term of $B_{n,p+1} \times P_0$ are contained in $B_n \times P_0$; however, this term is $B_p \times A \times o^{p-1} \times P_0 = B_p \times B_n \times P_0 \subset B_p \times P_0$. A consequence of (13) and (14) is
for arbitrary \( p, q \).

Observe now that \( 0 \in X_0 \) and \( B_n \times O \subseteq B_n \times X_0 \subseteq X_p \), \( n \geq p \); therefore, \( \text{diam} \ (b \times P_0) \leq 2^{-n}, b \in B_n \) implies \( P_p \subseteq U(X_p, 2^{-p+1}) \subseteq U(X_p, 2^{-p+1}) \). Firstly, we conclude (1.1.1) that \( P_p \) is compact because \( X_p \) is compact. Secondly, since \( X \subseteq \bigcap P_p \) (by (15)),

\[
X = \bigcap_{1}^{\infty} P_p.
\]

Analogous arguments show that \( Q_p \) is compact and

\[
Q = \bigcap_{1}^{\infty} Q_p.
\]

3. We list here several simple propositions needed in the sequel.

3.1. \( x = (x_1, \ldots, x_n) \in B_n \) implies \( x_k \in A_1 \cup \cdots \cup A_{n-k+1}, k = 1, \ldots, n \). Proof immediate by induction on \( n \).

3.2. \( x = (x_1, \ldots, x_n) \in B_n \) and \( x_q \neq o, 2 \leq q \leq n \), implies \( (x_1, \ldots, x_{q-1}) \in B_{q-1} \).

Proof of induction on \( n \geq q \) (\( q \) fixed). \( x \) can not belong to the last \( q-1 \) terms of (2) because the \( q \)th coordinate would be 0. Hence, \( x \in B_{n-k} \times A_k \times o^{k-1}, k \in \{1, \ldots, n-q+1\} \). If \( n-k < q \), then actually \( n-k = q-1 \) (otherwise we would have \( x_q = o \)). However, in this case \( x \in B_{q-1} \times A_{n-q+1} \times o^{n-q} \) and \( (x_1, \ldots, x_{q-1}) \in B_{q-1} \). In the remaining cases \( q \leq n-k \) and \( (x_1, \ldots, x_q, \ldots, x_{n-k}) \in B_{n-k} \) so that the hypothesis of induction is applicable.

3.3. For arbitrary \( q, n, B_q \times B_n \subseteq B_{q+n} \). Proof by induction on \( n \). Substitute (2) for \( B_n \), apply the inductive hypothesis and notice that the resulting expression gives the first \( n \) terms of (2) for \( B_{q+n} \).

3.4. If \( \emptyset \) denotes the empty set, then \( (B_p \times P_0) \cap (B_n \times P_0) = \emptyset \), for \( n > p \), and \( (B_n \times P_0) \cap (B_{m} \times P_0) = \emptyset \), for \( n > m \geq p \).

It suffices to prove the first assertion, because of \( B_n \times B_m \subseteq B_{n+m} \). Assume that \( x \in B_n \times P_0 \); there exists then an \( s \in \{n-p+1, \ldots, n\} \) (by (8)) such that \( x \in B_{n-s} \times A_s \times o^{s-1} \times P_0 \), hence \( x_{n-s+1} \in A_s, n-s+1 \leq p \). If at the same time \( x \in B_p \times P_0 \), then 3.1 would imply \( x_{n-s+1} \in A_1 \cup \cdots \cup A_{s-(n-p)} \). However, this set is disjoint with \( A_s \) (because of \( n > p \) and the definition of sets \( A_k \)), which presents a contradiction.

3.5. If \( q > p \), we have

\[
\left( \bigcup_{n=0}^{p-1} B_n \times Y_0 \right) \cap (B_q \times P_0) = B_q \times O.
\]

Indeed, if \( b \in B_q \times P_0 \), it follows immediately (by (8)), that \( b \times O \subseteq B_n \times Y_0 \), for an \( n \in \{0, \ldots, p-1\} \). On the other hand, for \( n \leq p-1 \), \( (B_n \times Y_0) \cap (B_q \times P_0) \)
= \emptyset by 3.4, so that \( x \in (B_n \times Y_0) \cap (B_{q,p} \times P_0) \) implies \( x \in B_{q-k} \times A_k \times o^{k-1} \times P_0 \), with \( 2 \leq q-p+1 \leq k \leq q-n \). Since \( o \in A_1 \) and \( x_{q-k+1} \in A_k \) (\( A_k \cap A_1 = \emptyset \)), we have \( x_{q-k+1} \neq o \), showing that at least one of the coordinates \( x_{p+1}, \ldots, x_p \) is \( \neq o \). However, \( x \in B_n \times Y_0 \) implies \( (x_{p+1}, \ldots, x_p, x_{p+1}, \ldots) \in Y_0 \) and thus \( (x_{p+1}, \ldots) = O \) (see 2.1); a fortiori \( (x_{q+1}, \ldots) = O \).

3.6.

\[
\begin{align*}
(19) \quad Y_{p+1} \cap (B_p \times P_0) &= B_p \times Y_1, \\
(20) \quad Q_{p+1} \cap (B_p \times P_0) &= B_p \times Q_1, \\
& \quad P_{p+1} \cap (B_p \times P_0) = B_p \times P_1.
\end{align*}
\]

Notice first that \( x \in Y_1 \) implies \((x_2, x_3, \cdots) \in Y_0 \subseteq Y_1 \). Therefore, \( x \in Y_{p+1} \) implies \((x_{p+1}, \cdots) \in Y_1 \) (see (7)); this proves (19). In order to prove the first relation in (20) (proof of the second relation is analogous), notice first that, for \( n \geq p+1 \), \( B_{n, p+1} \times P_0 = (B_n \times P_0) \cup (B_p \times A_{n-p} \times o^{n-p-1} \times P_0) \). Using 3.4, we conclude that \( x \in (B_{n, p+1} \times P_0) \cap (B_p \times P_0) \) implies \( x \in B_p \times A_{n-p} \times o^{n-p-1} \times P_0 \subseteq B_p \times Q_1 \). If on the other hand \( x \in (B_n \times Y_0) \cap (B_p \times P_0) \), \( n \leq p \), then \((x_{p+1}, \cdots) \in Y_0 \) and thus \( x \in B_p \times Y_0 \subseteq B_p \times Q_1 \); this proves \( \subseteq \) in (20). The other inclusion follows from the fact that, for \( n \geq 1 \), \( B_p \times B_{n+1} \times P_0 = B_p \times A_n \times o^{n-1} \times P_0 \) is the first term of \( B_{p+n, p+1} \times P_0 \subset Q_{p+1} \).

3.7.

\[
\begin{align*}
(21) \quad Y_{p+1} &= (Y_p \setminus (B_p \times P_0)) \cup (B_p \times Y_1), \\
Q_{p+1} &= (Q_p \setminus (B_p \times P_0)) \cup (B_p \times Q_1), \\
P_{p+1} &= (P_p \setminus (B_p \times P_0)) \cup (B_p \times P_1).
\end{align*}
\]

(21) is an immediate consequence of (7) and (19). To prove the first relation of (22) (the second is proved analogously) notice that the first summand in (12) is also contained in the expansion for \( Q_{p+1} \). Furthermore, for \( n \geq p+1 \), \( B_{n, p} \times P_0 \subseteq B_{n, p+1} \times P_0 \subset Q_{p+1} \). Since the only remaining term in (12) is \( B_p \times P_0 \), we conclude that \( Q_p \setminus (B_p \times P_0) \subseteq Q_{p+1} \). This and (20) prove \( \subseteq \) in (22). The other inclusion follows from (14) and (20).

3.8. The following sets (23) and (24) are compact, \( q \geq p \),

\[
\begin{align*}
(23) \quad (Q_p \setminus (B_{q,p} \times P_0)) &\cup (B_{q,p} \times O), \\
(24) \quad (Y_p \setminus (B_{q,p} \times P_0)) &\cup (B_{q,p} \times O).
\end{align*}
\]

It suffices to prove that (23) is compact, the assertion for (24) will then follow (using the fact that \( Y_p \) is compact and \( Y_p \subset Q_p \)).

Given a sequence \( x^1, \cdots, x^k, \cdots \) of points of \( (Q_p \setminus (B_{q,p} \times P_0)) \) we can assume that it converges towards a limit \( x \in Q_p \) (because \( Q_p \) is compact); we have to show that \( x \) belongs to the set (23). This is certainly the case if \( x \) is not in \( B_{q,p} \times P_0 \). Assume therefore that \( x \in Q_p \cap (B_{q,p} \times P_0) \). If \( x^k \in b \times P_0 \), \( b \in B_{m,p}, m \geq p, m \neq q \), replace \( x^k \) (in the sequence) by \( y^k = b \times O \subseteq U_{n-1}^{p+1} (B_n \times Y_0) \) (see 8)); notice also that \( b \times O \in B_{m,p} \times P_0 \) and thus does not belong to \( B_{q,p} \times P_0 \) (see 3.4). There can only be finitely many terms \( x^k \) in a given
If \(q > p\), our assertion follows immediately from (25) and (18). In the case \(q = p\), we have to prove that \(x_{p+1} = x_{p+2} = \cdots = o\). Suppose on the contrary that there is an \(r \geq 1\) with \(x_{p+r} \neq o\). Let \(k\) be so large that \(y^k_{p+r+1} \neq o\), too. Since \(y^k_{n+1} \in B_n \times Y_0\), for some \(0 \leq n \leq p-1\), it follows that \((y^k_{n+1}, \ldots, y^k_{p+r}, \ldots) \in Y_0\) and thus \(y^k_{n+1} = \cdots = y^k_{p+r-1} = o\). Hence, \(y^k \in B_n \times o^{p-n} \times o^p \times P_0 \subset B_p \times P_0\), contradicting the fact that \(y^k\) does not belong to \(B_p \times P_0\).

3.10.

(26) \((B_p \times P_0) \cap \overline{Y} = B_p \times \overline{Y}\).

Let \(x \in (B_p \times P_0) \cap \overline{Y}\). Since \(\overline{Y} \subset Q_{p+1}\) we conclude (from (20)) that \(x \in B_p \times Q_1\). Hence, \(x\) is either in \(B_p \times Y_0 \subset B_p \times \overline{Y}\) or in \((B_p \times B_{n_1} \times P_0) \cap \overline{Y}\), for an \(n_1 \geq 1\). Since also \(\overline{Y} \subset Q_{p+1+n_1}\), we see that, in the second case, \(x \in B_p \times B_{n_1} \times X_0\) (notice that, by 3.3, \(B_p \times B_{n_1} \subset B_p \times B_{n_1} \subset B_{n_1+n_1}\)) and thus either \(x \in B_p \times B_{n_1} \times Y_0 \subset B_p \times \overline{Y}\) or \(x \in (B_p \times B_{n_1} \times B_{n_2} \times P_0) \cap \overline{Y}\), for an \(n_2 \geq 1\). Continuing this argument we conclude that either \(x \in B_p \times \overline{Y}\) or there is a sequence \(n_1, n_2, \ldots \geq 1\), such that \((x_{p+1}, \ldots, x_{n_k}) \in B_{n_1+\cdots+n_k}\). However, in this last case, points \((x_{p+1}, \ldots, x_{n_k}, o, o, \cdots) \in B_{n_1+\cdots+n_k} \times O \in \overline{Y}\) converge to \((x_{p+1}, \cdots, \cdots)\), proving again that \(x \in B_p \times \overline{Y}\). In order to prove the other inclusion in (26) it suffices to observe that \(B_p \times Y \subset (B_p \times P_0) \cap Y\) is an immediate consequence of (4) and 3.3.

4. Lemma 6. Every loop \(f\) in \(\overline{Y}\) (based at \(O\)) can be deformed in \(X\) into a standard loop (contained in \(Y_0 \subset X_0\)).

4.1. According to (17), \(f\) can be considered as a loop of \(Q_p\), for every \(p = 0, 1, 2, \cdots (Q_0 = P_0)\). We shall define now deformations \(F^p\) of \(f\) (in \(Q_p\)) such that

(i) \(p f(x) \in b \times P_0\), \(b \in B_n p\), \(n \geq p \geq 1\), implies \(F^p(x, t) \in b \times P_0\) and \(F^p(x) = F^p(x, 1) \in b \times Y_0 \subset \overline{Y}_p\).

(ii) \(p f(x) \in Q_p \setminus (\bigcup_{n \geq p} B_{n_p} \times P_0)\), \(p \geq 1\), implies \(F^p(x, t) = f(x) \in Y_p\), requiring in addition that \(f^0\) be standard. (i) and (ii) imply \(\text{diam } F^p \leq 2^{-p}\) and thus \(\lim F^p = f\). The next step will consist in defining homotopies \(G^p_{p+1}\), connecting
4.2. $F^0$ exists by Lemma 3. For $p \geq 1$, let $R_p = (Q_p \setminus (B_p \times P_0)) \cup (B_p \times O)$ and let $b \in B_p$. Obviously $Q_p = [R_p \cup ((B_p \setminus \{b\}) \times P_0)] \cup (b \times P_0)$; both summands are compact (see 3.8) and their intersection is the single point $b \times O$. Since $f \subseteq Q_p$, the set $U = \{x \mid f(x) \in (b \times P_0) \setminus \{b \times O\}\} \subseteq I$ is open and $f(\overline{U} \setminus U) = b \times O$. If $V$ is any one of the components of $U$, then $V \setminus V \subseteq \overline{U} \setminus U$, so that $f |_{V}$ is a loop in $b \times P_0$, based at $b \times O$. We can apply now Lemma 5 (the part concerning $P_0$) to obtain homotopies deforming loops $f |_{V}$ into loops of $b \times Y_0$ in such a way that 1.1.4 is applicable and produces a deformation of $f$, defined over the entire interval $I$. Repeating the process with all $b$ of the finite set $B_p$, we arrive at a deformation, satisfying $(i)_p$, for $n = p$, and having the following property (“approximating” property $(ii)_p$): for $f(x) \in Q_p \setminus (B_p \times P_0)$, the deformation equals $f(x)$.

4.3. Now repeat the process described in 4.2, this time applied to the loop we obtained in 4.2 and to all $b \in B_{p+1}$ (we consider $R_{p+1} = (Q_p \setminus (B_{p+1} \times P_0)) \cup (B_{p+1} \times O)$ and the decomposition $Q_p = [R_{p+1} \cup ((B_{p+1} \setminus \{b\}) \times P_0)] \cup (b \times P_0)$). The resulting deformation affects only the set $B_{p+1} \times P_0$ (disjoint to $B_p \times P_0$) and does not interfere with the gain (in the direction of obtaining $(i)_p$ and $(ii)_p$) achieved in the preceding step. Defining in this manner a sequence of deformations and passing finally to the limit (1.1.2), one arrives at a deformation $F_p$, satisfying $(i)_p$ and $(ii)_p$ (1.1.2 is applicable because the diameter of the deformation in the step involving $B_n \times P_0$ is $\leq 2^{-n}$).

4.4. We proceed now to define $G^{p+1}$. Consider again $b \in B_p$ and the sets $R_p$ and $U$, defined as above. Points of $\overline{U} \setminus U$ can be approached arbitrarily close from $U$ as well as from $\cap U$. Since $F^p$ maps $U$ in $b \times P_0$ and $\cap U$ in $(Q_p \setminus (b \times P_0)) \cup (b \times O)$ (due to $(i)_p$ and $(ii)_p$), and these two sets are compact (see 3.8), we conclude that $F^p((\overline{U} \setminus U) \times I)$ is contained in their intersection, i.e.

$$F^p((\overline{U} \setminus U) \times I) = b \times O, \quad b \in B_p.$$  

In a similar way, using $(i)_{p+1}$ and $(ii)_{p+1}$, one can see that $F^{p+1}(U \times I) \subseteq b \times P_0$ and $F^{p+1}((\cap U) \times I) \subseteq (Q_p \setminus (b \times P_0)) \cup (b \times O)$ and therefore

$$F^{p+1}((\overline{U} \setminus U) \times I) = b \times O, \quad b \in B_p.$$  

Now let $V$ be any one of the components of $U$. Then $F^p |_{\overline{V} \times I}$ and $F^{p+1} |_{\overline{V} \times I}$ are homotopies in $b \times P_0$, connecting the loop $f |_{\overline{V}}$ with the loops $f^p |_{\overline{V}}$ and $f^{p+1} |_{\overline{V}}$ respectively; these loops are therefore homotopic in $b \times P_0$. Moreover, $f^p(\overline{V}) \subseteq b \times Y_0 \subseteq b \times X_0$, by $(i)_p$, while $(i)_{p+1}$ and $(ii)_{p+1}$ imply $f^{p+1}(\overline{V}) \subseteq Y_{p+1}$. Applying (19) we conclude that actually $f^{p+1}(\overline{V}) \subseteq b \times Y_1 \subseteq b \times X_0$. It follows (Theorem 1) that $f^p |_{\overline{V}}$ and $f^{p+1} |_{\overline{V}}$ are homotopic already in $b \times X_0$.

Notice now that $f^p$ and $f^{p+1}$ are uniformly continuous on $I$ and therefore,
for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \text{diam } V \leq \delta \) implies \( f^p \subset (b \times X_0) \cap U(b \times O, \epsilon) \) and \( f^{p+1} \subset (b \times X_0) \cap U(b \times O, \epsilon) \). Now take into account Lemma 5 (the part concerning \( X_0 \)) and Lemma 2. It is clear that we can define homotopies \( F_V \), connecting \( f^p \mid V \) and \( f^{p+1} \mid V \) in \( b \times X_0 \), for every \( V \), in such a way that 1.1.4 is applicable (with \( M = Y_p \cup (b \times X_0) \), base point \( b \times O \), open set \( U \), mapping \( f^p : I \rightarrow M \) and homotopies \( F_V \)), producing a homotopy in \( Y_p \cup (b \times X_0) \subset X_p \), defined over \( I \times I \). Repeating the whole construction for every \( b \in B_p \), we arrive at a homotopy contained in \( Y_p \cup (B_p \times X_0) \subset X_p \) and equal to \( f^p(x) \) on \( \{ x | f(x) \in (B_p \times P_0) \} \); \( f^p \) is deformed by this homotopy into a map which coincides with \( f^{p+1} \) on \( \{ x | f(x) \in (B_p \times P_0) \} \).

4.5. Repeat now the process described in 4.4 with all \( b \in B_{p+1} \) \( (R_p \) has to be replaced by \( R_{p+1} \), \( U = \{ x | f(x) \in (b \times P_0) \} \}) and apply 1.1.4 to the loop obtained from \( f^p \) as the result of the deformation described in 4.4. Continue this process for \( B_{p+2} \), \ldots. The step involving \( B_k \), \( k \geq p \), affects only the set \( \{ x | f(x) \in (B_k \times P_0) \} \) and has a diameter \( \leq 2^{-k} \); the resulting loop coincides with \( f^{p+1}(x) \) on \( \{ x | f(x) \in \bigcup_{n=p}^{k} (B_n \times P_0) \} \). Applying 1.1.2 (and (i), (ii), (i)_p, (ii)_p, (ii)_p+1, (ii)_p+1) we conclude, finally, that there is a homotopy \( G^p \) contained in \( X_p \) and connecting \( f^p \) with \( f^{p+1} \); if \( f(x) \in b \times P_0 \), \( b \in B_n \times P_0 \), \( n \geq p \), then \( G^p(x, t) = f^p(x) = f^{p+1}(x) \). Consequently, \( \text{diam } G^p \leq 2^{-p} \), so that 1.1.2 is applicable.

Notice that the deformation \( G^p \), that one obtains applying 1.1.2 to the sequence \( G^{p+1}, G^{p+2}, \ldots \), has some special properties that we state here (for future usage):

**Lemma 7.** Given any loop \( f \) in \( \overline{Y} \) (based at \( 0 \)) and any integer \( p \geq 0 \), there is a loop \( f^p \subset Y_p \) and a homotopy \( G^p \subset \overline{X} \), connecting \( f^p \) and \( f \), and having the property that, for \( f(x) \in b \times P_0 \), \( b \in B_n \times P_0 \), \( n \geq p \), we have \( G^p(x, t) \subset b \times P_0 \), while otherwise \( G^p(x, t) = f^p(x) = f(x) \).

5. If a sequence of (Euclidean) cells in a metric space has the property that the diameters of the cells tend to zero, we shall speak of a 0-sequence of cells.

**Lemma 8.** \( \overline{X} \) can be obtained from \( \overline{Y} \) by attaching a 0-sequence of disjoint 2-dimensional cells.

We precede the proof by some consequences.

**Lemma 9.** Every loop \( f \) in \( \overline{X} \) (based at \( 0 \)) can be deformed in \( \overline{X} \) into a loop of \( \overline{Y} \).

A proof follows from Lemma 8 and Propositions 1.1.3 and 1.1.2.

**Theorem 3.** \( \overline{X} \) is an arcwise connected subset of \( P_0 \). The inclusions \( X_0 \subset \overline{X} \subset P_0 \) induce isomorphisms of the corresponding fundamental groups.

Proof follows from Lemmas 9 and 6 and Theorem 1.
Proof of Lemma 8. If \( \sigma \) is a 2-cell (open) of \( K \), then \( \sigma \times O \) is a 2-cell imbedded in \( P_o \) and \( \cap K \times O \subset X_0 \). Let \( L_n \) be the subdivision of \( L \) obtained by considering all points of \( A_1 \cup \cdots \cup A_n \) as vertexes of \( L_n \), \( n \geq 1 \). If \( \sigma \) is a 1-cell (open) of \( L_n \) and \( \tau \) a 1-cell (open) of \( L \), then \( \sigma \times O^{n-1} \times \tau \times O \) is a 2-cell imbedded in \( P_o \) and \( \cap L \times O^{n-1} \times L \times O \subset X_0 \). The described 2-cells will be referred to in the sequel as 2-cells of the first and of the second kind respectively. It is not difficult to see that these cells are disjoint one from each other and from \( Q_i \), while their boundaries lie in \( Y_i \subset Q_i \), e.g. in the case of cells of the second kind, the boundary is lying in \( U_{k<\eta} (A_k \times O^{n-1} \times L \times O) \cap (L \times O) \subset Y_i \). Moreover, it is easy to see that all the described 2-cells can be ordered in a sequence \( e_1, e_2, \cdots \) with \( \lim \text{diam } e_n = 0 \). (Observe that the set \( A \) from §3, (1) is dense in \( L \) and that there are only finitely many cells of the first kind.) Finally,

\[
P_1 = Q_1 \cup \bigcup_{n=1}^{\infty} e_n,
\]

showing that \( P_1 \) is obtained from \( Q_1 \) by attaching the described 0-sequence of cells. We prove next

\[
P_{p+1} = Q_{p+1} \cup (\bigcup e_n) \cup (\bigcup B_1 \times e_n) \cup \cdots \cup (\bigcup B_p \times e_n).
\]

The inclusion \( \supset \) is immediate because of \( e_n \subset X_0 \). The inclusion \( \subset \) can be proved by induction on \( p \), using (30) and both relations in (22). Finally,

\[
\overline{X} = \overline{V} \cup (\bigcup e_n) \cup (\bigcup B_1 \times e_n) \cup \cdots \cup (\bigcup B_p \times e_n) \cup \cdots.
\]

Recall the relations (16) and (17). If \( x \in \overline{X} \backslash \overline{V} \), let \( p+1 \) be the smallest integer such that \( x \) does not belong to \( Q_{p+1} \). Since \( x \in \overline{X} \subset P_{p+1} \), it follows from (31) that \( x \) belongs to the set on the right side of (32). The other inclusion is obvious, since \( e_n \subset X_0 \).

Observe now that \( Q_i \cap e_n = \emptyset \) implies (by (20)) that \( \overline{V} \cap (B_p \times e_n) \subset Q_{p+1} \cap (B_p \times e_n) = \emptyset \). It implies also \( (B_p \times e_n) \cap (B_q \times e_m) = \emptyset \), for \( p > q \). Indeed, if \( x \in (B_q \times e_m) \), then \( (x_{q+1}, \cdots) \in e_m \) and thus obviously \( (x_{q+1}, \cdots) \in Y_0 \). Furthermore, \( p > q \) implies \( (x_{p+1}, \cdots) \in Y_0 \subset Q_i \), while \( x \in (B_p \times e_n) \) would imply \( (x_{p+1}, \cdots) \in e_n \). The boundary of \( e_n \) lies in \( Y_i \), therefore, the boundary of \( B_p \times e_n \) lies in \( B_p \times Y_i \subset Y_p \subset \overline{V} \). Finally, since \( \text{diam } e_n \) tends towards zero, the cells appearing in (32) can be ordered into a 0-sequence.

Notice that Lemma 8 proves also that points of \( \overline{X} \backslash \overline{V} \) have Euclidean 2-neighborhoods.

4. Local properties of \( \overline{X} \). 4.1. We shall now consider particular open sets of \( \overline{X} \), referred to in the sequel as standard open sets. A standard open set of \( \overline{X} \) is the intersection of \( \overline{X} \) and an open set \( U \) of \( P_9 \) of the form \( U = U_1 \times \cdots \times U_q \times P_9 \), where \( U_n \) are open in \( K \), provided that one can find a point \( b \times O \), \( b \in B_p \), \( p \leq q \), contained in \( U \). Moreover, if \( b_n \) denotes the \( n \)th coordinate of \( b \times O \), \( U_n \cap K \) should admit a cell-preserving (with respect to \( K \) deform-
tion retraction to \( U_n \cap L \) and \( U_n \cap L \) should be contractible to \( b_n \); for \( n < p \), this contraction should be cell-preserving with respect to the subdivision \( L_{p-n} \). Notice that these requirements imply that, for \( b_n \neq o \), \( U_n \) can not contain \( o \) and that, for \( n < p \), \( U_n \) can not contain points of \( A_1 \cup \cdots \cup A_{p-n} \) except \( b_n \), which may belong to that set.

4.2. **Lemma 10.** Standard open sets of \( \overline{X} \) form a basis of neighborhoods at every point \( x \) belonging to \( \overline{Y} \).

If \( x \in \overline{Y} \) and \( W \) is an open set of \( P_0 \), \( x \in W \), we have to find a standard open set \( U \cap \overline{X} \) such that \( x \in U \subseteq W \). Clearly, we can find \( V = V_1 \times \cdots \times V_q \times P_0, x \in V \subseteq W \), such that \( V_x \) is open and admits a cell-preserving deformation retraction of \( V_x \cap K \) to \( V_x \cap L \) and a contraction of \( V_x \cap L \) into \( x_n \), \( n < q \) (see 1.2).

Assume now first that \( x \in Y \) or more precisely that \( x \in B_p \times o^{m} \times L \times O \), see § 3, (4) and § 2, (3). We can also assume that \( q \geq p + m + 1 \). Let \( r \) be such an integer that \( (A_1 \cup \cdots \cup A_r) \cap V_{p+m+1} \neq \emptyset \) (\( A \) is dense in \( L \)). If \( x_{p+m+1} \) does not belong to \( A_1 \cup \cdots \cup A_r \), it belongs to a 1-cell of \( L_r \) and one of the endpoints of that 1-cell has to be in \( V_{p+m+1} \); denote that endpoint by \( a \). If \( x_{p+m+1} \in A_1 \cup \cdots \cup A_r \), put \( a = x_{p+m+1} \); thus, in all cases \( a \in A_k, k \leq r \). It is now possible to choose a new neighborhood \( U_{p+m+1} \subseteq V_{p+m+1} \) around \( a \), containing \( x_{p+m+1} \) and satisfying the requirements concerning retraction and contraction with respect to \( L_k \). Let \( b = (x_1, \cdots, x_{p+m}) \times a \times o^{k-1} \subseteq B_{p+m} \times A_k \times o^{k-1} \subseteq B_{p+m+k} \). Replace \( V_1, \cdots, V_{p+m}, V_{p+m+2}, \cdots, V_q \) by smaller neighborhoods \( U_1, \cdots, U_{p+m}, U_{p+m+2}, \cdots, U_q \) around \( x_1 = b_1, \cdots, x_{p+m} = b_{p+m}, x_{p+m+2} = b_{p+m+2}, \cdots, x_q = b_q \); these neighborhoods should be chosen so as to fulfill the requirements in the definition of a standard open set. If necessary, one can replace a few terms \( K \) in \( V \) by similar neighborhoods in order to achieve that \( U = U_1 \times \cdots \times U_q \times P_0 \) and \( q \geq p + m + k \).

Assume now that \( x \in \overline{Y} \setminus Y \). Since \( \overline{Y} \subseteq Q_q \), it follows (by (12)) that \( x \in b \times P_0 \), where \( b \in B_n \times B_s, n \geq q \). \( x \) and \( b \times O \) coincide in the first \( n \) coordinates, it is therefore easy to replace \( V_1, \cdots, V_q, K, \cdots, K \) by smaller neighborhoods \( U_1, \cdots, U_q, U_{q+1}, \cdots, U_n \) (containing \( x_1, \cdots, x_n \) respectively) in such a way that \( \overline{X} \cap (U_1 \times \cdots \times U_q \times P_0) \) is a standard neighborhood centered at \( b \times O \) and containing \( x \).

**Lemma 10** and the following **Theorem 4** prove the assertions of **Theorem 2** concerning neighborhoods of points in \( \overline{Y} \):

**Theorem 4.** A standard open set \( U \cap \overline{X} \) is connected. The inclusion \( U \cap \overline{X} \subseteq \overline{X} \) induces a monomorphism of corresponding fundamental groups. The image of \( \pi_1(U \cap \overline{X}) \) in \( \pi_1(\overline{X}) \) under this monomorphism is isomorphic to \( \pi_1(P_0) \cong \pi_1(\overline{X}) \).

The proof is based on two lemmas.
4.3. Lemma 11. Consider all the cells \( b \times e_n, b \in B_r, r = 0, 1, \ldots \), which have points in common with \( U \cap \overline{X} \), but \( U \cap \overline{X} \) does not contain their entire closure \( b \times (e^n) \). The set obtained from \( U \cap \overline{X} \) by removing exactly these cells is a deformation retract of \( U \cap \overline{X} \).

The fact that these cells are disjoint and can be ordered in a sequence with diameters tending to zero, makes it sufficient (see 1.1.2) to prove the corresponding proposition involving the removal of only one such cell, denoted henceforth by \( c \times e, c \in B_r \) (\( U \) and \( b \in B_p \) as in 4.1).

We assume that \( r < q \), the other case being trivial. If \( c = \sigma \times O \), i.e. of the first kind, we have \( (c \times e) \cap U = c \times (\sigma \cap U_{r+1}) \times O \). It suffices now to subject \( U_{r+1} \) to a (cell-preserving) deformation retraction into \((\sigma \cap U_{r+1}) \), \( \sigma \) being the boundary of \( \sigma \).

If \( c = \sigma \times o^{n-1} \times \tau \times O \), i.e. of the second kind, we have either:

1. \( r + n + 1 \leq q, (c \times e) \cap U = c \times (\sigma \cap U_{r+1}) \times o^{n-1} \times (\tau \cap U_{r+n+1}) \times O \) or
2. \( r + n + 1 > q, (c \times e) \cap U = c \times (\sigma \cap U_{r+1}) \times o^{n-1} \times \tau \times O \).

Observe that \( \sigma \cap U_{r+1} \) and \( \tau \cap U_{r+n+1} \) are simple arcs, while \( \tau \) is a simple closed curve. Therefore, it is an elementary task to verify that if in the case (1) \( o \in U_{r+n+1} \), then \( (c \times e) \cap U \) admits a deformation retraction to \( (c \times \hat{e}) \cap U \), where \( \hat{e} \) is the boundary of \( e \). Similarly, if \( U_{r+1} \) contains exactly one end-point of \( \sigma \), then (for (1) as well as for (2)), \( (c \times e) \cap U \) admits a deformation retraction into \( (c \times \hat{e}) \cap U \). We shall show now that at least one of the two cases described is always present.

Assume first (1). Let \( b_i \) denote the \( i \)-th coordinate of \( b \times O \subseteq U, b \in B_p \). If \( b_{r+n+1} = o \), then \( o \subseteq U_{r+n+1} \), because \( b_{r+n+1} \subseteq U_{r+n+1} \). Suppose now that \( b_{r+n+1} \neq o \) and thus \( r + n + 1 \leq p \). By 3.3.2 we conclude that \( (b, \ldots, b_{r+n}) \in B_{r+n} \), so that 3.3.1 gives \( b_{r+1} \in A_1 \cup \cdots \cup A_n \), showing that \( b_{r+1} \) is a vertex of \( L_n \).

However, \( \sigma \) is by supposition a 1-cell of \( L_n \), so that \( b_{r+1} \) does not belong to \( \sigma \).

Since \( b_{r+1} \in U_{r+1} \) and \( U_{r+1} \cap \sigma \neq \emptyset, U_{r+1} \) contains at least one end-point of \( \sigma \). On the other hand, \( U_{r+1} \) can contain at most one point of the set \( A_1 \cup \cdots \cup A_{p-r} \) (see 4.1), while both end-points of \( \sigma \) belong to its subset \( A_1 \cup \cdots \cup A_n \) \((n \leq p-r-1) \).

Assume now (2). If \( p \leq r \), then \( b_{r+1} = o \) is disjoint with \( \sigma \), hence, \( U_{r+1} \) contains at least one end-point of \( \sigma \). However, if \( U_{r+1} \) would contain both end-points, i.e. entire \( \sigma \), then \( U \) would contain entire \( c \times \hat{e} \), contrary to our assumption. Suppose now that \( p > r \). \( b_{r+1} \) is now the \((r+1)\)-th coordinate of \( b \in B_p \) and thus 3.3.1 gives \( b_{r+1} \in A_1 \cup \cdots \cup A_{p-r} \). Since, in this case, \( n > q-r-1 \geq p-r-1 \) or \( n \geq p-r \), we see that \( b_{r+1} \) is a vertex of \( L_n \) and thus disjoint with \( \sigma \). The rest of the argument is as above.

(1) For the definition of cells \( e_m \), see the proof of Lemma 8.
4.4. Lemma 12. If \( U \cap \overline{X} \) is a standard open set, then \( U \cap \overline{Y} \) and \( U \cap \overline{X} \) are connected. Every loop \( f \) in \( \overline{X} \cap U \) can be deformed, inside \( \overline{X} \cap U \), into a loop \( g \) of \( b \times o^{q-p} \times Y_0 \), such that \( (g_{s+1}, \ldots) \) is a standard loop of \( Y_0 \).

**Proof.** In view of Lemma 11, it suffices to prove that \( \overline{Y} \cap U \) is connected and that every loop \( f \) of \( \overline{Y} \cap U \) admits a deformation of the kind required by Lemma 12 (in order to “push” the loop out of the cells \( c \times e \) whose closure is in \( \overline{X} \cap U \), apply 1.1.3 and 1.1.2). Observe now that \( \overline{Y} \subseteq \overline{Q} \), and if \( f \) has points in \( c \times \overline{P}_0 \), \( c \in \overline{B}_{n-q} \), \( n \geq q \), then \( (c_1, \ldots, c_q) \in U_1 \times \cdots \times U_q \), hence, \( (c \times \overline{P}_0) \subseteq U \). By 3.3.9, we know that \( (c \times \overline{P}_0) \cap \overline{Y} = (c \times \overline{Y}) \cap \overline{U} \cap \overline{Y} \) is connected. Since \( c \times \overline{O} \subseteq (\bigcup_{n \leq q-1} B_n \times \overline{Y}_0) \cap U \), the connectedness of \( \overline{Y} \cap U \) will follow from that of \( (\bigcup_{n \leq q-1} B_n \times \overline{Y}_0) \cap U \). As to the deformation of \( f \), apply Lemma 7 to \( f \) and \( q \); the resulting deformation \( G^q \) takes place in \( \overline{X} \cap U \) (due to special properties of \( G^q \) listed in Lemma 7) and enables us to assume hereafter that \( f \subseteq Y_q \cap U \).

Assume now that \( b_q = o \) (qth coordinate of \( b \times O \), \( b \in B_p \)). If \( c \in B_{q-1} \) and \( (c \times \overline{Y}_0) \cap U \neq \emptyset \), then it follows easily that \( (c \times \overline{Y}_0) \cap U = (c \times (L \cap U_q) \times O) \cup (c \times o \times \overline{Y}_0) \) (observe that \( o = b_q \in U_q \) and \( c \in U_1 \times \cdots \times U_{q-1} \)). Both of the terms are, obviously, connected and have in common the point \( c \times O \subseteq \bigcup_{n \leq q-2} B_n \times \overline{Y}_0 \). The question is thus reduced to proving that \( (\bigcup_{n \leq q-2} B_n \times \overline{Y}_0) \cap U \) is connected. We can continue this process one step further if \( b_{q-1} = o \).

We now distinguish two cases. Either we meet a coordinate \( b_r \neq o \), \( r \geq 2 \), and have to prove that \( (\bigcup_{n \leq r-1} B_n \times \overline{Y}_0) \cap U \) is connected (obviously, \( r \leq p \)), or we have to prove the obvious statement that \( \overline{Y}_0 \cap U \) is connected (in the last case \( b \times O = b_1 \times O \subseteq U \), \( b_1 \in L \)).

In order to prove our assertion in the first case, let us prove that \( b_r \neq o \), \( 2 \leq r \leq p \), \( b \in B_p \), implies

\[
Y_n \cap U \subseteq (Y_{n-1} \cap U) \cup (b_1 \times \cdots \times b_{r-1}) \times (L \cap U_r) \times O,
\]

\( n = 1, \ldots, r - 1. \)

Indeed, let \( x \in (Y_n \cap U) \setminus (Y_{n-1} \cap U) \subseteq (B_{n-1} \times Y_1) \cap U \) (see §3,(21)). Then \( (x_1, \ldots, x_{n-1}) \in B_{n-1} \) and thus (by 3.3.1) \( x_s \in A_1 \cup \cdots \cup A_{n-s} \subseteq A_1 \cup \cdots \cup A_{p-s}, s \leq n-1 \). Since \( x_s \in U_s \), it follows from 4.1 that \( x_s = b_s, s = 1, \ldots, n-1 \). Assume now more precisely that

\[
x \in B_{n-1} \times A_k \times o^{k-1} \times Y_0,
\]

\( k \in \{0, 1, \ldots\} \).

Since \( x_r \in U_r \) and \( b_r \neq o \), 4.1 implies that \( x_r \neq o \). We infer from (4) that this is possible only if \( n + k \leq r \) and that \( (x_{r+1}, \ldots, x_r, \ldots) \in Y_0 \). Therefore, \( x_n \in A_k \cup A_1 \cup \cdots \cup A_{r-n} \subseteq A_1 \cup \cdots \cup A_{p-n} \). This fact, together with \( x_n \in U_n \), proves that \( x_n = b_n \) (by 4.1). Finally, \( x_r \neq o \) implies that all other coordinates of \( (x_{r+1}, \ldots) \) equal \( o \). To \( x_{r+1}, \ldots, x_{r-1} \), we apply again the argument involving 4.1 and obtain (3). It is easy to see that (3) remains valid for \( n = 0 \) if we put \( Y_{-1} = \emptyset \). Applying (3) subsequently with \( n = r - 1, \ldots, 1, 0 \), we obtain
Notice now that \((b_1 \times \cdots \times b_{r-1}) \in B_{r-1}\) (see 3.3.2) so that the set on the right side of (5) is contained in \((\bigcup_{n \leq r-1} B_n \times Y_0) \cap U \subset Y_{r-1} \cap U\) and we obtain

\[
\bigcup_{n \leq r-1} B_n \times Y_0 \cap U = Y_{r-1} \cap U = b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times O;
\]

the examined set is thus an arc and therefore is connected. This completes the proof of the connectedness of \(Y \cap U\) and \(X \cap U\).

Consider now the loop \(f \subset (Y_q \cap U)\) and suppose that \(b_q = 0\). Observe that for \(c \in B_{q-1}\), \((c \times Y_1) \cap U \subset c \times (L \cap U_q) \times Y_0\) and that \((c \times Y_1) \cap U \neq \emptyset\) implies \(c \times (L \cap U_q) \times Y_0 \subset (c \times X_0) \cap U \subset \bar{X} \cap U\). Define now a deformation of the set \((Y_q \setminus (c \times P_0)) \cup (c \times (L \cap U_q) \times Y_0)\) by taking identity on the first summand, on the second summand we keep all the coordinates fixed except the \(q\)-th which we subject to a contraction of \(L \cap U_q\) to the point \(b_q = 0\), this point being kept fixed during the deformation \((c \times O\) is the only common point of the two summands). The described deformation induces a deformation of the loop \(f\), which takes place in \(\bar{X} \cap U\) and brings \(f\) into \((Y_q \setminus (c \times P_0)) \cup (c \times O \times Y_0)\). Repeating the process for all \(c \in B_{q-1}\), we obtain a deformation of \(f\) in \(\bar{X} \cap U\), giving a loop in \(Y_{q-1} \cap U\) (see §3,(21)). We can continue this reducing process one step further if \(b_{q-1} = 0\) (by similar arguments), etc. If there is no \(b_r \neq 0\), \(r \geq 2\), then we have only to see that a loop \(f \subset Y_0 \cap U\) can be brought to the required form. Suppose now that there is a \(b_r \neq 0\), \(r \geq 2\). Then we can assume that \(f \subset (Y_q \cap U)\). Since

\[
Y_r \cap U = \left[\left(Y_{r-1} \setminus (B_{r-1} \times P_0)\right) \cap U\right] \cup \left[(B_{r-1} \times Y_1) \cap U\right]
\]

and in this case \(b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times O \subset (B_{r-1} \times Y_0) \subset (B_{r-1} \times P_0) \cap Y_{r-1}\), we infer from (6) that the first term in (7) is empty and thus \(Y_r \cap U = (B_{r-1} \times Y_1) \cap U\). However, \(f\) being connected, it has to lie entirely in a set \(c \times Y_1\), \(c \in B_{r-1}\). Since \(b \times O\) is the base point of \(f\) we conclude that \(c = (b_1, \cdots, b_{r-1})\), hence, \(f \subset (b_1 \times \cdots \times b_{r-1} \times Y_1) \cap U \subset b_1 \times \cdots \times b_{r-1} \times (L \cap U_r) \times (Y_0 \cap (U_{r+1} \times \cdots \times U_q \times P_0)) \subset \bar{X} \cap U\). A deformation of this set, determined by a contraction of \(L \cap U_r\) to \(b_q\), induces a deformation of \(f\), in \(\bar{X} \cap U\), into a loop of

\[
(b_1 \times \cdots \times b_r \times Y_0) \cap U \subset (b_1, \cdots, b_r) \times [(L \cap U_{r+1}) \times O) \cup (O \times (L \cap U_{r+2}) \times O) \cup \cdots \cup ((O^{\sigma^{-1}} \times (L \cap U_q) \times O) \cup (O^{\sigma^{-r}} \times Y_0)].
\]

All the terms of this set, except the last one, can be contracted to their only common point \(b_1 \times \cdots \times b_r \times O\). These contractions induce a deformation of \(f\) into a loop in \(b_1 \times \cdots \times b_r \times O \cap Y_0 = b \times O^{\sigma^{-r}} \times Y_0\). Since \(b \times O^{\sigma^{-r}} \times X_0 \subset U \cap \bar{X}\), we can apply Lemma 4 to obtain, finally, a loop as required by Lemma 12.

4.5. **Proof of Theorem 4.** Let \(f\) be a loop of \(U \cap \bar{X}\), which is homotopic to \(b \times O\) in \(P_0\). \(f\) can be deformed in \(U \cap \bar{X}\) to a loop \(g\) as in Lemma 12. Clearly,
(g_{q+1}, \cdots) is homotopic to O in P_0 and thus (by Theorem 1) homotopic to O already in X_0. Consequently, f is homotopic to b \times 0 in (b \times O \times X_0) \cap U \subset \overline{X} \cap U. \pi_1(\overline{X} \cap U) \rightarrow \pi_1(P_0) is thus a monomorphism.

Now associate with every loop of \overline{X} \cap U a "standard" loop g of b \times O \times X_0. Two loops, homotopic in \overline{X} \cap U, give rise to loops which are homotopic in b \times O \times X_0. This defines a monomorphism \pi_1(\overline{X} \cap U) \rightarrow \pi_1(b \times O \times X_0), which is clearly an epimorphism, because every loop of b \times O \times X_0 can be deformed, in b \times O \times X_0, into a "standard" loop g (see Lemma 3), which belongs to \overline{X} \cap U. Since b \times O \times X_0 is homeomorphic to P_0, we obtain

\[ \pi_1(\overline{X} \cap U) \cong \pi_1(P_0). \]

4.6. Dimension of X and Y. To complete the proof of Theorem 2, we now prove

\[ \dim Y = 1, \]

\[ \dim X = 2. \]

Since K has at least one 1-cell and \overline{Y} \supseteq Y_0 \supseteq L \times O, we have \dim Y \geq 1. Similarly, \dim X \geq 2, because of \overline{X} \supseteq Y_0 \supseteq L \times L \times O. \dim X \leq 2 is an easy consequence of \dim Y \leq 1 and Lemma 8 (apply the sum theorem of dimension theory). To prove that \dim Y \leq 1, consider open sets \( U = U_1 \times \cdots \times U_q \times P_0 \) of P_0, where \( U_n \) is open in K and \((U_n \setminus U_n)\) intersect L in a finite set, which is disjoint with the countable set A. Sets \( U \cap \overline{Y} \), obviously, form a basis of open sets for \overline{Y}. Since the boundary of \( U \cap \overline{Y} \) (with respect to \overline{Y}) is contained in \((U \setminus U) \cap \overline{Y}\), it suffices to show that \((U \setminus U) \cap \overline{Y}\) is a finite set. Notice now that

\[ \bigcup_{n=1}^q \bigcup_{n=1}^{n-1} \bigcup_{n=1}^{n+1} \bigcup_{n=1}^{q} U_n \times P_0. \]

It is clear that our assertion will follow from this proposition: given a fixed point \( a \in L \setminus A \) and an integer \( p \geq 1 \), the set of all \( x \in \overline{Y} \) with \( x_p = a \) is a finite set. In order to prove this proposition, observe that the \( p \)th coordinate of a point from \( B_{n, p} \times \overline{Y}, n \geq p, \) belongs to \( A \). Therefore, our set has to be contained in \( \bigcup_{n=1}^{p-1} B_n \times Y_0 \) (see §3, (27)). However, if \( c \in B_n, n \geq p \), and \( x \in c \times Y_0 \), then \( (x_{n+1}, \cdots, x_p, \cdots) \in Y_0 \), and since \( x_p = a \) is not in \( A \), \( x_p \) must be different from \( a \), hence, \( (x_{n+1}, \cdots, x_p, \cdots) = o^{p-n-1} \times a \times O \), showing that there is only one such \( x \). This proves the assertion.

5. First singular homology group of the infinite Cartesian product. 5.1. The first singular homology group (with integer coefficients) \( H_1(X) \) of an arcwise connected space \( X \) is the factor group of \( \pi_1(X) \) by the commutator subgroup (Theorem of Poincaré). \( H_1(X) \) is zero if and only if \( \pi_1(X) \) is a perfect group\(^{(8)}\). If \( G_1, G_2, \cdots \) is a sequence of groups, let \( \prod G_n \) denote their (complete) direct product\(^{(9)}\); if \( G_1 = G_2 = \cdots = G \), we use the notation \( \prod G \).

\(^{(8)}\) I.e. a group coinciding with its commutator subgroup.

\(^{(9)}\) Elements of the product are sequences \((g_1, g_2, \cdots)\), \( g_n \in G_n \), all \( g_n \) can be different from the unit; \((g_1, g_2, \cdots) \cdot (h_1, h_2, \cdots) = (g_1 h_1, g_2 h_2, \cdots) \) (see [8, p. 122]).
$X_1, X_2, \cdots$ is a sequence of arcwise connected spaces, then $\prod X_n$ is arcwise connected and $\pi_1(\prod X_n) \approx \prod(\pi_1(X_n))$.

**Lemma 13.** If $G$ is a (nontrivial) perfect finite group, then $\prod G$ is also (nontrivial) perfect.

Since $G$ is finite and perfect, there is an integer $p$, such that every element of $G$ is a product of $p$ commutators (some of which may be trivial, i.e., of type $e e e^{-1} e^{-1}$, $e$ being the unit of $G$). Let $g = (g_1, g_2, \cdots) \in G$ and let

$$g_n = a_{n1}b_{n1}a_{n1}^{-1}b_{n1}^{-1} \cdots a_{np}b_{np}a_{np}^{-1}b_{np}^{-1}, \quad n = 1, 2, \cdots.$$  

Furthermore, let

$$a_k = (a_{1k}, a_{2k}, \cdots), \quad b_k = (b_{1k}, b_{2k}, \cdots), \quad k = 1, 2, \cdots, p.$$  

Then

$$a_k^{-1} = (a_{1k}^{-1}, a_{2k}^{-1}, \cdots), \quad b_k^{-1} = (b_{1k}^{-1}, b_{2k}^{-1}, \cdots),$$

and it is readily verified that

$$g = a_1b_1a_1^{-1}b_1^{-1} \cdots a_pb_p a_p^{-1}b_p^{-1};$$

every $g \in G$ is thus a product of $p$ commutators.

Examples of nontrivial finite perfect groups are provided by the alternating group $A_n$ of degree $n \geq 4$ (see [3, p. 38]); another example is the “binary icosahedral group” (see [11, p. 218]) defined by two generators $a, b$ and relations $a^b = b^a = (ab)^2$.

5.2. If $G_n$ is a sequence of perfect groups (possibly $G_n = G$, for all $n$) and $G_n$ has at least one element $h_n \in G_n$, which is not a product of fewer than $n$ commutators, then $\prod G_n$ is not perfect. It suffices to see that the element $h = (h_1, h_2, \cdots) \in \prod G_n$ is not a product of finitely many commutators. The assumption that $h$ is a product of, say, $r$ commutators, would imply that $h_n$ is a product of $r$ commutators for all $n$. However, if $n > r$, this is in contradiction with the choice of $h_n$.

An example of such a situation is provided as follows. Let $G$ be a perfect nontrivial group (possibly finite); let $G_n$ be the $n$-fold free product $G_n = G * \cdots * G$ and let $h_n \in G_n$ be given by $h_n = g_1 g_2 \cdots g_n$, where $g_k \in G$ and is different from the unit of $G$, $k = 1, \cdots, n$. A theorem, due to H. B. Griffiths [4, p. 245], asserts that $h_n$ is not a product of fewer than $n$ commutators in $G_n$.

Here is a geometric consequence.

**Theorem 5.** There exists a sequence of (connected 2-dimensional) finite polyhedra $P_n, n = 1, 2, \cdots$, with vanishing homology groups $H_q(P_n) = 0$, $q = 1, 2, \cdots$, and such that the first singular homology group of the infinite Cartesian product $H_1(\prod P_n) \neq 0$.
Let $P$ be the 2-skeleton of the well-known "Poincaré space" described in [11, p. 216]. It is known that $H_2(P) = 0$ and that $\pi_1(P)$ is the "binary icosahedral group." Take now for $P_n$ $n$ copies of $P$ attached at a single common point. Obviously, $\pi_1(P_n) = \pi_1(P) \ast \cdots \ast \pi_1(P)$; this group is perfect, because $\pi_1(P)$ is a perfect group. Moreover, $H_2(P_n) = 0$, so that all the hypotheses of Theorem 5 are fulfilled. However, by the above remarks, $\pi_1(\prod P_n) = \prod (\pi_1(P_n))$ is not perfect and thus $H_1(\prod P_n) \neq 0$.

It is well-known that the singular homology groups of the Cartesian product of finitely many spaces are completely determined by the homology groups of these spaces. Theorem 5 shows that this is not the case for infinite products.

6. **Main theorem and lc$_1$ spaces which fail to be LC$^1$.** 6.1. Given any finitely presented\(^{(10)}\) group $G$, there exists a finite (2-dimensional) cell complex $K$, having a single vertex $o$ and satisfying $\pi_1(K) = G$ (see [12]). Assigning to $G$ such a $K$ and to $K$ the continuous curves $\overline{X}$ and $\overline{Y}$ described in 3.1, we derive from Theorem 2 our main result:

**Theorem 6.** Given any finitely presented group $G$, there exist a 2-dimensional continuous curve $C(G)$ and a 1-dimensional continuous curve $D(G) \subseteq C(G)$, having the following properties: $\pi_1(C(G)) \approx \prod G$; every point $x \in C(G) \setminus D(G)$ has neighborhoods homeomorphic to the Euclidean plane and every point $x \in D(G)$ has a basis of connected (open) neighborhoods $U(x)$ in $C(G)$ such that $U(x) \subseteq C(G)$ induces a monomorphism of $\pi_1(U(x))$ into $\pi_1(C(G))$ with an image isomorphic to $\prod G$.

6.2. Now take for $G$ a nontrivial perfect finite group. Then $\prod G$ is nontrivial and perfect (see Lemma 13). Therefore, every $x \in C(G)$ has a basis of connected neighborhoods $U(x)$ with $H_1(U(x)) = 0$, showing that $C(G)$ is a 2-dimensional continuous curve, everywhere lc$_1$. On the other hand, if $x \in D(G)$, $\pi_1(U(x)) \approx \prod G$ and thus nontrivial. Since $\pi_1(U(x)) \to \pi_1(C(G))$ is a monomorphism it follows that the space is not semi-1-LC at the points of $D(G)$\(^{(11)}\); a fortiori it is not LC$^1$ in those points. This proves

**Theorem 7.** Every nontrivial perfect finite group gives rise to a 2-dimensional continuous curve which is lc$_1$, but fails to be LC$^1$ in a subset of dimension 1.

**Conjecture.** A continuous curve which is everywhere lc$_1$ can not fail to be LC$^1$ in exactly one point.

This statement, if true, should explain why the examples exhibited in this paper are of a rather involved nature.

6.3. We now state (proof is easily supplied using Lemma 13).

\(^{(10)}\) I.e. group defined by a finite number of generators and relations.

\(^{(11)}\) A space $X$ is semi-1-LC at $x \in X$ if there is a neighborhood $V$ of $x$ such that the image of $\pi_1(V)$ in $\pi_1(X)$ (under the homomorphism induced by $V \subseteq X$) is trivial, i.e. the unit subgroup of $\pi_1(X)$.
Theorem 8. If $K$ is a finite complex with a single vertex $o$ and $\pi_1(K)$ is a nontrivial finite perfect group, then $\prod K$ is an infinite dimensional continuous curve, everywhere $lc^1$ and nowhere $LC^1$.

References


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