ON THE ASYMPTOTIC SOLUTIONS OF A CLASS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

I. EXISTENCE OF REGULAR FORMAL SOLUTIONS

BY

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1. **Introduction.** This paper is concerned with the asymptotic solutions of the linear differential equation of the fourth order

$$\mathcal{L}(\varphi) = 0 \tag{1.1}$$

when the parameter λ is large in the differential operator

$$\mathcal{L}(\varphi) = \frac{d^4\varphi}{dx^4} + \lambda^2 \left\{ P(x, \lambda) \frac{d^2\varphi}{dx^2} + Q(x, \lambda) \frac{d\varphi}{dx} + R(x, \lambda)\varphi \right\}. \tag{1.2}$$

In the above expression, $P(x, \lambda)$, $Q(x, \lambda)$ and $R(x, \lambda)$ are analytic functions of the complex variable x, and they depend on λ in such a manner that the following asymptotic expansions hold:

$$P(x, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} P_n(x), \tag{1.3a}$$

$$Q(x, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} Q_n(x), \tag{1.3b}$$

$$R(x, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} R_n(x). \tag{1.3c}$$

Especially we are interested in the behavior of the solution in the neighborhood of a *simple* zero of the function $P_0(x)$. Such a point is called a *turning point* (of the first order). It is of special interest because asymptotic solutions of (1.1) in the form

$$\varphi = e^{\lambda S(x)} \sum_{n=0}^{\infty} \lambda^{-n} \varphi_n(x) \tag{1.4}$$

exhibit singularities at such a point even though the point is a regular point of the differential equation (1.1). Other solution forms must therefore be found, if we wish to find a complete set of asymptotic solutions *uniformly* valid in a neighborhood containing the turning point. It is the principal aim of the present investigation to find such solutions.

Received by the editors, October 31, 1958.

(1) This work overlaps in part with the doctoral dissertation of the junior author at the Massachusetts Institute of Technology, June, 1958.

The interest in the equation (1.1) has been especially stimulated by the controversy surrounding the Orr-Sommerfeld equation of hydrodynamic stability, which plays a central role in the dynamics of a viscous fluid and belongs to the type (1.1) under consideration. We shall not go into further details of the discussion of the issues here(2) but merely wish to observe that one principal point in the controversy concerns the role of the *inviscid equation* obtained by formally putting the viscosity coefficient equal to zero. This corresponds to formally letting $\lambda \to \infty$ in (1.1) and leads to the *reduced equation*

$$P_0(x) \frac{d^2\varphi}{dx^2} + Q_0(x) \frac{d\varphi}{dx} + R_0(x)\varphi = 0. \tag{1.5}$$

This equation, in contrast to the full equation (1.1), does have a regular singularity at the zero of $P_0(x)$, except in the most unlikely case that both $Q_0(x)$ and $R_0(x)$ also vanish there. We shall see that the nature of the solutions of the reduced equation also plays an important role in the present mathematical theory. Although the general outline of the theory (cf. §2) is not greatly affected, the actual construction of the solutions (cf. §§6, 7, 8) can vary substantially with the nature of the reduced equation.

There have been many papers published in the literature of hydrodynamics and differential equations that deal with the problem at hand. Especially we wish to mention the theories of Wasow [12] and of Langer [4; 5], with which the present paper shares some common ideas but also has some essential differences. Langer's theory is based on his earlier work for a differential equation of the third order (Langer [2]) which in turn depends on the solution of the special differential equation (Langer [3]).

$$v''' + \lambda^2(zv' + \alpha v) = 0, \tag{1.6}$$

where α is a parameter which may depend on λ in the manner

$$\alpha = \sum_{n=0}^{\infty} \lambda^{-n}\alpha^{(n)}. \tag{1.7}$$

The present work is based on the special differential equation of the fourth order

$$L(u) \equiv u^{\text{iv}} + \lambda^2(zu'' + \alpha u' + \beta u) = 0 \tag{1.8}$$

where α and β may depend on λ asymptotically in the following manner:

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(2) For a discussion of the physical and mathematical issues involved in the hydrodynamical problem, see Lin [6, Chapter VIII]. Other references to the hydrodynamical problem are given at the end of this section (§1.)

$$\alpha = \sum_{n=0}^{\infty} \lambda^{-n} \alpha^{(n)}, \tag{1.9a}$$

$$\beta = \sum_{n=0}^{\infty} \lambda^{-n} \beta^{(n)}. \tag{1.9b}$$

The differential equation (1.8), which we shall call the *basic reference equation*, as distinct from the approximating *related equation*, is a special case of the given equation (1.1). It is in general not a close approximation to (1.1), but does retain the essential feature of having a turning point. Furthermore, the properties of its solutions can be studied in detail by the method of Laplace transformation, and this has been carried out by one of us (Rabenstein [10]). In this phase of our work, the ideas of Wasow [11] were used, but the greater generality of the equation brings new interesting features. The main differences between the present work and that of Wasow lie in the manner of construction of the solutions of (1.1) (cf. Wasow [13]), and in Wasow's restriction of (1.8) to the case $\alpha = 0$. Certain limitation in Wasow's theory are thereby removed.

To summarize, it is perhaps fair to say(3) that the present work follows more closely the spirit of Langer's theory for equations of the lower orders than either the theory of Wasow or of Langer for the fourth order case. We have also cast the general ideas of the theory in terms of a system of equations of the first order to make it more perspicuous. But by far the most important features of the present work are the following. In the first place, we have carried out the proof of the existence theorem for a *complete* complex neighborhood of the turning point. It is not difficult to prove that the formal solution is asymptotic to a true solution in every fan shaped sector centered at the origin, provided the size of the sector is limited. The difficult part is to prove that, in a *complete* complex neighborhood, there is a *single* solution of the given equation that is asymptotically represented by the *same* formal expansion(4). We shall see the contrast between these two types of theorems in the second part of this paper. The other important feature is its contrast with theories for equations of lower orders. This lies in the central role played by the reduced equation (1.5) and the reduced equation

$$zu'' + \alpha^{(0)} u' + \beta^{(0)} u = 0 \tag{1.10}$$

of the basic reference equation (1.8). Both of them have a regular singularity, and it is important, as we shall see below, to choose $\alpha^{(0)}$ and $\beta^{(0)}$ so that the essential characters of the solution of (1.5) are reproduced in the solutions of (1.10). In the case of equations of the third order (Langer [2]), a similar step has also been taken; but in that case, the reduced equations are of the

(3) An observation made by Professor Norman Levinson.

(4) As far as we know, this has not even been proved for equations of lower orders.

first order, the nature of their solutions are consequently much simpler; and there is no conspicuous effort for such a determination. In the case of second order equations (Langer [1]), the reduced equations become trivial. Thus, the present work also points a way to the development of theories for equations of higher orders: the basic reference equation should be chosen so that the solutions of its reduced equation reproduce the essential characteristics of the solutions of the reduced equation of the given differential equation.

On the basis of the above remarks, a close relationship can already be expected to exist between the theory of Langer and the present work in a fairly wide class of problems. If, in the reduced equation (1.10), $\alpha^{(0)}$ is not an integer, positive, negative or zero, the character of its solution is essentially determined by $\alpha^{(0)}$ and is independent of $\beta^{(0)}$. Consequently, we may put $\beta^{(0)}=0$ in (1.10) and it would then be natural also to try to put $\beta=0$ in the basic reference equation (1.8). This equation then becomes identical with Langer's equation (1.6) provided $u'=v$. The two theories should therefore yield comparable results. On the other hand, if $\alpha^{(0)}$ is an integer, positive, negative, or zero, (as it is in the hydrodynamical problem), the properties of the solution of (1.10) depend essentially on $\beta^{(0)}$, and the results of the two theories may not be directly comparable. In particular, when $\alpha^{(0)}$ is thus specified, the present theory is not essentially modified. On the other hand, there may exist, in Langer's theory, a distinctive class of problems that require special treatment. This class is at the present characterized in terms of certain features that occur in the process of the construction of the solution. As no corresponding features occur in the present theory, it appears that a proper comparison of the two theories can be made only after the distinctive class in Langer's has been characterized directly in terms of the original differential equation (1.1). At the time of this writing, Professor Langer has informed us that he is contemplating such an investigation.

The presentation of the present theory naturally divides itself into three parts:

PART I. Existence of formal asymptotic solutions with regular coefficients.

PART II. Existence of actual solutions approximated by these formal asymptotic solutions.

PART III. Applications to the hydrodynamical problem.

The new features of the present problem, in comparison with cases of lower orders, exhibit themselves most clearly in the development of Part I of the theory. In this sense, it is the most important part. The existence of actual solutions can be proved by following the usual method of constructing a related equation, which goes back to Liouville. The new feature of the present work, as mentioned before, is the existence theorem for a *complete* neighborhood. This is by no means a trivial application of the old ideas. There are certain steps which are quite distinctive for the problem at hand. Furthermore, it is necessary to make certain mathematical investigations which are interesting by themselves, such as a complete discussion of a fundamental system of solutions of the basic reference equation and its adjoint. These studies, made in the spirit of the classical analysis of special functions, are also essential for the final physical applications.

Since the types of mathematical considerations in Parts I and II are very much different from each other, it is thought desirable to present them separately. Accordingly, the present paper deals only with Part I; a sequel will deal with Part II. To provide an overall picture, an outline of the theory is given in §2. Part III will be published in the literature of mechanics(5), but perhaps a word is in order here about the significance of the present work in regard to the studies of the hydrodynamical problem. While the past controversies will no doubt be resolved by any kind of a uniformly valid solution, the extent of the usefulness of such solutions still depend on the solution being given in some convenient form. If our aim is merely to compute the stability characteristics of the flow on the basis of the Orr-Sommerfeld equation, this is still not a very serious point, for the lack of convenience of the solution form can be compensated, at least partly, by speedy modern computing methods. However, the issue goes deeper. The ultimate purpose of the theory of hydrodynamic stability is to clarify the mechanism of transition of the laminar flow into turbulence. Here, recent investigations, both theoretical and experimental, reveal new important roles of the *critical layer*—which corresponds to the turning point in the mathematical theory. It is thus essential that the nature of the solutions in the neighborhood of the turning point be fully understood. Specifically, it is desirable that the theory should yield correct third order derivatives in this neighborhood, while in the stability problem only derivatives of the first order are needed. For more detailed discussion of the hydrodynamic issues, the reader is referred to the earlier reports of the senior author (Lin [7; 8; 9]). In those reports, the inherent significance of the nature of the solutions of the reduced equation (1.8) has not been as explicitly brought out as in the present paper, but their application for the construction of useful solutions for the purpose of the hydrodynamical problem has been emphasized.

ACKNOWLEDGMENT. The authors wish to thank Professor K. O. Friedrichs, Norman Levinson, and especially R. E. Langer and Wolfgang Wasow for their interest in this work and their discussions during various stages of its completion. The main part of this work was completed at the Massachusetts Institute of Technology under the sponsorship of the Office of Naval Research. The final revision of the paper was made at the Mathematics Research Center, University of Wisconsin.

2. **Outline of the theory.** Although the asymptotic theory for the fourth-order equation (1.1) follows the general outline of the theory for lower orders, there are some significant deviations in the manner in which the solution is

(5) Preliminary forms of such work have already been reported (Lin, [7; 8; 9]).

constructed. As mentioned above, this is mainly due to the fact that the reduced equation (1.5) is of the second order. We shall describe in this section the major steps in the theory so that the outline will stand out more clearly.

The usual method of solution is to construct formal solutions of (1.1) in terms of the solutions of (1.8) in the following form

$$\varphi = c_0 u + c_1 u' + c_2 u'' + c_3 u''' \tag{2.1}$$

where the variables z and x are connected by a suitable analytic function, and $c_0(x, \lambda)$, $c_1(x, \lambda)$, $c_2(x, \lambda)$, $c_3(x, \lambda)$ are expected *not* to be all of the same order in λ. Rather, we might expect c_m to be of the order $O(\lambda^{-m})$, if we take the analogy with equations of lower orders. However, in the present case, it will turn out that

$$c_0,\ c_1 = O(1), \tag{2.2}$$

$$c_2,\ c_3 = O(\lambda^{-2}). \tag{2.3}$$

The fact that we need *two* coefficients of the order unity can be expected from the second-order nature of the reduced equation. For if we consider the *two* solutions $u_1^{(0)}$ and $u_2^{(0)}$ of the reduced equation (1.10), we should expect to get *two* solutions $\varphi_1^{(0)}$, $\varphi_2^{(0)}$ of the reduced equation (1.5) in the initial approximation. If we now think of the connection in terms of two first order systems, we should expect these solutions to be related to each other in the form

$$\overset{(0)}{\varphi} = \overset{(0)}{c_0}\,\overset{(0)}{u} + \overset{(0)}{c_1}\,\overset{(0)'}{u}. \tag{2.4}$$

Such a form can follow from (2.1) only if (2.2) is satisfied.

Detailed calculations (cf. Appendix A) show that in the present case, we should actually consider relations between vectors of the forms

$$\begin{aligned} \boldsymbol{\phi} &= (\varphi, \varphi', \lambda^{-1}\varphi'', \lambda^{-2}\varphi'''), \\ \boldsymbol{u} &= (u, u', \lambda^{-1}u'', \lambda^{-2}u'''). \end{aligned} \tag{2.5}$$

(It is immaterial here whether the dashes on φ denote differentiation with respect to z or x.)

The critical step in the solution is to show that we can indeed find functions

$$c_i(x, \lambda) = \lambda^{-m_i} \sum_{n=0}^{\infty} \lambda^{-n} \overset{(n)}{c_i}(x) \tag{2.6}$$

with analytic coefficients $c_i^{(n)}(x)$. In the case of equations of lower orders, it has always been possible to prove this by explicit quadrature. This is no longer true in the present case. As we shall see in §3, we have to depend on relations of the type (2.4) to overcome this difficulty. Indeed, we have to modify the form of the expansion somewhat for certain special cases (§8).

In order to gain maximum simplicity and to bring out the significance of the reduced equation, we shall deviate somewhat from the above described method of construction of formal solutions. We first introduce a minimum amount of transformation of variables to bring the equation (1.1) into a form having *exactly the same type of reduced equation* as (1.8). This is done in two steps.

FIRST TRANSFORMATION. We introduce into (1.1) the new independent variable

$$z = \left[\frac{3}{2}\int_{x_0}^{x} P_0^{1/2}(x)dx\right]^{2/3} \tag{2.7}$$

and the new dependent variable

$$\psi(z, \lambda) = \varphi(x, \lambda)[P_0(x)/z]^{3/4}. \tag{2.8}$$

Then the equation (1.1) becomes

$$\psi^{\text{iv}} + \lambda^2(p\psi'' + q\psi' + r\psi) = 0, \tag{2.9}$$

where p, q, r have behaviors identical with that specified in (1.3), i.e.,

$$p(z, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} p_n(z), \tag{2.10a}$$

$$q(z, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} q_n(z), \tag{2.10b}$$

$$r(z, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} r_n(z). \tag{2.10c}$$

The change of independent variable has the purpose of making (2.9) resemble (1.8) more closely in that

$$p_0(z) = z. \tag{2.11}$$

SECOND TRANSFORMATION. Next we consider a transformation of the form

$$\chi = A(z, \lambda)\psi + B(z, \lambda)\psi' + \lambda^{-1}C(z, \lambda)\psi'' + \lambda^{-2}D(z, \lambda)\psi'', \tag{2.12}$$

where A, B, C, D have the behavior (1.3). By formal differentiation, we can then obtain the transformation in the vector form

$$\boldsymbol{\chi} = G\boldsymbol{\psi}, \tag{2.13}$$

where

$$\boldsymbol{\chi} = (\chi, \chi', \lambda^{-1}\chi'', \lambda^{-2}\chi'''), \tag{2.14}$$

$$\boldsymbol{\psi} = (\psi, \psi', \lambda^{-1}\psi'', \lambda^{-2}\psi'''), \tag{2.15}$$

and $G(z, \lambda)$ is a nonsingular matrix with asymptotic dependence on λ in the manner

$$G(z, \lambda) = \sum_{n=0}^{\infty} \lambda^{-n} G^{(n)}(z). \tag{2.16}$$

We expect $G^{(n)}(z)$ to be all analytic. It will be seen that this form of $G(z, \lambda)$ is obtained provided certain simple conditions are fulfilled by the coefficients in (2.12).

It is then possible to show that we can *explicitly* determine the functions $A(z, \lambda)$, $B(z, \lambda)$, $C(z, \lambda)$, and $D(z, \lambda)$ as polynomials in λ^{-1} in such a manner that χ satisfies an equation of the form

$$\begin{aligned} L_0(\chi) &= \chi^{\mathrm{iv}} + \lambda^2(z\chi'' + \alpha_0\chi' + \beta_0\chi) \\ &= \lambda(\tilde{a}\chi + \tilde{b}\chi' + \lambda^{-1}\tilde{c}\chi'' + \lambda^{-2}\tilde{d}\chi''') \end{aligned} \tag{2.17}$$

where the coefficients $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ have the behavior (1.3). This is a major step in the theory and will be treated in detail in the next section. The form (2.17) will be referred to as the *normal form* of the equation.

In the process of carrying out this transformation, it will become clear that the constant α_0 is indeed completely specified by the given equation if the transformation is to be regular, and that the constant β_0 is also determined in the cases where α_0 is an integer, positive, negative, or zero.

The construction of asymptotic solutions of the normalized equation (2.17) then follows the usual methods. If we introduce the vector

$$\boldsymbol{u} = (u, u', \lambda^{-1}u'', \lambda^{-2}u'''), \tag{2.18}$$

where u is a solution of the equation (1.8), then we may expect it possible, by proper determination of α and β to obtain formal solutions of the structure

$$\boldsymbol{\chi} = H\boldsymbol{u} \tag{2.19}$$

and with H expected to be of the form[6]

$$H = I + \sum_{m=1}^{\infty} \lambda^{-m} h^{(m)}(z), \tag{2.20}$$

where I is the identity matrix. Combining (2.12) and (2.19), we may write

$$\mathrm{X} = G\Psi = HU, \tag{2.21}$$

or

$$\Psi = G^{-1}HU \tag{2.22}$$

where U, Ψ, X are fundamental matrix solutions of the systems of equations corresponding to (1.8), (2.9) and (2.17). For convenience of reference, we list below the equations for $\boldsymbol{u}$ and $\boldsymbol{\chi}$ explicitly. The equation for $\boldsymbol{u}$ may be written

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(6) Deviations from this expectation that occur in certain special cases will be discussed in §8; but the following outline will not be essentially altered.

$$\frac{du}{dz} = Mu, \tag{2.23}$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ -\beta & -\alpha & -\lambda z & 0 \end{bmatrix}. \tag{2.24}$$

The equation for χ may be written as

$$\frac{d\chi}{dz} = (M_0 + \epsilon)\chi \tag{2.25}$$

where M_0 is obtained from M by putting $\alpha = \alpha_0$, $\beta = \beta_0$ and

$$\epsilon = \lambda^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{a} & \bar{b} & \bar{c} & \bar{d} \end{bmatrix}. \tag{2.26}$$

If we terminate the series (2.20) with $h^{(n)}(z)$ as the last term(7) we obtain the nth approximation

$$\chi_n = H_n u \tag{2.27}$$

which satisfies the differential equation

$$\frac{d\chi_n}{dz} = (M_0 + \epsilon_n)\chi_n, \tag{2.28}$$

with

$$\epsilon - \epsilon_n = \lambda^{-(n+1)}\Delta, \tag{2.29}$$

where Δ is a matrix of the same form as $\lambda\epsilon$. The equation (2.28) is the *related equation* of (2.25). To prove that there exists a solution χ approximated by each solution χ_n of the related equation, we consider the difference vector

$$\delta = \chi - \chi_n \tag{2.30}$$

which satisfies the differential equation

$$\frac{d\delta}{dz} = M_0\delta + \epsilon\delta + \lambda^{-(n+1)}\Delta\chi_n. \tag{2.31}$$

(7) To be more precise, we do this for the first row of the matrix and let the other rows be derived by differentiation of χ_n.

Thus, $\mathfrak{d}$ is given by the solution of the integral equation(8).

$$
\begin{aligned}
\mathfrak{d}(z, \lambda) &= \int^{z} K(z, \zeta; \lambda)\epsilon(\zeta, \lambda)\mathfrak{d}(\zeta, \lambda)d\zeta \\
&\quad + \lambda^{-(n+1)} \int^{z} K(z, \zeta; \lambda)\Delta(\zeta, \lambda)\chi_n(\zeta, \lambda)d\zeta
\end{aligned}
\tag{2.32}
$$

where the kernel $K(z, \zeta; \lambda)$ is given by

$$K(z, \zeta; \lambda) = U_0(z, \lambda)U_0^{-1}(\zeta, \lambda), \tag{2.33}$$

and $U_0(z, \lambda)$ is a fundamental matrix solution of the differential system

$$\frac{dU_0}{dz} = M_0U_0. \tag{2.34}$$

This kernel can be calculated explicitly, since we know the matrix functions $U_0(z, \lambda)$ and its inverse. The appraisal of the order of magnitude of $\mathfrak{d}$ then follows the conventional methods of choosing the proper paths of integration and using the known asymptotic behavior of the functions $u(z, \lambda)$.

Variations of the above method might be found desirable. For example, one may wish to consider, instead of

$$\mathfrak{d} = \chi - \chi_n, \tag{2.35}$$

the quantity

$$H_n^{-1}\mathfrak{d} = \tilde{u}_n - u, \qquad \tilde{u}_n = H_n^{-1}\chi. \tag{2.36}$$

In this manner, we are comparing the function $\tilde{u}_n$ (which is explicitly related to the exact solution χ) with the exact solution of the basic reference equation (2.23). Obviously, $\tilde{u}_n$ may be expected to satisfy an equation very close to (2.23), and the usual method of appraisal of errors can be made. The advantage of such an approach is that the solutions of the equation (2.23) itself are well-known. The details of these modified methods will be discussed in Part II when the actual proofs are carried out.

For practical construction of solutions, the method used for developing the theory is not suitable. Rather, once we know the existence of the solutions of the form (2.1), the connection coefficients can be calculated by using the formal asymptotic solutions of φ and u of the type (1.4). This method was used in the hydrodynamical applications (Lin [9]).

3. **Reduction of the differential equation into the normal form.** We shall now carry out the second transformation that will bring the given differential equation in the form (2.9) into the form (2.17). The detailed calculations in-

(8) It is sometimes desirable to use a somewhat different approach and obtain an integral equation where the first term on the right-hand side is also small to higher orders of λ^{-1}. For these discussions, see Part II.

volved are very lengthy and are given in detail in appendices (A) and (B). Here, we shall recapitulate the major steps with frequent reference to the equations derived there.

If we introduce [cf. (A3)]

$$\chi = A\psi + B\psi' + \lambda^{-1}C\psi'' + \lambda^{-2}D\psi''' \tag{3.1}$$

as our new variable, where $A(z, \lambda)$, $B(z, \lambda)$, $C(z, \lambda)$ and $D(z, \lambda)$ have the asymptotic behavior (1.3), then all the derivatives of χ take on the form [cf. (A4), (A8), etc.]

$$\chi^{(i)} = A_i\psi + B_i\psi' + \lambda^{-1}C_i\psi'' + \lambda^{-2}D_i\psi'''. \tag{3.2}$$

However, the matrix G connecting the vectors $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$, as defined by (2.13) and (2.14), will not be of the order of unity unless we impose the conditions [cf. (A6)]

$$B - pD = \lambda^{-1}E, \quad \text{and} \quad C = \lambda^{-1}F \tag{3.3}$$

with E and F having the asymptotic behavior (1.3). The form of G is then given by [cf. (A12)]

$$G = \sum_{n=0}^{\infty} \lambda^{-n}G^{(n)} \tag{3.4}$$

with [cf. (A13)]

$$G^{(0)} \equiv \begin{bmatrix} A_0^{(0)} & B_0^{(0)} & 0 & D^{(0)} \\ A_1^{(0)} & B_1^{(0)} & C_1^{(0)} & D_1^{(0)} \\ 0 & 0 & B_1^{(0)} - p_0D_1^{(0)} & C_1^{(0)} \\ 0 & 0 & -p_0C_1^{(0)} & B_1^{(0)} - p_0D_1^{(0)} \end{bmatrix}. \tag{3.5}$$

It is necessary that $|G^{(0)}(z)| \neq 0$ and in particular this should hold at $z=0$. For this to be true, we must have [cf. (A14)].

$$A_0^{(0)}(0) \neq 0, \qquad B_1^{(0)}(0) \neq 0. \tag{3.6}$$

If we now form the operator $L_0(\chi)$, with α_0 and β_0 as yet unspecified, we find that an equation of the form (2.17) is satisfied, with the coefficients having the specified orders, provided the following conditions are satisfied [cf. (B10)]:

$$-r_0B_1^{(0)} + zA_1^{(0)\prime} + \alpha_0A_1^{(0)} + \beta_0A^{(0)} = 0, \tag{3.7a}$$

$$-q_0B_1^{(0)} + zA_1^{(0)} + zB_1^{(0)\prime} + \alpha_0B_1^{(0)} + \beta_0B^{(0)} = 0, \tag{3.7b}$$

$$-p_1(B_1^{(0)} - zD_1^{(0)}) - (q_0 + 1)C_1^{(0)} - 2zC_1^{(0)\prime} + \alpha_0C_1^{(0)} = 0, \tag{3.7c}$$

$$-zC_1^{(0)} + 2B_1^{(0)\prime} + B_2^{(0)} - 2(zD_1^{(0)})' + \alpha_0D_1^{(0)} + \beta_0D^{(0)} = 0, \tag{3.7d}$$

where [cf. (A6), (A7) and (A9)]

$$B_0^{(0)} = zD_0^{(0)}, \tag{3.8}$$

$$A_1^{(0)} = A^{(0)\prime} - r_0 D^{(0)}, \tag{3.9a}$$

$$B_1^{(0)} = B^{(0)\prime} + A^{(0)} - q_0 D^{(0)}, \tag{3.9b}$$

$$C_1^{(0)} = E^{(0)}, \tag{3.9c}$$

$$D_1^{(0)} = D^{(0)\prime} + F^{(0)}, \tag{3.9d}$$

and

$$B_2^{(0)} = B_1^{(0)\prime} + A_1^{(0)} - q_0 D_1^{(0)}. \tag{3.10}$$

Thus, we are faced with the problem of solving the differential equations (3.7)–(3.10), for the variables $A^{(0)}$, $B^{(0)}$, $D^{(0)}$, $E^{(0)}$ and $F^{(0)}$ under the restriction (3.6). The constants α_0 and β_0 are at our disposal. They should obviously be chosen toward making the solution functions regular at $z=0$. If these functions can be found, the conditions (3.6) indeed guarantee that $|G^{(0)}(z)|$ is a constant [cf. (A16) and the discussion following]. We may then write (3.1) in the form

$$\chi = A^{(0)}\psi + [B^{(0)} + \lambda^{-1}(E^{(0)} + p_1 D^{(0)})]\psi' + \lambda^{-2}[F^{(0)}\psi'' + D^{(0)}\psi'''] \tag{3.11}$$

where the higher approxinations in the coefficients are omitted, since they do not enter the conditions for the function (3.11) to satisfy an equation of the form (2.17).

The system of equations to be solved can be divided into two groups. If we consider D_0 as eliminated from (3.9a) and (3.9b) by the use of (3.8), then the four equations (3.7a), (3.7b), (3.9a), (3.9b) form a system of four linear differential equations of the first order in the dependent variables $A^{(0)}$, $B^{(0)}$, $A_1^{(0)}$, $B_1^{(0)}$. After this system is solved, the equations (3.7c) and (3.7d) may be regarded as a system of two equations of the first order for the variables $C_1^{(0)}$ and $D_1^{(0)}$, or equivalently the variables $E^{(0)}$ and $F^{(0)}$ (cf. (3.9c), (3.9d)). They take on the form

$$2zE^{(0)\prime} + (q_0 + 1 - \alpha_0)E^{(0)} - p_1 zF^{(0)} = g_1(z), \tag{3.12}$$

$$2zF^{(0)\prime} + (q_0 + 2 - \alpha_0)F^{(0)} - zE^{(0)} = g_2(z), \tag{3.13}$$

where $g_1(z)$ and $g_2(z)$ are known analytic functions of z.

The system of four equations for $A^{(0)}$, $B^{(0)}$, $A_1^{(0)}$, $B_1^{(0)}$ appears to be very difficult to solve. However, one may anticipate the answers from other considerations and then verify the conjecture. From (3.11), it is clear that the solutions $\chi^{(0)}$ and $\psi^{(0)}$ of the reduced equations

$$z\chi^{(0)\prime\prime} + \alpha_0\chi^{(0)\prime} + \beta_0\chi^{(0)} = 0, \tag{3.14}$$

$$z\psi^{(0)\prime\prime} + q_0\psi^{(0)\prime} + r_0\psi^{(0)} = 0 \tag{3.15}$$

must satisfy the relation

$$\chi^{(0)} = A^{(0)}\psi^{(0)} + B^{(0)}\psi^{(0)\prime}. \tag{3.16}$$

From this, we can calculate $\chi^{(0)\prime}$ and obtain

$$\chi^{(0)\prime} = A_1^{(0)}\psi^{(0)} + B_1^{(0)}\psi^{(0)\prime}. \tag{3.17}$$

We expect that $A_1^{(0)}$ and $B_1^{(0)}$ will satisfy (3.9a) and (3.9b), if $D^{(0)}$ is eliminated by using (3.8). To verify the above conjecture, we need only show that (3.7a) and (3.7b) are indeed satisfied, provided $A^{(0)}$, $B^{(0)}$, $A_1^{(0)}$, $B_1^{(0)}$ satisfy (3.16) and (3.17). This can be done with a little calculation (§4), if we keep the equations (3.14) and (3.15) in mind. These calculations also show that there are indeed four sets of solutions, which agrees with the fact that equations (3.7a), (3.7b), (3.9a), (3.9b) form a system of four equations of the first order. Moreover, one of the solutions can be shown to be regular with $B^{(0)}(0)=0$ if the parameters α_0 and β_0 are properly chosen. The details of these discussions will be given in the next section.

Having found such regular solutions for $A^{(0)}$, $B^{(0)}$, $A_1^{(0)}$, $B_1^{(0)}$, we may turn back to the discussion of (3.12) and (3.13). It will be found (cf. §4) that

$$\alpha_0 = q_0(0) \tag{3.18}$$

so that we know exactly the values of $E^{(0)}$ and $F^{(0)}$ in (3.12) at $z=0$. It will then not be difficult to verify that a power series solution exists for the system (3.12) and (3.13). That such a formal series solution is a true solution depends on the fact that the system is of the Fuchsian type (Coddington and Levinson, 1955, p. 117).

We may also prove the existence of a regular solution by rewriting the differential system (3.12), (3.13) into an integral system:

$$\hat{E}(z) = [R(z)]^{-1}z^{-1/2}\int_0^z z^{3/2}R(z)[h_1(z) + p_1\hat{F}]dz, \tag{3.19a}$$

$$\hat{F}(z) = [R(z)]^{-1}z^{-1/2}\int_0^z z^{2}R(z)[h_2(z) + \hat{E}]dz, \tag{3.19b}$$

where

$$R(z) = \exp\left[\int_0^z \frac{1}{2z}(q_0 - \alpha_0)dz\right], \tag{3.20}$$

and

$$\hat{E}(z) = E^{(0)}(z) - E^{(0)}(0), \qquad \hat{F}(z) = F^{(0)}(z) - F^{(0)}(0), \tag{3.21a}$$

$$h_1(z) = \frac{1}{z}[g_1(z) - g_1(0)], \qquad h_2(z) = \frac{1}{z}[g_2(z) - g_2(0)]. \tag{3.21b}$$

It is easy to see that an iteration method would converge and that only regular functions are involved in each step. Consequently, the solution obtained is also regular.

Thus, anticipating the results obtained from the analysis of the reduced equations in the next section, we may state the following lemma.

LEMMA. *The asymptotic theory for the differential equation* (1.1) *is equivalent to that for the special differential equation* (2.17), *in which the constant parameter* α_0 *is fixed by the given differential equation, but the constant* β_0 *is at our disposal except in the cases where* α_0 *is an integer, positive, negative, or zero. In the latter case,* β_0 *is also fixed by the given differential equation. The coefficients* $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ *in* (2.17) *have the asymptotic behavior in* λ *similar to those in the given differential equation as specified by* (1.3).

4. **The solutions of the reduced equations.** As we have just seen, the success of the desired transformation depends on obtaining regular solutions of the system of differential equations (3.7a), (3.7b), (3.9a) and (3.9b) for the functions $A^{(0)}$, $B^{(0)}$, $A_1^{(0)}$ and $B_1^{(0)}$. We have also noticed that these functions should be the same as those that appear in the connection formulae

$$\chi^{(0)} = A^{(0)}\psi^{(0)} + B^{(0)}\psi^{(0)\prime}, \tag{4.1}$$

$$\chi^{(0)\prime} = A_1^{(0)}\psi^{(0)} + B_1^{(0)}\psi^{(0)\prime}, \tag{4.2}$$

between the solutions of the equations

$$z\chi^{(0)\prime\prime} + \alpha_0\chi^{(0)\prime} + \beta_0\chi^{(0)} = 0, \tag{4.3}$$

and

$$z\psi^{(0)\prime\prime} + q_0\psi^{(0)\prime} + r_0\psi^{(0)} = 0. \tag{4.4}$$

To verify these conclusions, let us differentiate (4.1) with respect to z to obtain

$$\chi^{(0)\prime} = A^{(0)\prime}\psi^{(0)} + (A^{(0)} + B^{(0)\prime})\psi^{(0)\prime} + B^{(0)}\psi^{(0)\prime\prime},$$

and hence, by using (4.4),

$$\chi^{(0)\prime} = (A^{(0)\prime} - r_0B^{(0)}/z)\psi^{(0)} + (B^{(0)\prime} + A^{(0)} - q_0B^{(0)}/z)\psi^{(0)\prime}. \tag{4.5}$$

Thus, a comparison of (4.2), (4.5) shows that (3.9a) and (3.9b) are satisfied. To verify that (3.7a) and (3.7b) are also satisfied, we have to differentiate (4.2) once more. We obtain (cf. (4.5))

$$\chi^{(0)\prime\prime} = (A_1^{(0)} - r_0B_1^{(0)}/z)\psi^{(0)} + (B_1^{(0)\prime} + A_1^{(0)} - q_0B_1^{(0)}/z)\psi^{(0)\prime}. \tag{4.6}$$

Substituting (4.1), (4.2) and (4.6) into (4.3), we find that the coefficient of $\psi^{(0)}$ is precisely the right-hand side of (3.7a) and the coefficient of $\psi^{(0)\prime}$ is precisely that of (3.7b). The above contentions are therefore verified.

Having established the relations (4.1) and (4.2), we can make use of them to determine the desired solutions in a purely algebraic manner. Consider *any two independent* solutions $\psi_1^{(0)}$ and $\psi_2^{(0)}$ of (4.4) and *any* two solutions $\chi_1^{(0)}$ and $\chi_2^{(0)}$ of (4.3), and *define* the functions $A^{(0)}$ and $B^{(0)}$ by the pair of relations

$$\chi_1^{(0)} = A^{(0)}\psi_1^{(0)} + B^{(0)}\psi_1^{(0)\prime}, \tag{4.7}$$

$$\chi_2^{(0)} = A^{(0)}\psi_2^{(0)} + B^{(0)}\psi_2^{(0)\prime}. \tag{4.8}$$

These functions and their associated functions $A_1^{(0)}$ and $B_1^{(0)}$ will satisfy all the desired requirements. Solving for $A^{(0)}$ and $B^{(0)}$, we obtain

$$A^{(0)} = W_0^{-1}(\chi_1^{(0)}\psi_2^{(0)\prime} - \chi_2^{(0)}\psi_1^{(0)\prime}), \tag{4.9}$$

$$B^{(0)} = -W_0^{-1}(\chi_1^{(0)}\psi_2^{(0)} - \chi_2^{(0)}\psi_1^{(0)}) \tag{4.10}$$

and

$$W_0 = \psi_1^{(0)}\psi_2^{(0)\prime} - \psi_2^{(0)}\psi_1^{(0)\prime}. \tag{4.11}$$

Now, if we let

$$\psi_1^{(0)} = \psi_{\mathrm{I}}^{(0)}, \qquad \psi_2^{(0)} = \psi_{\mathrm{II}}^{(0)} \tag{4.12}$$

be a *specific* pair of *independent* solutions, and $\chi_1^{(0)}$ and $\chi_2^{(0)}$ be an *arbitrary* pair expressed in terms of a *specific* pair of independent solutions,

$$\chi_1^{(0)} = c_{11}\chi_{\mathrm{I}}^{(0)} + c_{12}\chi_{\mathrm{II}}^{(0)}, \qquad \chi_2^{(0)} = c_{21}\chi_{\mathrm{I}}^{(0)} + c_{22}\chi_{\mathrm{II}}^{(0)}, \tag{4.13}$$

then (4.9) and (4.10) gives

$$A^{(0)} = W_0^{-1}\{c_{11}\chi_{\mathrm{I}}^{(0)}\psi_{\mathrm{II}}^{(0)\prime} + c_{12}\chi_{\mathrm{II}}^{(0)}\psi_{\mathrm{II}}^{(0)\prime} - c_{21}\chi_{\mathrm{I}}^{(0)}\psi_{\mathrm{I}}^{(0)} - c_{22}\chi_{\mathrm{II}}^{(0)}\psi_{\mathrm{I}}^{(0)\prime}\}, \tag{4.14}$$

$$B^{(0)} = W_0^{-1}\{c_{11}\chi_{\mathrm{I}}^{(0)}\psi_{\mathrm{II}}^{(0)} + c_{12}\chi_{\mathrm{II}}^{(0)}\psi_{\mathrm{II}}^{(0)} - c_{21}\chi_{\mathrm{I}}^{(0)}\psi_{\mathrm{I}}^{(0)} - c_{22}\chi_{\mathrm{II}}^{(0)}\psi_{\mathrm{I}}^{(0)}\}. \tag{4.15}$$

Thus, we have four independent solutions, since the constants c_{11}, c_{12}, c_{21}, and c_{22} are entirely arbitrary. (They need not be chosen to make $\chi_1^{(0)}$ and $\chi_2^{(0)}$ independent solutions of (4.3). For example, if $\chi_1^{(0)}$ and $\chi_2^{(0)}$ are both zero, we simply obtain the trivial solution $A^{(0)}=0$, $B^{(0)}=0$, which is perfectly valid.)

We now wish to show that *at least one set of the solutions is a nontrivial regular set with the further property* $A_0^{(0)} \neq 0$, $B_1^{(0)} \neq 0$, *and* $B_0^{(0)}=0$ *at* $z=0$. This requires a detailed examination of the solutions of the reduced equations (4.3) and (4.4), and it is here that the coefficients α_0 and β_0 are determined.

The properties of the equations of the type (4.4) are well-known[9]. It has a regular singularity at the point $z=0$, with indices $\rho=0$ and $1-q_0(0)$.

[9] Equation (4.3) is merely a special case of (4.4).

Let us denote the indices by ρ_1 and ρ_2, with $\rho_2-\rho_1 \leqq 0$. Obviously, for our purposes, we must take $\alpha_0 = q_0(0)$ so that the indices of (4.3) and (4.4) are the same. Further discussions depend on whether $\rho_2-\rho_1$ is an integer.

If $\rho_2-\rho_1$ *is not an integer*, then two linearly independent solutions of (4.4) may be denoted by ([10])

$$\psi_1 = z^{\rho_1}(1 + \cdots), \tag{4.16}$$

$$\psi_2 = z^{\rho_2}(1 + \cdots) \tag{4.17}$$

where the expressions in the parentheses are regular functions. The Wronskian determinant has the value $\rho_2-\rho_1$. If we take χ_1 and χ_2 to be two corresponding solutions of (4.3), we may eliminate the powers z^{ρ_1} and z^{ρ_2} from these equations, and deal only with regular functions. Specifically, we have the equations

$$(z^{-\rho_1}\chi_1) = A^{(0)}(z^{-\rho_1}\psi_1) + (B^{(0)}/z)(z^{-\rho_1+1}\psi_1'), \tag{4.18}$$

$$(z^{-\rho_2}\chi_2) = A^{(0)}(z^{-\rho_2}\psi_2) + (B^{(0)}/z)(z^{-\rho_2+1}\psi_2') \tag{4.19}$$

to solve for $A^{(0)}$ and $B^{(0)}/z$. The coefficient determinant can be easily shown to have the constant value $\rho_2-\rho_1 \neq 0$, and the functions $A^{(0)}$ and $B^{(0)}$ thus obtained have the required properties. Indeed, a little calculation will show that $A^{(0)}(0) = B_1^{(0)}(0) = 1$, and consequently the restrictions (3.6) are satisfied.

If $\rho_2-\rho_1$ *is an integer*, then the linearly independent solutions of (4.4) take on the form

$$\psi_1 = z^{\rho_1}(1 + \cdots), \tag{4.20}$$

$$\psi_2 = \tilde{\psi}_2 + k\psi_1 \log z, \qquad \tilde{\psi}_2 = z^{\rho_2}(1 + \cdots) \tag{4.21}$$

where k is a constant, which is in general not equal to zero but may vanish under special circumstances. We must then choose not only α_0 to make (4.3) have the same indices as (4.4), but also β_0 to make the two solutions of (4.3) have the same character as (4.20) and (4.21). Thus, if k happens to be zero, we must also take $\beta_0 = 0$, which is the condition for the special equation (4.3) to have no logarithmic terms in the solutions. Thus, we may write the solutions of (4.3) in the form

$$\chi_1 = z^{\rho_1}(1 + \cdots), \tag{4.22}$$

$$\chi_2 = \tilde{\chi}_2 + k\chi_1 \log z, \qquad \tilde{\chi}_2 = z^{\rho_2}(1 + \cdots), \tag{4.23}$$

where the coefficient of k can be made to be the same as in (4.21).

The relations (4.7) and (4.8) then become

$$\chi_1 = A^{(0)}\psi_1 + B^{(0)}\psi_1',$$
$$\tilde{\chi}_2 = A^{(0)}\psi_2 + B^{(0)}(\tilde{\chi}_2' + k\psi_1/z)$$

or

([10]) For convenience of notation, the superscript (0) is temporarily dropped in the following paragraphs of this section.

$$(4.24) \qquad (z^{-\rho_1}\chi_1) = A^{(0)}(z^{-\rho_1}\psi_1) + (B^{(0)}/z)(z^{-\rho_1+1}\psi_1'),$$

$$(4.25) \qquad (z^{-\rho_2}\tilde{\chi}_2) = A^{(0)}(z^{-\rho_2}\psi_2) + (B^{(0)}/z)(z^{-\rho_2+1}\tilde{\psi}_2' + kz^{-\rho_2}\psi_1).$$

These are equations for $A^{(0)}$ and $B^{(0)}/z$ involving only regular functions. The function

$$z^{-\rho_2} = z^{\rho_1-\rho_2}(1 + \cdots)$$

is regular because $\rho_1 \geqq \rho_2$. The coefficient determinant for the above equations (4.24), (4.25) is a constant, by a consideration of the Wronskian determinant. That it is not equal to zero can also be directly verified at $z=0$. At that point, we obtain the value $\rho_2-\rho_1$ except when this is zero, and the value $(\rho_2-\rho_1)-k$ when $\rho_2-\rho_1=0$. In the latter case, k never vanishes. Thus we have indeed a nonvanishing constant in all nontrivial cases.

We have now completed the proof of the feasibility of the reduction of the given equation (1.1) into the normalized form (2.17), with α_0 and β_0 completely specified in the case α_0 is an integer[11]. If α_0 is not an integer, there is no restriction on β_0, and we may take $\beta_0=0$. The implications of these differences have been discussed in the introduction.

In the above analysis, we may have arbitrary multiples of χ_1 and χ_2 in (4.18), (4.19) or (4.24), (4.25) and thus obtain answers involving two parameters. Thus, in general there are actually *two* linearly independent regular solutions.

If the equation (4.4) has *two* regular solutions, and this can happen only if the two indices $\rho=0$ and $\rho=1-q_0$ are both integers, positive or zero, then equation (4.3) takes on the form

$$(4.26) \qquad z\chi'' - n\chi' = 0,$$

i.e., we have $\alpha_0=-n$, $\beta_0=0$. The solutions of (4.26) are 1 and z^n. We may take in this case

$$\chi_1 = c_1 + c_2 z^n,$$
$$\chi_2 = \qquad c_3 z^n$$

and thereby obtain a *three-parameter* family of regular solutions from equations (4.24) and (4.25). These facts will prove useful in later discussions.

5. **Solution of the normalized differential equation.** We have shown in the previous sections that the asymptotic theory of the equation (1.1) can be reduced to that for equation (2.17), which is now redesignated as

$$(5.1) \quad L_0(\chi) = \chi^{\mathrm{iv}} + \lambda^2(z\chi'' + \alpha_0\chi' + \beta_0\chi) = \lambda(\bar{a}\chi + \bar{b}\chi' + \lambda^{-1}\bar{c}\chi'' + \lambda^{-2}\bar{d}\chi'''),$$

where $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ have the behavior (1.3). This last equation can be further transformed to eliminate the term in χ''', but this is not necessary in view of

[11] The difference of the coefficients $\rho_2-\rho_1$ is equal to $-|\alpha_0-1|$.

the concise matrix form (2.25) into which (5.1) can be put. To solve the above equation, we write

$$\chi = \mu u + \nu u' + \lambda^{-1}\sigma u'' + \lambda^{-2}\tau u''', \tag{5.2}$$

where u is a solution of the fundamental reference equation, and μ, ν, σ, τ are expected to have the behavior (1.3). This expression has exactly the same character as (3.1), since the equation for u is a special case of the equation for ψ. Thus, all the formulae in §3 and in the Appendices apply, except that we should now put

$$p = z, \text{ i.e., } p_0 = z,\ p_1 = 0,\ p_k = 0,\ k = 2, 3, \cdots, \tag{5.3a}$$

$$q = \alpha, \quad \text{and} \quad r = \beta. \tag{5.3b}$$

From (B10) it follows that if χ is to satisfy (2.17), we must have

$$z\mu_1' + \alpha_0\mu_1 + \beta_0\mu - \beta\nu_1 = \lambda^{-1}\overline{A}, \tag{5.4a}$$

$$z\nu_1' + \alpha_0\nu_1 + \beta_0\nu + z\mu_1 - \alpha\nu_1 = \lambda^{-1}\overline{B}, \tag{5.4b}$$

$$-2z\sigma_1' + [\alpha_0 - \alpha - 1]\sigma_1 + \beta_0\sigma = \lambda^{-1}\overline{C}, \tag{5.4c}$$

$$-2z\tau_1' + [\alpha_0 - \alpha - 2]\tau_1 + \beta_0\tau = \lambda^{-1}\overline{D} - (3\nu_1' + \mu_1) \tag{5.4d}$$

where $\overline{A}, \overline{B}, \overline{C}, \overline{D}$ are given by (C3) with (A, B, C, D) replaced by (μ, ν, σ, τ), and

$$\nu - z\tau = \lambda^{-1}\tilde{\sigma}, \qquad \sigma = \lambda^{-1}\tilde{\tau}; \tag{5.5}$$

$$\mu_1 = \mu' - \beta\tau, \tag{5.6a}$$

$$\nu_1 = \nu' + \mu - \alpha\tau, \tag{5.6b}$$

$$\sigma_1 = \tilde{\sigma}, \tag{5.6c}$$

$$\tau_1 = \tau' + \tilde{\tau}. \tag{5.6d}$$

We shall now solve this set of equations in terms of asymptotic expansions in inverse powers of λ. Then it is clear that the exact forms of the expressions on the right-hand side of (5.4) do not matter. Only the forms of the operator on the left are important. It is also convenient to think of σ_1 and τ_1 rather than $\tilde{\sigma}$ and $\tilde{\tau}$ as the variable sought, and rewrite (5.5) as

$$\nu - z\tau = \lambda^{-1}\sigma_1, \qquad \sigma = \lambda^{-1}(\tau_1 + \tau'). \tag{5.7}$$

By using (5.5) and (5.7), we may rewrite (5.4) as

$$[z\mu'' + \alpha_0\mu' + (\beta_0 - \beta)\mu] - \beta[2z\tau' + \tau + (\alpha_0 - \alpha)\tau] = O(\lambda^{-1}), \tag{5.8a}$$

$$\begin{aligned}[z^2\tau'' + (\alpha_0 - 2\alpha + 2)z\tau' + (\alpha_0 - \alpha)(1 - \alpha)\tau + (\beta_0 - \beta)z\tau]&\\ + [2z\mu' + (\alpha_0 - \alpha)\mu] &= O(\lambda^{-1}),\end{aligned} \tag{5.8b}$$

$$-2z\sigma_1' + (\alpha_0 - \alpha - 1)\sigma_1 = O(\lambda^{-1}), \tag{5.8c}$$

$$\begin{aligned}-2z\tau_1' + (\alpha_0 - \alpha - 2)\tau_1&\\ = O(\lambda^{-1}) - [4\mu' + 3z\tau'' + 3(1 - 2\alpha)\tau' + (\beta_0 - \beta)\tau]&\end{aligned} \tag{5.8d}$$

where $O(\lambda^{-1})$ denotes terms of that order in the variables μ, τ, σ_1 and τ_1, and the parameters α and β. Clearly, in solving for successively higher approximations, the equations (5.8a) and (5.8b) may be first solved for μ and τ and then the equations (5.8c) and (5.8d) solved for σ_1 and τ_1. It is to be noted that α and β are also unknowns to be determined, but we may expect α and β to be equal to α_0 and β_0 in the limit $\lambda \to \infty$.

The solution of these equations will now be discussed in the next three sections for three different cases, depending on the nature of solutions of the reduced equation

$$z\chi'' + \alpha_0\chi' + \beta_0\chi = 0. \tag{5.9}$$

Case I. If the indices $\rho=0$ and $\rho=1-\alpha_0$ of equation (5.9) do not differ by an integer, positive, negative or zero, the solutions of (5.9) *do not* contain logarithmic terms. Indeed, their essential characteristics do not depend on β_0. This is the simplest case.

Case II. If the indices differ by an integer, positive, negative or zero, and $\beta_0 \neq 0$, one of the solutions of (5.9) definitely contains a logarithmic term. This case is still relatively simple.

Case III. If the indices differ by an integer, positive, negative or zero, and $\beta_0=0$, the solutions of (5.9) are explicitly known to be

$$\chi = 1 \quad \text{and} \quad z^{-\alpha_0+1} \qquad \text{if } \alpha_0 \neq 0 \tag{5.10a}$$

or

$$\chi = 1 \quad \text{and } z \qquad \text{if } \alpha_0 = 0. \tag{5.10b}$$

This is the most complicated case, because the simplicity of the initial approximation makes them less useful for controlling the behavior of the higher approximations. Indeed, it will be found that the coefficients (μ, ν, σ, τ) in (5.1) have the usual behavior (1.3) only under very special circumstances. In general, they must be taken to have the form

$$\mu(z, \lambda) = \sum_{p=0}^{\infty} \lambda^{-p/(n+1)}\mu^{(p)}(z). \tag{5.11}$$

It is clear that the general outline of the theory, as presented in §2, will be somewhat modified. But the essential spirit of the analysis will not [cf. footnote 6].

6. **Existence of regular formal solutions**—Case I. We shall now consider the solution of the system of equations (5.8) in the simplest case when the equation (5.9) has one nonintegral index. The method that we discuss in some detail will be the one that may be used without any essential change in the next two cases. Other methods will be briefly indicated at the end of this section.

We consider expansions of the type

$$(6.1)\qquad \alpha = \sum_{j=0}^{\infty} \lambda^{-j}\alpha^{(j)}, \qquad \beta = \sum_{j=0}^{\infty} \lambda^{-j}\beta^{(j)},$$

$$(6.2)\qquad \mu = \sum_{j=0}^{\infty} \lambda^{-j}\mu^{(j)},$$

and similar equations for ν, σ and τ. We naturally attempt to set

$$(6.3)\qquad \alpha^{(0)} = \alpha_0, \qquad \beta^{(0)} = \beta_0.$$

The initial approximation for (5.8) then becomes

$$(6.4a)\qquad [z\mu^{(0)''} + \alpha_0\mu^{(0)'}] - \beta_0[2z\tau^{(0)'} + \tau^{(0)}] = 0,$$

$$(6.4b)\qquad z[z\tau^{(0)''} + (2 - \alpha_0)\tau^{(0)'} + 2\mu^{(0)'}] = 0,$$

$$(6.4c)\qquad 2z\sigma_1^{(0)'} + \sigma_1^{(0)} = 0,$$

$$(6.4d)\qquad 2z\tau_1^{(0)'} + 2\tau_1^{(0)} = [4\mu^{(0)'} + 3z\tau^{(0)''} + 3(1 - 2\alpha_0)\tau^{(0)'}].$$

We expect the initial approximation to be $\mu^{(0)}=1$, $\nu^{(0)}=\sigma^{(0)}=\tau^{(0)}=0$; and this is indeed a possible solution for (6.4). The higher approximations are given by

$$(6.5a)\qquad [z\mu^{(j)''} + \alpha_0\mu^{(j)'}] - \beta_0[2z\tau^{(j)'} + \tau^{(j)}] = \beta^{(j)} + R_j(z),$$

$$(6.5b)\qquad z[z\tau^{(j)''} + (2 - \alpha_0)\tau^{(j)'} + 2\mu^{(0)'}] = \alpha^{(j)} + R_j(z),$$

$$(6.5c)\qquad 2z\sigma_1^{(j)'} + \sigma_1^{(j)} = R_j(z),$$

$$(6.5d)\qquad 2z\tau_1^{(j)} + 2\tau_1^{(j)'} = R_j(z) + [4\mu^{(j)'} + 3z\tau^{(j)''} + 3(1 - 2\alpha_0)\tau^{(j)'}],$$

where $R_j(z)$ is the generic symbol for a regular function which is known when the $(j-1)$th approximations are all known. After regular solutions of the equations (6.5a) and (6.5b) are found, we may integrate (6.5c) and (6.5d) as follows:

$$(6.6)\qquad \sigma_1^{(j)} = z^{-1/2}\int_0^z z^{1/2}R_j(z)dz,$$

$$(6.7)\qquad \tau_1^{(j)} = z^{-1}\int_0^z 2^{-1}\tilde{R}_j(z)dz$$

where $\tilde{R}_j(z)$ denotes the right-hand side of (6.5d).

To solve the equations (6.5a) and (6.5b), we first adopt the simpler notation $(\bar{\ })$ for all unknowns with an index j, and use different symbols $R_1(z)$ and $R_2(z)$ for the two regular functions; thus

$$(6.8)\qquad [z\bar{\mu}'' + \alpha_0\bar{\mu}'] - \beta_0[2z\bar{\tau}' + \bar{\tau}] = \bar{\beta} + R_1(z),$$

$$(6.9)\qquad z[z\bar{\tau}'' + (2 - \alpha_0)\bar{\tau}' + \bar{\mu}'] = \bar{\alpha} + R_2(z).$$

Clearly, to have regularity, we must take

(6.10) $$\bar{\alpha} = -R_2(0)$$

so that (6.9) becomes

(6.11) $$z\bar{\tau}'' + (2 - \alpha_0)\bar{\tau}' + 2\bar{\mu}' = R_3(z).$$

This equation can be integrated once more to give

(6.12) $$z\bar{\tau}' + (1 - \alpha_0)\bar{\tau} + 2\bar{\mu} = R_4(z)$$

where $R_4(z)$ has an additive constant of integration which is at our disposal.

We shall now attempt to solve $\bar{\mu}$ and $\bar{\tau}$ from (6.8) and (6.12) as regular power series. This method is based on the fact that the system is Fuchsian, as can be easily verified by eliminating $z\bar{\tau}'$ from (6.8) by means of (6.12), and consider the system of equations of the first order with $(\bar{\mu}, \bar{\mu}', \bar{\tau})$ as the three dependent variables. Thus, formal power series solutions (possibly including logarithmic terms) are true solutions.

Let us write

(6.13) $$\bar{\mu} = \sum_{k=0}^{\infty} \bar{\mu}_k z^k, \qquad \bar{\tau} = \sum_{k=0}^{\infty} \bar{\tau}_k z^k.$$

Then (6.8) leads to

(6.14) $$-\beta_0(2k+1)\bar{\tau}_k + (k+\alpha_0)(k+1)\bar{\mu}_{k+1} = R_{1,k} - \bar{\beta}\delta_{k0}, \quad k = 0, 1, 2, \cdots$$

and (6.12) leads to

(6.15) $$2\bar{\mu}_k + [k + (1-\alpha_0)]\bar{\tau}_k = R_{4,k}, \qquad k = 0, 1, \cdots .$$

The latter may be solved for $\bar{\mu}_k$ in terms of $\bar{\tau}_k$. To determine $\bar{\tau}_k$, write (6.15) as

(6.16) $$2\bar{\mu}_{k+1} + [k + (2-\alpha_0)]\bar{\tau}_{k+1} = R_{4,k+1}, \qquad k = -1, 0, 1, \cdots$$

and eliminate $\bar{\mu}_{k+1}$ from (6.4). Thus

(6.17) $$2\beta_0(2k+1)\bar{\tau}_k + (k+1)(k+\alpha_0)[k + (2-\alpha_0)]\bar{\tau}_{k+1} = S_k + 2\bar{\beta}\delta_{k0},$$

for $k=0, 1, 2, \cdots$, where S_k is a known constant. Equation (6.17) enables us to solve all the $\bar{\tau}_k$'s in terms of the leading coefficient $\bar{\tau}_0$, unless the coefficient for one of the unkowns $\bar{\tau}_{k+1}$ vanishes; i.e., when

(6.18) $$(k+\alpha_0)(k+2-\alpha_0) = 0$$

for some integral value of $k>0$. This is possible only when

(6.19) $$\alpha_0 = n + 2 \text{ or } -n$$

for some value of $n=0, 1, 2, \cdots$. Thus, if the parameter α_0 is not an integer, or if it takes on the special integral value

(6.20) $$\alpha_0 = 1,$$

the above method can be carried through. The condition (6.20) implies that the indices of the reduced equation

$$z\chi'' + \alpha_0\chi' + \beta_0\chi = 0 \tag{6.21}$$

are both equal to zero, and the operator $L_0(\chi)$ is self-adjoint. One of the solutions of (6.21) contains logarithmic terms no matter what the value of β_0 may be. This is however really an exceptionally simple case of Case II, and will be included in the next section. We may therefore formulate the following theorem.

THEOREM I. *If the indices $\rho=0$ and $\rho=1-\alpha_0$ of the reduced equation* (6.21) *of the normalized differential equation* (5.1) *do not differ by an integer, then the latter has formal solutions of the form* (5.2) *where the coefficients μ, ν, σ, τ are given by formulas of the form* (6.2) *and $u(z, \lambda)$ is any solution of the differential equation* (1.8) *with coefficients α and β in the form* (1.9). *Furthermore, we have*

$$\mu^{(0)} = 1, \ \nu^{(0)} = 0, \ \sigma^{(0)} = 0, \ \tau^{(0)} = 0, \ \alpha^{(0)} = \alpha_0, \ \beta^{(0)} = \beta_0;$$

and that all the coefficients $\alpha^{(j)}$ $(j=1, 2, \cdots)$ must be chosen in a manner prescribed by the original differential equation; the coefficients $\beta^{(j)}$ $(j=1, 2, \cdots)$ are however arbitrary.

There are at least two other ways of treating the system of equations (6.8) and (6.9). By analogy with §§3, 4, we see that the solutions of the homogeneous system are known. They are simply the connection coefficients of the solutions of the reduced equation (6.1) with themselves. Thus, they are indeed explicitly expressible in terms of Bessel functions. We may then write the solutions of the nonhomogeneous system by the method of variation of parameters and use the disposable constants to make the solution regular.

Another method is to use the form (6.8) and (6.12) and eliminate $z\bar{\tau}'$ from the former by using the latter. It is then easy to reduce this system into an integral system by solving (6.8) for $\bar{\mu}$ and (6.12) for $\bar{\tau}$. The reasoning here runs parallel to that used at the end of §3.

7. **Existence of regular formal solutions**—CASE II. We shall now apply the method used above to consider the case where α_0 is an integer and $\beta_0 \neq 0$. The case $\beta_0=0$ will be considered in the next section. We shall prove the following theorem.

THEOREM II. *If α_0 is an integer, positive, negative, or zero, and $\beta_0 \neq 0$,* Theorem I *still holds, but only when the coefficients $\beta^{(j)}$ $(j=1, 2, 3, \cdots)$ are also chosen in a manner prescribed by the original differential equation. Only the case $\alpha_0=1$ is free from this requirement.*

To prove this theorem, let us write out equation (6.17) explicitly as follows:

$$
(7.1)\quad
\begin{aligned}
&k = 0, && 2\beta_0\bar{\tau}_0 + \alpha_0(2 - \alpha_0)\bar{\tau}_1 = R_0^{(3)} + 2\bar{\beta},\\
&k = 1, && 6\beta_0\bar{\tau}_1 + 2(1 + \alpha_0)(3 - \alpha_0)\bar{\tau}_2 = R_1^{(3)},\\
&k = 2, && 10\beta_0\bar{\tau}_2 + 3(2 + \alpha_0)(4 - \alpha_0)\bar{\tau}_3 = R_2^{(3)},\\
& && \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&k = n, && 2\beta_0(2n + 1)\bar{\tau}_n + (n + 1)(n + \alpha_0)(n + 2 - \alpha_0)\bar{\tau}_{n+1} = R_n^{(3)}.
\end{aligned}
$$

If we regard these as a system of equations for $\bar{\tau}_1, \cdots, \bar{\tau}_{n+1}$, the coefficient matrix has the structure in the form of two diagonal terms

$$
\begin{bmatrix}
\alpha_0(2-\alpha_0) & & & & \\
6\beta_0 & 2(1+\alpha_0)(3-\alpha_0) & & & \\
 & 10\beta_0 & 3(2+\alpha_0)(4-\alpha_0) & & \\
 & & \ddots & \ddots & \\
 & & & 2\beta_0(2n+1) & (n+1)(n+\alpha_0)(n+2-\alpha_0)
\end{bmatrix}
$$

The determinant can vanish only if one of the terms in the main diagonal vanishes and this can happen only when (6.19) is satisfied, and only once. We now consider the matrix for which the lower right corner term vanishes, and this will still be denoted by the above matrix. Then, in order to be able to solve for $\bar{\tau}_1, \cdots, \bar{\tau}_{n+1}$, the matrix obtained by replacing the last column by the right-hand side column of the system of equations (7.1) must have a vanishing determinant. This can always be made to be true *if the coefficient of* $2\bar{\beta}$ *does not vanish.* This coefficient is the $n\times n$ determinant at the lower left side of the above matrix (up to a sign), and its value is clearly $(2\beta_0)^n(2n+1)(2n-1)\cdots 5.3$. If $\beta_0\neq 0$, then, the augumented matrix can be made to have the same rank as the coefficient matrix, and there is a one-parameter family of solutions for $\bar{\tau}_1, \cdots, \bar{\tau}_{n+1}$. The equations for $k=n+1, \cdots$ then determine all the other coefficients. We have thus indeed found regular solutions by proper choice of the coefficients $\beta^{(i)}$, and the theorem is proved.

8. **Existence of regular formal solutions**—CASE III. We shall now consider the apparently simpler but actually more complicated case where $\beta_0=0$, and the reduced equation (6.21) takes the form

$$
(8.1)\qquad z\chi'' + \alpha_0\chi' = 0
$$

where α_0 is an integer, positive, negative, or zero. We shall restrict our discussion only to cases

$$
(8.2)\qquad \alpha_0 = -n, \qquad n = 0, 1, 2, \cdots,
$$

since the adjoint equation of (5.1) has the corresponding coefficient given by[12]

[12] Cf. Langer [4].

$$\bar{\alpha}_0 = 2 - \alpha_0 = 2, 3, 4, \cdots. \tag{8.3}$$

The gap left by (8.2) and (8.3) is the special case

$$\alpha_0 = \bar{\alpha}_0 = 1, \tag{8.4}$$

when the operator $L_0(\chi)$ is self-adjoint. This case has been covered by Theorems I and II.

Let us first notice the similarity and the differences between the present case and the above cases. The equations (5.8) remain unchanged. Clearly, if a method of successive approximations is used, Equations (5.8c) and (5.8d) can be treated in exactly the same manner as in §6. Thus, the problem again focuses itself on the first two equations (5.8a) and (5.8b). In the special case considered here, these may be written in the following convenient form:

$$z\mu'' - n\mu' = (\beta - \beta_0)(\mu + 2z\tau' + \tau - (\alpha - \alpha_0)\tau) + O(\lambda^{-1}), \tag{8.5}$$

$$\begin{aligned} z[z\tau'' + (n+2)\tau'] = &- 2z\mu' + (\alpha - \alpha_0)(\mu + 2z\tau' + \tau - \alpha\tau) \\ &+ (\beta - \beta_0)z\tau + O(\lambda^{-1}), \end{aligned} \tag{8.6}$$

where we have put $\alpha_0 = -n$ and $\beta_0 = 0$ in some of the expressions, but have retained the differences $\alpha - \alpha_0$, $\beta - \beta_0$ to suggest that they are terms of an order higher. In Equation (8.6), we put μ' on the right-hand side, because it is supposed to be obtained from Equation (8.5) in a process of successive approximations. As before, the initial approximation is given by $\mu^{(0)} = 1$, $\tau^{(0)} = 0$.

It is easy to see from Equation (8.5) that the usual expansion of μ and τ in inverse powers of λ would not work, except under extremely special circumstances; for logarithmic terms tend to appear. The successive approximations for μ, beyond the initial one, satisfy the inhomogeneous differential equation

$$z\bar{\mu}'' - n\bar{\mu}' = \bar{R}(z), \tag{8.7}$$

where $\bar{R}(z)$ is a regular function determined by the lower approximations. The solution of (8.7) can be easily written down. It is[13]

$$\bar{\mu}' = z^n \left\{ c - \frac{1}{n} \int z^{-(n+1)} \bar{R}(z) dz \right\}, \tag{8.8a}$$

$$\bar{\mu} = c_1 + c_2 z^{n+1} - \int \frac{z^n}{n} dz \int z^{-(n+1)} \bar{R}(z) dz. \tag{8.8b}$$

Thus, in general, if $\bar{R}(z)$ contains a term $\bar{R}_m z^m$, there is a corresponding term $-\bar{R}_m\{(m-n)(m+1)n\}^{-1} z^{m+1}$ occurring in $\bar{\mu}$. However, if $m = n$, i.e., if $\bar{R}(z)$ should contain a term $\bar{R}_n z^n$, the integral of (8.7) will contain a term of the form

([13]) We are treating the case $n \neq 0$ here. The case $n = 0$ requires some obvious minor modifications.

$$\bar{\mu}' = cz^n + \bar{R}_n z^n \log z, \tag{8.9a}$$

$$\bar{\mu} = c_1 + c_2 z^{n+1} \frac{\bar{R}_n}{n+1} z^{n+1}(\log z - 1). \tag{8.9b}$$

The occurrence of logarithmic terms must therefore be prevented, if they tend to occur, by trying to remove terms proportional to z^n on the right-hand side of (8.7).

There is no such difficulty associated with (8.6). As in §6, we can choose α properly to counteract the factor z on the left-hand side of the equation. After this is done, we are faced with solving equations of the type

$$z\bar{\tau}'' + (n+2)\bar{\tau}' = \bar{R}(z), \tag{8.10}$$

whose regular solution is given by

$$\bar{\tau}' = (n+2)^{-1} z^{-(n+2)} \left[\int_0^z z^{(n+1)} \bar{R}(z) dz \right], \tag{8.11a}$$

and

$$\bar{\tau}' = (n+2)^{-1} \left\{ \int_0^z z^{-(n+2)} \left[\int_0^z z^{(n+1)} \bar{R}(z) dz \right] dz + c \right\}, \tag{8.11b}$$

where c is an arbitrary constant of integration. Thus, a positive power term $\bar{R}_m z^m$ in $\bar{R}(z)$ would always produce a positive power term

$$\bar{R}_m \{(m+n+2)(m+1)(n+2)\}^{-1} z^{m+1} \text{ in } \bar{\tau}.$$

Removal of the logarithmic term by fractional powers. To be able to remove the logarithmic terms in any scheme of successive approximation it is then necessary to be able to control an explicit term cz^n which enters $\bar{R}(z)$ in (8.7) and whose coefficient c is at our disposal. If we attempt to solve (8.5) and (8.6) by expanding μ and τ in the usual form

$$\mu = \sum_{j=0}^{\infty} \lambda^{-j} \mu^{(j)} \quad \text{and} \quad \tau = \sum_{j=0}^{\infty} \lambda^{-j} \tau^{(j)} \tag{8.12}$$

with corresponding expansions for α and β, we do not have such a control even at the stage of determining $\mu^{(1)}$ and $\tau^{(1)}$. Let us assume, for definiteness, that the logarithmic terms would arise in the first approximation by using the above expansion (8.12).

Clearly, some modification of this expansion is needed. Heuristically, one may reason as follows to arrive at the form of this expansion. The occurrence of the logarithmic term indicates that μ is *not* of the form

$$\mu = \mu^{(0)} + O(\lambda^{-1}), \qquad \mu^{(0)} = 1,$$

but rather that $\mu - \mu^{(0)}$ must be *larger* than $O(\lambda^{-1})$. This immediately suggests the use of fractional powers. The next question is "Which fraction?"

Let us consider an expansion in the parameter $\epsilon=\lambda^{-(1/N)}$ where $N>1$, but is otherwise unspecified. We note from (8.5) that, before terms of the order of λ^{-1} are included, the right-hand sides of (8.7) and (8.10) can only arise from the lower approximations. We have also noticed that a term in z^m on the right-hand sides of these equations gives rise to a term in z^{m+1} in the solution.

Since $\mu^{(0)}=1$, $\tau^{(0)}=1$, we may expect that z^i is precisely the significant power in $\mu^{(i)}$ and $\tau^{(i)}$. Thus, the nth approximation should appear in the right-hand side of the equation which also includes the term $O(\lambda^{-1})$ in (8.5), i.e., we should have

$$\epsilon^{n+1} = \lambda^{-1},$$

or

$$\epsilon = \lambda^{-1/(n+1)}. \tag{8.13}$$

Having thus determined ϵ in a tentative manner, we may try to put

$$\mu = \sum_{p=0}^{\infty} \epsilon^i \mu^{(i)}(z), \qquad \mu^{(0)} = 1, \tag{8.14a}$$

$$\tau = \sum_{p=0}^{\infty} \epsilon^i \tau^{(i)}(z), \qquad \tau^{(0)} = 0, \tag{8.14b}$$

$$\alpha = \sum_{p=0}^{\infty} \epsilon^i \alpha^{(i)}, \qquad \alpha^{(0)} = \alpha_0 = -n, \tag{8.14c}$$

$$\beta = \sum_{p=0}^{\infty} \epsilon^i \beta^{(i)}, \qquad \beta^{(0)} = \beta_0 = 0 \tag{8.14d}$$

in (8.5) and (8.6). Then, we obtain for the first approximation

$$z\mu^{(i)''} - n\mu^{(i)'} = \beta^{(1)}, \tag{8.15}$$

and

$$z\left[z\tau^{(1)''} + (h+2)\tau^{(1)'}\right] = -2z\mu^{(1)'} + \alpha^{(1)}; \tag{8.16}$$

and for the higher approximations, $2\leqq i\leqq n-1$,

$$z\mu^{(i+1)'} - n\mu^{(i+1)} = \beta_1(\mu^{(i)} - 2z\tau^{(i)'} + \tau^{(i)}) + \beta^{(i+1)} + \cdots \tag{8.17}$$

and

$$\begin{aligned} &z\left[z\tau^{(i+1)''} + (n+2)\tau^{(i+1)'}\right] \\ &= -2z\mu^{(i+1)'} + \alpha_1(\mu^{(i)} + 2z\tau^{(i)'} + (n+1)\tau^{(i)}) + \alpha^{(i+1)} + \beta_1 z\tau^{(i)} + \cdots \end{aligned} \tag{8.18}$$

where the terms not written out explicitly involve only $\mu^{(i-1)}$, $\tau^{(i-1)}$, $\alpha^{(i)}$, $\beta^{(i)}$

and lower approximations. As we have noted before, the terms $\alpha^{(1)}$ in (8.16) and $\alpha^{(i+1)}$ in (8.18) must be chosen to make the solution regular.

Let us now examine the rise of power in the successive approximations. Clearly, we have from (8.15) [cf. (8.8b)]

$$\mu^{(1)} = c_1 + c_2 z^{n+1} - (\beta^{(1)}/n)z \tag{8.19}$$

and, from (8.16) [cf. (8.11b)].

$$\tau^{(1)} = c_3 - \frac{c_2}{(n+1)(n+2)} z^{n+1} + \frac{2\beta^{(1)}}{n(n+2)^2} z. \tag{8.20}$$

Terms with power higher than n will not interfere with the cancellation of the term z^n in the right-hand side of the $(n+1)$th approximation. We may therefore omit such terms by putting $c_2=0$. Thus, the leading powers in $\mu^{(1)}$ and $\tau^{(1)}$ indeed behave in the manner as expected. In the higher approximations, as one can see from (8.17) and (8.18), the leading powers would arise from the terms $\beta_1\mu^{(i)}$ and $-2z\mu^{(i+1)\prime}$ on the right-hand side of these equations. Thus, if we denote by C_i and $\overline{C}_i$ the coefficients of z^i in $\mu^{(i)}$ and $\tau^{(i)}$ respectively, we have the following relations for the determination of these coefficients:

$$(i+1)(i-n)C_{i+1} = \beta_1 C_i, \tag{8.21}$$

$$(i+n+2)\overline{C}_{i+1} = -2C_{i+1}, \tag{8.22}$$

where $2 \leqq i \leqq n-1$. Thus, the coefficient of z^n in $\mu^{(n)}$ is pr0portional to β_1^n. This contributes a term proportional to $\beta_1^{n+1}z^n$ in the right-hand side of (8.17) in the $(n+1)$th approximation. In this approximation, the right-hand side contains other terms in z^n through the term $O(\lambda^{-1})$ in (8.5), and β_1^{n+1} must be so chosen as to annihilate such terms. Note that these new terms to be annihilated will depend only on the initial approximation $(\mu^{(0)}, \tau^{(0)}) = (1, 0)$, and are not influenced by the intermediate approximations in fractional powers of λ.

A little calculation will show that as long as $\beta_1 \neq 0$, nothing difficult will arise in the higher approximations. Thus, we have covered the case when the logarithmic terms would appear in the approximation $O(\lambda^{-1})$. If this is not the case, we simply defer the introduction of the fractional powers until the occurrent of logarithmic terms is imminent. Actually, no change of procedure is necessary. If logarithmic terms do not tend to appear until the term $O(\lambda^{-2})$ is considered, the coefficients $\beta^{(1)}, \cdots, \beta^{(n)}$ would automatically be forced to equal to zero by the condition of regularity.

We may summarize our results in the following theorem.

THEOREM III. *If the reduced equation of the normalized differential equation* (5.1) *has two regular solutions, i.e., if* $\alpha_0 = -n$ $(n = 0, 1, 2, \cdots)$ *and* $\beta_0 = 0$, *then there exists a set of formal solutions in the form*

$$\chi = \mu u + \nu u' + \lambda^{-1}\sigma u'' + \lambda^{-2}\tau u''' \tag{8.23}$$

where u is any solution of the basic reference equation

$$u^{\mathrm{iv}} + \lambda^{-2}(zu'' + \alpha u' + \beta u) = 0, \tag{8.24}$$

and the coefficients $\mu(z, \lambda)$, $\nu(z, \lambda)$, $\sigma(z, \lambda)$, $\tau(z, \lambda)$ *and* $\alpha(\lambda)$, $\beta(\lambda)$ *have asymptotic expansions of the form* (8.14) *in which* ϵ *is given by* (8.13) *and the functions* $\mu^{(i)}(z), \cdots, \tau^{(i)}(z)$ *are all regular.*

Appendix A

In this appendix, we shall give some transformation formulae which will be found useful both for the reduction of the given equation into the normal form and for the solution of the normalized differential equation. We consider the transformation of the dependent variable $\psi(z, \lambda)$ in the differential equation

$$\psi^{\mathrm{iv}} + \lambda^2(p\psi'' + q\psi' + r\psi) = 0 \tag{A 1}$$

of the type (1.1), having in mind, however, that (A 1) is to be regarded as the system of differential equations

$$\frac{d}{dz}\begin{bmatrix} \psi \\ \psi' \\ \lambda^{-1}\psi'' \\ \lambda^{-2}\psi''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ -r & -q & -\lambda p & 0 \end{bmatrix}\begin{bmatrix} \psi \\ \psi' \\ \lambda^{-1}\psi'' \\ \lambda^{-2}\psi''' \end{bmatrix} \tag{A 2}$$

for the vector variable $\boldsymbol{\psi}(\psi, \psi', \lambda^{-1}\psi'', \lambda^{-2}\psi''')$. We introduce a new variable χ by the relation

$$\chi = A\psi + B\psi' + \lambda^{-1}C\psi'' + \lambda^{-2}D\psi''', \tag{A 3}$$

where (A, B, C, D) are connection coefficients with the type of asymptotic behavior (1.3). If we differentiate (A3) with respect to z, we find

$$\chi' = A_1\psi + B_1\psi' + \lambda^{-1}C_1\psi'' + \lambda^{-2}D_1\psi''', \tag{A 4}$$

where

$$A_1 = A' - rD, \tag{A 5a}$$

$$B_1 = B' + A - qD, \tag{A 5b}$$

$$C_1 = C' + \lambda(B - pD), \tag{A 5c}$$

$$D_1 = D' + \lambda C. \tag{A 5d}$$

If we now impose the condition that the connection coefficients (A_1, B_1, C_1, D_1) shall also have the type of behavior (1.3), then we must have

(A 6) $$B - pD = \lambda^{-1}E \quad \text{and} \quad C = \lambda^{-1}F,$$

and the relations (A5) become

(A 7a) $$A_1 = A' - rD,$$

(A 7b) $$B_1 = B' + A - qD,$$

(A 7c) $$C_1 = E + \lambda^{-1}F',$$

(A 7d) $$D_1 = D' + F.$$

The second derivative χ'' is given by

(A 8) $$\chi'' = A_2\psi + B_2\psi' + \lambda^{-1}C_2\psi'' + \lambda^{-2}D_2\psi''',$$

where

(A 9a) $$A_2 = A_1' - rD_1,$$

(A 9b) $$B_2 = B_1' + A_1 - qD_1,$$

(A 9c) $$C_2 = C_1' + \lambda(B_1 - pD_1),$$

(A 9d) $$D_2 = D_1' + \lambda C_1.$$

Note that C_2 and D_2 are of the order of λ, and hence the connection coefficients $\lambda^{-1}(A_2, B_2, C_2, D_2)$, which are appropriate for $\lambda^{-1}\chi''$, automatically have the behavior of the type (1.3).

The third derivative χ''' is given by

(A 10) $$\chi''' = A_3\psi + B_3\psi' + \lambda^{-1}C_3\psi'' + \lambda^{-2}D_3\psi''',$$

where

(A 11a) $$A_3 = A_2' - rD_2 = (A_2' - rD_1) - \lambda rC_1,$$

(A 11b) $$B_3 = B_2' + A_2 - qD_2 = (B_2' + A_2 - qD_1') - \lambda qC_1,$$

(A 11c) $$\begin{aligned} C_3 &= C_2' + \lambda(B_2 - pD_2) \\ &= C_1'' + \lambda[(B_1 - pD_1)' + (B_2 - pD_2')] - \lambda^2 pC_1, \end{aligned}$$

(A 11d) $$D_3 = D_2' + \lambda C_2 = D_1'' + 2\lambda C_1' + \lambda^2(B_1 - pD_1).$$

Note that the order of λ is indicated explicitly in the final form of the above expressions, and hence the connection coefficients $\lambda^{-2}(A_3, B_3, C_3, D_3)$, which are appropriate for $\lambda^{-2}\chi'''$, automatically have the behavior of the type (1.3).

Thus, the matrix G connecting the vectors $\boldsymbol{\chi}(\chi, \chi', \lambda^{-1}\chi'', \lambda^{-2}\chi''')$ and $\boldsymbol{\psi}(\psi, \psi', \lambda^{-1}\psi'', \lambda^{-2}\psi''')$ has the form

(A 12) $$G = \sum_{n=0}^{\infty} \lambda^{-n}G_n,$$

where G_0 has the form

$$\text{(A 13)} \qquad G_0 \equiv \begin{bmatrix} A_0 & B_0 & 0 & D_0 \\ A_{1,0} & B_{1,0} & C_{1,0} & D_{1,0} \\ 0 & 0 & B_{1,0} - p_0 D_{1,0} & C_{1,0} \\ 0 & 0 & -p_0 C_{1,0} & B_{1,0} - p_0 D_{1,0} \end{bmatrix}.$$

It is essential that $|G_0| = \det G_0 \neq 0$ so that the transformation of variables is nonsingular. In particular, this should hold for the point $z=0$. At that point, we have

$$\text{(A 14)} \qquad |G_0(0)| = A_0(0)\{B_{1,0}(0)\}^3,$$

and the following conditions are necessary and sufficient for the transformation to be nonsingular in a neighborhood[14] of $z=0$:

$$\text{(A 15)} \qquad A_0(0) \neq 0, \qquad B_{1,0}(0) = A_0(0) + D_0(0)[1 - q_0(0)] \neq 0,$$

where $B_{1,0}$ has also been expressed in terms of more elementary quantities by means of (A 7b) and (A 6).

As a matter of fact, we shall require the function χ to be a solution of a fourth order differential equation of the form (B 1) specified below, which is quite similar to (A 1). Thus, if we consider a fundamental matrix solution Ψ of the system (A 2), then

$$\text{(A 16)} \qquad \mathrm{X} = G\Psi$$

is a fundamental matrix solution of the corresponding system for $\boldsymbol{\chi}(\chi, \chi', \lambda^{-1}\chi'', \lambda^{-2}\chi''')$. The determinant of G thus appears as the ratio of two Wronskian determinants, and the leading term $|G_0|$ must be a constant. Thus, it would take on the constant value at $z=0$ as given by (A 14).

We note that the nature of the matrix G indicates that the above manner of defining the vectors $\boldsymbol{\chi}$ and $\boldsymbol{\psi}$ is the natural one. Other choices of powers of λ in the definition would not have been as convenient.

Appendix B

In this appendix, we shall derive the differential equation satisfied by the function χ defined in the last section, and then obtain the conditions which must be satisfied by the coefficients A, B, C, D so that χ satisfies a differential equation of the type

$$\text{(B 1)} \qquad L_0(\chi) = \lambda[\bar{a}\chi + \bar{b}\chi' + \bar{c}\lambda^{-1}\chi'' + \bar{d}\lambda^{-2}\chi''']$$

where

$$\text{(B 2)} \qquad L_0(\chi) \equiv \chi^{\mathrm{iv}} + \lambda^2(z\chi'' + \alpha_0\chi' + \beta_0\chi),$$

as defined in (2.17), and the coefficients $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ have the behavior (1.3).

[14] The zeros of the analytic function $|G_0(z)|$ are isolated.

To do this, we shall first calculate the fourth derivative of χ. We may write it in the form

(B 3) $$\chi^{\text{iv}} = A_4\psi + B_4\psi' + \lambda^{-1}C_4\psi'' + \lambda^{-2}D_4\psi''',$$

where the coefficients (A_4, B_4, C_4, D_4) are given by

(B 3a) $$A_4 = A_3' - rD_3,$$

(B 3b) $$B_4 = B_3' + A_3 - qD_3,$$

(B 3c) $$C_4 = C_3' + \lambda(B_3 - pD_3),$$

(B 3d) $$D_4 = D_3' + \lambda C_3.$$

After some calculation, we find that

(B 4a) $$A_4 = (A_1' - rD_1')' - rD_1'' - \lambda[(rC_1)' + 2rC_1'] - \lambda^2 r(B_1 - pD_1),$$

(B 4b) $$B_4 = (B_2' + A_2 - qD_1')' + (A_2' - rD_1') - qD_1'' - \lambda[(qC_1)' - rC_1 - 2qC_1'] - \lambda^2 q(B_1 - pD_1),$$

(B 4c) $$C_4 = C_1''' + \lambda[(B_1 - pD_1)'' + (B_2 - pD_1')' + B_2 + A_2 - qD_1' - pD_1''] + \lambda^2[-(pC_1)' - qC_1 - 2pC_1'] - \lambda^3 p(B_1 - pD_1),$$

(B 4d) $$D_4 = D_1''' + 3\lambda C_1'' + \lambda^2[2(B_1 - pD_1)' + (B_2 - pD_1')] - \lambda^3 pC_1.$$

If we now form the differential operator $L_0(\chi)$, we have

(B 5) $$L_0(\chi) = \tilde{A}\psi + \tilde{B}\psi' + \lambda^{-1}\tilde{C}\psi'' + \lambda^{-2}\tilde{D}\psi''',$$

where

(B 6a) $$\tilde{A} = A_4 + \lambda^2[zA_2 + \alpha_0 A_1 + \beta_0 A],$$

(B 6b) $$\tilde{B} = B_4 + \lambda^2[zB_2 + \alpha_0 B_1 + \beta_0 B],$$

(B 6c) $$\tilde{C} = C_4 + \lambda^2[zC_2 + \alpha_0 C_1 + \beta_0 C],$$

(B 6d) $$\tilde{D} = D_4 + \lambda^2[zD_2 + \alpha_0 D_1 + \beta_0 D].$$

Now, *if* the matrix G in the connection formula

(B 7) $$\boldsymbol{\chi} = G\boldsymbol{\psi}$$

for the vectors $\boldsymbol{\chi}(\chi, \chi', \lambda^{-1}\chi'', \lambda^{-2}\chi''')$ and $\boldsymbol{\psi}(\psi, \psi', \lambda^{-1}\psi'', \lambda^{-2}\psi''')$ is nonsingular in the manner specified in part A (cf. (A 10) and (A 11)), we may also write (B 5) in the final form (B 1), and obtain a differential equation for χ, where $(\lambda\tilde{a}, \lambda\tilde{b}, \lambda\tilde{c}, \lambda\tilde{d})$ are linear combinations of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with coefficients of the order of unity. Thus, in order to make the coefficients $(\lambda\tilde{a}, \lambda\tilde{b}, \lambda\tilde{c}, \lambda\tilde{d})$ small to a certain order, *all the coefficients* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *must be made uniformly of that order.*

The detailed expressions for $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are too long to be of interest. We shall merely give their leading terms in the case

(B 8) $$p = z + \sum_{n=1}^{\infty} \lambda^{-n} p_n, \qquad q = \sum_{n=0}^{\infty} \lambda^{-n} q_n, \qquad r = \sum_{n=0}^{\infty} \lambda^{-n} r_n.$$

Then

(B 9a) $$\tilde{A} = \lambda^2[-r_0 B_1 + zA_1' + \alpha_0 A_1 + \beta_0 A] + O(\lambda),$$

(B 9b) $$\tilde{B} = \lambda^2[-q_0 B_1 + zA_1 + zB_1' + \alpha_0 B_1 + \beta_0 B] + O(\lambda),$$

(B 9c) $$\tilde{C} = \lambda^2[-p_1(B_1 - zD_1) - (q^{(0)} + 1)C_1 - 2zC_1' + \alpha_0 C_1] + O(\lambda),$$

(B 9d) $$\tilde{D} = \lambda^2[-p_0 C_1 + 2B_1' + B_1 - 2(zD_1)' + \alpha_0 D_1 + \beta_0 D] + O(\lambda).$$

We notice that in the above expressions, only the ***known*** functions p and q are replaced by their asymptotic forms. The as yet unspecified functions $A, B, C, D; A_1, B_1, C_1, D_1$ are left as they are. This is important for examining the higher approximations. For the initial approximation, we may then make $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ all of the order λ by requiring

(B 10a) $$-r_0 B_{1,0} + zA_{1,0}'' + \alpha_0 A_{1,0} - \beta_0 A_0 = 0,$$

(B 10b) $$-q_0 B_{1,0} + zA_{1,0} + zB_{1,0}' + \alpha_0 B_{1,0} + \beta_0 B_0 = 0,$$

(B 10c) $$-p_1(B_{1,0} - zD_{1,0}) - (q_0 + 1)C_{1,0} - 2zC_{1,0}' + \alpha_0 C_{1,0} = 0,$$

(B 10d) $$-p_0 C_{1,0} + 2B_{1,0}' + B_{1,0} - 2(zD_{1,0})' + \alpha_0 D_{1,0} + \beta_0 D_0 = 0.$$

If these relations can be satisfied under the restrictions (A 15), then the coefficients ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$) and hence also ($\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$) have the behavior (1.3), as desired in the final equation (B1).

The restrictions (A 15) guarantee the validity of the above conclusions only in a small neighborhood of $z=0$. However, since all the functions involved are generally analytic, the conclusions would hold except at the singularities of the functions involved.

Appendix C

The above calculations can also be used, with rather minor changes, for expressing the solutions χ of (B 1) in terms of the solutions of the differential equation (A 1), if we regard the latter as known. In such a case, instead of calculating ($\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$) in terms of ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$), we express the latter in terms of the former. We express the right-hand side of (B 1) in terms of ψ, ψ', $\lambda^{-1}\psi''$, $\lambda^{-2}\psi'''$, by means of (A 3), (A 4), (A 8) and (A 10) so that the differential equation (B 1) is satisfied if

(C 1a) $$\tilde{A} = \lambda(\tilde{a}A + \tilde{b}A_1 + \tilde{c}A_2 + \tilde{d}A_3),$$

(C 1b) $$\tilde{B} = \lambda(\tilde{a}B + \tilde{b}B_1 + \tilde{c}B_2 + \tilde{d}B_3),$$

(C 1c) $$\tilde{C} = \lambda(\tilde{a}C + \tilde{b}C_1 + \tilde{c}C_2 + \tilde{d}C_3),$$

(C 1d) $$\tilde{D} = \lambda(\tilde{a}D + \tilde{b}D_1 + \tilde{c}D_2 + \tilde{d}D_3).$$

Equating the corresponding expressions in (B 6) and (C 1), we obtain a set of homogeneous equations for A, B, C, D. For convenience of reference, these are listed as follows:

$$zA_1' + \alpha_0 A_1 + \beta_0 A - rB_1 = -rD_1(p - z) + \lambda^{-1}\overline{A}, \tag{C 2a}$$

$$zB_1' + \alpha_0 B_1 + \beta_0 B + zA_1 - qB_1 = -qD_1(p - z) + \lambda^{-1}\overline{B}, \tag{C 2b}$$

$$(z - 3p)C_1' + (\alpha_0 - q - p')C_1 + \beta_0 C = -\lambda(B_1 - pD_1)(p - z) + \lambda^{-1}\overline{C}, \tag{C 2c}$$

$$\begin{aligned}(z - 3p)D_1' &+ (\alpha_0 - q - 2p')D_1 + \beta_0 D \\ &= -\lambda C_1(p - z) - (3B_1' + A_1) + \lambda^{-1}\overline{D},\end{aligned} \tag{C 2d}$$

in which $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ are given by

$$\overline{A} = \lambda^{-1}\tilde{A} + (rC_1') + 2rC_1' + \lambda^{-1}(A' - rD_1'), \tag{C 3a}$$

$$\begin{aligned}\overline{B} = \lambda^{-1}\tilde{B} &- qC_1' + (q' - r)C_1 \\ &+ \lambda^{-1}[B_2'' + 2A_2' - 2qD_1'' - (q' + r)D_1'],\end{aligned} \tag{C 3b}$$

$$\begin{aligned}\overline{C} = \lambda^{-1}\tilde{C} + [3B_1'' &+ 3A_1' - 3pD_1'' - (3p' + 3q)D_1' \\ &- (p'' + 2q' + r)D_1] + \lambda^{-1}C_1''',\end{aligned} \tag{C 3c}$$

$$\overline{D} = \lambda^{-1}\tilde{D} + 3C_1'' + \lambda^{-1}D_1''', \tag{C 3d}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are in turn given by (C 1). We have put some of the apparently important terms on the right-hand side of (C 2), because we are interested in the special case when the coefficients in (A 1) are given by

$$p = z, \qquad q = \alpha, \qquad r = \beta \tag{C 4}$$

where α and β are given by (1.8).

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