ON THE SPECTRAL THEORY OF SYMMETRIC
FINITE OPERATORS

BY

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Let \( A \) be a linear operator defined on a linear system \( X \) and let \( N(A - \lambda I) \)
be the null space, \( R(A - \lambda I) \) the range of \( A - \lambda I \), and \( \lambda \) an arbitrary complex
number. We call \( A \) a finite operator if for each \( \lambda \neq 0 \) the dimensions of
\( N(A - \lambda I) \) and \( X/R(A - \lambda I) \) are finite and equal. The present paper is con-
cerned with an iteration method for determining characteristic values and
characteristic elements of symmetric finite operators on a not necessarily
complete Hilbert space \( X \) and with the structure of the spectrum of such
operators. The following two theorems are the basis of our exposition.

Theorem 1. If \( A \) is a symmetric finite operator on \( X \) and \( \text{Co}(A) \) its con-
tinuous spectrum, then \( \text{Co}(A) - \{0\} \) consists of all the limit points of character-
istic values of \( A \) which are different from zero and no characteristic values them-
selves\(^{(*)} \).

Theorem 2. If \( A \) is a symmetric finite operator on \( X \) and \( A \neq 0 \), then \( A \)
has a characteristic value different from zero and each element \( Ax \) can be ex-
panded in a series

\[
Ax = \sum_{e \in E} (Ax, e)e = \sum_{e \in E} \lambda(x, e)e,
\]

where \( E \) is a complete orthonormal system of characteristic elements of \( A \) cor-
responding to the characteristic values different from zero\(^{(3)} \).

Theorem 2 gives rise to a convenient definition. We say that a number
\( \lambda \neq 0 \) contributes to the element \( x \) if \( \lambda \) actually appears in the series \( (1) \). It is
readily seen that this definition does not depend on the particular system, \( E \),
chosen.

In the following, it will be convenient to suppose that \( A \) is not only sym-
metric and finite, but also bounded and positive. Excluding the trivial case
\( A = 0 \), we further assume throughout \( A \neq 0 \). Under these assumptions, the

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\(^{(*)} \) The research reported in this article was done in the Range Instrumentation Develop-
ment Division, White Sands Missile Range, New Mexico.

\(^{(3)} \) A proof of this theorem can be found in [1]. The results of this thesis will be published
in a forthcoming paper in the Mathematische Zeitschrift.

\(^{(3)} \) This theorem was first proved by Professor H. Wielandt in a lecture given at the Uni-
versity of Tübingen in the summer of 1952. Another proof can be found in [1]. It is understood
that the series \( (1) \) contains only those terms for which \( (x, e) \neq 0 \). The number of those terms is at
least enumerable.
set of characteristic values $\lambda \neq 0$ of $A$ is a nonempty bounded set of positive numbers. At the end of this paper it will be shown how to eliminate the hypothesis $A \geq 0$.

After these preliminary remarks we can prove the following theorem:

**Theorem 3.** Let $x$ be an element of $X$ with $Ax \neq 0$. Then at least one characteristic value of $A$ contributes to $x$, and $\lim_{k \to \infty} \left( \|A^k x\| / \|A^{k-1} x\| \right)$ exists and is equal to the least upper bound of all characteristic values contributing to $x$.

**Proof.** The first assertion follows directly from Theorem 2. Now let $e_1, e_2, \cdots$ be the sequence of elements of $E$ for which $(x, e) \neq 0$, $\lambda_1, \lambda_2, \cdots$ the sequence of the corresponding characteristic values, and let $\mu$ be the least upper bound of these $\lambda_i$. Consider the series

$$
\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\xi \lambda_i)^k \left( x, e_i \right)^2.
$$

If $|\xi| < 1/\mu$, then the series $\sum_{i=1}^{\infty} |\xi \lambda_i|^k (x, e_i)^2$ is obviously convergent and has the sum $\left( (|\xi \lambda_i|/(1 - |\xi \lambda_i|)) \right) \left( x, e_i \right)^2$. Since $|\xi \lambda_i|/(1 - |\xi \lambda_i|) \leq |\xi| \mu/(1 - |\xi| \mu)$, and since the series $\sum_{i=1}^{\infty} |(x, e_i)|^2$ converges by virtue of Bessel's inequality, it follows that the series $\sum_{i=1}^{\infty} \left( |\xi \lambda_i|/(1 - |\xi \lambda_i|) \right) \left( x, e_i \right)^2$ is also convergent. Hence, by Cauchy's theorem, the rearranged series

$$
\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (\xi \lambda_i)^k \left( x, e_i \right)^2
$$

converges in the circle $|\xi| < 1/\mu$. Since by Theorem 1

$$
A^k x = \sum_{i=1}^{\infty} \lambda_i (x, e_i) e_i
$$

and therefore

$$
(A^k x, x) = \sum_{i=1}^{\infty} \lambda_i^k \left( x, e_i \right)^2,
$$

we see that the power series

$$
\sum_{k=1}^{\infty} (A^k x, x) \xi^k
$$

converges in $|\xi| < 1/\mu$. On the other hand, this series cannot converge for any value of $\xi$ with $|\xi| > 1/\mu$, since otherwise all the series $\sum_{i=1}^{\infty} \lambda_i (x, e_i)$ would be convergent by the same type of argument used above, which contradicts the fact that there is a $\lambda_i$ with $|\xi \lambda_i| > 1$. Therefore the radius of convergence of the power series (3) equals $1/\mu$ from which we infer that $\mu = \lim_{k \to \infty} \sup_{k \to \infty} (A^k x, x)^{1/k}$. Now by the generalized Schwarz inequality [3, p. 260]
\[(A^k x, x) = (A^{k-1} x, Ax) \leq (A^{k-1} x, A x)(A^{k-1} A x, A x) = (A^{k-1} x, (A^{k+1} x, x)\]

and therefore

\[\frac{(A^k x, x)}{(A^{k-1} x, x)} \leq \frac{(A^{k+1} x, x)}{(A^k x, x)}\]

from which it follows that the sequence \((A^k x, x)/(A^{k-1} x, x)\) converges. This implies convergence of the sequence \((A^k x, x)^{1/k}\), so that

\[\mu = \lim_{k \to \infty} (A^k x, x)^{1/k}.\]

It follows that

\[\mu = \lim_{k \to \infty} (A^{2k} x, x)^{1/(2k)} = \lim_{k \to \infty} (A^k x, A^k x)^{1/(2k)} = \lim_{k \to \infty} \left(\|A^k x\|\right)^{1/k}\]

and since the sequence \(\|A^k x\|/\|A^{k-1} x\|\) converges \([3, p. 238]\), \(\mu\) must equal \(\lim_{k \to \infty} (\|A^k x\|/\|A^{k-1} x\|)\), which completes the proof.

Theorem 3 does not tell us whether or not \(\mu\) contributes to \(x\). The next theorem will close this gap.

**Theorem 4.** Let \(A x\) be different from zero and let

\[\mu = \lim_{k \to \infty} \frac{\|A^k x\|}{\|A^{k-1} x\|}.\]

Then

\[\lim_{k \to \infty} \frac{\|A^k x\|}{\lambda^k} = 0 \quad \text{for } \lambda > \mu;\]

\[\lim_{k \to \infty} \frac{\|A^k x\|}{\mu^k} = 0, \quad \text{when } \mu \text{ does not contribute to } x;\]

\[\lim_{k \to \infty} \frac{\|A^k x\|}{\mu^k} = \rho \neq 0, \quad \text{when } \mu \text{ contributes to } x; \rho\]

equals the length of the projection of \(x\) on \(N(A - \mu I)\) along \(R(A - \mu I)\);

\[\lim_{k \to \infty} \frac{\|A^k x\|}{\lambda^k} = \infty \quad \text{for } 0 < \lambda < \mu.\]

**Proof.** By Theorem 2 we have

\[A x = \sum_{i=1}^{\infty} \lambda_i (x, e_i e_i^*, (x, e_i) \neq 0 \quad \text{for } i = 1, 2, \cdots,\]

from which it follows that for any \(\lambda_i \neq 0\)
\begin{align}
(5) \quad \frac{\|A^k x\|^2}{\lambda^{2k}} &= \frac{(A^k x, A^k x)}{\lambda^{2k}} = \frac{(A^{2k} x, x)}{\lambda^{2k}} = \sum_{i=1}^{\infty} \left(\frac{\lambda_i}{\lambda}\right)^{2k} |(x, e_i)|^2.
\end{align}

Now let \(\lambda > \lambda_i\), so that \(0 < (\lambda_i/\lambda) < 1\) for \(i = 1, 2, \ldots\). Given an arbitrary number \(\epsilon > 0\) there exists a number \(N(\epsilon)\) such that
\[
\sum_{i=N+1}^{\infty} |(x, e_i)|^2 < \frac{\epsilon}{2}.
\]

Obviously, since \(0 < (\lambda_i/\lambda) < 1\),
\begin{align}
(6) \quad \sum_{i=N+1}^{\infty} \left(\frac{\lambda_i}{\lambda}\right)^{2k} |(x, e_i)|^2 < \frac{\epsilon}{2} \quad \text{for } k = 1, 2, \ldots,
\end{align}

and
\begin{align}
(7) \quad \sum_{i=1}^{N} \left(\frac{\lambda_i}{\lambda}\right)^{2k} |(x, e_i)|^2 < \frac{\epsilon}{2} \quad \text{for } k > k_0(\epsilon) \geq N(\epsilon).
\end{align}

It follows from (5), (6), and (7) that
\[
\frac{\|A^k x\|^2}{\lambda^{2k}} < \epsilon \quad \text{for } k > k_0(\epsilon);
\]
so that \(\lim_{k \to \infty} (\|A^k x\|/\lambda^k) = 0\). Observing that by Theorem 3 \(\lambda_i < \mu\) when \(\mu\) does not contribute to \(x\) the first two assertions of Theorem 4 follow.

Now let \(\mu\) contribute to \(x\) and for the sake of simplicity, let
\[
\mu = \lambda_1 = \lambda_2 = \cdots = \lambda_n, \mu > \lambda_i \quad \text{for } i > n.
\]

By (5) we have
\[
\frac{\|A^k x\|^2}{\mu^{2k}} = \sum_{i=1}^{n} |(x, e_i)|^2 + \sum_{i=n+1}^{\infty} \left(\frac{\lambda_i}{\mu}\right)^{2k} |(x, e_i)|^2.
\]

By the same argument as above it is seen that the last term of this equation tends to zero as \(k\) tends to infinity so that
\begin{align}
(8) \quad \lim_{k \to \infty} \frac{\|A^k x\|}{\mu^k} = \left(\sum_{i=1}^{n} |(x, e_i)|^2\right)^{1/2} = \rho > 0
\end{align}

which proves the third assertion of Theorem 4.

If, finally, \(0 < \lambda < \mu\), there exists by Theorem 3 a characteristic value, say \(\lambda_1\), such that \(\lambda < \lambda_1\), and since by (5)
\[
\frac{\|A^k x\|}{\lambda^k} \geq \left(\frac{\lambda_1}{\lambda}\right)^{k} |(x, e_1)|,
\]
the last assertion of our theorem follows readily(4).

We see by Theorem 4 that \( \mu \) can be characterized as the greatest lower bound of all real \( \lambda \) for which \( \lim_{k \to \infty} \left( \frac{\|A^k x\|}{\|A^{k-1} x\|} / \lambda^k \right) = 0 \).

The proof of Theorem 3 depends essentially on the fact that for \( |\xi| < 1/\mu \) the quantities \( |\xi \lambda_i| / (1 - |\xi \lambda_i|) \) have a finite upper bound. This no longer needs to be true when \( \xi \) equals \( 1/\mu \). In this case Theorem 3 does not provide any information about the series (4). But by a closer inspection of the operator \( A \) we can prove the following theorem relating the convergence of the series (4) for \( \xi = 1/\mu \) to the contribution of \( \mu \) to \( x \).

**Theorem 5.** Let \( Ax \neq 0 \) and \( \mu = \lim_{k \to \infty} \left( \frac{\|A^k x\|}{\|A^{k-1} x\|} \right) \). Then the series

\[
\sum_{k=1}^{\infty} \frac{(A^k x, x)}{\mu^k}
\]

converges if and only if \( \mu \) does not contribute to \( x \).

We first take up the easier part of the proof. If the series (9) converges then the sequence \( (A^k x, x)/\mu^k \) tends to zero as \( k \to \infty \) and so does the sequence \( (A^{2k} x, x)/\mu^{2k} = \left( \frac{\|A^k\|}{\mu^k} \right)^2 \). Therefore by Theorem 4 we can infer that \( \mu \) does not contribute to \( x \).

Now let the real number \( \lambda_0 \neq 0 \) be in the resolvent set or the continuous spectrum of \( A \) and let \( \overline{X} \) be the closure of \( X \). The adjoint transformation \( A^* \) is a self-adjoint extension of \( A \), defined on \( \overline{X} \), and \( \lambda \) is in the resolvent set or the continuous spectrum of \( A^* \) [1, Paragraph 5]. Therefore \( (A^*-\lambda_0 I)^{-1} \) exists and, by the definition of \( A \), \( X \) lies in the domain of \( (A^*-\lambda_0 I)^{-1} \). By [3, pp. 342 and 346] we see that

\[
\int_{-\infty}^{\infty} \left( \frac{1}{\lambda - \lambda_0} \right)^2 d(E_\lambda x, x)
\]

exists for every \( x \) in \( X \), where \( E_\lambda \) is the resolution of the identity corresponding to \( A^* \). Since the system of characteristic elements of \( A^* \) is complete in \( \overline{X} \) and since the characteristic manifolds of \( A^* \) and \( A \) which correspond to the same characteristic value \( \lambda \neq 0 \) coincide [1, Paragraph 5], it follows that the series

\[
\sum_{i=1}^{\infty} \left( \frac{1}{\lambda_i - \lambda_0} \right)^2 \left| (x, e_i) \right|^2
\]

converges for every \( x \) in \( X \); \( \{e_k\} \) is the sequence of elements of \( E \) for which \( (e, x) \neq 0 \).

Now let \( \lambda_0 \neq 0 \) be a characteristic value of \( A \) which does not contribute to \( x \). It follows that \( x \) is orthogonal to the characteristic manifold \( N(A-\lambda_0 I) \). Since by the definition of \( A \), \( X \) is the direct sum

(4) An analogous theorem for arbitrary symmetric operators is proved in [4].
we infer that $x$, as well as all the characteristic elements corresponding to characteristic values $\lambda \neq \lambda_0$, lies in the linear system $R(A - \lambda_0 I)$. Now $A$ is a symmetric finite operator $A'$ on $R(A - \lambda_0 I)$ [1, Paragraph 2] and by Theorem 1 $\lambda_0$ is in the resolvent set or the continuous spectrum of $A'$. Since each characteristic value $\lambda \neq \lambda_0$ of $A$ is a characteristic value of $A'$ and vice versa and since the corresponding characteristic manifolds are equal, it follows that the series (10) converges for each $x$ orthogonal to $N(A - \lambda_0 I)$. We may summarize these results by stating that the series (10) converges for any $x$ to which $\lambda_0 \neq 0$ does not contribute.

If $\lambda_0 \neq 0$ does not contribute to $x$, then $\lambda_0$ does not contribute to $Ax$ either. Therefore, the series

$$
\sum_{i=1}^{\infty} \left( \frac{1}{\lambda_i - \lambda_0} \right)^2 |(Ax, e_i)|^2 = \sum_{i=1}^{\infty} \left( \frac{\lambda_i}{\lambda_i - \lambda_0} \right)^2 |(x, e_i)|^2
$$

is convergent. It follows that the series

$$
(11) \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_0} |(x, e_i)|^2
$$

is also convergent. Suppose now that $\lambda_0 = \mu$ and that $\mu$ does not contribute to $x$. By Theorem 3 we have $\lambda_i < \mu$ for all characteristic values $\lambda_i$ contributing to $x$, therefore the series (11) converges absolutely for $\lambda_0 = \mu$ and we have

$$
\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i - \mu} |(x, e_i)|^2 = -\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{\lambda_i}{\mu} \right)^k |(x, e_i)|^2.
$$

By rearranging this series and by using the identity (3) we see that the series (9) converges.

This completes the proof.

The following theorem shows that the usual iteration method [2; 3, p. 237; 5] for determining characteristic values and characteristic elements can be successfully applied if the iteration sequence $\|Ax\|/\|Ax\|$ converges.

**Theorem 6.** Let $Ax$ be different from zero. The sequence $\|Ax\|/\|Ax\|$ converges to an element $h \in X$ if and only if $\mu = \lim_{k \to \infty} (\|Ax\|/\|Ax-kx\|)$ contributes to $x$. In this case $h$ is a normed characteristic element corresponding to the characteristic value $\mu$. If $\mu$ does not contribute to $x$, then $\|Ax\|/\|Ax\|$ converges weakly to zero.

**Proof.** Let $\mu$ contribute to $x$ and for the sake of simplicity, let $\mu = \lambda_1 = \lambda_2 = \cdots = \lambda_n$, $\mu > \lambda_i$ for $i > n$, in the expansion

$$
Ax = \sum_{i=1}^{\infty} \lambda_i(x, e_i)e_i.
$$

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We then have

\[ \frac{A^kx}{\| A^kx \|} = \frac{A^kx}{\mu^k} \frac{\mu^k}{\| A^kx \|} = \frac{\mu^k}{\| A^kx \|} \left[ \sum_{i=1}^{n} (x, e_i)e_i + \sum_{i=n+1}^{\infty} \left( \frac{\lambda_i}{\mu} \right)^k(x, e_i)e_i \right]. \]

Now as \( k \to \infty \) the sequence \( \sum_{i=n+1}^{\infty} (\lambda_i/\mu)^k(x, e_i)e_i \) tends to zero (see the proof of Theorem 4) and the sequence \( \mu^k/\| A^kx \| \) tends to \( \left( \sum_{i=1}^{n} (x, e_i)^2 \right)^{-1/2} \) by Theorem 4. Therefore \( A^kx/\| A^kx \| \) converges to a normed characteristic element corresponding to \( \mu \).

Now we suppose that \( \mu \) does not contribute to \( x \). Then all the characteristic values \( \lambda_i \) in (11) are less than \( \mu \) by virtue of Theorem 3. If \( e \) is an element of \( E \), we have

\[ \left( \frac{A^kx}{\| A^kx \|}, e \right) = \frac{1}{\| A^kx \|} \left( \sum_{i=1}^{\infty} \lambda_i^k(x, e_i)e_i, e \right) = \begin{cases} 0 \text{ in case } e \neq e_i, & i = 1, 2, \ldots, \\ \lambda_j^k/\| A^kx \| (x, e_j) \text{ in case } e = e_j. \end{cases} \]

Since \( 0 < \lambda_j < \mu \), the sequence \( \lambda_j^k/\| A^kx \| \) converges to zero as \( k \to \infty \) by Theorem 4. Therefore we have

\[ \lim_{k \to \infty} \left( \frac{A^kx}{\| A^kx \|}, e \right) = 0 \quad \text{for every } e \in E. \]

Recalling Theorem 2 we infer that \( (A^kx/\| A^kx \|, y) \to 0 \) for every \( y \) in the closure \( Y \) of \( AX \). Obviously \( (A^kx/\| A^kx \|, z) = 0 \) for every \( z \) in the orthogonal complement of \( Y \) in \( X \), thus it follows that for every element \( \tilde{x} \in X \) \( \lim_{k \to \infty} (A^kx/\| A^kx \|, \tilde{x}) = 0 \). Hence \( A^kx/\| A^kx \| \) converges weakly to zero. We see by this result, that if \( \mu \) does not contribute to \( x \) the sequence \( A^kx/\| A^kx \| \) cannot converge strongly, because otherwise its limit would equal the weak limit 0 which is impossible, since \( A^kx/\| A^kx \| \) is a normed element.

This completes the proof of Theorem 6.

**Corollary.** Suppose \( Ax \neq 0 \). Then \( \mu = \lim_{k \to \infty} (\| A^kx \|/\| A^{k-1}x \|) \) contributes to \( x \) if and only if there is a positive constant \( \alpha \) (which depends only on \( x \) but not on \( k \)) such that

\[ (13) \quad \| A^kx \| \leq \alpha \| A^kx \|^2 \quad \text{for } k = 1, 2, \ldots. \]

**Proof.** Let \( \mu \) contribute to \( x \). Then by Theorem 6 the sequence \( A^kx/\| A^kx \| \) converges to a characteristic element \( h \) corresponding to \( \mu \). Therefore

\[ \lim_{k \to \infty} \left( \frac{A^kx}{\| A^kx \|}, x \right) = \lim_{k \to \infty} \frac{1}{\| A^kx \|} (A^kx, A^kx) = \lim_{k \to \infty} \frac{\| A^kx \|^2}{\| A^kx \|} = (h, x). \]

\( (h, x) \) is different from zero, otherwise \( (h, A^kx/\| A^kx \|) = (1/\| A^kx \|)(A^k, x) = (\mu^k/\| A^kx \|)(h, x) \) would equal zero and, by Theorem 4, \( \lim_{k \to \infty} (h, A^kx/\| A^kx \|) = (h, h) \) would equal zero contrary to \( h \neq 0 \). Therefore it follows that the
sequence $\|A^kx\|^2/\|A^{2k}x\|$ has a positive lower bound $1/\alpha$ so that (13) holds. If on the other hand (13) is valid, then $(A^{2k}x/\|A^{2k}x\|, x)$, and therefore $(A^kx/\|A^kx\|, x)$ cannot converge to zero. Hence, by Theorem 6, $\mu$ contributes to $x$.

For the rest of this paper we need a concept first introduced by Wavre [4]. We call a bounded symmetric operator $B$ regular, if for each element $x$ of its domain with $Bx \neq 0$, $\lim_{k \to \infty} (\|B^kx\|/\mu_k^*)$ is different from zero (where $\mu_x = \lim_{k \to \infty} (\|B^kx\|/\|B^{k-1}x\|)^*(\cdot)$).

**Theorem 7.** $A$ is regular if and only if for each $x \in X$ with $Ax \neq 0$ the characteristic values contributing to $x$ can be arranged in a nonincreasing sequence.

**Proof.** Suppose first that the characteristic values $\lambda_i$ contributing to $x$ can be arranged in a nonincreasing sequence $\lambda_1 = \lambda_2 = \cdots = \lambda_n > \lambda_{n+1} \geq \cdots$. Then $\sup \lambda_i = \lambda_1$, so that the least upper bound of the $\lambda_i$'s contributes to $x$. By Theorems 3 and 4 we have therefore $\lim_{k \to \infty} (\|A^kx\|/\mu_k^*) \neq 0$, hence $A$ is regular.

Now let $A$ be regular and $Ax \neq 0$. Then with $\mu = \mu_x$ we have by Theorems 3 and 4

$$Ax = \sum_{i=1}^{n} \mu(x, e_i)e_i + \sum_{i=n+1}^{\infty} \lambda_i(x, e_i)e_i, \quad \lambda_i < \mu \quad \text{for } i = n + 1, n + 2, \cdots.$$  

Since, by the definition of $A$, $x = e + f$, $e \in N(A - \mu I), f \in \mathcal{R}(A - \mu I)$, it follows that

$$Ae - \sum_{i=1}^{n} \mu(x, e_i)e_i = \sum_{i=n+1}^{\infty} \lambda_i(x, e_i)e_i - Af.$$  

The first term of this equation is an element of $\mathcal{N}(A - \mu I)$, the second term is an element of $\mathcal{R}(A - \mu I)$, but because these two linear systems are orthogonal to each other both terms must vanish. Therefore

$$Af = \sum_{i=n+1}^{\infty} \lambda_i(e + f, e_i)e_i = \sum_{i=n+1}^{\infty} \lambda_i(f, e_i)e_i,$$  

since $(e, e_i) = 0$ for $i \geq n + 1$. $\mu_f = \sup_{i \geq n + 1} \lambda_i$ is equal to one of the $\lambda_i$'s, $i \geq n + 1$, because $A$ is regular, and therefore $\mu_f < \mu_x$. The proof can now be finished by mathematical induction.

Next we consider the relation between the regularity of $A^*$ and the spectrum of $A$. In order to state Theorem 8 it is convenient to introduce the following definition.

We say that $A$ has a band spectrum if every limit point $\lambda_0 \neq 0$ of characteristic values of $A$ can be approximated only by characteristic values greater

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*(*) The existence of these limits is proved in [4].
than \( \lambda_0 \) (so that in a left-hand neighborhood \( \lambda_0 - \epsilon < \lambda < \lambda_0 \) there are no characteristic values).

It follows immediately that the number of these limit points is at most enumerable.

**Theorem 8.** \( A \) has a band spectrum if and only if \( A^* \) is regular.

**Proof.** Let \( A^* \) be regular and let \( \lambda_0 \neq 0 \) be a limit point of characteristic values of \( A^* \). Suppose there is a sequence of different characteristic values \( \lambda_i \neq 0 \) of \( A^* \), \( i = 1, 2, \ldots \), with \( \lambda_i < \lambda_0 \), \( \lambda_i \to \lambda_0 \) for \( i \to \infty \). Let \( \alpha_1, \alpha_2, \ldots \), be an arbitrary sequence of complex numbers with \( \sum_{i=1}^{\infty} |\alpha_i|^2 < +\infty \), \( \alpha_i \neq 0 \), and consider the element \( x = \sum_{i=1}^{\infty} \alpha_i e_i \) of \( X \), where \( e_i \) is a normed characteristic element corresponding to \( \lambda_i \). Since \( \lambda_i \alpha_i \neq 0 \) we have

\[
A^* x = \sum_{i=1}^{\infty} \lambda_i \alpha_i e_i 
eq 0.
\]

The proofs of the Theorems 3 and 4 were based only on the expansion (1) and the positiveness of \( A \). Here we have the expansion (14), and the positiveness of \( A^* \) follows readily from the positiveness of \( A \). Hence both theorems hold for \( A^* x \) and it results that

\[
\lim_{k \to \infty} \frac{\| (A^*)^k x \|}{\| (A^*)^{k-1} x \|} = \sup \lambda_i = \lambda_0, \quad \lim_{k \to \infty} \frac{\| (A^*)^k x \|}{\lambda_0^k} = 0.
\]

But the last equation contradicts the regularity of \( A^* \). Hence \( \lambda_0 \) cannot be approximated by characteristic values less than \( \lambda_0 \). Therefore \( A^* \) has a band spectrum. Since the characteristic values \( \neq 0 \) of \( A \) and \( A^* \) are the same by [1, Paragraph 5], it follows that \( A \) has also a band spectrum.

Now suppose that \( A \) has a band spectrum and let \( x \) be an arbitrary element in \( X \) with \( A^* x \neq 0 \). Then the expansion

\[
A^* x = \sum_{i=1}^{\infty} \lambda_i (x, e_i) e_i
\]

is valid, where the \( e_i \)'s are in \( E \) and the \( \lambda_i \)'s are the corresponding characteristic values of \( A \) (see [1, Paragraph 5]). By Theorem 3 we infer from this expansion that \( \mu x = \lim_{k \to \infty} (\| (A^*)^k x \|)/(\| (A^*)^{k-1} x \|) \) equals \( \sup \lambda_i \). But since \( A \) has a band spectrum, \( \sup \lambda_i \) is one of the characteristic values \( \lambda_1, \lambda_2, \ldots \), hence, by Theorem 4, \( \lim_{k \to \infty} (\| (A^*)^k x \|/\mu^2) \neq 0 \), i.e. \( A^* \) is regular.

The regularity of \( A \) does not imply that \( A \) has a band spectrum. Consider for example the space \( X \) of all sequences \( \{\alpha_1, \alpha_2, \ldots\} \) of complex numbers where only finitely many \( \alpha_i \) are different from zero and define the linear operations and the inner product in the usual way. Define an operator \( A \) on \( X \) by the diagonal matrix

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\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
0 & \ddots
\end{pmatrix},
\]

\(\lambda_n = 1 - 1/n\). \(A\) is obviously symmetric, positive, finite and regular but has no band spectrum because the characteristic values \(\lambda_n\) approximate their limit point 1 from the left side.

Suppose now that the operator \(A\) on \(X\) is finite, symmetric and bounded, but not necessarily positive. Then \(A^2\) is finite by [1, Paragraph 2] and obviously symmetric, bounded and positive. Thus all our theorems can be applied to \(A^2\) and we can deduce from them, in the conventional way, the corresponding theorems for \(A\). Without going into detail we state only the following theorem.

**Theorem 9.** Let \(A\) be a finite, symmetric and bounded operator on \(X\) and \(Ax \neq 0\). Then the sequence \(A^{2^k}/\|A^{2^k}x\|\) converges to an element \(h \in X\) if and only if \(\mu^{1/2}\) or \(-\mu^{1/2}\) is a characteristic value of \(A\) contributing to \(x\), where \(\mu = \lim_{k \to \infty} (\|A^{2^k}x\|/\|A^{2^{k-2}}x\|)\). If \(h\) exists, at least one of the elements

\[e' = h + (1/\mu^{1/2})Ah, \quad e'' = h - (1/\mu^{1/2})Ah\]

is a characteristic element of \(A\) corresponding to the characteristic value \(\mu^{1/2}\) or \(-\mu^{1/2}\) respectively.

This theorem follows immediately from Theorems 2 and 6, since \(Ae' = \mu^{1/2}e'\), \(Ae'' = -\mu^{1/2}e''\) and at least one of the elements \(e', e''\) is different from zero, because \(e' + e'' = h\) and \(h \neq 0\).

**References**