COMBINATORY METHODS AND STOCHASTIC KOLMOGOROV EQUATIONS IN THE THEORY OF QUEUES WITH ONE SERVER

BY

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1. **Introduction.** There is a queue in front of a single server, and the waiting customers are served in order of arrival, with no defections from the queue. We are interested in the waiting-times of customers.

As a mathematical idealization of the delays to be suffered in the system, we use the virtual waiting-time $W(t)$, which can be defined as the time a customer would have to wait for service if he arrived at time t. $W(t)$ is continuous from the left; at epochs of arrival of customers, $W(t)$ jumps upward discontinuously an amount equal to the service-time of the arriving customer; otherwise $W(t)$ has slope -1 while it is positive. If it reaches zero, it stays equal to zero until the next jump.

It is usual to define the stochastic process $W(t)$ in terms of the arrival epoch t_k and the service-time S_k of the kth arriving customer, for $k=1, 2, \cdots$. However, the following procedure is a little more elegant; we describe the service-times and the arrival epochs simultaneously by a single function $K(t)$, which is defined for $t \geqq 0$, left-continuous, nondecreasing, and constant between successive jumps. The locations of the jumps are the epochs of arrivals, and the magnitudes are the service-times. $K(t)$ can be interpreted as the "work load" submitted to the server during the interval $[0, t)$, and $W(t)$ can be thought of as the amount of work remaining to be done at time t. In terms of this interpretation, one can see that the work remaining at t equals the work offered up to t minus the elapsed time t plus the total time during which the server was idle in $(0, t)$. Formally, then, $W(t)$ is defined in terms of $K(t)$ by the nonlinear integral equation

$$W(t) = K(t) - t + \int_0^t U(-W(u))du, \qquad t \geqq 0, \tag{1}$$

where $U(\cdot)$ is the unit step-function, i.e., $U(x)=1$ for $x \geqq 0$, and $U(x)=0$ otherwise. For simplicity we have set $W(0)=K(0)$.

The literature of applied probability theory contains many investigations of waiting-times (for one server); however, these studies have depended essentially on assumptions of statistical independence or special distributions. In the papers of Khintchine [5], Kendall [4], Bailey [1], Takács [10], and Beneš [2], the arrivals form a Poisson process, and the service-times are inde-

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pendent of each other and of the arrival process. In the work of Smith [9] and Lindley [6], it is assumed that the interarrival times and the service-times are (independent) renewal processes.

Many useful and interesting results have been obtained under these assumptions, which probably include most cases of practical interest. We believe, though, that the assumptions have tended to obscure the stochastic process of interest (the waiting-times) with analytical detail; it is not always possible to separate the essential features of the stochastic process from those which only reflect the strength and analytic nature of the hypotheses.

From a theoretical viewpoint, the assumptions made in the literature have been inadmissibly strong. The queue defines a mapping, expressed by equation (1), from a stochastic process $K(t)$ of service-times and interarrival times to another stochastic process $W(t)$ of waiting-times. The general theoretical question is: What is the form of the operator that gives the distribution of $W(t)$ in terms of distributions associated with $K(u)$ for $u \leqq t$? To answer this question is our principal aim. The functional we seek should depend only on (i) the integral equation relating $W(t)$ and $K(t)$, and (ii) the fact that $K(t)$ is a nondecreasing step-function; it should depend on no special characteristics of the probability measure for $K(t)$ other than those implied by (ii). Accordingly, the present work involves no independence assumptions and no special distributions.

One approach to the waiting-times, used really in all the papers cited previously, is to solve the Kolmogorov equations for the distributions of a Markov process. Since our assumptions do not necessarily give rise to a Markov process, this approach is not sufficiently general to solve our problem.

Another possibility is first to solve the equation (1) and then to try to express the distribution of the solution $W(t)$ in terms of the probability measure for $K(u)$, $u \leqq t$. However, the solution of (1) involves the supremum functional, and so this approach incurs directly the notorious difficulties associated with the distribution of a supremum.

In outline, our procedure is as follows: We first obtain a representation of the random variable $\exp\{-sW(t)\}$, $\operatorname{Re}(s)>0$; the expectation of this variate is the Laplace-Stieltjes transform of the distribution of $W(t)$; from this expectation we derive a formula for $\Pr\{W(t)\leqq w\}$ by inversion; the formula expresses the functional which we seek.

This paper is not a complete monograph on queues with one server. Rather it is an account of principal results deduced by methods that are relatively new in queueing theory. Several of these results have been included because they show that the structure of the problem in the most general case is the same as in the special cases considered to date by means of Markov processes. Our effort to dispense with assumptions of independence and special distributions has been particularly stimulated by the work of Reich [7; 8] on the integrodifferential equation of Takács [10].

2. **Summary.** Let $\mathfrak{F}_t$ be the Borel field corresponding to the knowledge for $K(u)$ for $u \leqq t$. In §3 we exhibit the solution $W(t)$ of the integral equation (1) as a definite functional of $K(\cdot)$, and we show that $W(t)$ is measurable on $\mathfrak{F}_t$, and hence is a well-defined stochastic process (measurable function).

We prove two main results in this paper. The first of these, in §4, states that with probability one $\exp\{-sW(t)\}$ can be represented as

$$
\begin{aligned}
&\exp\{-sW(t)\} \\
(2) \qquad &\quad = \exp\{-s[K(t) - t]\}\left\{1 - s\int_0^t \exp\{s[K(u) - u]\}P(u, 0)du\right\}
\end{aligned}
$$

where $P(u, 0)$ is 1 if $W(u) = 0$, and 0 otherwise. The representation (2) embodies a combinatory result that is independent of the queueing problem.

In §5 we show that $\exp\{-sW(t)\}$ is almost surely the unique solution of a stochastic integral equation, which can be interpreted as a stochastic Kolmogorov-type equation for conditional expectations. Under this interpretation the stochastic equation is closely analogous to the (genuine Kolmogorov) equation (8) of Takács [10]. The stochastic equation implies (2), and can be derived independently of (2) by purely probabilistic methods that parallel the usual infinitesimal arguments for Markov processes. §6 describes a stochastic Volterra equation of the first kind, satisfied by $P(t, 0)$, and similar to an equation of Reich for $\Pr\{W(t) = 0\} = E\{P(t, 0)\}$.

The second main result, in §7, is a formula for $\Pr\{W(t) \leqq w\}$; this result solves the general problem stated in §1. The operator which gives $\Pr\{W(t) \leqq w\}$ in terms of distributions associated with $K(u)$ for $u \leqq t$ is linear, and depends only on $\Pr\{K(t) - t \leqq w\}$ and for each $u \leqq t$, on the conditional distribution of $K(t) - K(u) - t + u$ relative to knowledge that

$$
\sup_{0<y<u} [K(u) - K(y) - u + y] \leqq 0.
$$

The formula for $\Pr\{W(t) \leqq w\}$ is obtained by first integrating (2) with respect to the basic probability measure, and then using the inversion formula for Laplace-Stieltjes transforms. The formula depends on prior determination of $\Pr\{W(t) = 0\}$ via a Volterra equation. Use is made of the definition of conditional probability (Radon-Nikodym theorem) to split a joint probability into a product.

3. **Measurability of** $W(t)$. We assume as given a space Ω of points ω, a Borel field $\mathfrak{F}$ of sets $A \subseteq \Omega$, a probability measure $P(\cdot)$ of $\mathfrak{F}$-sets, and functions $K(t, \omega)$, $t \geqq 0$, such that

(i) $K(t, \cdot)$ is measurable on $\mathfrak{F}$ for $t \geqq 0$.

(ii) Except on a null set, $K(\cdot, \omega)$ is left-continuous, non-negative, and constant except for jumps; the jumps form an (at most) denumerable set, with no finite limit point. Condition (ii) implies that $K(t)$ is a separable process.

Let $\mathfrak{F}_t$ be the Borel field generated by $K(u)$ for $u \leqq t$. To prove that $W(t)$ is measurable on $\mathfrak{F}_t$ we use the following result:

LEMMA 1. *Let z be the first zero of $K(t)-t$, if there be one, and ∞ otherwise. Then z is $\mathfrak{F}$-measurable. On the ω-set $\{z>t\}$ we have $W(t)=K(t)-t$, while on $\{z \leqq t\}$ the solution of the integral equation* (1) *is*

$$W(t) = \sup_{0<u<t} [K(t) - K(u) - t + u]. \tag{3}$$

Proof. $K(t)$ is separable, so the measurability of z follows from

$$\{z > \lambda\} = \left\{ \inf_{0<u<\lambda} [K(u) - u] > 0 \right\},$$

which also shows that

$$\{z > t\} \in \mathfrak{F}_t. \tag{4}$$

If $z>t$, the integral in (1) is zero, and $W(t)=K(t)-t$; but if $z \leqq t$, the argument used by Reich to prove Lemma 5.1 of his [7] proves equation (3). The measurability of $W(t)$ on $\mathfrak{F}_t$ follows from (3), (4), and the separability condition.

4. **A representation for** $\exp\{-sW(t)\}$. From Theorem 9.4, p. 29 of Doob [3], we deduce that there is a function $P(t, w, \omega)$, measurable on $\mathfrak{F}_t$, such that

(i) $P(t, \cdot, \omega)$ is a distribution function for all t, ω.

(ii)
$$\begin{aligned} P(t, w, \omega) &= \Pr\{W(t) \leqq w \mid \mathfrak{F}_t\}, \\ &= \chi_{\{W(t) \leqq w\}}(\omega), \end{aligned}$$

almost surely

(iii) $P(\cdot, w, \omega)$ is a Borel measurable function of t almost surely.

We omit ω-dependence henceforth. It follows from (ii) or from the measurability of $W(t)$ on $\mathfrak{F}_t$, that

$$E\{\exp\{-sW(t)\} \mid \mathfrak{F}_t\} = \exp\{-sW(t)\}$$

with probability one; hence $P(t, \cdot)$ must be a degenerate distribution in w, consisting of a single step of unity at $w = W(t)$.

LEMMA 2. *With probability one, for* $\operatorname{Re}(s)>0$,

$$\exp\left\{-s\int_0^t P(u, 0)du\right\} = 1 - s\int_0^t \exp\{s[K(u) - u]\} P(u, 0)du. \tag{5}$$

Proof. It is sufficient to consider real s; let $[x_i, y_i)$ be the ith interval during which $W(t)$ is continuously zero. Equation (5) is true for $t \leqq x_1$; assume that (5) holds for $t \leqq x_n$. In the interval $x_n < t < y_n$, $K(\cdot)$ has no jumps, so that

$$K(t) - t = K(x_n) - t;$$

also, $W(\cdot)$ becomes equal to zero at x_n, so that by the integral equation (1),

$$K(x_n) - x_n + \int_0^{x_n} P(u, 0)du = 0;$$

hence for $x_n < t < y_n$,

$$\begin{aligned} \exp\left\{sK(x_n) - st\right\} &= \exp\left\{-s\int_0^{x_n} P(u, 0)du - s(t - x_n)\right\} \\ &= \exp\left\{-s\int_0^t P(u, 0)du\right\}. \end{aligned}$$

Since $P(\cdot, 0)$ vanishes on (y_n, x_{n+1}), we can conclude that $x_n < t \leqq x_{n+1}$ implies

$$(6) \qquad -sP(t, 0)\exp\left\{-s\int_0^t P(u, 0)du\right\} = -sP(t, 0)\exp\left\{s[K(t) - t]\right\}.$$

Equation (6) says that the t-derivatives of the two sides of equation (5) agree on $x_n < t \leqq x_{n+1}$, so that (6) holds for $t \leqq x_{n+1}$. Lemma 2 follows by induction.

The proof of Lemma 2 just given depends on properties of $W(t)$, particularly on the integral equation (1). However, Lemma 2 embodies a general combinatory result valid for any function which is the characteristic function of the union of a set of disjoint intervals. The only connection with $W(t)$ is that the intervals on which $W(t) = 0$ form such a set, and $P(t, 0)$ is such a characteristic function.

LEMMA 3. Let

$$0 \leqq x_1 < y_1 < \cdots < x_i < y_i < x_{i+1} < \cdots$$

be a sequence of extended real numbers. Let $f(\cdot)$ be the characteristic function of the set

$$(7) \qquad \bigcup_{i=1}^{\infty} [x_i, y_i),$$

and let $U(\cdot)$ be that of $[0, \infty]$. Let $F(t) = \int_0^t f(u)du$, or alternately let $F(t)$ be the Lebesgue measure of (7) $\cap (0, t)$. Then for any $x \geqq 0$, $t \geqq 0$,

$$(8) \qquad \max\left\{0, x - F(t)\right\} = x - \int_0^t U(x - F(y))dF(y).$$

Proof. The right-hand side σ of equation (8) can be rewritten, with

$$\lambda_i = \max\{0, \min\{t - x_i, y_i - x_i\}\},$$

as

(9) $$\sigma = x - \sum_{i=1}^{\infty} \int_0^{\lambda_i} U(x - w - F(x_i))dw,$$

or as

(10) $$\sigma = x - \sum_{i=1}^{\infty} \max\{0, \min\{t - x_i, y_i - x_i, x - F(x_i)\}\}.$$

If $F(\cdot)$ is unbounded, there exist integers $n = n(t)$, $k = k(x)$, defined by the respective conditions

$$x_n < t \leqq x_{n+1},$$
$$F(x_k) < x \leqq F(x_{k+1}).$$

Thus

$$\sigma = x - \sum_{i=1}^{n-1} \max\{0, \min\{y_i - x_i, x - F(x_i)\}\}$$
$$- \max\{0, \min\{t - x_n, y_n - x_n, x - F(x_n)\}\}.$$

There are three cases: $k < n$, $k = n$, and $k > n$. If $k < n$, then $x_{k+1} \leqq x_n$ and $F(x_{k+1}) \leqq F(x_n) \leqq F(t)$; but $x \leqq F(x_{k+1})$; therefore $x < F(t)$. Also, $k < n$ implies that $x - F(x_i)$ is negative for $i > k$, so that

$$\sigma = x - \sum_{i=1}^{k-1} (y_i - x_i) - x + F(x_k) = 0.$$

If $k = n$, then $x_n = x_k$, and

$$\sigma = x - \sum_{i=1}^{n-1} (y_i - x_i) - \min\{t - x_n, y_n - x_n, x - F(x_n)\},$$
$$= x - F(x_n) - \min\{t - x_n, y_n - x_n, x - F(x_n)\}.$$

Two subcases arise: first, if $x > F(t)$, then

$$x - F(x_n) > F(t) - F(x_n) = \min\{t - x_n, y_n - x_n\}$$

so that in this subcase $\sigma = x - F(t)$; second, if $x \leqq F(t)$, then

$$x - F(x_n) \leqq F(t) - F(x_n) = \min\{t - x_n, y_n - x_n\}$$

and $\sigma = 0$. There remains the case $k > n$; then $x_k \geqq x_{n+1}$, $F(x_k) \geqq F(x_{n+1})$, so that $x > F(t)$. If $k > n$, then $x - F(x_i)$ is positive for $i \leqq n$, and

$$\sigma = x - \sum_{i=1}^{n} \min\{t - x_i, y_i - x_i\} = x - F(t).$$

The equation $\sigma = \max\{0, x - F(t)\}$ now follows by a dilemma argument. The proof for bounded $F(\cdot)$ is similar.

To apply Lemma 3 to the queueing problem we interpret $f(\cdot)$ as $P(\cdot, 0)$, and $[x_i, y_i)$ as the ith interval during which the system is continuously empty. If we set

$$T(t) = \int_0^t P(u, 0)du = \text{total idle time to } t,$$

then Lemma 3 gives

$$\max\{0, x - T(t)\} = x - \int_0^t U(x - T(y))dT(y).$$

However, each x_i is a point at which $W(\cdot)$ reaches 0; that is, by equation (1),

$$0 = W(x_i) = K(x_i) - x_i + T(x_i).$$

Since $K(\cdot)$ has no jumps at interior points of the support S of $T(\cdot)$, it follows that for y in S

$$T(y) = y - K(y) \geqq 0.$$

Hence

$$\max\{0, x - T(t)\} = x - \int_0^t U(x - y + K(y))dT(y). \tag{11}$$

We now take the Laplace-Stieltjes transform of equation (11) with respect to $x \geqq 0$; this yields

$$s^{-1} \exp\left\{-s\int_0^t P(u, 0)du\right\} = s^{-1} - \int_0^t \exp\{s[K(u) - u]\}P(u, 0)du,$$

which is equivalent to Lemma 2.

Our first main result is a representation for $\exp\{-sW(t)\}$.

THEOREM 1. *With probability one, for* $\operatorname{Re}(s) > 0$,

$$\exp\{-sW(t)\} \\ = \exp\{-s[K(t) - t]\}\left\{1 - s\int_0^t \exp\{s[K(u) - u]\}P(u, 0)\,du\right\}. \tag{12}$$

This is an immediate consequence of Lemma 2 and the basic integral equation (1). The equation (12) is an analogue, for conditional expectations taken relative to $\mathfrak{F}_t$, of the formula (13) for $E\{\exp\{-sW(t)\}\}$ in the case of Poisson arrivals and independent service-times, given by Takács in [10].

5. **General stochastic analogues of Takács' equations.** In [10], L. Takács derived and solved a differential equation for $E\{\exp\{-sW(t)\}\}$, and, equivalently, an integrodifferential equation for $\Pr\{W(t) \leqq w\}$, in the case of Poisson arrivals and independent service-times. In this case $W(t)$ is a

Markov process, and the equation for $\Pr\{W(t) \leqq w\}$ is a Kolmogorov equation. We next show that in general $\exp\{-sW(t)\}$ and $P(t, w)$ themselves satisfy close stochastic analogues of Takács' equations. Let

$$J(u) = \inf_{1>\epsilon>0} [K(u+\epsilon) - K(u)],$$

$$N(u) = \text{number of arrivals in } (0, u).$$

$J(u)$ is the jump magnitude of $K(\cdot)$ at u, and is zero if there is no jump.

THEOREM 2. *With probability one*, $\exp\{-sW(t)\}$ *is the unique solution of the equation*

$$\begin{aligned}\phi(t, s) &= s\int_0^t [\phi(u, s) - P(u, 0)]du \\ &\quad - \int_{0+}^{t-} [1 - \exp\{-sJ(u)\}]\phi(u, s)dN(u) + \exp\{-sK(0)\}.\end{aligned} \tag{13}$$

Proof. Let (14) be the result of substituting $\exp\{-sW(\cdot)\}$ for $\phi(\cdot, s)$ in (13). Except possibly at the jumps of $K(\cdot)$, each side of (14) has a well-defined t-derivative. That of the left-hand side is

$$-se^{-sW(t)}dW/dt = se^{-sW(t)}[1 - P(t, 0)],$$

since $dW/dt = 0$ if $W(t) = 0$, a.e., and is -1 otherwise. The t-derivative of the right-hand side of (14) is

$$se^{-sW(t)} - sP(t, 0) = se^{-sW(t)}[1 - P(t, 0)].$$

Since $W(t+0) = W(t) + J(t)$, it is evident that the effect of the integral with respect to $N(\cdot)$ in (14) is just to ensure that (14) holds at $(t+0)$ if it holds at t; an induction on the number of jumps in $(0, t)$ proves that $\exp\{-sW(t)\}$ satisfies (13). A similar inductive procedure establishes that any solution $\phi(t, s)$ of (13) is equal to

$$\exp\{-s[K(t) - t]\}\left\{1 - s\int_0^t \exp\{s[K(u) - u]\}P(u, 0)du\right\}, \tag{15}$$

and so equals $\exp\{-sW(t)\}$, by Theorem 1.

By a choice of suitably detailed Borel fields for conditioning, the equation (13) of Theorem 2 may be interpreted as a stochastic equation for conditional expectations, or as a (Laplace-Stieltjes transformed) equation for conditional probabilities. The simplest, though in a sense the most wasteful, choice of field consists in using the basic (and very fine) field $\mathfrak{F}$ uniformly for all the random variables involved. For then

$$E\{\exp\{-sW(t)\} \mid \mathfrak{F}\} = \exp\{-sW(t)\} = \int_{0-}^{\infty} e^{-sw}d_wP(t, w), \tag{16}$$

with probability one, with a similar equation holding for $J(t)$. Equation (13) is then a t-integrated analogue of Takács' equation (8), with $\exp\{-sJ(t)\}$ replacing the transform of the service-time distribution, and integration with respect to $N(\cdot)$ replacing integration with respect to the Poisson average $\Lambda(\cdot)=E\{N(\cdot)\}$.

If, under the interpretation just given, we apply Laplace-Stieltjes inversion to equation (13), we can prove:

THEOREM 3. *Let $B(u, w)=\Pr\{J(u)\leq w|\mathfrak{F}\}$. Then with probability one,*

$$P(t, w) = \Pr\{W(t) \leq w | \mathfrak{F}\},$$

and $P(t, w)$ satisfies the equation

$$\begin{aligned}\int_0^w P(t, x)dx = \int_0^t &[P(u, w) - P(u, 0)]du \\ &- \int_0^w \int_{0+}^{t-} [P(u, x) - P(u, x) * B(u, x)]dN(u)dx \\ &+ \int_0^w P(0, x)dx, \qquad t \geq 0, w \geq 0.\end{aligned} \tag{17}$$

Equation (17) of Theorem 3 is a doubly integrated (on t and w) *stochastic* analogue of the equation (3) of Takács [10]. It is a Kolmogorov equation for the conditional probability $\Pr\{W(t)\leq w|\mathfrak{F}\}$. The t- and w-integrations are both used here because although $P(t, w)$ is differentiable a.e. with respect to either variable, the resulting derivatives are zero a.e.

It may be worth remarking that our historically first proof of Theorem 1 consisted in deriving the equation (13) by a stochastic analogue of Takács' infinitesimal argument in [10], and showing that the solution was given by (15). The method is entirely probabilistic in character, and may be useful in problems other than queueing. It has the advantage that it is motivated by a strict though delicate analogy with the usual infinitesimal arguments for the distributions of a Markov process. However, compared with the methods used to prove Theorem 1 in §4, it is very cumbersome.

6. A Volterra equation for $P(t, 0)$.

THEOREM 4. *The function $P(t, 0)$ satisfies (a.e. in ω) the stochastic Volterra equation of the first kind*

$$\max\{0, t - K(t)\} = \int_0^t \Gamma(t, u)P(u, 0)du, \tag{18}$$

where the kernel $\Gamma(\cdot, \cdot)$ is given by

$$\Gamma(t, u) = \begin{cases} 1 \text{ if } K(t) - K(u) - t + u < 0, \\ 1/2 \text{ if } K(t) - K(u) - t + u = 0, \\ 0 \text{ if } K(t) - K(u) - t + u > 0. \end{cases}$$

Proof. We choose $c>0$, divide equation (12) of Theorem 1 by $2\pi i s^2$, and integrate along $\mathrm{Re}(s)=c$. By Theorem 7.6a of Widder [11], we find that for $t>0$,

$$\lim_{x\to\infty} \frac{1}{2\pi i}\int_{c-ix}^{c+ix} \phi(s, t)s^{-2}e^{s0}ds = 0,$$

since $s^{-1}\phi(s, t)$ is the Laplace-Stieltjes transform of

$$\int_0^w P(t, y)dy.$$

Similarly

$$\lim_{x\to\infty} \frac{1}{2\pi i}\int_{c-ix}^{c+ix} \exp\{-sK(t) + st\}s^{-2}ds = \int_0^t U(y - K(t))dy,$$
$$= \max\{0, t - K(t)\}.$$

Finally, for $0<u<t$,

$$\lim_{x\to\infty} \frac{1}{2\pi i}\int_{c-ix}^{c+ix} \exp\{-s[K(t) - K(u) - t + u]\}s^{-1}ds = \Gamma(t, u).$$

The result follows by the theorem of bounded convergence. Equation (18) is a general stochastic analogue of the Equation 2.4 of Reich [7].

7. **Probabilities.** From the representation of $\exp\{-sW(t)\}$ given by Theorem 1, we can now obtain probabilities and expectations by integration with respect to the basic probability measure $P(\cdot)$. By Fubini's theorem, we find that Theorem 1 implies:

THEOREM 5. *For* $\mathrm{Re}(s)>0$,

$$E\{\exp\{-sW(t)\}\} = E\{\exp\{-sK(t) + st\}\}$$
$$- s\int_0^t E\{\exp\{-s[K(t) - K(u) - t + u]\}P(u, 0)\}du. \tag{19}$$

By applying the inversion formula for Laplace-Stieltjes transforms to Theorem 5, we can prove:

THEOREM 6. *For* $y\geqq 0$

$$\int_0^y \Pr\{W(t) \leqq u\}du = \int_0^{t+y} \Pr\{K(t) \leqq u\}du$$
$$- \int_0^t \Pr\{K(t) - K(u) - t + u \leqq y \,\&\, W(u) = 0\}du; \tag{20}$$

for $-t\leqq y\leqq 0$, *however*,

$$(21)\quad \begin{aligned} 0 = &\int_0^{t+y} \Pr\{K(t) \leqq u\} du \\ &- \int_0^{t+y} \Pr\{K(t) - K(u) - t + u \leqq y \,\&\, W(u) = 0\} du. \end{aligned}$$

Details of the proof of Theorem 6 are as follows: choose $c>0$, divide the equation (19) of Theorem 5 by $2\pi i s^2 \exp\{-sy\}$, and integrate along $\mathrm{Re}(s) = c$. By Theorem 7.6a of Widder [11], we find that for $y \geqq 0$,

$$\lim_{x\to\infty} \frac{1}{2\pi i} \int_{c-ix}^{c+ix} E\{\exp\{-sW(t) + sy\}\} s^{-2} ds = \int_0^y \Pr\{W(t) \leqq u\} du.$$

By the same argument, for $y \geqq 0$,

$$(22)\quad \lim_{x\to\infty} \frac{1}{2\pi i} \int_{c-ix}^{c+ix} E\{\exp\{-sK(t) + s(t+y)\}\} s^{-2} ds = \int_0^{t+y} \Pr\{K(t) \leqq u\} du,$$

$$(23)\quad \begin{aligned} &\lim_{x\to\infty} \frac{1}{2\pi i} \int_{c-ix}^{c+ix} E\{\exp\{-s[K(t) - K(u) - t + u - y]\} P(u, 0)\} s^{-1} ds \\ &\qquad = \Pr\{K(t) - K(u) - t + u \leqq y \,\&\, W(u) = 0\}. \end{aligned}$$

Since

$$E\{\exp\{-(c+ix)[K(t) - K(u)]\} P(u, 0)\} \frac{\exp\{(c+ix)(t-u+y)\}}{c+ix}$$

is bounded in absolute value, we can use the theorem of bounded convergence (with the previous result) to conclude that inversion of the last term of (19) leads to the last term of (20). This proves Theorem 6 for $y \geqq 0$. The proof for $-t \leqq y \leqq 0$ is similar; the left-hand side vanishes because $W(t) \geqq 0$, a.e.; and for $y < u - t$ the limit in (23) is zero, and this circumstance leads to the upper limit $(t+y)$, y negative, rather than t, in (21).

From Theorem 6 we shall now obtain, as our second principal result, the following characterization of $\Pr\{W(t) \leqq w\}$:

THEOREM 7. *Let $\mathcal{G}_u$ be the Borel field generated by the event $\{W(u) = 0\}$, and let $R(t, u, w)$ be the almost everywhere constant value of the conditional probability*

$$\Pr\{K(t) - K(u) - t + u \leqq w \mid \mathcal{G}_u\}, \qquad u < t,$$

on the set $\{W(u) = 0\}$. The probability $\Pr\{W(t) \leqq w\}$ is given in terms of the basic functions

$$(24)\quad \begin{aligned} &\Pr\{K(t) \leqq w\}, \\ &R(t, u, w), \qquad u < t, \end{aligned}$$

by the formulae

$$\Pr\{W(t) \leqq w\} = 0, \qquad \text{for } w < 0, \tag{25}$$

$$\Pr\{W(t) \leqq w\} = \Pr\{K(t) - t \leqq w\} - \frac{\partial}{\partial w}\int_0^t \Pr\{K(t) - K(u) - t + u \leqq w \,\&\, W(u) = 0\} du, \tag{26}$$

$$\text{for } w \geqq 0.$$

In terms of the basic functions (24), *the integrand in* (26) *is given by*

$$\Pr\{K(t) - K(u) - t + u \leqq w \,\&\, W(u) = 0\} = R(t, u, w)\Pr\{W(u) = 0\}, \tag{27}$$

and for each $t \geqq 0$, *and* $0 \geqq y \geqq -t$, *the chance* $\Pr\{W(u) = 0\}$ *that the system be empty at u satisfies*

$$\begin{aligned} E\{\max\{0, t + y - K(t)\}\} &= \int_0^{t+y} R(t, u, y)\Pr\{W(u) = 0\} du \\ &= \int_0^{t+y} \Pr\{K(t) \leqq u\} du. \end{aligned} \tag{28}$$

Proof. By definition of conditional probability,

$$\begin{aligned} &\Pr\{K(t) - K(u) - t + u \leqq w \,\&\, W(u) = 0\} \\ &= \int_{\{W(u)=0\}} R(t, u, w) P(d\omega), \\ &= R(t, u, w)\Pr\{W(u) = 0\}, \end{aligned} \tag{29}$$

since the integrand is constant a.e. on the domain of integration. From (29), (21), and (18) we deduce (27) and (28). Finally, (25) is obvious, and we may differentiate equation (20) of Theorem 6 to obtain (26).

Although the definition of $R(t, u, w)$ we have given involves a reference to $W(u)$, this reference is only a convenience; it can be eliminated by characterizing the event $\{W(u) = 0\}$ purely in terms of $K(\cdot)$, as follows: we note that

$$\{W(u) = 0\} \subseteq \{z \leqq u\},$$

so that by Lemma 1,

$$\{W(u) = 0\} = \left\{\sup_{0<y<u} [K(u) - K(y) - u + y] \leqq 0\right\}.$$

Theorem 7 answers the general question posed in the Introduction, §1: what is the form of the operator that gives the distribution of $W(t)$ in terms of distributions associated with $K(u)$ for $u \leqq t$? The operator is linear, and acts only on the distribution of $K(t)-t$, and, for each $u \leqq t$, on the conditional distribution of $K(t)-K(u)-t+u$ relative to the knowledge that

$$\sup_{0<y<u} [K(u) - K(y) - u + y] \leqq 0.$$

Thus Theorem 7 gives insight into the type of correlation, or conditioning, the type of "Nachwirkung" or after-effect that is relevant to the analysis of the $W(t)$ process. For example, it can be seen that the only gain in generality obtained by letting $K(\cdot)$ have independent increments, rather than restricting it to the compound Poisson form of Takács and Reich, consists in allowing a different service-time distribution for each arrival epoch. The arrivals still form a Poisson process; the service-times are still independent of each other and of the arrivals, but need not be identically distributed.

Theorem 7 also generalizes results of Reich, proven in [7] for Poisson arrivals and independent service-times. Reich, however, did not give explicit forms and "physical" interpretations to his analogues of the basic functions (24), leaving them as inversion integrals. For $y=0$, equation (28) is a Volterra equation of the first kind in t, analogous to Reich's equation 2.4, which is actually a special case of (28).

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