Introduction. In the study of the complex finite-dimensional semi-simple Lie algebras a crucial role is played by the fundamental bilinear form \( \langle x, y \rangle = \text{Tr}(\text{ad}(x)\text{ad}(y)) \). Since the definition is meaningless when the restriction of finite-dimensionality is removed, if any of the highly desirable properties of the form are to be retained in this case they must necessarily be given a priori. By reconsidering the finite-dimensional situation it is possible to formulate suitable conditions in a more convenient form. To see this let \( L \) be a complex finite-dimensional semi-simple Lie algebra and let \( L_0 \) be a compact real form for \( L \) with \( \sigma \) as the associated involution (conjugation). If we let \( x^* = -\sigma(x) \) and \( <x, y> = \langle x, y^* \rangle \) then \( L \) becomes a finite-dimensional Hilbert space, the mapping \( x \) into \( x^* \) is a Hilbert space conjugation, and the connecting property \( (<[x, y], z> = \langle y, [x^*, z]\rangle \) holds for all \( x, y, z \). An \( L^* \) algebra as defined here is simply a Lie algebra whose vector space is a Hilbert space such that the connecting property above holds. This paper is a study of such algebras with emphasis, of course, on the infinite-dimensional ones. For finite dimensions nothing new is obtained and it is shown here that in this case every semi-simple \( L^* \) algebra arises essentially from a construction like that above (see the remark after 2.5).

There is an associative algebra analogue of this problem in the paper of Ambrose [1] on \( H^* \) algebras and some of his results are used here. Any \( H^* \) algebra gives rise to an \( L^* \) algebra by letting \( [x, y] = xy - yx \) and the only known examples of \( L^* \) algebras are those obtained as Lie subalgebras of \( H^* \) algebras.

The main result of this paper is a classification of the (separable) simple \( L^* \) algebras which have Cartan decompositions (see §2) and it is shown that this class coincides with the simple self-adjoint Lie subalgebras of a (separable) simple \( H^* \) algebra. The results turn out to be the natural extensions of the finite-dimensional theory.

Associated with each of the Lie algebras considered here there is a gener-
alized analytic group nucleus. For a discussion of this relationship one may refer to the paper of Birkhoff [2].

1. Preliminaries.

**Definition.** An $L^*$ algebra is defined as a Lie algebra $L$ over the complex field such that the vector space of $L$ is a Hilbert space and for each $x \in L$ there is an $x^*$ in $L$ with $([x, y], z) = (y, [x^*, z])$ for all $y, z$ in $L$.

**Examples.** Let $A$ be an $H^*$ algebra and let $[x, y] = xy - yx$. Any closed Lie subalgebra of $A$ which is closed under the operation of taking adjoints is then an $L^*$ algebra. Any complex finite-dimensional semi-simple Lie algebra is an $L^*$ algebra. The Hilbert space direct sum of $L^*$ algebras defines an $L^*$ algebra in the obvious way.

**Definitions and Remarks.** $L$ will represent an $L^*$ algebra. For subsets $M, N$ of $L$ let $M^* = \{ m^*: m \in M \}$, $M^\perp = \{ x: (x, m) = 0 \text{ for all } m \in M \}$, $[M, N]$ = the closed subspace spanned by $\{ [m, n]: m \in M, n \in N \}$. For subspaces $S_1$ and $S_2$ of $L$ the notation $S_1 + S_2$ will be used only when $S_1 \perp S_2$.

For $x$ in $L$ let $D_x$ denote the linear operator $D_xy = [x, y]$. Then $D_x, D^*$ are everywhere defined (this implies both are bounded) and $D^*_x = D_{x^*}$. By using the principle of uniform boundedness it is not hard to show that the mapping $x$ to $D_x$ is continuous from $L$ into the space of bounded operators on $L$ under the uniform norm. Furthermore we may assume that $\|D_x\| \leq \|x\|$.

$L$ will be called semi-simple if and only if $L = [L, L]$ and this is equivalent to the mapping $x$ to $D_x$ being one-one. $L$ will be called simple if and only if there are no nontrivial closed ideals. It is a simple argument to show that a closed subspace $I$ of $L$ is an ideal of $L$ if and only if $I^\perp$ is an ideal. Using this one obtains the result that every $L^*$ algebra is the direct sum of an abelian ideal (the center) and a semi-simple ideal (the derived algebra, $[L, L]$). Hence an $L^*$ algebra is necessarily reductive in the sense of [3, Exposé 7].

From now on we will assume $L$ is semi-simple. Using the fact that the adjoint representation is then faithful and the properties of adjoints for operators it follows that the mapping $x$ to $x^*$ is involutory, conjugate linear, and $[x, y]^* = [y^*, x^*]$. Then the connecting property implies $(x, [y, z]) = ([y, z]^*, x^*)$ for all $x, y, z$. By semi-simplicity, $(x, y) = (y^*, x^*)$ for all $x, y$ so that the $*$ mapping is a Hilbert space conjugation. $L$ is then the complexification of the real Lie algebra formed by the skew-adjoint elements. It can be proved from all of this that every closed ideal of $L$ is an $L^*$ algebra.

A Cartan subalgebra of a semi-simple $L$ is defined as a maximal self-adjoint abelian subalgebra. An application of Zorn's Lemma shows that every $x \in L$ with $[x, x^*] = 0$ is contained in a Cartan subalgebra. A Cartan subalgebra is necessarily closed.

1.1. Let $H$ be a Cartan subalgebra of $L$. Then $H$ is maximal abelian and $H^\perp = [H, L]$.

**Proof.** Suppose $[H, x] = 0$. Then $[H, x^*] = [H^*, x^*] = [H, x]^* = 0$. Hence $[H, x + x^*] = [H, x - x^*] = 0$. Since $H$ is maximal self-adjoint abelian this im-
plies $x + x^*$ and $x - x^*$ are in $H$, hence $x \in H$ and $H$ is maximal abelian. If $h_1, h_2 \in H$ and $x \in L$ then $(h_1, [h_2, x]) = ([h_2^*, h_1], x)$ implies $[h_2, x] \in H$ and $[H, L] \subseteq H^*$. If $x \in [H, L]^*$ then $(x, [h^*, y]) = 0$ for all $y$ implies $([h, x], y) = 0$ so that $[H, x] = 0$ and $x \in H$.

In the event that $L$ is finite-dimensional a Cartan subalgebra $H$ as defined here is a Cartan subalgebra in the usual sense. For $H$ is maximal abelian and for each $h \in H$, $[h, h^*] = 0$ implies $D_h$ is normal, hence diagonalizable. These two properties characterize the Cartan subalgebras of $L$ (see [3, Exposé 9]). Conversely, if $L$ is semi-simple and finite-dimensional, a Cartan subalgebra $H$ of $L$ in the sense of [3] is one in our sense for a suitable $*$ mapping and inner product, for by Exposé 11 of [3] there is a compact real form $L_0$ of $L$ with associated involution $\sigma$ such that $\sigma(H) = H$. Applying the construction used in the introduction gives the result.

1.2. Theorem 1. Let $L$ be a semi-simple $L^*$ algebra. Then there exist simple closed $L^*$ ideals $L_j$, indexed by some set $J$, such that $L = \sum_{j \in J} L_j$, the sum being the usual Hilbert space direct sum. Every closed ideal of $L$ is obtained by summing the $L_j$ over some subset of $J$.

Outline of Proof. Let $H$ be a Cartan subalgebra of $L$ and $B$ the $C^*$ algebra generated by $\{D_h : h \in H\}$. $B$ is then topologically and algebraically isomorphic with the algebra of all continuous complex-valued functions vanishing at infinity on the locally compact space $\Delta$ of all homomorphisms of $B$ onto the complex numbers. Each $\alpha \in \Delta$ defines a bounded linear functional on $H$ and hence there is a unique $h_\alpha \neq 0$ in $H$ such that $\alpha(D_h) = (h, h_\alpha)$ for all $h$. Then $\|h_\alpha\| \leq 1$ and $\alpha(D_h^*) = [\alpha(D_h)]^*$ implies $h_\alpha^* = h_\alpha$. For $\alpha, \beta \in \Delta$ let $(\alpha, \beta) = (h_\alpha, h_\beta)$ and define $\alpha \perp \beta$ if and only if $(\alpha, \beta) = 0$. A subset $M$ of $\Delta$ will be called indecomposable if $M$ cannot be written as the union of nonempty orthogonal subsets. Then each $\alpha \in \Delta$ is contained in a unique maximal indecomposable subset $M_\alpha$. Then either $M_\alpha = M_\beta$ or $M_\alpha \perp M_\beta$. Let $\{M_j : j \in J\}$ be the set of the distinct $M_\alpha$'s. For each $j$ let $H_j$ be the span of the $h_\alpha$ where $\alpha$ runs over $M_j$ and let $L_j = H_j + [H_j, L]$. By a proof like that used in the finite-dimensional case each $L_j$ is a simple closed ideal of $L$ and $L_j \perp L_k$ for $j \neq k$. If $K = \sum L_j$ then $K$ is a closed ideal containing $H$ (the $h_\alpha$'s span $H$) and hence $[K^\perp, H] = 0$ implies $K^\perp = 0$ so that $L = \sum L_j$. The last statement is a consequence of the way the decomposition is obtained.

2. Roots and Cartan decompositions.

Definition. For this section $L$ is a semi-simple $L^*$ algebra with $H$ as a Cartan subalgebra. For a linear mapping $\alpha$ of $H$ into the complex numbers let $V_\alpha = \{v : [h, v] = \alpha(h)v \text{ for all } h \in H\}$. Then $V_\alpha$ is a closed subspace of $L$ and $\alpha$ will be called a root (relative to $H$) if and only if $V_\alpha \neq 0$. The zero function is a root and $V_0 = H$. If $\alpha$ is a root then necessarily it corresponds to a homomorphism of the operator algebra generated by $\{D_h : h \in H\}$. Hence $\alpha$ is bounded and $\alpha(h^*) = [\alpha(h)]^*$. As in the proof of Theorem 1 there is a unique
\( h_a \) in \( H \) with \( \|h_a\| \leq 1 \), \( h_a^* = h_a \), and \( \alpha(h) = (h, h_a) \) for all \( h \). From this it follows that if \( \alpha \) is a root \( -\alpha \) is also one and \( V_\alpha^* = V_{-\alpha} \). If \( \alpha, \beta \) are distinct then \( V_\alpha \perp V_\beta \).

By the Jacobi identity \([V_\alpha, V_\beta] = V_{\alpha + \beta}\).

Let \( K = \sum V_\alpha \), the sum being taken over the distinct roots relative to \( H \). Then \( K \) is a closed \( L^* \) subalgebra of \( L \) with \( H \subseteq K \subseteq L \). We will say that \( L \) has a Cartan decomposition (relative to \( H \)) if and only if \( K = L \), i.e. if and only if the set \( \{ D_h : h \in H \} \) is simultaneously diagonalizable. It is an open question as to whether or not every \( L \) has such a decomposition; however, I hope to have more complete results to be given in a later paper. Theorem 2 below settles the question if \( L \) is embedded in an \( H^* \) algebra and the later classification theory shows this is necessary as well as sufficient, at least when every simple ideal component of \( L \) is separable.

2.1. Let \( L \) be a simple \( L^* \) algebra and suppose \( \phi \) is a continuous linear mapping of \( L \) into a Hilbert space \( K \) with \( \langle \phi([x, y]), \phi(z) \rangle = \langle \phi(y), \phi([x^*, z]) \rangle \) for all \( x, y, z \) in \( L \). Then there is an \( \epsilon \geq 0 \) such that \( \langle \phi(x), \phi(y) \rangle = \epsilon(x, y) \) for \( x \) and \( y \) in \( L \).

**Proof.** Since \( \phi \) is bounded there is a bounded operator \( B \) on \( L \) such that \( \langle \phi(x), \phi(y) \rangle = (Bx, y) \). Then \( B \geq 0 \) implies \( B \) is self-adjoint. The assumption on \( \phi \) implies \( B \) commutes with every \( D_x \); by the spectral theorem every projection in the spectral resolution of \( B \) commutes with every \( D_x \). The range of such a projection is then a closed ideal of \( L \), hence is either 0 or all of \( L \) so that \( B = \epsilon 1 \) for some \( \epsilon \geq 0 \).

2.2. **Theorem 2.** Suppose \( L \) is a semi-simple \( L^* \) subalgebra of an \( H^* \) algebra \( A \) and \( H \) is a Cartan subalgebra of \( L \). Then \( L \) has a Cartan decomposition relative to \( H \).

**Proof.** Let \( L = \sum L_j \) where each \( L_j \) is a simple closed ideal. If \( H_j = H \cap L_j \) it is easily seen that \( H_j \) is a Cartan subalgebra of \( L_j \). Hence it will be sufficient to prove the theorem when \( L \) is simple.

If \( I \) is a simple (associative) ideal of \( A \) the restriction to \( L \) of the projection \( P \) of \( A \) onto \( I \) satisfies the hypotheses of 2.1 and hence there is an \( \epsilon \geq 0 \) such that \( \langle Px, Py \rangle = \epsilon(x, y) \) for all \( x, y \) in \( L \). Since \( A \) is a direct sum of such simple ideals there must be some \( I \) such that the corresponding \( \epsilon \) is positive. Thus \( L \) is topologically and algebraically isomorphic with a Lie subalgebra of \( I \) so that we may assume \( A \) itself is simple. Then by [1], \( A \) is the set of all Hilbert-Schmidt operators on some Hilbert space \( \mathcal{H} \).

The set \( H \) is then a collection of commutative completely continuous normal operators on \( \mathcal{H} \) and hence can be simultaneously diagonalized. Using a basis of \( \mathcal{H} \) composed of common eigenvectors for \( H \) and regarding \( A \) as the algebra of square-convergent matrices relative to this basis, \( H \) becomes a subset of the diagonal matrices. For \( h \in H \) and \( y \in A \) let \( T_h y = hy - yh \). Then, as in the finite-dimensional case, the operators \( T_h \) can be simultaneously
diagonalized. Since $L$ is an invariant subspace under the set of all $T_h$ and the restriction of $T_h$ to $L$ is $D_h$ then $L$ has a Cartan decomposition relative to $H$.

For the remainder of this section we will assume only that $L$ is semisimple and $H$ is a Cartan subalgebra.

2.3. If $\alpha$ is a nonzero root $V_\alpha$ is one-dimensional.

**Proof.** Choose $v_1 \in V_\alpha$ with $||v_1|| = 1$. Let $v_2 \in V_\alpha$ with $(v_1, v_2) = 0$. It is sufficient to show that this implies $v_2 = 0$. For any $v \in V_\alpha$ we have $[v_1, v^*] \in H$. For any $h \in H$, $(h, [v_1, v^*]) = ([h, v_1], v_1) = (h, h_\alpha)(v, v_1)$ implies $[v_1, v^*] = (v_1, v)h_\alpha$ so that $[v_1, v^*] = 0$. The same argument can be used to show that $[v_2, v^*_2] = [v_2, v^*_1] = 0$. Then $[v_2, v^*_2] = 0$ so that $[v_2, v^*_1] = 0$ and $v_2 = 0$.

**Definition.** Let $R$ be the set of nonzero roots relative to $H$. By Zorn's lemma it is possible to decompose $R$ as $R = R_1 \cup R_2$ where $R_1, R_2$ are disjoint and $\alpha \in R_1$ if and only if $-\alpha \in R_2$. For each $\alpha \in R_1$ choose $e_\alpha \in V_\alpha$ such that $||e_\alpha|| = 1$. Then $e_\alpha \in V - \alpha$ and $||e_\alpha|| = 1$. For $\alpha \in R_2$ let $e_\alpha = e^*_\alpha$. Thus $e_\alpha = e^*_\alpha$ for all $\alpha$ in $R$ and the set $\{e_\alpha\}$ is an orthonormal set. By the proof of 2.2, $[e_\alpha, e^*_\alpha] = h_\alpha$.

Suppose $\alpha, \beta \in R$ and $\beta = -\alpha$. If $\alpha + \beta$ is a root let $c_{\alpha, \beta}$ be defined by the equation $[e_\alpha, e^*_\beta] = c_{\alpha, \beta}e_{\alpha + \beta}$, otherwise let $c_{\alpha, \beta} = 0$ and $e_{\alpha + \beta} = 0$.

If $\beta$ is any root and $\alpha$ a nonzero root the sequence $\{\beta - k\alpha: k = 0, \pm 1, \ldots\}$ contains only finitely many roots for if $\beta - k\alpha$ is a root then $1 \geq ||h_{\beta - k\alpha}|| = ||h_\beta - k\alpha|| \geq ||k|| ||h_\alpha|| - ||h_\beta||$. Thus it is possible to define the integers $k_1(\alpha, \beta)$ and $k_2(\alpha, \beta)$ by the conditions $\beta + k\alpha$ is a root for $-k_1 \leq k \leq k_2$ while $\beta - (k_1 + 1)\alpha$ and $\beta + (k_2 + 1)$ are not roots. Then, by the same proof as used in [3], $(h_\alpha, h_\beta) = (1/2) [k_1(\alpha, \beta) - k_2(\alpha, \beta)] ||h_\alpha||^2$ for any roots $\alpha, \beta$ with $\alpha \neq 0$.

2.4. Suppose $\alpha_1, \ldots, \alpha_k \in R$. Let $M$ be the set of all roots which are linear combinations with integral coefficients of $\alpha_1, \ldots, \alpha_k$. Let $V$ be the span of the $e_\alpha$'s where $\alpha \in M$ and let $H_1$ be the span of $h_{\alpha_1}, \ldots, h_{\alpha_k}$. Then $L_1 = H_1 + V$ is a finite-dimensional semi-simple $L^*$ algebra with $H_1$ as a Cartan subalgebra and $M$ is the complete set of roots relative to $H_1$.

**Proof.** The proof is straightforward except, perhaps, for the statement that the dimension of $L_1$ is finite. Since $\dim H_1 \leq k < \infty$, $\dim L_1$ is infinite if and only if $\{e_\alpha: \alpha \in M\}$ is infinite and this can occur only if $\{h_\alpha: \alpha \in M\}$ is infinite. In this event the latter set is an infinite bounded set in the unitary space $H_1$ and must then contain an infinite convergent sequence $h_n$. Letting $k_i = 2||h_{\alpha_i}||^{-1}h_{\alpha_i}$ for $i = 1, \ldots, k$, $\langle h_{\alpha_i}, h_i \rangle$ is an integer for all $n$ and $i$ and $H_1$ is spanned by $h_1, \ldots, h_k$. From this it is clear that no such sequence exists and consequently $L_1$ is finite-dimensional.

2.5. Suppose $L$ is finite-dimensional and simple. Let $\langle x, y \rangle = \text{Tr}(D_x D_y)$ for all $x, y$. Then there is an $\varepsilon > 0$ such that $\langle x, y \rangle = \langle x, y^* \rangle$.

**Proof.** Define the operator $B$ on $L$ by the equation $\langle Bx, y \rangle = \langle x, y^* \rangle$ for all
Then \( (Bx, x) = \text{Tr}(D_x D_x^*) \) implies \( B \) is positive definite. The condition \( \langle [x, y], z \rangle = \langle x, [y, z] \rangle \) implies \( B \) commutes with every \( D_x \) and, by the argument used in 2.1, \( B \) must be a positive multiple of the identity.

**Remark.** The result of 2.5 justifies the remarks of the introduction for finite-dimensional \( L^* \) algebras. If \( L \) is simple and \( \epsilon \) is as in 2.5 let \( L_0 \) be the set of skew-adjoint elements of \( L \) and \( \sigma(x) = -x^* \) for all \( x \). Then \( \sigma \) is an involution and 2.5 shows that \( L_0 \) is a compact real form for \( L \). The extension to semi-simple algebras is immediate. An immediate consequence of this relationship is the result 2.6 below which will be needed in the later classification theory.

**2.6.** Let \( L \) be finite-dimensional and simple with \( H \) as a Cartan subalgebra.

(i) If \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( \alpha + \beta + \gamma = 0 \) then \( c_{\alpha, \beta} = c_{\beta, \gamma} = c_{\gamma, \alpha} \).

(ii) If \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \), \( \alpha + \beta + \gamma + \delta = 0 \), and the sum of no pair is zero, then \( c_{\alpha, \beta} c_{\gamma, \delta} + c_{\beta, \gamma} c_{\alpha, \delta} + c_{\gamma, \alpha} c_{\beta, \delta} = 0 \).

(iii) If \( \alpha, \beta \in \mathbb{R} \) and \( \beta \neq -\alpha \) then

\[
    c_{\alpha, \beta} c_{-\alpha, -\beta} = -\left(\frac{1}{2}\right) k_2(\alpha, \beta) (1 + k_1(\alpha, \beta)) \|h_0\|^2.
\]

**Proof.** See [3, Exposé 11, Lemmas 1, 2, 3].

**3. The classification theory.** For this section \( L \) will be a simple infinite-dimensional \( L^* \) algebra and \( H \) a Cartan subalgebra such that \( L \) has a Cartan decomposition relative to \( H \). We further require that the space of \( L \) be separable.

**Definition.** For a finite subset \( F = \{\alpha_1, \ldots, \alpha_k\} \) of \( \mathbb{R} \) let \( L(F) \) denote the finite-dimensional semi-simple algebra defined in 2.4. Then \( F_1 \subset F_2 \) implies \( L(F_1) \subset L(F_2) \). A subset \( G \) of \( \mathbb{R} \) will be called a root system if and only if \( \alpha \in G \) implies \( -\alpha \in G \) and \( \alpha, \beta \in G, \alpha + \beta \in \mathbb{R} \) implies \( \alpha + \beta \in G \). Then \( L(F) \) is the subalgebra generated by the \( e_\alpha \) where \( \alpha \) ranges over the root system generated by \( F \). Using the notion of indecomposability as in the proof of Theorem 1, if \( F \) is an indecomposable finite subset of \( \mathbb{R} \) then the root system generated by \( F \) is indecomposable and \( L(F) \) is simple. Furthermore it is clear that \( R \) is indecomposable since \( L \) is simple. A subset \( \alpha_0, \ldots, \alpha_n \) of \( R \) will be called a chain from \( \alpha_0 \) to \( \alpha_n \) if \( (h_{\alpha_i-1}, h_{\alpha_i}) \neq 0 \) for \( i = 1, \ldots, n \). Since \( R \) is indecomposable any \( \alpha, \beta \in R \) must be connected by a finite chain. Any chain is indecomposable.

**3.1.** For any finite subset \( F \) of \( \mathbb{R} \) there exists a finite indecomposable root system containing \( F \).

**Proof.** Let \( F = \{\alpha_1, \ldots, \alpha_n\} \). For each \( i, 1 \leq i \leq n-1 \), let \( F_i \) be a chain from \( \alpha_i \) to \( \alpha_{i+1} \). Let \( F_1 = \bigcup F_i \). Then \( F_1 \) is indecomposable and finite. If \( F_2 \) is the root system generated by \( F_1, F_2 \) is indecomposable and 2.4 implies \( F_2 \) is finite.

**Definition.** Since \( L \) is separable the orthonormal set \( \{e_\alpha: \alpha \in \mathbb{R}\} \) is count-
able and hence $R$ is countably infinite. Let $R = \{\alpha_1, \alpha_2, \cdots \}$ and let $F_n = \{\alpha_1, \cdots , \alpha_n\}$.

3.2. There is a sequence $G_n$ of finite subsets of $R$ such that the following are true:

(i) $F_n \subset G_n \subset G_{n+1}$.
(ii) $G_n$ is an indecomposable root system.
(iii) $R = \bigcup G_n$.
(iv) The simple subalgebras $L(G_n)$ form a strictly increasing sequence with $L =$ closure of $\bigcup L(G_n)$. All of the $L(G_n)$ are of the same Cartan type $A$, $B$, $C$, or $D$.

Proof. The sequence $\{G_n\}$ can be defined inductively. Let $G_1$ be a finite indecomposable root system containing $F_1$. Having chosen $G_1, \cdots , G_{n-1}$ satisfying (i) and (ii) let $F = G_{n-1} \cup F_n$ and choose $G_n$ to be a finite indecomposable root system containing $F$. The $G_n$ obtained in this way will then satisfy (i) and (ii). Since $R = \bigcup F_n$, (iii) will hold and $G_n \subset G_{n+1}$ implies $L(G_n) \subset L(G_{n+1})$. An $h \in H$ such that $(h, h_n) = 0$ for all $\alpha$ in $R$ would then have $D_h = 0$, hence $h = 0$ and $H$ is spanned by the set of $h_n$. Since the set of $\varepsilon_\alpha$ spans $H^\perp$ then $L =$ closure of $\bigcup L(G_n)$. Now dim $L$ is infinite and each $L(G_n)$ is finite-dimensional so that there are infinitely many distinct $L(G_n)$. Then any infinite subsequence of the $G_n$ will also satisfy (i), (ii), and the first part of (iv). By passing to subsequences if necessary it is possible to eliminate any duplications and furthermore obtain a sequence whose elements are all of the same type. Since their dimensions are unbounded there can be no exceptional algebras.

Definition. Let $K_n$ be the real linear subspace of the conjugate space of $H$ spanned by $\{a : a \in G_n\}$. Let $p_1 = \dim K_1$ and $p_n = \dim (K_n/K_{n-1})$ for $n = 2, 3, \cdots$. Then each $p_i$ is a positive integer and the rank of the simple algebra $L(G_n)$ is $p_1 + \cdots + p_n$.

Since $G_1$ is a root system for $L(G_1)$ there exist $\alpha_{1,1}, \cdots , \alpha_{1,p_1}$ in $G_1$ which form a linear basis of $K_1$. Since $G_2$ is a root system for $L(G_2)$ there exist $\alpha_{2,1}, \cdots , \alpha_{2,p_2}$ in $G_2$ such that $\alpha_{1,1}, \cdots , \alpha_{1,p_1}, \alpha_{2,1}, \cdots , \alpha_{2,p_2}$ form a linear basis for $K_2$. Necessarily $\alpha_{2,i} \in G_1$. Continuing this process we can find, for each $n \geq 2$, $\alpha_{n,1}, \cdots , \alpha_{n,p_n}$ in $G_n - G_{n-1}$ such that the set

$$\{\alpha_{i,j} : i = 1, \cdots , n; j = 1, \cdots , p_i\}$$

is a linear basis for $K_n$. Order this basis as follows:

$$\alpha_{n,p_n}, \cdots , \alpha_{n,1}, \alpha_{n-1,p_{n-1}}, \cdots , \alpha_{n-1,1}, \cdots , \alpha_{2,1}, \alpha_{1,p_1}, \cdots , \alpha_{1,1}.$$
ing of $G_n$. By the choice of basis for each $K_n$, for integers $n$, $m$ and $\alpha, \beta \in G_n \cap G_m$, $\alpha > \beta$ in the ordering of $G_n$ if and only if $\alpha > \beta$ in the ordering of $G_m$.

Now suppose $\alpha, \beta$ are any roots. Choose $n$ such that $\alpha, \beta \in G_n$ and define $\alpha > \beta$ if and only if they are so related in the ordering of $G_n$. This gives a well-defined total ordering on the set of all roots and has the following properties:

(i) $\alpha > 0$ implies $-\alpha < 0$.
(ii) $\alpha > 0, \beta > 0$ implies $\alpha + \beta > 0$.
(iii) If $\alpha > 0$ and $\alpha \in G_n$ then $\alpha > \beta$ for every $\beta \in G_n$.
(iv) The ordering induced on $G_n$ is a lexicographical ordering with respect to a basis of roots.

Let $R^+$ be the set of positive roots. Then, since $G_n$ is finite, property (iii) implies that $R^+$ is well-ordered. An $\alpha \in R^+$ will be called simple if $\alpha$ cannot be written as the sum of two positive roots. Let $S$ denote the set of all simple roots.

3.3. (1) $\Delta \cap G_n$ is a complete set of simple roots (in the sense of [3]) for $L(G_n)$.
(2) For $\alpha, \beta$ in $S$, $\alpha - \beta$ is a root only if $\alpha = \beta$. Thus $k_1(\alpha, \beta) = k_1(\beta, \alpha) = 0$.
(3) $S$ is linearly independent over the reals and every $\alpha$ in $R^+$ is a linear combination of elements of $S$ with non-negative integral coefficients which are almost all zero.
(4) If $\tau = \sum n_i \alpha_i$ where $\alpha_i \in S$ and almost all $n_i$ are zero there is an algorithm to determine whether or not $\tau$ is a root. To apply the algorithm it is sufficient to know $(h_\alpha, h_\beta)$ for all $\alpha, \beta \in S$.

Proof. (1) If $\alpha, \beta, \gamma \in R^+$ and $\alpha = \beta + \gamma$ then $\alpha > \beta > 0$ and $\alpha > \gamma > 0$. If $\alpha \in G_n$ then (iii) of the definition above implies $\beta, \gamma \in G_n$. Hence an $\alpha \in G_n$ is simple in $G_n$ if and only if $\alpha$ is simple in $R$.

(2) and (3) can be deduced from the corresponding properties for the finite-dimensional case proved in [3, Exposé 10].

(4) Choose $n$ such that $\tau \in K_n$. The statement then follows from the result proved in [3, Exposé 16], applied to the algebra $L(G_n)$, using the fact that the fundamental bilinear form is determined up to a constant multiple.

Definition. Define the graph of $S$ to be the set $G$ of all $(h_\alpha, h_\beta)$ where $\alpha, \beta$ vary over $S$. Then knowing the graph is equivalent to determining $\|h_\alpha\|$ and $k_\alpha(\alpha, \beta)$ for $\alpha, \beta \in S$. If $L, L'$ are two algebras of the type considered in this section with $H$ and $H'$ as Cartan subalgebras and $G, G'$ as the corresponding graphs we will say that $G$ is isomorphic to $G'$ if and only if there is a mapping $\alpha$ to $\alpha'$ of $S$ onto $S'$ with $(h_\alpha, h_\beta) = (h'_\alpha, h'_\beta)$ for all $\alpha, \beta \in S$.

3.4. Let $L, L'$ be as above and suppose $G$ is isomorphic to $G'$. Then there is an algebraic isomorphism $\phi$ of $L$ onto $L'$ such that:

(1) $\phi(h_\alpha) = h'_\alpha$, for all $\alpha \in R$.
(2) $\phi(x)^* = \phi(x^*)$ for all $x \in L$.
(3) $(\phi(x), \phi(y)) = (x, y)$ for all $x, y$ in $L$.

Proof. By using the algorithm of 3.3, (4) it is possible to extend the map of $S$ onto $S'$ to a mapping $\alpha$ to $\alpha'$ of $R$ onto $R'$ which preserves inner products.
for the $h_a$. This mapping then necessarily preserves all of the algebraic structure of $R$. For a complex linear combination $h = \sum c_i h_{a_i}$, where $\alpha_i \in R$, let $\phi(h) = \sum c_i h'_{a_i}$. Then $\phi$ is well-defined and preserves inner products so that it extends uniquely to an isometry of $H$ onto $H'$ and satisfies (1). Since $\phi(h^*_a) = \phi(h_a)^*$ for all $h_a$, $\phi$ will satisfy (2) for any $x \in H$.

Let $\{f_{a'}: \alpha' \in R'\}$ be a fixed set of elements in $L'$ with $f_{a'} \in V_{a'}$, $\|f_{a'}\| = 1$, and $f_{a'}^* = f_{-a'}$. Let $c_{a', \beta'}$ be the structure constants for $L'$ defined by the set of $f_{a'}$. To extend $\phi$ to all of $L$ with the required properties it is then sufficient to find a set $\{e_{\alpha}: \alpha \in R\}$ in $L$ with $e_{\alpha} \in V_{\alpha}$, $\|e_{\alpha}\| = 1$, $e_{\alpha}^* = e_{-\alpha}$, and such that the structure constants $c_{a, \beta}$ for $L$ defined by this set satisfy $c_{a, \beta} = c_{a', \beta'}$.

Thus the problem is reduced to finding the set of $e_{\alpha}$. A corresponding result appears in [3, Exposé 11, Théorème 1]. An examination of the proof there shows the essential features are a well-ordering of $R^+$ compatible with the algebraic structure and the relations on the structure constants which were proved here in 2.6. (These hold for $L$ since there is always an $n$ such that $\alpha, \beta, \gamma, \delta$ all lie in $G_n$.) Using these, the proof in [3] can be repeated here word for word.

3.5. Because of 3.4 it only remains to determine the possible graphs for $L$ and give examples of each type in order to complete the classification.

First, suppose all of the $L(G_n)$ in 3.2 are of type A. Then the root diagram for the simple system $S \cap G_1$ has the form:

$\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_p$

This means, of course, that $k_2(\alpha_i, \alpha_{i-1}) = k_2(\alpha_i, \alpha_{i+1}) = 1$ and otherwise $k_2(\alpha_i, \alpha_j) = 0$. Furthermore, by the remark after 2.5, $\|h_{\alpha_i}\| = \|h_{\alpha_j}\|$ for $1 \leq i, j \leq p$. Now let $S \cap G_2$ be written as $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p$. After any necessary reordering the diagram for $S \cap G_2$ will have the form:

$\beta_1 \beta_k \alpha_1 \alpha_2 \alpha_{p_1} \beta_{k+1} \cdots \beta_{p_2}$

It is possible that all the $\beta_j$ may be at one end of the chain. Again it follows that $\|h_{\alpha_i}\| = \|h_{\beta_j}\|$ for $\alpha_i, \beta_j$. Continuing this process and introducing the necessary new notation for the $\alpha$'s in $S$ we will obtain one of the following two possibilities:

Type A.

$\alpha_1 \alpha_2 \alpha_n$

or

Type A'.

$\alpha_n \alpha_{n-1} \cdots \alpha_1 \cdots \alpha_n$
In either case \( \|a_i\| = \|a_j\| \) for all \( i, j \), \( k_2(\alpha_i, \alpha_j) = 0 \) for \( j \neq i - 1, i + 1 \) while \( k_2(\alpha_i, \alpha_{i-1}) = k_2(\alpha_i, \alpha_{i+1}) = 1 \). Thus the graph is completely determined up to a constant multiple.

 Entirely similar arguments for the other possibilities give the following types:

Type B.

\[
\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_n
\]

Here \( 2^{1/2} \|a_i\| = \|a_i\| \) for \( i = 2, 3, \ldots \) and \( k_2(\alpha_1, \alpha_2) = 1, k_2(\alpha_2, \alpha_1) = 1 \) while otherwise \( k_2(\alpha_i, \alpha_j) \) is as above.

Type C.

\[
\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_n
\]

Here \( \|a_i\| = 2^{1/2} \|a_i\| \) for \( i = 2, 3, \ldots \) and \( k_2(\alpha_1, \alpha_2) = 2, k_2(\alpha_2, \alpha_1) = 1 \) while otherwise \( k_2(\alpha_i, \alpha_j) \) is as above.

Type D.

\[
\alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_n
\]

Here \( \|a_i\| = \|a_i\| \) for all \( i \) and \( j \) and \( k_2(\alpha_1, \alpha_2) = k(\alpha_2, \alpha_1) = 0, k_2(\alpha_3, \alpha_1) = 1 \) while otherwise \( k_2(\alpha_i, \alpha_j) \) is as above.

3.6. In this paragraph it will be shown that each of the five types A, A', B, C, D occurs as the graph of an \( L^* \) algebra. However, these algebras are not all distinct and give rise to only three nonisomorphic types. More explicitly, A and A' are isomorphic and so are B and D.

All of these examples are Lie subalgebras of the associative \( H^* \) algebra \( K \) of Hilbert-Schmidt operators on a separable Hilbert space \( \mathcal{H} \). For descriptive purposes it is convenient to choose an orthonormal basis of \( \mathcal{H} \) and regard \( K \) as a matrix algebra relative to this basis. In each case a Cartan subalgebra \( H \) is obtained by taking the intersection of the algebra in question with the set of diagonal matrices. Having done this we will let \( \lambda_i \) denote the linear functional on \( H \) which assigns the \( i \)th diagonal entry to every element of \( H \). Determination of a set of simple roots and the associated graph is analogous to the finite-dimensional case and the computations will be omitted here. After choosing the proper norm on \( L \) an application of 3.4 and 3.5 will show that \( L \) is isomorphic in all respects to one of the algebras described here.

In the following discussion a conjugate linear transformation \( J \) of \( \mathcal{H} \) onto
3C such that \((Jx, Jy) = (y, x)\) will be called a conjugation if \(J^2 = 1\) and an anti-conjugation if \(J^2 = -1\).

**Type A.** Let \(\{\phi_n : n = 1, 2, \cdots\}\) be a basis of 3C and let \(A\) be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. \(A\) is simple since the center of \(K\) is trivial. A simple system of roots is given by

\[\{\lambda_i - \lambda_{i+1} : i = 1, 2, \cdots\}\].

**Type A'.** Let \(\{\phi_n : n = 0, \pm 1, \pm 2, \cdots\}\) be a basis of 3C and let \(A'\) be the Lie algebra of all Hilbert-Schmidt matrices relative to this basis. A simple system of roots is given by \(\{\lambda_i - \lambda_{i+1} : i = 0, \pm 1, \pm 2, \cdots\}\).

The algebras \(A\) and \(A'\) are isomorphic since there is a unitary operator \(U\) on \(3C\) such that \(X \in A\) if and only if \(UXU^{-1}\) is in \(A'\).

**Type B.** Let \(\{\phi_n : n = 0, \pm 1, \pm 2, \cdots\}\) be a basis of 3C and let \(J_1\) be the conjugation of 3C such that \(J_1\phi_n = \phi_{-n}\). Let \(B\) be the set of Hilbert-Schmidt operators \(T\) such that \(T^*J_1 = -J_1T\). If \(\langle x, y \rangle\) is defined by \(\langle x, y \rangle = \langle x, T^*y \rangle\) for \(x, y \in \mathfrak{C}\) then \(\langle , \rangle\) is a symmetric bilinear form and \(B\) is the set of \(T\) in \(K\) which are skew-adjoint with respect to this form. A simple system of roots is given by \(\{\lambda_i - \lambda_{i+1} : i = 1, 2, \cdots\}\).

**Type D.** Let \(\{d_n : n = \pm 1, \pm 2, \cdots\}\) be a basis of 3C and let \(J_2\) be the conjugation on 3C such that \(J_2\phi_n = \phi_{-n}\). Let \(D\) be the set of \(T\) in \(K\) such that \(T^*J_2 = -J_2T\). A simple system of roots is given by

\[\{\lambda_1 + \lambda_2, \lambda_i - \lambda_{i+1} : i = 1, 2, \cdots\}\].

Since \(J_1\) and \(J_2\) are two conjugations of 3C there is a unitary \(U\) on 3C such that \(UJ_1 = J_2U\). Then for any \(T \in K\), \(T \in B\) if and only if \(UTU^{-1}\) is in \(D\). Hence \(B\) is isomorphic to \(D\).

**Type C.** Let \(\{\phi_n : n = \pm 1, \pm 2, \cdots\}\) be a basis of 3C. Let \(J\) be the anti-conjugation on 3C such that \(J\phi_n = -\phi_{-n}\) for all positive \(n\). Let \(C\) be the set of all Hilbert-Schmidt operators on 3C such that \(T^*J = -JT\). Then \(C\) is the set of all \(T \in K\) which are skew-symmetric with respect to the skew-symmetric form \(\langle x, y \rangle = \langle x, Jy \rangle\). A simple system of roots is given by

\[\{2\lambda_1, \lambda_i - \lambda_{i+1} : i = 1, 2, \cdots\}\].

**3.7. Theorem 3.** Let \(L\) be a separable simple \(L^*\) algebra which has a Cartan decomposition relative to some Cartan subalgebra. Then (up to a multiple of the inner product on \(L\)) \(L\) is isomorphic to one of the following algebras:

1. \(A\), the algebra of all Hilbert-Schmidt operators on a separable Hilbert space \(3C\).

2. \(B\), the algebra of all Hilbert-Schmidt operators \(T\) on 3C such that \(T^*J = -JT\) for some fixed conjugation \(J\) of 3C.

3. \(C\), the algebra of all Hilbert-Schmidt operators \(T\) on 3C such that \(T^*J = -JT\) for some fixed anti-conjugation \(J\) of 3C.

**Remark.** It still should be shown that the remaining three algebras \(A\),
$B, C$ are nonisomorphic. For two algebras $L, L'$ of the type described in 3.7 and acting on the same space $\mathcal{C}$ let $L$ be equivalent to $L'$ if and only if there is a unitary $U$ on $\mathcal{C}$ with $ULU^{-1}=L'$. We will show that $L$ and $L'$ are isomorphic only if they are equivalent. Since $A, B,$ and $C$ are clearly not equivalent this will be sufficient.

An $x \in L$ will be called primitive if (i) $x=x^* \neq 0$, (ii) $D^2_x = D_x$, and (iii) $x$ cannot be written $x=y+z$ where $y$ and $z$ satisfy (i) and (ii). By using the fact (see the proof of Theorem 2) that every Cartan subalgebra of $L$ is a set of diagonal matrices relative to some basis of $\mathcal{C}$ it follows that each such subalgebra has a basis of primitive elements and the vectors $h_\alpha$ are obtained from these by linear operations in a unique way according to the type of the associated graph. Since any isomorphism of $L$ will preserve primitive elements the set $\{h_\alpha : \alpha \in R\}$, and hence the graph of $L$, is determined up to equivalence and the same will then hold for $L$.

4. Some remarks on derivations.

Definition. Let $L$ be a semi-simple $L^*$ algebra. A bounded operator $D$ on $L$ will be called a derivation of $L$ if and only if $D[x, y] = [Dx, y] + [x, Dy]$ for all $x, y$ in $L$.

If $\dim L$ is finite it is known that every derivation of $L$ is inner, i.e. equal to $D_x$ for some $x \in L$ [3, Exposé 7]. However, this is not true in general. To see this let $A$ be the $L^*$ algebra of all Hilbert-Schmidt operators on a separable infinite-dimensional Hilbert space. Then $A$ is an associative ideal in the algebra of all bounded operators (see [4, pp. 73–75]). For a bounded operator $B$ let $T_B$ be the operator on $A$ defined by $T_BX = BX - XB$. Then $T_B$ is a bounded derivation of $A$ and $T_B = 0$ if and only if $B$ is a scalar multiple of the identity. Hence $T_B$ is inner only if it differs from a Hilbert-Schmidt operator by a multiple of the identity and this implies $A$ has outer derivations. Similar arguments can be used for $B$ and $C$.

The same example can be used to show that the image of $L$ under the adjoint representation need not be closed. By the closed graph theorem this is equivalent to proving that the norms on $L$ and its image are not equivalent. Letting $L=A$ as above and regarding $A$ as a matrix algebra with the usual unit matrices as a basis let $X_k = k^{-1/2} \sum_i E_{ii}$. Then $\|X_k\| = 1$ while $\|D_{X_k}\| = k^{-1/2}$. Thus $\|D_{X_k}\|$ tends to zero as $k$ becomes large.

Bibliography


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