LORENTZ STRUCTURES ON THE PLANE

BY

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1. **Introduction.** Very little is known at present concerning the global properties of Lorentz manifolds. One observes that most of the methods which have proved fruitful for Riemannian geometry in the large depend in an essential way on the positive definite character of the metric tensor. They are therefore quite inapplicable to Lorentz geometry, and one is faced with the problem of devising new methods of approach.

Among the global questions which arise for spaces with indefinite metric, a few have analogues in Riemannian geometry while many others do not. To illustrate the former, we can cite a basic theorem of Riemannian geometry(2) which asserts that every point pair of a complete manifold is connected by a geodesic. Although it is not quite clear what "completeness" should mean for a Lorentz space, we shall show in the sequel that if this notion is defined in a relatively natural way, the corresponding theorem fails to hold even for Lorentz structures on the plane. Here we have a global problem which has an analogue in Riemannian geometry, although the results turn out to be different. On the other hand, a Lorentz metric gives rise to a distinction between space-like, time-like and null-vectors in the tangent vector spaces, and it is evident that questions pertaining to the behavior of time-like curves or space-like hypersurfaces are indeed peculiar to Lorentz geometry. For example, one observes that the existence of a closed time-like curve in a simply connected Lorentz manifold is something of an anomaly. Although such curves can exist even on the n-cell for $n > 2$, they require a considerable "twisting" of the Lorentz structure. One would like to know in how many different ways a Lorentz structure can be twisted, and how this affects the curvature properties of the space.

It would take us too far afield to say something concerning global problems of Lorentz geometry which arise from the general theory of relativity. The fundamental existence and uniqueness questions of relativity clearly fall into this category, and it appears to us that so far only local results have been obtained. Apart from this, recent developments(3) have indicated that

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(2) Cf. Hopf and Rinow [2].

(3) Cf. Misner and Wheeler [4].

charges and particles may be associated with *global* properties of space-time, a viewpoint which marks a radical departure from the predominantly local thinking which has been characteristic in that subject. At this point we wish only to suggest that the global geometric questions which arise in this context tend to fall beyond the scope of existing mathematical methods, and that their effective study must await the development of Lorentz geometry in the large.

A small contribution to this end is made in the present paper, which investigates the simplest case: Lorentz structures on the plane. After noting the most basic global properties of these spaces, we begin in §3 by defining a "normal" space, and we show that normal spaces are geodesically connected. The question arises then what can prevent a space from being normal, and the answer is found in certain structures which will be referred to as "barriers." After introducing a notion of completeness in §4, we establish a connection between barriers and curvature for a complete space. Some examples are considered in the last section.

2. **Some basic properties.** A second order covariant tensor field on a differentiable manifold is said to give a *Riemannian* or *Lorentz structure*, depending on whether the signature of the tensor field is zero or one. Each of these structures defines an inner product on the tangent vector spaces, and there exists a uniquely determined affine connection without torsion which preserves this inner product. Curvature, geodesics and so forth are now well defined. A Lorentz structure also defines a classification of nonzero tangent vectors into three types: *time-like*, *null* and *space-like*, depending on whether the "length" of the vector is real, zero or imaginary. This leads to the notion of a *homogeneous curve* whose tangent vectors are all of one type. Every geodesic is clearly a homogeneous curve, since its tangent vectors all have equal length. The last basic notion required—which also has no analogue in Riemannian geometry—is "Lorentz orientation": Two time-like vectors belonging to the same tangent vector space may be called equivalent if one can be transformed into the other by an element of the proper Lorentz group. This gives two equivalence classes. A *Lorentz orientation* is a function which, to every point of the manifold, assigns an equivalence class in a "continuous" manner. Continuity may be understood to mean that the preferred equivalence classes are preserved by parallel transport.

We henceforth specialize the discussion to the case where the underlying manifold of the Lorentz structure is homeomorphic to the real number plane R^2. We also agree that all manifolds considered shall be of differentiability class C^∞, and that the word "curve" shall refer to a C^∞-curve. The Lorentz structures considered will likewise have differentiability class C^∞. It should be noted that every point now lies on exactly two null-curves, which are also geodesics. Some basic properties of these spaces may be summarized in the following propositions, frequently needed in the sequel.

THEOREM I. *Let V be a plane with a Lorentz structure. Then*

(i) *V is Lorentz orientable*;

(ii) *homogeneous curves do not have multiple points*;

(iii) *a time-like curve can intersect a space-like curve in one point at most*;

(iv) *a homogeneous curve can intersect*(4) *a null-geodesic in one point at most*;

(v) *if $\mathfrak{A}_1$ intersects $\mathfrak{A}_2$ and $\mathfrak{A}_2$ intersects $\mathfrak{A}_3$, the $\mathfrak{A}_i$ being null-geodesics, then $\mathfrak{A}_1$ does not intersect $\mathfrak{A}_3$*;

(vi) *geodesic rays end at infinity, i.e. if $\mathfrak{A}$ is a geodesic ray emanating from a point O and S is a compact set containing O, then all points of $\mathfrak{A}$ sufficiently far from O belong to the complement of S. (In other words, $\mathfrak{A}^{-1}(S)$ is compact.)*

To prove (i), one needs only to show that the parallel translate of a time-like vector with respect to a closed curve belongs to the same equivalence class. But this is clear, since the holonomy transformation for a null-homotopic path belongs to the connected component of the identity—in this case, it must be an element of the proper Lorentz group.

Propositions (ii) to (v) can be established by a simple homotopy argument, based on the following idea. Let η be a continuous field of time-like vectors on V. To construct such a field, one may first put a Lorentz orientation and a Riemannian structure on V, and then define η at P to be the time-like vector of unit length (in Lorentz or Riemannian sense) which belongs to the preferred equivalence class and bisects the (Riemannian) angle subtended by the null-directions at P. Let $\mathfrak{C}$ be a simple closed path in V having a continuous tangent ξ except at isolated points, where the tangent may undergo a jump. We now put a second Riemannian structure on V, which need not coincide with the first. Let θ denote the angle measured from η to ξ at regular points of $\mathfrak{C}$, determined by the second Riemannian structure (modulo integral multiples of 2π). We assert now that

$$\oint_{\mathfrak{C}} d\theta = \pm 2\pi, \tag{2.1}$$

the left side being suitably interpreted as a Stieltjes integral. We shall refer to it as the *rotation* with respect to $\mathfrak{C}$. It will involve contributions due to regular points as well as contributions $\Delta\theta$ due to singular points of $\mathfrak{C}$, where the former are uniquely determined as ordinary integrals. To determine the quantities $\Delta\theta$, we impose the condition $-\pi < \Delta\theta < \pi$. This defines the Stieltjes integral completely, and it is clear that its value must be an integral multiple of 2π. We therefore recognize that the rotation is a homotopy invariant and can be evaluated by considering a very small simple closed path. For such a

(4) When we speak of two curves as *intersecting*, it should be understood that their tangents at the point of intersection are distinct.

path, however, the integral measures only the turning of the tangent, which gives $\pm 2\pi$.

To establish propositions (ii) to (v), one can show that the contrary assumption gives a closed path for which the rotation is strictly less than 2π in absolute value. To avoid excessive detail, we shall be content to illustrate the method by considering one or two representative cases.

Let us consider proposition (iii), for instance. We suppose that $\mathfrak{C}_t$ and $\mathfrak{C}_s$ are time-like and space-like curves, respectively, having common end points P_1 and P_2. The first Riemannian structure, which is used to define η, can be chosen so that

$$\eta \text{ is parallel to } \xi_t \text{ at } P_1 \text{ and } P_2. \tag{2.2}$$

The second Riemannian structure, which is used to define θ, can be chosen so that

$$\eta \text{ is perpendicular to } \xi_s \text{ at } P_1 \text{ and } P_2. \tag{2.3}$$

Letting $\mathfrak{C}$ denote the closed path $\mathfrak{C}_t - \mathfrak{C}_s$, one has

$$\oint_{\mathfrak{C}} d\theta = \int_{\mathfrak{C}_t} d\theta - \int_{\mathfrak{C}_s} d\theta + \Delta\theta_1 + \Delta\theta_2 \tag{2.4}$$

for the rotation over $\mathfrak{C}$, where $\Delta\theta_i$ denotes the contribution at P_i. To estimate the first integral on the right side of Equation (2.4), we observe that the maximum variation of θ along $\mathfrak{C}_t$ must be strictly less than π in absolute value. This implies by condition (2.2) that the integral over $\mathfrak{C}_t$ must vanish. Similarly, using condition (2.3), one finds that the integral over $\mathfrak{C}_s$ must likewise vanish. For the contributions $\Delta\theta_i$ one obtains $\pm\pi/2$, in virtue of conditions (2.2) and (2.3). It follows now that the rotation cannot exceed π in absolute value, which contradicts Equation (2.1).

As a second and final illustration, we consider proposition (v). Let it be supposed that $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ intersect so as to constitute a geodesic triangle with vertices P_1, P_2 and P_3. We can choose the second Riemannian structure, which determines θ, so that

(2.5) adjacent sides subtend right angles at P_1, P_2 and P_3.

Choosing the first Riemannian structure to coincide with the second, it follows that

(2.6) the angles measured from η to ξ_i at the end points of $\mathfrak{A}_i$ have the form $(\pi/2)(n+1/2)$, n being an integer and $i=1, 2, 3$.

For the rotation with respect to the boundary $\mathfrak{C}$ of our geodesic triangle one has again

$$\oint_{\mathfrak{C}} d\theta = \sum_{i=1}^{3} \int_{\mathfrak{A}_i} d\theta + \Delta\theta_i. \tag{2.7}$$

To estimate the contribution from $\mathfrak{A}_i$, we note that the maximum variation of θ along a null-geodesic is strictly less than $\pi/2$ in absolute value, which implies by condition (2.6) that the integral over $\mathfrak{A}_i$ must vanish. For the contributions $\Delta\theta_i$ one obtains $\pm\pi/2$ by condition (2.5). The rotation (2.7) therefore cannot exceed $3\pi/2$ in absolute value.

It remains now to prove (vi). We must show that there cannot exist a divergent sequence on $\mathfrak{A}$ which lies entirely in S. Suppose $\{P_n\}$ is such a sequence. It must have a limit point P in S. Let S^* denote a closed set containing P which is bounded by four null-geodesic segments. We can take S^* so small that every geodesic ray emanating from an interior point of S^* must intersect the boundary of S^*. Since $\{P_n\}$ diverges on $\mathfrak{A}$ and has an infinite number of points in S^*, $\mathfrak{A}$ must intersect the boundary of S^* an infinite number of times. On the other hand, it follows by proposition (iv) that $\mathfrak{A}$ can intersect the boundary of S^* four times at most. This gives the desired contradiction.

3. **Normal spaces and barriers**[5]. The space V is said to be *normal* if there exists a diffeomorphism ϕ of V onto the real number plane R^2 which takes every null-geodesic into an axis-parallel line.

THEOREM II. *A normal space is geodesically connected, i.e. every point pair is connected by a geodesic.*

The proof is simple, and proceeds as follows: If V is normal, we may identify V with R^2 under ϕ. Let O and P be distinct points of V. We can assume without loss of generality that O is the origin and P belongs to the first quadrant. For if P would lie on one of the two coordinate axes, P would be connected to O by a null-geodesic. To fix our ideas, we may further suppose that tangent vectors at O pointing into the first quadrant are time-like. Let X denote the open line segment through P having slope -1 and bounded by the coordinate axes. Clearly X is space-like. Let the time-like geodesic rays emanating from O into the first quadrant be parametrized in some way, the parameter θ ranging over some open interval I. A mapping ψ: $I\to X$ is defined as follows: The closed triangular region bounded by X and the coordinate axes being compact, it follows by proposition (vi) that every geodesic entering at O must intersect the boundary of this triangle again at some point R. By proposition (ii), R cannot be O; by proposition (iv), it cannot lie on either coordinate axis; hence R must lie on X. It follows now by proposition (iii) that every geodesic entering the triangle at O must intersect X exactly once. This defines our mapping ψ. One concludes from standard theorems concerning ordinary differential equations that ψ is continuous, and that $\psi(\theta)$ approaches opposite end points of X as θ approaches opposite end points of I. Hence ψ is onto, and there exists a geodesic connecting O and P.

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(5) The results of this section are closely related to a paper by L. Markus [3]. We are indebted to the referee for pointing this out to us.

In the sequel we shall need to speak of null-geodesics which belong to a "one-sided neighborhood" of a given null-geodesic $\mathfrak{A}$. It is rather evident what this should mean. To define it precisely, let P be a point of $\mathfrak{A}$ and let $\mathfrak{B}$ denote the null-geodesic intersecting $\mathfrak{A}$ at P. Let $\mathfrak{B}'$ denote an open connected subset of $\mathfrak{B}$ having P as one of its end points. The totality of null-geodesics intersecting $\mathfrak{B}'$ is said to constitute a *one-sided neighborhood* of $\mathfrak{A}$. A null-geodesic $\mathfrak{A}$ will be called a *barrier* if there exists a null-geodesic disjoint from $\mathfrak{A}$, which intersects every null-geodesic belonging to a one-sided neighborhood of $\mathfrak{A}$.

THEOREM III. *The space is normal if and only if it has no barriers.*

To prove the second part of this theorem, one needs only to observe that if the space is normal, the null-geodesics may be regarded as axis-parallel lines of R^2, and it is then evident that there are no barriers.

Conversely, we suppose now that V is not normal. Let $\mathfrak{A}$ and $\mathfrak{B}$ be a pair of intersecting null-geodesics, and let there be given parametrizations of $\mathfrak{A}$ and $\mathfrak{B}$, i.e. homeomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ onto the real line R. Let S denote the set of all points P of V such that each of the two null-geodesics through P intersects $\mathfrak{A}$ or $\mathfrak{B}$. We now define a mapping $\phi: S \to R^2$ as follows: Given P in S, let $\mathfrak{C}$ and $\mathfrak{D}$ denote the null-geodesics through P. By definition of S, $\mathfrak{C}$ must intersect either $\mathfrak{A}$ or $\mathfrak{B}$, and likewise for $\mathfrak{D}$. If $\mathfrak{C}$ intersects $\mathfrak{A}$, one can conclude from propositions (iv) and (v) that it intersects $\mathfrak{A}$ exactly once and fails to intersect $\mathfrak{B}$. Moreover, $\mathfrak{D}$ must then intersect $\mathfrak{B}$ exactly once, and must fail to intersect $\mathfrak{A}$. These intersections determine therefore a unique pair of parameter values (x, y), and we may set $\phi(P) = (x, y)$.

One can conclude at once that ϕ is one-to-one, and that both ϕ and ϕ^{-1} are C^∞. The former follows by proposition (iv), the latter by standard theorems which assert that solutions of ordinary differential equations (with smooth coefficients) depend differentiably on the initial conditions.

Since V is not normal, one of the following cases must hold.

CASE A. *S is a proper subset of V.* Clearly S is open. If P belongs to the boundary of S, then P is not in S and at least one of the null-geodesics through P—call it $\mathfrak{C}$—intersects neither $\mathfrak{A}$ nor $\mathfrak{B}$. Let P_0 be a point of S sufficiently near to P so that one of the null-geodesics through P_0—call it $\mathfrak{D}$—intersects $\mathfrak{C}$ at a point P^*. Since P_0 belongs to S, $\mathfrak{D}$ must intersect one of the null-geodesics $\mathfrak{A}$ or $\mathfrak{B}$—say it intersects $\mathfrak{A}$. We wish now to prove that every null-geodesic intersecting $\mathfrak{D}$ between P_0 and P^* must intersect $\mathfrak{B}$. This would imply that $\mathfrak{C}$ is a barrier.

We suppose, to the contrary, that the null-geodesic intersecting $\mathfrak{D}$ at a point P_1 between P_0 and P^* fails to intersect $\mathfrak{B}$. Let P_2 be an arbitrary point of $\mathfrak{D}$ such that P_1 lies between P_0 and P_2, and let $\mathfrak{C}_0$, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ denote the null-geodesics intersecting $\mathfrak{D}$ at P_0, P_1 and P_2, respectively. $\mathfrak{C}_0$ intersects $\mathfrak{B}$ at some point R_0, since P_0 belongs to S, and $\mathfrak{C}_1$ fails to intersect $\mathfrak{B}$. We assert that $\mathfrak{C}_2$

must likewise fail to intersect $\mathfrak{B}$. For if $\mathfrak{C}_2$ would intersect $\mathfrak{B}$ at some point R_2, then $P_0P_2R_2R_0$ constitutes a geodesic parallelogram which, in virtue of propositions (iv) and (v), is in fact a closed Jordan curve. It would follow by the Jordan curve theorem and proposition (vi) that $\mathfrak{C}_1$ must intersect the parallelogram at a second point R_1. By propositions (ii) and (iv), R_1 cannot lie on $\mathfrak{D}$; by proposition (v), it cannot lie on $\mathfrak{C}_0$ or $\mathfrak{C}_2$; hence it must lie on $\mathfrak{B}$, which contradicts our assumption.

We see, therefore, that if $\mathfrak{C}_1$ fails to intersect $\mathfrak{B}$, all null-geodesics belonging to a neighborhood of $\mathfrak{C}$ likewise fail to intersect $\mathfrak{B}$. Since these null-geodesics all intersect $\mathfrak{D}$ which intersects $\mathfrak{A}$, we can conclude by proposition (v) that they all fail to intersect $\mathfrak{A}$ as well. But this would imply that P has a neighborhood belonging to the complement of S, which is impossible (because P lies on the boundary of S).

CASE B. *$S=V$ and $\phi(S)$ is a proper subset of R^2.* There exist then points A on $\mathfrak{A}$ and B on $\mathfrak{B}$ such that the null-geodesic $\mathfrak{C}$ intersecting $\mathfrak{A}$ at A fails to intersect the null-geodesic intersecting $\mathfrak{B}$ at B. Since $S=V$, every null-geodesic intersecting $\mathfrak{C}$ must intersect $\mathfrak{B}$. This determines a mapping of $\mathfrak{C}$ into $\mathfrak{B}$ which we recognize to be a homeomorphism of $\mathfrak{C}$ onto an open connected subset $\mathfrak{B}^*$ of $\mathfrak{B}$. Since B does not belong to this subset, $\mathfrak{B}^*$ has an end point B^* in $\mathfrak{B}$. The null-geodesic intersecting $\mathfrak{B}$ at B^* is therefore a barrier.

4. **Barriers, completeness and curvature.** A barrier was defined to be a null-geodesic $\mathfrak{B}$, such that every null-geodesic $\mathfrak{C}$ belonging to a one-sided neighborhood of $\mathfrak{B}$ intersects a certain null-geodesic $\mathfrak{B}'$ which is disjoint from $\mathfrak{B}$. If P is a point of $\mathfrak{B}$ and Q is sufficiently near to P, the null-geodesic intersecting $\mathfrak{C}$ at Q will likewise intersect $\mathfrak{B}$. When this happens, Q will have a neighborhood on $\mathfrak{C}$ such that every null-geodesic intersecting $\mathfrak{C}$ at points of the neighborhood will also intersect $\mathfrak{B}$. As Q is moved along $\mathfrak{C}$ towards the intersection of $\mathfrak{C}$ with $\mathfrak{B}'$, there must consequently exist a first position Q^* such that the null-geodesic $\mathfrak{B}^*$ intersecting $\mathfrak{C}$ at Q^* fails to intersect $\mathfrak{B}$. $\mathfrak{B}^*$ is again a barrier, and one may say that $\mathfrak{B}^*$ is *associated* with $\mathfrak{B}$. This implies that $\mathfrak{B}$ is associated with $\mathfrak{B}^*$. On the other hand, $\mathfrak{B}$ may be associated with several other barriers. It is advantageous at this point to introduce the notion of a *directed barrier* as an ordered pair $(\mathfrak{B}, \mathfrak{B}^*)$, where $\mathfrak{B}$ and $\mathfrak{B}^*$ are associated barriers. Directed barriers thus occur in conjugate pairs, $(\mathfrak{B}^*, \mathfrak{B})$ being called *conjugate* to $(\mathfrak{B}, \mathfrak{B}^*)$.

For a directed barrier $(\mathfrak{B}, \mathfrak{B}^*)$ one can define preferred directions on $\mathfrak{B}$ and $\mathfrak{B}^*$ as follows: Given $\mathfrak{C}$, Q and Q^* as above, the ordered pair (Q, Q^*) determines a direction on $\mathfrak{C}$ which in turn determines a direction on $\mathfrak{B}$. A direction on $\mathfrak{B}^*$ is defined by the same construction for the conjugate directed barrier $(\mathfrak{B}^*, \mathfrak{B})$. We observe now that a Lorentz orientation of V permits one to distinguish between positive and negative directions on the null-geodesics. A directed barrier will therefore belong to one of four possible orientation classes: $(+, +)$, $(-, -)$, $(+, -)$, $(-, +)$; depending on whether the preferred directions on $\mathfrak{B}$ and $\mathfrak{B}^*$ are positive or negative. Since a reversal of the Lorentz orientation just interchanges the first two and the last two orientation classes, one can associate with every directed barrier an algebraic sign in the following way: Given a Lorentz orientation of V, we define the sign of $(\mathfrak{B}, \mathfrak{B}^*)$ to be *positive* or *negative*, depending on whether the preferred directions on $\mathfrak{B}$ and $\mathfrak{B}^*$ have equal or opposite signs. We observe that conjugate barriers carry the same sign.

The question arises now whether there exists a connection between barriers and the curvature properties of V. One may suspect that there could be a relation between the algebraic signs of the Gaussian curvature and of the directed barriers on V. Clearly no such connections can exist unless one introduces an appropriate notion of completeness and restricts one's attention to complete spaces. For if V is an arbitrary space—say, a flat space—one can cut out a disc from V and obtain thus a new space having many barriers. In fact, every null-geodesic of the new space could be a barrier.—It is not clear what "completeness" should mean in Lorentz geometry. In Riemannian geometry this notion may be expressed by several equivalent statements(6), one of which asserts that every geodesic can be extended to infinite arc length. Clearly this has no exact analogue for Lorentz spaces; along every null-geodesic, for instance, the arc length remains zero. The number of possible definitions is limited by the natural requirement that every proper subset of a complete space should fail to be complete. Apart from this, one seeks a definition that will lead to some good theorems. We now propose one for which the first requirement, at least, is satisfied.

By a canonical parametrization of a geodesic we mean one which causes the tangent vectors at distinct points to be parallel with respect to the geodesic. Two canonical parametrizations can differ only by an affine transformation. We may define completeness to mean that every null-geodesic shall be extendible to infinite values of the canonical parameter for an arbitrary canonical parametrization. This suffices to insure that every proper subset of a complete space is not complete. It does not insure, however, that every time-like or space-like geodesic shall be extendible to infinite arc length. One could adopt a stronger condition of completeness, which demands that all geodesics be infinitely extendible with respect to their canonical parametrizations. This condition, which asserts that the affine connection determined by the Lorentz structure is complete in the standard sense, is however not easily verified in given cases, and also one is not certain whether there would exist many complete spaces. We therefore adopt the first definition given above, which seems natural in the present context. It enables us to prove the following result:

THEOREM IV. *A complete space with Gaussian curvature non-negative* (*non-*

(6) Cf. Hopf and Rinow [2].

positive) outside a compact set does not admit negative (positive) directed barriers. If V is complete and the curvature integral over V converges absolutely, then V has no barriers. V is then normal and geodesically connected.

We suppose that $(\mathfrak{A}, \mathfrak{A}^*)$ is a directed barrier. To prove the theorem, it suffices to show that, for every positive number M, there exists a region $\mathfrak{R}$ of V such that the curvature integral over $\mathfrak{R}$ is greater than M or smaller than $-M$, depending on whether the sign of $(\mathfrak{A}, \mathfrak{A}^*)$ is positive or negative.

We begin by defining a suitable coordinate system in a neighborhood of $\mathfrak{A}$. Let P be a point of $\mathfrak{A}$, let $\mathfrak{B}$ denote a null-geodesic intersecting $\mathfrak{A}$ at P, and let S denote the open set defined previously in the proof of Theorem III. Again we choose parametrizations of $\mathfrak{A}$ and $\mathfrak{B}$, this time with more care. Let x be a canonical parameter on $\mathfrak{A}$ tending to plus infinity in the preferred direction(7), and let y be a differentiable parameter on $\mathfrak{B}$. This determines a homeomorphism ϕ of S onto a subset of R^2, which in fact constitutes an isothermic coordinate system on S.

Let $\mathfrak{C}$ be a neighboring null-geodesic to $\mathfrak{A}$ which intersects $\mathfrak{B}$ at Q and $\mathfrak{A}^*$ at Q^*. From the definition of $\mathfrak{A}^*$ we may conclude that every null-geodesic intersecting $\mathfrak{C}$ on the open segment QQ^* intersects $\mathfrak{A}$. This open segment is therefore covered by our coordinate system. We assume now, without loss of generality, that P has coordinates $(0, 0)$ and Q coordinates $(0, \pm 1)$, depending on whether the sign of $(\mathfrak{A}, \mathfrak{A}^*)$ is positive or negative. This means that time-like vectors belong to the first and third quadrants, as is easily verified.

Let $\{e_1, e_2\}$ denote the natural reference frame(8) associated with the given coordinate system, and let ω^i, ω_j^i denote the corresponding 1-forms of the affine connection. This leads to the following formulas, which are easily verified:

$$(4.1)\qquad \begin{aligned} &e_1\cdot e_1 = 0 \qquad e_2\cdot e_2 = 0 \qquad e_1\cdot e_2 = g^{(9)}\\ &\overset{1}{\omega} = dx \qquad \overset{2}{\omega} = dy\\ &\overset{1}{\omega}_2 = \overset{2}{\omega}_1 = 0\\ &\overset{1}{\omega}_1 = \frac{\partial}{\partial x}\log g\, dx \qquad \overset{2}{\omega}_2 = \frac{\partial}{\partial y}\log g\, dy. \end{aligned}$$

Let s denote a canonical parameter on a null-geodesic with y=constant, and let differentiation with respect to s be denoted by a dot. The tangent vectors $\dot{x}e_1$ are parallel with respect to the geodesic, which means that $d(\dot{x}e_1)$ must vanish. Using Equations (4.1), one thus obtains

$$\ddot{x} + \left(\frac{\partial}{\partial x}\log g\right)\dot{x}^2 = 0.$$

(7) This is the only place in the proof where completeness is invoked.
(8) We are following the method and notation of Elie Cartan [1].
(9) Our assumptions imply $g>0$.

This equation can be integrated to give

$$s = \alpha \int_0^x g(t, y)dt + \beta, \tag{4.2}$$

where α and β are constants. Since x is a canonical parameter on $\mathfrak{A}$, one can conclude from Equation (4.2) that

$$g(x, 0) = \text{constant}. \tag{4.3}$$

Next we assert that the parameter x tends to infinity on QQ^* as Q^* is approached. In other words, we claim that given $x>0$, the null-geodesic intersecting $\mathfrak{A}$ at $(x, 0)$ must also intersect $\mathfrak{C}$. This requires proof. We know that every null-geodesic intersecting $\mathfrak{C}$ on the open segment QQ^* also intersects $\mathfrak{A}$, and this determines a homeomorphism of QQ^* onto an open connected subset $\mathfrak{A}'$ of $\mathfrak{A}$. If our assertion is false, then $\mathfrak{A}'$ must have an upper end point P^* in $\mathfrak{A}$, and one concludes from the known behavior of integral curves that the null-geodesic $\mathfrak{B}^*$ intersecting $\mathfrak{A}$ at P^* must also intersect $\mathfrak{C}$ at Q^*. But the two null-geodesics through Q^* are $\mathfrak{A}^*$ and $\mathfrak{C}$. $\mathfrak{B}^*$ cannot be $\mathfrak{A}^*$, because $\mathfrak{A}$ and $\mathfrak{A}^*$ are disjoint; and it cannot be $\mathfrak{C}$, because then $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ would constitute a triangle, contradicting proposition (v). This proves our assertion. We can now conclude from Equation (4.2) that

$$\int_0^\infty g(x, \pm 1)dx < \infty,$$

and we observe that given $\epsilon>0$, there must exist a positive number $x(\epsilon)$ such that

$$\left| g(x(\epsilon), \pm 1) \right| < \epsilon. \tag{4.4}$$

Let us now evaluate the curvature integral over the region $\mathfrak{R} = [0, x(\epsilon)] \times [0, \pm 1]$. From the second structure equation and Equation (4.1) one can conclude that the curvature 2-form Ω_1^1 is just the exterior derivative of ω_1^1, and therefore

$$\int \Omega_1^1 = \int d\omega_1^1 = \int \omega_1^1,$$

the last integral being extended over the boundary of $\mathfrak{R}$. It is easily evaluated, and one obtains

$$\begin{aligned} \int \Omega_1^1 = \mp \{&\log g(x(\epsilon), \pm 1) - \log g(0, \pm 1) \\ &+ \log g(0, 0) - \log g(x(\epsilon), 0)\}. \end{aligned} \tag{4.5}$$

Equations (4.3), (4.4) and (4.5) together imply that the curvature integral can assume arbitrarily large positive (negative) values for a suitable region $\mathfrak{R}$ when V admits a positive (negative) directed barrier, as was to be proved.

5. **Examples.** To construct an example of a geodesically disconnected space, one naturally looks for a Lorentz structure with a considerable "twist." In terms of given coordinates (x, y), one is prompted to write an expression such as

$$(5.1)\qquad \begin{pmatrix} -\cos 2x & -\sin 2x \\ -\sin 2x & \cos 2x \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

for the metric tensor field. We show now that this does give a geodesically disconnected space.

One can put a flat Riemannian structure on the (x, y)-plane with respect to which x and y are rectangular cartesian coordinates, and this enables one to define a time-like vector field η as in §2. The inclination of η with respect to the coordinate axes is thus a function of x, and we observe that η rotates as one moves in the x-direction. For $x=0$ and $x=\pi$, η is parallel to the y-axis but points in opposite directions. It is therefore not difficult to see that one can connect an arbitrary point on the line $x=0$ with an arbitrary point on the line $x=\pi$ by a time-like curve. This implies by propositions (iii) and (iv) that it is impossible to go from $x=0$ to $x=\pi$ by a space-like or a null-curve. A similar consideration shows that it is possible to go from $x=\pi/2$ to $x=3\pi/2$ by a time-like curve, proving our assertion.

Since the space is geodesically disconnected, there must exist barriers. These do show up quite clearly if one examines the null-geodesics, and one finds that they are just the lines

$$x = \frac{1}{2}\,\pi\left(n + \frac{1}{2}\right), \qquad n = 0, \pm 1, \pm 2, \cdots$$

Moreover, each barrier belongs to exactly two directed barriers of opposite sign, and it is therefore not surprising, in virtue of Theorem IV, that the Gaussian curvature changes sign across the barriers.

One can arrive at this example in a different way, which is more illuminating[10]. Let V^* denote the (z, w)-plane with the origin removed, and let a Lorentz structure be given on V^* by the equation

$$(5.2)\qquad ds^2 = g\, dzdw.$$

It is evident that V^* has exactly four barriers, which are the four semi-axes of the (z, w)-plane. Lifting the given metric onto the universal covering space V of V^*, one obtains a Lorentz structure on the plane. Our example (5.1) arises from the choice

$$(5.3)\qquad g = \frac{1}{z^2 + w^2}$$

[10] This was pointed out to the author by Dr. Y. H. Clifton.

for the conformal factor in Equation (5.2).

We observe that the null-geodesics on V^* and V are unaltered by a change in the conformal factor g, indicating that V is geodesically disconnected for every choice of g. Taking $g=1$ gives a flat connection on V^* which is complete at infinity but incomplete at the origin, as follows from Equation (4.2). The choice (5.3), on the other hand, makes V^* complete at the origin but incomplete at infinity. One can therefore take

$$g = 1 + \frac{1}{z^2 + w^2}$$

to obtain a complete space V^*. The universal covering plane V is now complete and geodesically disconnected.

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