ON THE COHOMOLOGY OF THE REAL GRASSMANN COMPLEXES AND THE CHARACTERISTIC CLASSES OF n-PLANE BUNDLES

BY

EMERY THOMAS

1. Introduction. This paper deals with two things: first the cohomology of the real Grassmann spaces; and secondly, relations between the various characteristic classes of n-plane bundles. Let $\xi^n$ be a real n-plane bundle over a paracompact space $B$. With $\xi^n$ we associate the Stiefel-Whitney characteristic class

$$W(\xi^n) = 1 + W_1(\xi^n) + \cdots + W_n(\xi^n),$$

where $W_i(\xi^n) \in H^i(B; Z_r)$ ($Z = \text{integers, } Z_r = \text{integers mod } r$). We refer to Milnor [4] for the properties of n-plane bundles and for the axioms which the Stiefel-Whitney classes satisfy.

Similarly, let $\omega^n$ be a complex n-plane bundle over a paracompact space $X$. With $\omega^n$ we associate the Chern characteristic class

$$c(\omega^n) = 1 + c_1(\omega^n) + \cdots + c_n(\omega^n),$$

where $c_i(\omega^n) \in H^{2i}(X; Z)$. The Chern characteristic classes are axiomatized by Hirzebruch in [3].

The remaining characteristic classes we consider are the Pontrjagin classes. Given a real n-plane bundle $\xi$, the Pontrjagin class $p(\xi)$ is defined by

$$(1.1) \quad p_i(\xi) = (-1)^i c_{2i}(\xi^c) \quad (i = 0, 1, \cdots, [n/2]),$$

where $\xi^c$ is the complexification of the bundle $\xi$ (see [4, XII] for details). Thus, $p_i(\xi) \in H^{4i}(B; Z)$, where $B$ is the base of $\xi$.

For the results which follow we will need the following relationship between the Pontrjagin and Chern classes. Suppose that $\omega^n$ is a complex n-plane bundle. Each fibre of $\omega^n$ is then an n-dimensional complex vector space. By restricting attention to the real numbers as scalars, we see that each fibre is also a 2n-dimensional real vector space. Thus, $\omega^n$ gives rise to a real 2n-plane bundle, $\omega^n_R$. We then have the following result relating the classes of these two bundles (see Wu [9] and [4, §XII]):

Received by the editors December 18, 1958 and, in revised form, August 20, 1959.

(1) This research has been supported partly by U. S. Air Force contract AF 49(638)-79, and partly by a postdoctoral National Science Foundation fellowship.
(1.2) Theorem.

\[\mathcal{P}_i(\omega_R^n) = (c_i(\omega^n))^2 + \sum_{j=0}^{i-1} (-1)^{i+j} 2c_j(\omega^n)c_{2i-j}(\omega^n).\]

**Grassmann complexes.** In order to obtain further results on the characteristic classes we will study the cohomology of the real Grassmann complexes, \(G_n (n = 1, 2, \cdots)\). These are defined as follows. Let \(R^\infty\) denote the countable direct sum of the real numbers: i.e., a point in \(R^\infty\) is a sequence \((r_1, r_2, \cdots, r_n, \cdots)\), \(r_i \in \mathbb{R}\), where all but a finite number of \(r_i\) are zero. Define \(G_n\) to be the set of all \(n\)-dimensional linear subspaces of \(R^\infty\). Milnor shows that \(G_n\) is in fact a CW-complex (see [4, V]), and exhibits the cell-decomposition.

With each complex \(G_n\) we associate a canonical \(n\)-plane bundle, \(\gamma^n\), by defining

\[(1.3) \quad \gamma^n = (E_n, \pi_n, G_n),\]

where

\[E_n = \{(V, e) \mid V \text{ is an } n\text{-dimensional subspace of } R^\infty, e \in V\}; \pi_n(V, e) = V.\]

Notice that \(G_1 = P^\infty\), the infinite dimensional real projective space. The importance of the spaces \(G_n\) is underlined by the following classification theorem (see [5, 19.3]):

(1.4) Theorem. Let \(X\) be a paracompact space. Then, the equivalence classes of real \(n\)-plane bundles over \(X\) are in 1-1 correspondence with the homotopy classes of maps \(X \to G_n\).

The cohomology of the space \(G_n\) is therefore important, since it is mapped into the cohomology of the base space of each real \(n\)-plane bundle. We state here the facts which are known about this cohomology.

(1.5) Theorem. Let \(n\) be a positive integer and set

\[W_i = W_i(\gamma^n) \quad (i = 0, 1, \cdots, n).\]

Then,

\[H^*(G_n; Z_2) = Z_2[W_1, W_2, \cdots, W_n].\]

For a proof see Chern [2] and Milnor [4].

(1.6) Theorem. Let \(p\) be an odd prime and set

\[\bar{p}_i = p_i(\gamma^n) \mod p \quad (i = 0, 1, \cdots, [n/2]).\]

Then,

\[H^*(G_n; Z_p) = Z_p[\bar{p}_1, \cdots, \bar{p}_q] \quad (q = [n/2]).\]
This follows from the work of Borel (see [1]). As a consequence of Theorem (1.6) we have:

(1.7) Theorem. Let $R_0$ be the rational numbers and denote by $\rho_0$ the cohomology homomorphism induced by the inclusion $Z \to R_0$. Set

$$\rho_i^0 = \rho_0(\phi_i(\gamma^n)) \quad (i = 0, 1, \ldots, \lceil n/2 \rceil).$$

Then,

$$H^*(G_n; R_0) = R_0[\rho_0, \ldots, \rho_q^0] \quad (q = \lceil n/2 \rceil).$$

2. Statement of results. We discuss the cohomology of the complexes $G_n$ from two points of view: first, we give results which partially characterize the algebraic structure of the cohomology groups; and secondly, we give mappings which isomorphically embed the cohomology of $G_n$ into the cohomology of simpler spaces.

We begin by describing certain cohomology homomorphisms. Define

$$\theta_i: Z_2 \to Z_2^i \quad \rho_j: Z \to Z_j \quad (i = 1, 2, \ldots; j = 2, 3, \ldots)$$

by

$$\theta_i(1 \mod 2) = 2^{i-1} \mod 2^i; \quad \rho_j(1) = 1 \mod j.$$

Let $\delta_*$ be the Bockstein coboundary associated with the exact sequence

$$0 \to Z \to Z \xrightarrow{\rho_2^0} Z_2 \to 0,$$

and set

$$\beta_k = \rho_k \delta_* \quad (k = 2, 3, \ldots).$$

Recall that the coboundary $\beta_2$ is a derivation—i.e., $\beta_2(uv) = \beta_2(u)v + u\beta_2(v)$ for any mod 2 cohomology classes $u$ and $v$.

Now let $n$ be a fixed positive integer and consider the cohomology ring $H^*(G_n; Z_2)$. Since $\beta_2$ is a derivation, the kernel of $\beta_2$ is a subring of $H^*(G_n; Z_2)$. In §3 we show that

(2.3) Proposition. Kernel $\beta_2 = P_2 \oplus T_2$ (vector space direct sum) where

$$P_2 = Z_2[W_2^2, W_4^2, \ldots, W_{2q}^2] \quad \text{and} \quad T_2 = \text{Image } \beta_2.$$

Here $W_i = W_i(\gamma^n) \quad (i = 1, 2, \ldots, n)$ and $q = \lceil n/2 \rceil$. Proposition 2.3 leads to a description of the integral cohomology ring of $G_n$.

(1) For an independent proof, see [4, XV].

(2) If $\theta$ is any coefficient group homomorphism, we will denote by the same symbol the cohomology group homomorphism induced by $\theta$.
Theorem A. $H^\ast(G_n; Z) = P \oplus T$ (group direct sum) where $P = \mathbb{Z}[\bar{p}_1, \ldots, \bar{p}_q]$ ($\bar{p}_i = p_i(\gamma^n)$), $T$ = ideal of torsion elements.

We next describe the cohomology of $G_n$ with coefficients mod $2^i$, $i = 1, 2, \ldots$. We may regard $H^\ast(G_n; Z)$ simply as a vector space over the field $Z_2$. Since $\beta_2$ is a linear vector homomorphism, the kernel of $\beta_2$ is a linear subspace of $H^\ast(G_n; Z_2)$. We obtain a direct sum (vector) splitting of $H^\ast(G_n; Z_2)$ by choosing a complement to Kernel $\beta_2$. That is, we write

$$H^\ast(G_n; Z_2) = \text{Kernel } \beta_2 \oplus S_2 \text{ (vector direct sum).}$$

Let $P_2$ be the summand defined in Proposition (2.3), and let $Q_2$ be the summand of $H^\ast(G_n; Z_2)$ spanned by all the monomials (in $W_1, \ldots, W_n$) not belonging to $P_2$. Clearly, $\beta_2 Q_2 \subseteq Q_2$; and hence, Image $\beta_2 \subseteq Q_2$. Thus, we may choose the summand $S_2$ so that Image $\beta_2 \oplus S_2 \subseteq Q_2$. We assume that such a choice has been made.

For each positive integer $i$ define

$$S_{2^i} = \theta_i(S_2), \quad T_{2^i} = \rho_{2^i}(T),$$

where $T$ is given in Theorem A. $S_{2^i}$ and $T_{2^i}$ are each subgroups of $H^\ast(G_n; Z_{2^i})$, and we in fact have

Theorem B. $H^\ast(G_n; Z_{2^i}) = P_{2^i} \oplus T_{2^i} \oplus S_{2^i}$ ($i = 1, 2, \ldots$) where

(a) $P_{2^i} = \rho_{2^i}(P) = \mathbb{Z}_i[\bar{p}_1, \ldots, \bar{p}_q]$;
(b) $\rho_{2^i}$ maps $T$ isomorphically onto $T_{2^i}$;
(c) $\theta_i$ maps $S_2$ isomorphically onto $S_{2^i}$.

Here $\bar{p}_j = \rho_{2^i}(p_j(\gamma^n))$ ($j = 1, \ldots, q$), and the splitting is simply a module splitting over the ring $Z_{2^i}$. These results do not completely characterize the cohomology of $G_n$, since the algebraic structure of the rings $T$, $T_{2^i}$ is not given, nor is any information about $S_2$, $S_{2^i}$ given. In a subsequent paper Theorems A and B will be used to complete the characterization of $H^\ast(G_n; Z)$. For the purpose of this paper, however, this lack of information is compensated for in the following way.

Let $P^\infty$ and $P^\infty(C)$ be respectively the real and complex infinite dimensional projective spaces. Let $n$ be a fixed, positive integer, and set $q = \lceil n/2 \rceil$. In §4 we define a real $n$-plane bundle $\xi^n$ over $(P^\infty)^n$ ($n$-fold cartesian product) and a real $n$-plane bundle $\eta^n$ over $(P^\infty(C))^q$ ($q$-fold cartesian product). Let

$$(2.4) \quad f_n: (P^\infty)^n \to G_n, \quad g_n: (P^\infty(C))^q \to G_n$$

be mappings which induce the respective bundles $\xi^n$, $\eta^n$ from the universal bundle $\gamma^n$ over $G_n$ (see 1.4). We then have the following results: consider the cohomology homomorphisms $f_n^\ast$, $g_n^\ast$ on the group $H^\ast(G_n; Z)$ (see Theorem A).
Theorem A'. (a) $f_n^*$ restricted to $T$ is a monomorphism, mapping $T$ into $H^*((P^n)_n; Z)$;
(b) $g_n^*$ restricted to $P$ is a monomorphism, mapping $P$ into $H^*((P^n(C)_n; Z)$.

Next, consider the homomorphisms $f_n^*$, $g_n^*$ on the group $H^*(G_n; Z_2)$, for $i = 1, 2, \ldots$; (see Theorem B).

Theorem B'. (a) $f_n^*$ restricted to $T^i_2 \otimes S^i_2$ is a monomorphism, mapping $T^i_2 \otimes S^i_2$ into $H^*((P^n)_n; Z_2)$;
(b) $g_n^*$ restricted to $P^i_2$ is a monomorphism, mapping $P^i_2$ into $H^*((P^n(C)_n; Z_2)$.

The original motivation for this research was to understand the result of Wu (see [8] and Theorem C below). This led to the above results, and in §10 we make use of Theorem B' to give a new proof of Wu's theorem. To state this, let $\beta_2$ be the Pontrjagin square cohomology operation (see [6]), and $\beta_q^i$ the Steenrod square cohomology operation ($i = 0, 1, \ldots$). Let $\theta_4$ and $\beta_4$ be the operations defined respectively in (2.1) and (2.2). We then have:

Theorem C (Wu). Let $\xi$ be a real $n$-plane bundle over a space $X$. Set $W_i = W_i(\xi)$ ($i = 1, 2, \ldots, n$), and $\beta_j = p_j(\xi)$ ($j = 1, 2, \ldots, \lfloor n/2 \rfloor$). Then,

(a) $\beta_2(W_{2i+1}) = \beta_4 S^i_2 W_{2i+1} + \theta_2(W_1 S^i_2 W_{2i+1})$;
(b) $\beta_2(W_{2i}) = \rho_2 \beta_i + \theta_2 \left[ W_1 S^i_2 W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j} \right]$.

The paper is organized as follows: in §3 we give the proof of Proposition 2.3, while in §§4, 5 we establish some facts about the cohomology of the spaces $(P^n)_n$ and $(P^n(C)_n$. In §6 we discuss the cohomology of a certain type of space of which the complexes $G_n$ are a special case. Finally, the proof of Theorem A is given in §7, the proof of Theorem B in §8, the proofs of Theorems A' and B' in §9, and the proof of Theorem C in §10.

I should like to thank Professor S. S. Chern for putting me in touch with Wu's paper and for suggesting to me that a different proof of Wu's theorem might be found.

3. A direct sum splitting of $H^*(G_n; Z_2)$. The proof of Proposition 2.3 will follow at once from a theorem about polynomial algebras. Let $n$ be a positive integer, and denote by $A^{(n)}$ the polynomial algebra over $Z_2$ in indeterminates $w_1, \ldots, w_n$ with unit 1. Regarding $A^{(n)}$ as a vector space over $Z_2$, we

---

(4) By monomorphism we mean a homomorphism whose kernel is zero.
(5) This statement of the theorem differs from that given by Wu, as he uses a slightly different definition of the Pontrjagin classes. For an interesting application of the theorem see R. Bott and J. Mihlb, On the parallelisability of the spheres, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 87-89.
assume in addition that $A^{(n)}$ has a linear endomorphism $\beta$ with the following properties:

(3.1) $\beta(uv) = \beta(u)v + u\beta(v)$  \hspace{1cm} u, v \in A^{(n)}$

(3.2) $\beta \circ \beta = 0$;

(3.3) $\beta(w_i) = w_i^2$, $\beta(w_{2i}) = w_{2i+1}$ (1 \leq i \leq [(n - 1)/2]),

$\beta(w_n) = w_1w_n$ (n even).

By 3.1 we see that $\beta$ is a derivation. Thus, the kernel of $\beta$ is a subring of $A^{(n)}$; and Image $\beta$ is an ideal in Kernel $\beta$. Setting $K^{(n)} = \text{Kernel } \beta$ we then have:

(3.4) **Theorem.** $K^{(n)} = P^{(n)} \oplus T^{(n)}$ (vector space direct sum) where $P^{(n)} = \mathbb{Z}_2[w_2, w_4, \cdots, w_{2q}]$ and $T^{(n)} = \text{Image } \beta$ ($q = [n/2]$).

We first show how Theorem 3.4 implies Proposition 2.3. Let $\gamma^n$ be the $n$-plane bundle over $G_n$ defined in 1.3, and let $W_1, \cdots, W_n$ be the Stiefel-Whitney classes of $\gamma^n$. Let $\beta_2$ be the Bockstein coboundary defined in 2.2. Then (see [8]),

$$\beta_2(W_{2i}) = W_1W_{2i} + W_{2i+1}, \quad \beta_2(W_{2i+1}) = W_1W_{2i+1}.$$  

We make a change of basis for $H^*(G_n; \mathbb{Z}_2)$ as follows (see [1]). For $1 \leq i \leq [n/2]$, $1 \leq j \leq [(n-1)/2]$, set

(3.5) $W_1^* = W_1$, $W_{2i}^* = W_{2i}$, $W_{2j+1}^* = W_1W_{2j} + W_{2j+1}$.

Clearly, we continue to have:

$$H^*(G_n; \mathbb{Z}_2) = \mathbb{Z}_2[W_1^*, \cdots, W_n^*].$$

Furthermore,

$$\beta_2(W_1^*) = W_1^*, \quad \beta_2(W_{2i}^*) = W_{2i+1}^* \quad (1 \leq i \leq [(n - 1)/2]),$$

$$\beta_2(W_n^*) = W_1W_n^* \quad (n \text{ even}).$$

Thus, $H^*(G_n; \mathbb{Z}_2)$ with generators $W_1^*, \cdots, W_n^*$ and linear homomorphism $\beta_2$ is a polynomial algebra of the type $A^{(n)}$ defined in 3.1–3.3. Since $W_{2i} = W_{2i}^*$, Proposition 2.3 follows at once from Theorem 3.4.

The proof of Theorem 3.4 falls into 2 cases: $n$ even and $n$ odd. Suppose first that $n$ is an even integer, say $n = 2q$ ($q \geq 1$). The proof is then a consequence of Theorem 2 in [7]. For setting

$$w_{2i} = x_i, \quad w_{2i+1} = y_i, \quad (1 \leq i \leq q - 1),$$

$$w_1 = u, \quad w_2 = v,$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
we obtain the algebra $B_q$ with the derivation $\beta^*$, considered in §3 of [7].

We now prove Theorem 3.4 for the case $n = 2q - 1$ ($q \geq 1$). To simplify the notation set $A^{(2q)} = A$. We regard $A^{(2q-1)}$ as the subspace, $A'$, of $A$ spanned by all the monomials in $w_1, w_2, \cdots, w_{2q-1}$. Set $I =$ ideal of $A$ generated by $w_{2q}$. Clearly, $A = A' \oplus I$ (vector direct sum). Since $\beta w_{2q} = w_1 w_{2q}$, it is clear that $\beta(I) \subset I$, $\beta(A') \subset A'$. Let $\beta' = \beta$ restricted to $A'$. Then,

Kernel $\beta' = \text{Kernel } \beta \cap A'$, \hspace{1cm} Image $\beta' = \text{Image } \beta \cap A'$.

Since

$$P^{(2q)} \cap A' = Z_2[w_2, w_4, \cdots, w_{2q-2}] = P^{2(q-1)},$$

we have proved Theorem 3.4 for the case $n = 2q - 1$ and hence for all $n$.

4. The spaces $(P^\infty)^n$, $(P^\infty(C))^n$. In order to continue the study of the cohomology of the complexes $G_n$, we first study the cohomology of two simpler spaces, stating here some well-known results (e.g., see [4]).

We have noted earlier that when $n = 1$, then $G_1 = P^\infty$, the infinite real projective space, and that the canonical 1-plane bundle $\gamma^1$ over $P^\infty$ has Stiefel-Whitney class

$$W(\gamma^1) = 1 + a,$$

where $a$ generates $H^1(P^\infty; Z_2)$.

Let $n$ be a fixed positive integer. We will need the characteristic classes of a certain bundle $\xi^n$ over $(P^\infty)^n$; that is, the $n$-fold cartesian product of the space $P^\infty$. Let $\pi_i : (P^\infty)^n \rightarrow P^\infty$ be the $i$th projection map. Let $\gamma^1_i$ be the bundle over $(P^\infty)^n$ induced by $\pi_i$ from $\gamma^1$, and let $a_i = \pi^*_i(a)$ ($i = 1, 2, \cdots, n$). Define

$$\xi^n = \gamma^1_1 \oplus \cdots \oplus \gamma^1_n.$$

Then, (see [4, VI]),

$$W_i(\xi^n) = \sigma_i, \text{ the } i\text{th elementary symmetric function in the classes } a_1, \cdots, a_n.$$

In order to determine the Pontrjagin classes of $\xi^n$ we first determine the Chern classes of $\xi^n$, the complexification of $\xi^n$. By reasoning very similar to that used to prove (4.2), one shows that

$$c_i(\xi^n) = \sigma_i, \text{ the } i\text{th elementary symmetric function in the classes } b_1, \cdots, b_n, \text{ where } b_i = \delta^*_w(a_i).$$

Therefore, by Definition 1.1 we have:

$$p_i(\xi^n) = \tau_{2i},$$

since $-\tau_{2i} = \tau_{2i}$.

We now look at $P^\infty(C)$, the infinite complex projective space. In a way entirely analogous to 1.3 one defines a canonical complex 1-plane bundle
ω^1 over \( P^\infty(C) \). Also, \( c(\omega^1) = 1 + d \), where \( d \) is the canonical generator of \( H^2(P^\infty(C); \mathbb{Z}) \). As is well known, \( H^*(P^\infty(C); \mathbb{Z}) \) is a polynomial ring in the class \( d \).

Let \( n \) again be a fixed integer \( > 1 \); set \( n = 2q + \epsilon \), where \( \epsilon = 0 \) or \( 1 \). As above let \( \pi_j: (P^\infty(C))^q \to P^\infty(C) \) be the projection map on the \( j \)th factor \( (j = 1, 2, \ldots, q) \). Let \( d_j = \pi_j^*(d) \). Then, \( H^*((P^\infty(C))^q; \mathbb{Z}) \) is a polynomial ring in the classes \( d_1, \ldots, d_q \). Let \( \omega^j_1 \) be the complex 1-plane bundle over \( (P^\infty(C))^q \) induced from \( \omega^1 \) by \( \pi_j \) (see 1.4 in the complex case). Then, exactly as in (4.2), one has

\[(4.5) \quad c_j(\theta^q) = \mu_j, \text{ the } j\text{th elementary symmetric function in the classes } d_1, \ldots, d_q, \text{ where } \theta^q = \omega^1_1 \oplus \cdots \oplus \omega^q_1.\]

Let \( \theta^b_2 \) be the real 2q-plane bundle obtained from \( \theta^q \). Using Theorem (1.2) one obtains

\[(4.6) \quad p_i(\theta^b_2) = \nu_i, \text{ the } i\text{th elementary symmetric function in the classes } d_1^2, \ldots, d_q^2.\]

The purpose of introducing the complex \( (P^\infty(C))^q \) is to study the cohomology of \( G_n \). To this end we introduce a real \( n \)-plane bundle \( \eta^n \) over \( (P^\infty(C))^q \) by defining

\[(4.7) \quad \eta^n = \begin{cases} \theta^q_1, & \text{if } n = 2q, \\ \theta^q_1 \oplus \tau^1, & \text{if } n = 2q + 1 \end{cases}\]

where \( \tau^1 \) is the trivial real 1-plane bundle over \( (P^\infty(C))^q \). We then have:

\[(4.8) \quad \text{Lemma. } p(\eta^n) = p(\theta^b_2), \quad W(\eta^n) = W(\theta^b_2).\]

The proof follows at once from the multiplicative properties of the Pontrjagin and Stiefel-Whitney classes.

\[(4.9) \quad \text{Corollary. } p_i(\eta^n) = \nu_i, \text{ (see 4.6).}\]

5. Cohomology operations and characteristic classes. From the standpoint of applications it is obviously very useful to have as many relations as possible between the various characteristic classes. It is the purpose of this section to exhibit such relations. These are known results, but we give simple proofs based on the methods of [4].

(5.1) Theorem. Let \( \xi^n \) be a real \( n \)-plane bundle, and let \( p_2 \) be the cohomology homomorphism induced by the factor map \( Z \to \mathbb{Z}_2 \). Then, (see [9]),

\[ p_2(p_i(\xi^n)) = W_{2i}(\xi^n)^2 \quad (1 \leq i \leq [n/2]). \]

Proof. As is the case for any theorem concerning characteristic classes, it suffices to prove the theorem in the universal case \( \xi^n = \gamma^n \), the canonical \( n \)-plane bundle over \( G_n \). Let \( p_i, \ W_j \) be the respective characteristic classes for the bundle \( \gamma^n \).

In §4 we defined an \( n \)-plane bundle \( \xi^n \) over \( (P^\infty)^n \) and calculated its char-
acteristic classes in 4.2 and 4.4. Using the notation of 4.2, we stated a well known fact (e.g., see [4, VII]) which we will need later.

(5.2) Lemma. Let $f_n$ be a map from $(P^\infty)^n$ to $G_n$ which induces $\xi^n$. Then, $f_n^*$ maps $H^*(G_n; \mathbb{Z}_2)$ monomorphically into $H^*((P^\infty)^n; \mathbb{Z}_2)$; the image of $f_n^*$ is $\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$.

Our goal is to show that $\rho_2(\phi_i) = W^{2i}_{2i}$. We do this by making a calculation in the bundle $\xi^n$. We write

$$p_i(\xi^n) = \sum \delta_*(a_1) \cdot \cdots \cdot \delta_*(a_{2i}),$$

$$W^{2i}_{2i}(\xi^n) = \sum a_1 \cdots a_{2i},$$

where the summation indicates that we are taking respective symmetric functions with the indicated elements as first terms. It follows that

$$\rho_2p_1(\xi^n) = \sum \rho_2\delta_*(a_1) \cdots \rho_2\delta_*(a_{2i})$$

$$= \sum \beta_2(a_1) \cdots \beta_2(a_{2i})$$

$$= \sum a_1^2 \cdots a_{2i}^2.$$

Here we have used the fact that $\beta_2 = Sq^1$, and $Sq^1(u) = u^2$ when dimension $u = 1$. But (see 4.2),

$$(W^{2i}_{2i}(\xi^n))^2 = (\sum a_1 \cdots a_{2i})^2 = \sum a_1^2 \cdots a_{2i}^2,$$

since $2a_j = 0$ ($j = 1, \ldots, n$). Thus,

$$\rho_2p_1(\xi^n) = W^{2i}_{2i}(\xi^n)^2 \quad (1 \leq i \leq [n/2]).$$

Now,

$$f_n^*\rho_2(\phi_i) = \rho_2p_1(\xi^n), \quad f_n^*(W^{2i}_{2i}) = W^{2i}_{2i}(\xi^n)^2.$$

Therefore, $f_n^*[\rho_2(\phi_i) - W^{2i}_{2i}] = 0$. But by 5.2, $f_n^*$ is a monomorphism. Consequently,

$$\rho_2(\phi_i) = W^{2i}_{2i},$$

as was to be proved.

Now, let $\omega^n$ be an $n$-dimensional complex plane bundle; and denote by $\omega^n_R$ the real $2n$-dimensional plane bundle determined by $\omega^n$. We then have the following result (see [9]):

(5.3) Theorem. $\rho_2(c_i(\omega^n)) = W_{2i}(\omega^n_R)$ ($i = 1, \ldots, n$).

Proof. We use the formula stated in Theorem 1.2: namely,

$$p_i(\omega^n_R) = (c_i(\omega^n))^2 + \sum_{j=0}^{i-1} (-1)^{i+j} 2c_j(\omega^n)c_{2i-j}(\omega^n).$$
Thus,

\[ \rho_2(\phi_i(\omega_R^n)) = \rho_2(c_i(\omega^n)) = (\rho_2c_i(\omega^n))^2. \]

Using 5.1, we have

\[ (W_2(\omega_R^n))^2 = (\rho_2c_i(\omega^n))^2. \]

The fact that this implies

\[ W_2(\omega_R^n) = \rho_2c_i(\omega^n), \]

follows at once using the universal bundle over \((P^\infty(C))^n\); the details are left to the reader.

The relations between characteristic classes obtained so far have used only the coefficient group homomorphism \(\rho_2\). In order to get further relations on the classes \(W_i\), we must use cohomology operations defined on cohomology classes mod 2. We will use two types of such operations: the Steenrod squares and the Pontrjagin square.

For future reference we state a special case of a result of Wu:

(5.4) **Theorem (Wu).** Let \(Sq^i\) be the Steenrod square mapping \(H^q(X; \mathbb{Z}_2)\) to \(H^{q+i}(X; \mathbb{Z}_2)\). Let \(\xi\) be an \(n\)-plane bundle. Then,

\[ Sq^i W_{i+1}(\xi) = \sum_{j=0}^i W_j(\xi)W_{2i+1-j}(\xi). \]

In §10 we will determine the Pontrjagin square on the classes \(W_j\). Here we prove a special case which will be needed in that section:

(5.5) **Theorem.** Let \(\omega^n\) be a complex \(n\)-plane bundle. Let \(\Psi_2\) be the Pontrjagin square mapping \(H^{2q}(X; \mathbb{Z}_2)\) to \(H^{4q}(X; \mathbb{Z}_4)\). Let \(\rho_4\) be the homomorphisms defined in §2. Then,

\[ \Psi_2(W_{2i}(\omega_R^n)) = \rho_4(\phi_i(\omega_R^n)) + \theta_2 \left[ \sum_{j=0}^{i-1} W_{2j}(\omega_R^n)W_{4i-2j}(\omega_R^n) \right]. \]

**Proof.** We need the following fact about the Pontrjagin square (see [6, Theorem 1 and 2.1]).

(5.6) Let \(u \in H^{2q}(X; \mathbb{Z})\). Then, \(\Psi_2\rho_2(u) = \rho_4(u^2)\).

Now, let \(c = c(\omega^n), p = p(\omega_R^n), W = W(\omega_R^n)\). By 5.3, \(\rho_2(c_i) = W_{2i}\). Thus,

\[ \Psi_2(W_{2i}) = \Psi_2(\rho_2(c_i)) = \rho_4(c_i^2). \]

Recalling Theorem 1.2, we know
\[ \rho_4(c_i) = \rho_4(p_i) - \rho_4 \left( \sum_{j=0}^{i-1} (-1)^{i+j} 2c_jc_{2i-j} \right). \]

Let \( \eta_2 \) be induced by the factor map \( \mathbb{Z}_4 \to \mathbb{Z}_2 \). Then,
\[ 2\rho_4 = \theta_2\eta_2\rho_4 = \theta_2\rho_2. \]

Thus,
\[ \pm 2\rho_4(c_jc_{2i-j}) = \theta_2\rho_2(c_jc_{2i-j}) = \theta_2(\rho_2(c_j)\rho_2(c_{2i-j})) = \theta_2(W_2jW_{4i-2j}). \]

Consequently,
\[ (5.8) \quad \rho_4(c_i) = \rho_4(p_i) + \theta_2 \left( \sum_{j=0}^{i-1} W_2jW_{4i-2j} \right). \]

Combining 5.7 and 5.8 we have our result.

6. The cohomology of certain spaces. In order to continue the study of the cohomology of the complexes \( G_n \), we digress to study the cohomology of a general class of spaces which includes the spaces \( G_n \). That is, we study spaces \( X \) with the property that if \( T \) denotes the torsion subgroup of \( H^*(X; \mathbb{Z}) \), then there is a prime number \( p \) such that \( pT = 0 \). We see by [1, Theorem 24.7], that the complexes \( G_n \) are examples of such a space, with \( p = 2 \).

We need a bit of notation which we will keep throughout this section. Let \( X \) be a fixed space of the type described above. Set
\[ A = H^*(X; \mathbb{Z}), \quad A_n = H^*(X; \mathbb{Z}_n), \quad (n = 2, 3, \ldots). \]

Let \( \rho_n \) be the homomorphism of \( A \) to \( A_n \) induced by the factor map \( \mathbb{Z} \to \mathbb{Z}_n \). Let \( \beta_0 \) be the coboundary associated with the exact sequence
\[ (6.1) \quad 0 \to Z \xrightarrow{\rho_p} Z \xrightarrow{\beta_p} Z_p \to 0. \]

Set \( \beta_n = \rho_n \circ \beta_0 \), and notice that
\[ (6.2) \quad \beta_p \circ \beta_p = 0, \quad \beta_p \circ \rho_p = 0. \]

Using the exactness of 6.1 together with 6.2, the following facts are easily proved.

(6.3) **Lemma.** Let \( T \subseteq A \) be the torsion subgroup of \( A \). Then \( \rho_p \) restricted to \( T \) is a monomorphism mapping \( T \) into \( A_p \).

(6.4) **Lemma.**
\[ \beta_0(A_p) = T. \]

(6.5) **Lemma.**
\[ \rho_p(T) = \text{Image } \beta_p \subseteq A_p; \quad \rho_p(A) = \text{Kernel } \beta_p \subseteq A_p. \]
Now, \( A_p \) may be regarded as a vector space over \( Z_p \). Since \( \beta_p \) is a homomorphism of \( A_p \) into itself, Kernel \( \beta_p \) is a linear subspace of \( A_p \); set \( V_p = \text{Kernel } \beta_p \). By 6.5, \( V_p = \rho_p(A) \). Since \( A_p \) is a vector space, \( V_p \) is a direct summand; let \( S_p \) be a complement to \( V_p \). That is,

\[
(6.6) \quad A_p = V_p \oplus S_p,
\]
as a vector space.

Let \( i \) be a fixed integer \( \geq 1 \). Our goal is to determine the structure of \( A_{p^i} = H^*(X; Z_{p^i}) \). Let \( \theta_i \) be the homomorphism from \( A_p \) to \( A_{p^i} \) induced by the homomorphism from \( Z_p \) to \( Z_{p^i} \) which sends \( 1 \mod p \) into \( p^i \). Let

\[
V_{p^i} = \rho_{p^i}(A), \quad S_{p^i} = \theta_i(S_p).
\]
Then,

\[
(6.7) \text{Theorem. } A_{p^i} = V_{p^i} \oplus S_{p^i} \text{ (group direct sum).}
\]

We precede the proof by several lemmas. To begin with, consider the commutative diagram

\[
(\ast) \quad 0 \rightarrow Z \xrightarrow{\rho_p} Z \xrightarrow{\beta_p} Z_p \rightarrow 0
\]

\[
(\ast\ast) \quad 0 \rightarrow Z \xrightarrow{\rho_{p^i}} Z \xrightarrow{\beta_{p^i}} Z_{p^i} \rightarrow 0.
\]

Let \( \beta_0, \delta_0 \) be the Bockstein coboundary operators associated respectively with the exact sequences \((\ast)\) and \((\ast\ast)\). By the commutativity of the diagram one has

\[
(6.8) \quad \beta_0 = \delta_0 \theta_i.
\]

We use this to show:

\[
(6.9) \text{Lemma.} \quad V_{p^i} \cap S_{p^i} = 0.
\]

\textbf{Proof.} Let \( u \in V_{p^i} \cap S_{p^i} \); i.e., \( u = \rho_p(x) \) for \( x \in A \), and \( u = \theta_i(y) \), for \( y \in S_p \). Therefore,

\[
\delta_0(u) = \delta_0 \rho_p(x) = 0,
\]
by the exactness of \((\ast\ast)\). But we also have,

\[
\delta_0(u) = \delta_0 \theta_i(y) = \beta_0(y), \text{ by 6.8.}
\]

Thus \( \beta_0(y) = 0 \). Therefore, \( \beta_p(y) = \rho_p \beta_0(y) = 0 \). That is, \( y \in \text{Kernel } \beta_p = V_p \). However, by hypothesis, \( y \in S_p \), and by 6.6, \( V_p \cap S_p = 0 \). Therefore, \( y = 0 \) and hence \( u = \theta_i(y) = 0 \); this completes the proof.
One obtains similarly,

(6.10) **Lemma.** $\delta_0$ maps $S_p'$ isomorphically onto $T$.

As an immediate consequence we have

(6.11) **Lemma.** $\theta_i$ is an isomorphism mapping $S_p$ onto $S_p'$.

We now give the proof of Theorem 6.7. Notice that $\delta_0(A_p') \subset T$. Thus, by Lemma 6.10, we may define an inverse to $\delta_0$ restricted to $S_p'$. Let $\epsilon_0: T \rightarrow S_p'$ be the inverse; i.e., $\delta_0 \epsilon_0 = \text{identity}$. For any $u \in A_p'$, set $u_2 = \epsilon_0 \delta_0(u)$. Then,

$$\delta_0(u - u_2) = \delta_0(u) - \delta_0 \epsilon_0 \delta_0(u) = \delta_0(u) - \delta_0(u) = 0.$$

Thus, by exactness of (**), $u - u_2 = \rho_p'(v)$ for some $v \in A$. That is, $u - u_2 \in V_p'$. Set $u_1 = u - u_2$. Then,

$$u = u_1 + u_2, \quad u_1 \in V_p', \ u_2 \in S_p'.$$

Since $S_p' \cap V_p' = 0$, this splitting is unique. Thus, $A_p' = V_p' \oplus S_p'$ as asserted.

7. **The proof of Theorem A.** Denote by $P$ the subring of $H^*(G_n; \mathbb{Z})$ generated by $p_1, \cdots, p_q$. We first show that $P$ is in fact a polynomial ring. Let $R_0$ be the rational numbers, and let $p_0$ be the cohomology homomorphism induced by the inclusion $\mathbb{Z} \subset R_0$.

Let $\phi$ be a polynomial in $\mathbb{Z}[x_1, \cdots, x_q]$ and suppose that $\phi(p_1, \cdots, p_q) = 0$. We show that $\phi \equiv 0$. Let $\phi_0$ be the image of $\phi$ in $R_0[x_1, \cdots, x_q]$. Then, $0 = p_0(\phi(p_1, \cdots, p_q)) = \phi_0(p_1^0, \cdots, p_q^0)$. But by Theorem 1.7, this implies that $\phi_0 \equiv 0$. Now the injection $\mathbb{Z}[x_1, \cdots, x_q] \rightarrow R_0[x_1, \cdots, x_q]$ is a monomorphism. Thus, $\phi \equiv 0$; and hence, $P$ is a polynomial ring.

Since every element of $P$ has infinite order, it is clear that $P \cap T = 0$. Thus, Theorem A is proved when we show

(7.1) **Lemma.** Let $u \in H^*(G_n; \mathbb{Z})$. Then, there are elements $u_1 \in P$, $u_2 \in T$ such that $u = u_1 + u_2$.

To prove this consider the exact sequence

$$0 \rightarrow T \xrightarrow{\iota} H^*(G_n; \mathbb{Z}) \xrightarrow{p_0} H^*(G_n; R_0),$$

where $\iota$ is the inclusion homomorphism. It follows from Theorem (1.7) that for any class $u \in H^*(G_n; \mathbb{Z})$ we have

$$\rho_0(u) = \phi(p_1^0, \cdots, p_q^0),$$

where $\phi \in R_0[x_1, \cdots, x_q]$. If $\phi \equiv 0$, then $u \in T$ by the exactness of sequence (7.2). Suppose then that $\phi \neq 0$. Let $a_1, \cdots, a_N$ be the nonzero coefficients of $\phi$, where each $a_i$ is a rational number written in lowest form. Set

$$d_i = \text{denominator of the fraction } a_i,$$
and let

\[ m = \text{least common multiple of } d_1, \ldots, d_N. \]

Then, \( ma_1, ma_2, \ldots, ma_N \) are all integers. Furthermore, if \( p \) is a prime number dividing \( m \), then:

(7.3) There is at least one integer \( ma_i \) which \( p \) does not divide.

Let \( \psi = m\phi \); then \( \psi \) has integer coefficients and may be regarded as an element of \( Z[x_1, \ldots, x_q] \). Let

\[ u_1 = \psi(p_1, \ldots, p_q). \]

Clearly,

\[ \rho_0(u_1) = \psi(p_1^0, \ldots, p_q^0) = m\phi(p_1^0, \ldots, p_q^0) = m\rho_0(u). \]

Therefore, \( \rho_0(mu - u_1) = 0 \). Hence, by the exactness of the sequence (7.2), there is a class \( u_2 \in T \) such that

\[ mu = u_1 + u_2. \]

The proof of Lemma (7.1) consists simply in showing that \( m = 1 \). Suppose, to the contrary, that \( m \) contains a prime factor \( p \). Let \( \rho_p \) be the cohomology homomorphism induced by the reduction \( Z \to \mathbb{Z}_p \). Then, \( \rho_p(mu) = 0 \); i.e.,

(7.4) \[ \rho_p(u_1) + \rho_p(u_2) = 0. \]

We must consider two cases:

**Case I.** \( p > 2 \). Then, \( \rho_p(u_2) = 0 \), since \( 2T = 0 \). Hence, (7.4) implies that \( \rho_p(u_1) = 0 \). Recall that \( u_1 = \psi(p_1, \ldots, p_q) \). Now by (7.3), the polynomial \( \psi \) has at least one coefficient which is not divisible by \( p \). Thus, \( \psi_p \neq 0 \), where \( \psi_p \) denotes the image of \( \psi \) in \( \mathbb{Z}_p[x_1, \ldots, x_q] \). But,

\[ \rho_p(u_1) = \rho_p(\psi(p_1, \ldots, p_q)) = \psi_p(\bar{p}_1, \ldots, \bar{p}_q), \]

where \( \bar{p}_i = \rho_p(p_i) \). From Theorem (1.6) we see that \( \psi_p(\bar{p}_1, \ldots, \bar{p}_q) \neq 0 \). Thus, \( \rho_p(u_1) \neq 0 \), which is a contradiction. Consequently, \( m \) is not divisible by any odd prime.

**Case II.** \( p = 2 \). Then, (7.4) implies that

\[ \rho_2(u_1) = \rho_2(u_2). \]

Now from Theorem (5.1) and Proposition (2.3) it is clear that \( \rho_2(P) = P_2 \). Also, by Lemma (6.5), \( \rho_2(T) = T_2 \). Hence, \( \rho_2(u_1) \subseteq P_2 \cap T_2 \). But by Proposition (2.3), \( P_2 \cap T_2 = 0 \). Thus, \( \rho_2(u_1) = 0 \). But now the same argument as that given in Case I may be used to obtain a contradiction—using Theorem (1.5) in this case instead of Theorem (1.6), (together with the fact that \( \rho_2(p_i) = W_{2i}^{2i} \)). Hence, assuming that the integer \( m \) contains a prime factor always leads to a contradiction. Therefore, \( m = 1 \), which proves Lemma (7.1) and hence Theorem A.
8. The proof of Theorem B. Let us apply the results of §6 to the group \( H^*(G_n; \mathbb{Z}_2^i) \) \((i = 1, 2, \cdots)\). As in §2 we may choose a summand \( S_2 \) of \( H^*(G_n; \mathbb{Z}_2) \) such that

\[
H^*(G_n; \mathbb{Z}_2) = \text{Kernel } \beta_2 \oplus S_2 \text{ (vector direct sum)}.
\]

Thus, by Theorem 6.7, we have

\[
H^*(G_n; \mathbb{Z}_2^i) = V_2^i \oplus S_2^i,
\]

where \( V_2^i = \rho_2^i(H^*(G_n; \mathbb{Z})) \), \( S_2^i = \theta_i(S_2) \), and the direct sum is a module splitting over the ring \( \mathbb{Z}_2^i \). Now, from Theorem A (and Lemma 6.5) it is clear that \( V_2^i = \rho_2^i(P) \oplus \rho_2^i(T) = \rho_2^i(P) \oplus T_2^i \). From Lemma 6.11 we have that \( \theta_i \) maps \( S_2 \) isomorphically onto \( S_2^i \); and from Lemma 6.3 it follows that \( \rho_2^i \) maps \( T \) isomorphically onto \( T_2^i \). Thus, Theorem B is proved when we show that \( \rho_2^i(P) \) is a polynomial ring in \( \rho_2^i(p_1), \cdots, \rho_2^i(p_q) \) \((q = \left\lceil n/2 \right\rceil)\). But that follows at once from Theorem A and the fact that \( 2T = 0 \).

Before going on to the proofs of Theorems A' and B', we remark one more property of the groups \( H^*(G_n; \mathbb{Z}_2^i) \) which will be needed in §10. When \( i = 1 \) we have the splitting

\[
H^*(G_n; \mathbb{Z}_2) = P_2 \oplus T_2 \oplus S_2,
\]

given by Theorem B. Let \( \theta_i \) be the homomorphism defined in 2.1. From Lemma 6.11, we know that \( \theta_i \) maps \( S_2 \) isomorphically onto \( S_2^i \) \((i = 1, 2, \cdots)\).

(8.1) Lemma. \( \theta_i(T_2) = 0, \theta_i(P_2) \subset P_2^i \) \((i = 2, 3, \cdots)\).

Proof. We use the following fact: let \( \eta_2^i \) be the factor homomorphism \( \mathbb{Z}_2^i \rightarrow \mathbb{Z}_2 \). Then,

\[
\theta_i\eta_2^i(u) = 2^{i-1}(u), \quad u \in H^*(X; \mathbb{Z}_2^i).
\]

Thus,

\[
\theta_i(T_2) = \theta_i\rho_2(T) = \theta_i\eta_2^i\rho_2(T) = 2^{i-1}T_2^i = 0,
\]

since \( i \geq 2 \) and \( 2T_2^i = 0 \).

Similarly, \( \theta_i(P_2) = 2^{i-1}P_2^{i-1} \); which completes the proof of the lemma.

9. The proofs of Theorems A' and B'.

Proof of Theorem A'. (a) We are to show that \( f_n^* \) restricted to \( T \) is a monomorphism; i.e., given \( u \in T \) such that \( f_n^*(u) = 0 \), we are to show that \( u = 0 \). Let \( \rho_2 \) be the factor map defined in 2.1. Then, since \( f_n^*(u) = 0, \rho_2f_n^*(n) = 0 \). But \( \rho_2 \) is natural; thus, \( f_n^*\rho_2(u) = 0 \). Now, by 5.2, \( f_n^* \) is a monomorphism on \( H^*(G_n; \mathbb{Z}_2) \). Therefore, \( \rho_2(u) = 0 \). But by Lemma 6.3, this implies that \( u = 0 \).

(b) This follows at once from Corollary (4.9) and the fact that \( P = \mathbb{Z}[p_1, \cdots, p_q] \).

Proof of Theorem B'. (a) We give the proof in 2 parts. First, if \( u \in T_2^i \) and \( f_n^*(u) = 0 \), then the fact that \( u = 0 \) follows by an argument entirely similar
to (a) above. Suppose, secondly, that \( u \in S_i \) and \( f_n^*(u) = 0 \). We are to show that \( u = 0 \). Since \( u \in S_i \), by definition \( u = \theta_i(v) \) for some class \( v \in S_2 \). Thus, \( f_n^*(u) = 0 \) implies \( f_n^*\theta_i(v) = 0 \); i.e., by naturality, \( \theta_i f_n^*(v) = 0 \). Now consider the following commutative diagram,

\[
\begin{array}{ccc}
0 & \rightarrow & Z \rightarrow Z \rightarrow Z \rightarrow 0 \\
\downarrow \rho_2 & & \downarrow \rho_2^i \\
0 & \rightarrow & Z_i \rightarrow Z_i \rightarrow Z_i \rightarrow 0,
\end{array}
\]

where \( \xi \) is the natural factor map. Let \( \delta_0 \) be the coboundary associated with upper exact sequence, and \( \delta_2 \) the coboundary associated with the lower exact sequence. By commutativity we have \( \delta_2 = \rho_2 \delta_0 \). Since \( \theta_i f_n^*(v) = 0 \), by the exactness of (**) we have \( f_n^*(v) = \delta_2(w) \), for some class \( w \in H^*((P^n)^n; Z_{2^{i-1}}) \); hence, \( f_n^*(v) = \rho_2 \delta_0(w) \). Therefore,

\[
\beta_n f_n^*(v) = \beta_2 \rho_2 \delta_0(w) = 0,
\]

since \( \beta_2 \rho_2 = 0 \). Thus, by naturality, \( f_n^* \beta_2(v) = 0 \). But, by 5.2, \( f_n^* \) maps \( A_2 \) isomorphically into \( H^*((P^n)^n; Z_2) \). Thus, \( \beta_2(v) = 0 \); i.e., \( v \in \text{Kernel } \beta_2 \). But, by hypothesis, \( v \in S_2 \). From §2 we know \( \text{Kernel } \beta_2 \cap S_2 = 0 \); hence \( v = 0 \). Therefore, \( u = \theta_i(v) = 0 \), as was to be proved.

The proof of Theorem B'(b) is entirely similar. We leave the details to the reader.

10. The proof of Theorem C. We begin by giving some of the properties of the Pontrjagin square cohomology operation. This is a mapping \( \Psi_2 \) from \( H^q(X; Z_2) \) to \( H^{2q}(X; Z_2) \), with the following properties:

(10.1) \( \Psi_2 = f^* \Psi_2 \), where \( f^* \) is induced by a map \( f \).

(10.2) \( \eta_2 \Psi_2(u) = u^2 \), where \( u \in H^q(X; Z_2) \) and \( \eta_2 \) is induced by the factor homomorphism \( Z_2 \rightarrow Z_i \).

(10.3) \( \Psi_2(u + v) = \Psi_2(u) + \Psi_2(v) + \theta_2(uv) \), where \( u, v \in H^{2q}(X; Z_2) \) and \( \theta_2 \) is defined in 2.1.

(10.4) \( \Psi_2(u) = \beta_4 S^q(u) + \theta_2 S^q \beta_2(u) \), where \( u \in H^{2q+1}(X; Z_2) \), and \( \beta_n = \rho_n \delta_n \) (see 2.2).

(10.5) \( \Psi_2(uv) = \Psi_2(u) \Psi_2(v) + \theta_2 \left[ S^q(u) (v \beta_2(v)) + (u \beta_2(u)) S^q(u) \right] \), where \( u \in H^{2q+1}(X; Z_2) \), \( v \in H^{2q+1}(X; Z_2) \).

The above properties are proved as follows: 10.1, 10.3 in [6, Theorem I]; 10.2 in [6, Theorem II]; 10.4 in [6, Proposition 7.7] and 10.5 in [6, p. 75]. With the exception of 10.4, they are also noted by Wu in [8].

As usual, it suffices to prove the theorem in the universal case, where \( \xi^n = \gamma^n \), the \( n \)-plane bundle over \( G_n \). We prove part (a) of Theorem C by first observing some facts:
\( \beta_2(W_{2i+1}) = W_1W_{2i+1} \).

(10.7) \( Sq^*(uv) = Sq^*(u)v + Sq^{*-1}(u)\beta_2(v) \), if dimension \( v = 1 \).

(10.8) \[ \text{If } u, v \in H^*(X; \mathbb{Z}_2), \text{ then } \theta_2(u \cup \beta_2(v)) = \theta_2(\beta_2(u) \cup v). \]

(10.9) \( \beta_2 Sq^{2i+1} = 0 \) \( (r = 0, 1, \cdots) \).

Now by 10.4,
\[ \Psi_2(W_{2i+1}) = \beta_4 Sq^{2i}(W_{2i+1}) + \theta_2 Sq^{2i} \beta_2(W_{2i+1}). \]

Using 10.6-10.9 we have
\[ \theta_2 Sq^{2i} \beta_2(W_{2i+1}) = \theta_2 Sq^{2i}(W_1W_{2i+1}) \]
\[ = \theta_2 [Sq^{2i}(W_{2i+1})W_1 + Sq^{2i-1}(W_{2i+1})\beta_2(W_1)] \]
\[ = \theta_2 [W_1 Sq^{2i}(W_{2i+1})] + \theta_2 [\beta_2 Sq^{2i-1}(W_{2i+1})W_1] \]
\[ = \theta_2 [W_1 Sq^{2i}(W_{2i+1})]. \]

Thus,
\[ \Psi_2(W_{2i+1}) = \beta_4 Sq^{2i}(W_{2i+1}) + \theta_2 (W_3 Sq^{2i}W_{2i+1}), \]
as was to be proved.

To prove part (b) of Theorem C we proceed as follows: by 10.2 we know that
\[ \eta_2 \Psi_2(W_{2i}) = (W_{2i})^2. \]

By 5.1, we have
\[ \rho_2(\rho_i) = (W_{2i})^2. \]

But, \( \rho_2 = \eta_2 \rho_4 \), where \( \eta_2 \) is the factor map in the exact sequence
\[ (\star) \quad 0 \rightarrow Z_2 \overset{\theta_2}{\rightarrow} Z_4 \overset{\eta_2}{\rightarrow} Z_2 \rightarrow 0. \]

Thus,
\[ \eta_2 [\Psi_2(W_{2i}) - \rho_i(\rho_i)] = 0. \]

Consequently, by the exactness of (\( \star \)) there is a class \( X \in H^*(G_n; \mathbb{Z}_2) \) such that
\[ (10.10) \quad \Psi_2(W_{2i}) = \rho_i(\rho_i) + \theta_2(X). \]

From Theorem B we have
\[ X = X_1 + X_2 + X_3 \quad (X_1 \in P_2, X_2 \in T_2, X_3 \in S_2); \]
and by 8.1, \( \theta_2(X_2) = 0 \); thus,
\[ (10.11) \quad \theta_2(X) = \theta_2(X_1) + \theta_2(X_3). \]

We prove Theorem C by determining the classes \( \theta_2(X_1) \) and \( \theta_2(X_3) \).
Lemma. \[ \theta_2(X_1) = 0, \]
\[ \theta_2(X_3) = \theta_2 \left[ W_1 Sq^{2i-1} W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j} \right]. \]

Proof. We first obtain the class \( \theta_2(X_3) \). Let \( \xi^n \) be the bundle over \((P^n)^n \) defined in 4.1. Let \( f: (P^n)^n \to G_2 \) be the map which induces \( \xi^n \) from \( \gamma^n \). Then,
\[ f^*(W_i) = \sigma_i, \quad f^*(\hat{\rho}_i) = \tau_{2i}, \]
(see 4.2 and 4.4). We compute the operation \( \Psi_2 \) in \( H^*((P^n)^n; \mathbb{Z}_2) \).

Lemma. \[ \Psi_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2 \left[ \sigma_1 Sq^{2i-1}\sigma_{2i} + \sum_{j=0}^{i-1} \sigma_{2j}\sigma_{4i-2j} \right]. \]

The proof is given at the end of this section. We use Lemma 10.14 to compute the class \( \theta_2(X_3) \). By 10.10 we have:
\[ f^*\Psi_2(W_{2i}) = f^*\rho_4(\hat{\rho}_i) + f^*\theta_2(X). \]

Using 10.11, 10.13, and the fact that \( \Psi_2, \rho_4 \) and \( \theta_2 \) are cohomology operations, it follows that
\[ \Psi_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2(f^*X_1) + \theta_2(f^*X_3). \]
Comparing 10.14 and 10.15 we see that
\[ \theta_2(f^*X_1) + \theta_2(f^*X_3) = \theta_2 \left[ \sigma_1 Sq^{2i-1}\sigma_{2i} + \sum_{j=0}^{i-1} \sigma_{2j}\sigma_{4i-2j} \right]. \]

We first show
\[ \theta_2(f^*X_1) = 0. \]
Since \( X_1 \subset P_2 \) this is a consequence of the fact that
\[ \theta_2(\hat{P}_2) = 0, \]
where \( \hat{P}_2 = f^*(P_2) \). To show this recall that \( P_2 = \rho_2(P) \). Therefore,
\[ \theta_2(\hat{P}_2) = \theta_2(f^*P_2) = \theta_2(f^*\rho_2(P)) = \theta_2\rho_2(f^*(P)). \]
But,
\[ \theta_2\rho_2 = \theta_2\eta_2\rho_4 = 2\rho_4. \]
Thus,
\[ \theta_2\rho_2(f^*(P)) = 2\rho_4(f^*P) = \rho_4(2f^*(P)). \]
Since \( f^*(P) \subset H^*((P^n); Z) \) and \( 2H^*((P^n); Z) = 0 \), we have
\[
\theta_2(\hat{P}_2) = \theta_2 \rho_2(f^*(P)) = \rho_4(2f^*(P)) = 0,
\]
which proves 10.17. From 10.13 we have that \( f^*(W_i) = \sigma_i \). Using this, together, with 10.16, 10.17, we have

\[
(10.18) \quad f^*\left[ \theta_2(X_3) - \theta_2\left( W_i S_q^{2i-1} W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j} \right) \right] = 0.
\]

Consider the direct sum splitting given in Theorem B. By hypothesis, \( X_3 \subseteq S_2 \). But clearly, \( (W_i S_q^{2i-1} W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j}) \in T_2 \oplus S_2 \). This follows from Theorem (5.4), the choice of the summand \( S_2 \), and the fact that \( P_2 = Z_2 [W_2, \ldots, W_{2q}] \). Thus,
\[
\theta_2\left[ X_3 - \left( W_i S_q^{2i-1} W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j} \right) \right] \in S_4,
\]
since \( \theta_2(T_2) = 0 \), \( \theta_2(S_2) = S_4 \), by 8.1 and 6.11. Finally, using 10.18 we have,
\[
\theta_2\left[ (X_3) - \left( W_i S_q^{2i-1} W_{2i} + \sum_{j=0}^{i-1} W_{2j} W_{4i-2j} \right) \right] = 0,
\]
since \( f^* \) is a monomorphism on \( S_4 \) by Theorem B’ (a). Thus, the class \( \theta_2(X_3) \) is determined in Lemma 10.12.

We have left to show that \( \theta_2(X_1) = 0 \) to complete the proof of Lemma 10.12 and hence of Theorem C. To do this we use the model complex \( (P^*(C))^q \), where \( q = [n/2] \). Let \( g: (P^*(C))^q \rightarrow G_n \) be the map which induces the bundle \( \eta^n \) over \( (P^*(C))^q \) (see 4.7). Let
\[
W_i(\eta^n) = \tilde{W}_i, \quad \rho_i(\eta^n) = \tilde{\rho}_i \quad (i = 0, 1, \ldots).
\]

Then, by 4.8, we have
\[
\tilde{W}_i = W_i(\theta^q_R), \quad \tilde{\rho}_i = \rho_i(\theta^q_R),
\]
where \( \theta^q \) is the complex \( q \)-plane bundle over \( (P^*(C))^q \) defined in 4.5.

Using Theorem 5.5 we have

\[
(10.19) \quad \mathcal{F}_2(\tilde{W}_{2i}) = \rho_4(\tilde{\rho}_i) + \theta_2\left( \sum_{j=0}^{i-1} \tilde{W}_{2j} \tilde{W}_{4i-2j} \right).
\]

But,
\[
\tilde{W}_i = g^*W_i, \quad \tilde{\rho}_i = g^*\rho_i.
\]

Therefore, applying \( g^* \) to 10.10, we obtain:

\[
(10.20) \quad \mathcal{F}_2(\tilde{W}_{2i}) = \rho_4(\tilde{\rho}_i) + \theta_2(g^*X_1) + \theta_2(g^*X_3).
\]
Comparing 10.19 and 10.20, we see that:

\[ \theta_2(g^*X_1) + \theta_2(g^*X_3) = \theta_2 \left( \sum_{j=0}^{n-1} \tilde{W}_{2j} \tilde{W}_{4i-2j} \right). \]  

Now, by 10.12,

\[ \theta_2(g^*X_3) = g^*\theta_2(X_3) = \theta_2 \left( -\tilde{W}_1 S q^{2i-1} \tilde{W}_{2i} + \sum_{j=0}^{n-1} \tilde{W}_{2j} \tilde{W}_{4i-2j} \right). \]

But, \( \tilde{W}_1 = 0 \). Therefore,

\[ \theta_2(g^*X_1) = g^*\theta_2(X_1) = 0, \]  by 10.21.

Now, by hypothesis, \( X_1 \in P_2 \); hence, \( \theta_2(X_1) \in P_4 \), by 8.2. But, by Theorem B'(b), \( g^* \) restricted to \( P_4 \) is a monomorphism. Thus, \( g^*(\theta_2X_1) = 0 \) implies that \( \theta_2(X_1) = 0 \), as was asserted in 10.12. This, then, completes the proof of Lemma 10.12 and hence of Theorem C.

**Proof of Lemma 10.14.** The proof proceeds by induction on the integer \( n \), the dimension of the bundle \( \gamma^p \). When \( n = 1 \), \( (P^o)^n = P^o \). But \( H^*(P^o; Z_2) = Z_2[a_1] \). Thus, \( \sigma_{2i} = \tau_{2i} = 0 \) \( (i \geq 1) \). Hence, both sides of 10.14 are zero and the lemma is true for \( n = 1 \). Now, let \( n \) be a fixed integer \( > 1 \). Suppose that we have proved Lemma 10.14 for all integers \( j < n \). We prove the lemma for the integer \( n \); and hence, by induction, for all integers. Let \( q: (P^o)^n \to (P^o)^{n-1} \) be the map defined by \( q(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_1) \in (P^o) \). Then, \( q^* \) maps \( H^*((P^o)^{n-1}; Z_2) \) isomorphically into \( H^*((P^o)^n; Z_2) \). We agree to identify \( H^*((P^o)^{n-1}; Z_2) \) with its image under \( q^* \); i.e., we set

\[ H^*((P^o)^{n-1}; Z_2) = Z_2[a_1, \ldots, a_{n-1}], \]  (see 4.1).

Let \( \xi^{n-1} \) be the bundle over \( (P^o)^n \) induced by \( q \) from \( \xi^{n-1} \). Clearly

\[ W_i(\xi^{n-1}) = q^* W_i(\xi^{n-1}) = \sigma_i, \]

\[ p_\ell(\xi^{n-1}) = q^* p_\ell(\xi^{n-1}) = \tau_{2 \ell}, \]

where \( \sigma_i^* \) is the \( i \)th elementary symmetric function in \( a_1, \ldots, a_{n-1} \) and \( \tau_j^* \) is the \( j \)th elementary symmetric function in \( b_1, \ldots, b_{n-1} \) \( (b_i = \delta_i(a_i)) \). Using the monomorphism \( q^* \) we identify \( W_i(\xi^{n-1}) \) with \( \sigma_i^* \) and \( p_\ell(\xi^{n-1}) \) with \( \tau_{2 \ell}^* \). Thus, by 10.14 and the inductive hypothesis we have

\[ \theta_2(\sigma_{2i}) = \rho_4(\tau_{2i}) + \theta_2 \left[ \sigma_{2i}^* S q^{2i-1} \sigma_{2i}^* + \sum_{j=0}^{n-1} \sigma_j^* \sigma_{4i-2j} \right]. \]

Now, let \( \sigma_i \), (resp. \( \tau_j \)), be the symmetric functions in \( a_1, \ldots, a_n \) (resp. \( b_1, \ldots, b_n \)) as before. Then, it is easily verified that

\[ \sigma_i = \sigma_{i-1} a_n + \sigma_i^*, \quad \tau_j = \tau_{j-1} b_n + \tau_j^*. \]
for \( i, j = 0, 1, \ldots \). Here we adopt the convention that
\[
\sigma_{-1} = \tau_{-1} = \sigma_{n+1} = \tau_{n+1} = 0.
\]
Combining 10.23 and 10.3 we obtain,
\[
\mathcal{P}_2(\sigma_{2i}) = \mathcal{P}_2(\sigma_{2i-1}a_n) + \mathcal{P}_2(\sigma_{2i}) + \theta_2(\sigma_{2i-1} \sigma_{2i} a_n).
\]
By 10.5,
\[
\mathcal{P}_2(\sigma_{2i-1}a_n) = \mathcal{P}_2(\sigma_{2i-1}) \mathcal{P}_2(a_n) + \theta_2[\sigma_{2i-1}S\mathcal{P}^2(\sigma_{2i-1}a_n) + \sigma_{2i-1}(\beta_2 \sigma_{2i-1}) \mathcal{P}^0(a_n)].
\]
Thus,
\[
(10.24) \quad \mathcal{P}_2(\sigma_{2i}) = \mathcal{P}_2(\sigma_{2i-1}) \mathcal{P}_2(a_n) + \mathcal{P}_2(\sigma_{2i}) + \theta_2[\sigma_{2i-1}S\mathcal{P}^2(\sigma_{2i-1}a_n) + \sigma_{2i-1}(\beta_2 \sigma_{2i-1}) \mathcal{P}^0(a_n)] + (\sigma_{2i-1} \sigma_{2i} a_n).
\]
We prove Lemma 10.14 by analysing the right hand side of 10.24 term by term. Since \( \sigma_{2i-1} = \mathcal{W}_{2i-1}(\xi^{*n-1}) \), by Theorem C (a) we have
\[
(10.25) \quad \mathcal{P}_2(\sigma_{2i-1}) = \beta_4 S\mathcal{P}^2(\sigma_{2i-1}) + \theta_2(\sigma_4 S\mathcal{P}^2 \sigma_{2i-1}).
\]
By methods similar to those in §5, one may easily show
\[
(10.26) \quad \begin{align*}
(a) & \quad \delta_* S\mathcal{P}^2(\sigma_{2i-1}) = (\tau_{2i-1}), \\
(b) & \quad \beta_* S\mathcal{P}^2(\sigma_{2i-1}) = \rho_4(\tau_{2i-1}).
\end{align*}
\]
Next, consider the term \( \mathcal{P}_2(a_n) \) in the expression 10.24.
\[
(10.27) \quad \mathcal{P}_2(a_n) = \rho_4(b_n).
\]
**Proof.** Since \( a_n \) is 1-dimensional (and hence odd), 10.4 gives
\[
\mathcal{P}_2(a_n) = \beta_4 S\mathcal{P}^0(a_n) + \theta_2 S\mathcal{P}^0(\beta_2 a_n) = \beta_4(a_n) + \theta_2 \beta_2(a_n)
\]
since \( S\mathcal{P}^0 = \text{identity} \). But \( \beta_4 = \rho_4 \delta_* \) and \( \delta_*(a_n) = b_n \). Also, \( \theta_2 \beta_2 = 0 \) by exactness. Thus,
\[
\mathcal{P}_2(a_n) = \rho_4 \delta_*(a_n) + \theta_2 \beta_2(a_n) = \rho_4(b_n),
\]
as was asserted.
Finally, let \( u \in H^*(X; Z_2) \) and \( v \in H^*(X; Z) \) for any space \( X \). Then, as is easily shown,
\[
(10.28) \quad \theta_2(u)\rho_4(v) = \theta_2(u\rho_2(v)).
\]
Combining 10.25 through 10.28 we have

\[ \Psi_2(\sigma_i) = \beta_2(\sigma_i) + \sum_{j=0}^{i-1} \sigma_2 \sigma_{4i-2j} \]

since \( \beta_2(\sigma_i) = \beta_2(\sigma_i) \). Thus, combining 10.24, 10.25, and using the induction hypothesis 10.22 we have:

\[ \Psi_2(\sigma_i) = \rho_4(\tau_i) + \sum_{j=0}^{i-1} \sigma_2 \sigma_{4j-2j} \]

\[ + \left( \sigma_2 \sigma_i \sigma_{4i-2j} \right) + \sigma_i \sigma_{4j-2j} \]

(10.30)

Here we have used the following facts:

(a) \( \sigma_n(\beta_2(\sigma_i)) = \sigma_n(\beta_2(\sigma_i)) \)

(b) \( \sigma_{2i-1}(\beta_2(\sigma_{2i-1})) = \sigma_{2i-1}(\beta_2(\sigma_{2i-1})) \)

(c) \( \tau_{2i} = \tau_{2i} \)

Comparing 10.30 and 10.14 we see that Lemma 10.14 is proved when we show

(a) \( \rho_4(\tau_i) = \sigma_i \sigma_{4i-2j} + \sigma_i \sigma_{4i-2j} \)

(10.31)

(b) \( \rho_4(\tau_i) = \sigma_i \sigma_{4i-2j} + \sigma_i \sigma_{4i-2j} \)

The proof of this is a simple mechanical verification using 10.23, 10.7 and Theorem 5.5. We leave the details to the reader.

11. Appendix. The oriented case. So far in this paper we have considered only the unoriented Grassmann complexes. We remark here an analogous result for the oriented Grassmann complexes, \( G_n \). Let \( \gamma^n \) be the classifying bundle over \( G_n \) which is the analogue of \( \gamma^n \), and let \( q \) be a positive integer.

12.1 Theorem. (a) \( H^*(\tilde{G}_{2q+1}; Z) = \mathbb{P} \oplus T \), where \( P = Z \) and \( T = \text{ideal of torsion elements} \). (b) \( H^*(\tilde{G}_{2q}; Z) = \mathbb{P}^* \oplus T^* \), where \( P^* = Z \) and \( T^* = \text{ideal of torsion elements} \).

Here \( p_i = p_i(\gamma^n) \) and \( X \) is the Euler class of the bundle \( \gamma^{2q} \) (see [4, VIII]). The proof of the theorem is very similar to the proof of Theorem A given in
§7. We use the analogues of Theorems (1.5)–(1.7) for the oriented case (see [1] and [4]), together with Corollary 1 of [7]. The details are left to the reader.

BIBLIOGRAPHY


UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIFORNIA