DIFFERENTIATION OF SET FUNCTIONS USING VITALI COVERINGS

BY

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1. Introduction. The original motivation for this paper was a proposal by G. B. Price in which he suggested a way for basing a theory of surface area and more general theories on a measure-theoretic foundation. Previous writers, especially T. Radó [8, p. 552], had already commented on the lack of a measure-theoretic approach to surface area comparable to the Lebesgue theory and had indicated the desirability of such an approach.

Certain results of W. K. Moore [4] concerning generalized derivatives seemed to indicate that under reasonable conditions one could expect to be able to compute the area of a surface as the integral of the derivative of a set function connected with the surface, the integral to be taken with respect to a suitably defined measure.

In this paper only the abstract measure-theoretic tools used in the approach are developed; applications to surface area will be reserved for a later paper, where it will be shown that for an important class of surfaces (see Radó [8, p. 439]), our treatment gives the same result as the classical Lebesgue approach.

We now summarize the paper.

Throughout, $X$ will be a metric space, $\mathcal{B}$ will be the set of all Borel sets of $X$, and $\mu$ will be a measure on $\mathcal{B}$. Also, $\mathcal{C}$ will be a set of subsets of $X$, usually a $\mu$-Vitali covering of $X$ (cf. Definition 1 in §2), and $\lambda$ will be an extended-real-valued function whose domain includes $\mathcal{C}$. In §3 we define the lower measure $m_\lambda$ (whose domain includes $\mathcal{B}$) by applying Munroe's Method II (cf. [7, p. 105]) to $\lambda$ and $\mathcal{C}$. In §4 we define the upper measure $m_\lambda^*$ similarly by using open coverings directed by refinement (rather than the explicit metric on $X$). In §5 we use standard methods to prove that, under certain conditions, the $\mu$-nonsingular component of $m_\lambda$ is the indefinite integral of the lower derivative of $\lambda$ with respect to $\mu$ and $\mathcal{C}$ (cf. Definition 2 of §2 and Theorem 9 and Remark 10 of §5). In §6 we use standard methods to prove...
that, under certain conditions, \( m^o \) is the integral of the upper derivative of \( \lambda \) with respect to \( \mu \) and \( \mathcal{E} \) (cf. Definition 2 of §2 and Theorem 12 of §6). In §7 we investigate the relationship between \( m_0 \) and \( m^o \) and formulate conditions in terms of \( m_0 \) and \( m^o \) sufficient for the differentiability of \( \lambda \) with respect to \( \mu \) and \( \mathcal{E} \) almost everywhere (mod \( \mu \)) (cf. Theorem 15 of §7 and its corollary). We specialize our results to the case in which \( \lambda \) is a measure in Theorem 16. In §8 we specialize to the case in which \( X \) is a subset of Euclidean \( q \)-space and \( \mu \) is the appropriate restriction of Lebesgue measure.

2. Preliminaries. We let \( X \) be a fixed metric space with metric \( d \), and we let \( \mathcal{E}(X) \) be the class of all subsets of \( X \). \( \mathcal{B} \) is the class of all Borel sets in \( X \) and \( \mu \) is a measure on \( \mathcal{B} \) with a regular completion

\[
\bar{\mu}: \mathcal{B}^* \to \{ t \mid 0 \leq t \leq +\infty \}
\]

where \( \mathcal{B}^* \) is the domain of the completion of \( \mu \).

A Baire set (or Borel set) of \( X \) is any member of the sigma-ring generated by the set of open sets of \( X \). A Baire function is an extended-real-valued function \( f \) on \( X \) such that \( f^{-1}(U) \) is a Baire set for each open set \( U \) of real numbers.

Definition 1. A \( \mu \)-Vitali covering\(^{\text{(3)}} \) of \( X \) is a set \( \mathcal{C} \subseteq \mathcal{B}^* \) such that

1. \( \emptyset \subseteq \mathcal{C} \).
2. \( 0 < \bar{\mu}(C) < +\infty \) for each nonvoid \( C \subseteq \mathcal{C} \).
3. For each positive integer \( n \) the union of some countable subset of

\[
\mathcal{C}_n = \left\{ C \mid C \subseteq \mathcal{C}, \text{diam}(C) \leq \frac{1}{n} \right\}
\]

is \( X \).
4. If \( \mathcal{D} \subseteq \mathcal{C} \), \( \mathcal{D}_n = \mathcal{C}_n \cap \mathcal{D} \) for \( n = 1, 2, \ldots \), and \( A \subseteq \bigcup_{n=1}^{\infty} \bigcup_{D \in \mathcal{D}_n} D \), then there is a sequence \( \{ D_n \}_{n=1}^{\infty} \) in \( \mathcal{D} \) such that \( D_m \cap D_n = \emptyset \) for \( m \neq n \) and such that \( \mu(A - \bigcup_{n=1}^{\infty} D_n) = 0 \).

On the line, the set of all closed intervals is an \( L \)-Vitali covering, where \( L \) is the Lebesgue measure in \( R^1 \).

Remark 1. The existence of a \( \mu \)-Vitali covering implies that \( X \) is separable and that \( \mu \) is sigma-finite\(^{\text{(3)}} \). To prove \( \mu \) is sigma-finite, one could first show from (3) that \( \bar{\mu} \) is sigma-finite. It would then follow that \( \mu \) is sigma-finite, for, given \( B^* \subseteq \mathcal{B}^* \), there is a \( B \subseteq \mathcal{B} \) such that \( B^* \subseteq B \) and \( \bar{\mu}(B^*) = \mu(B) \).

Remark 2. Suppose that \( X \) is a Borel subset of \( R^q \), suppose \( d \) is the Euclidean metric, and suppose \( \mu \) is the restriction of Lebesgue measure to the set of Borel sets of \( X \). Suppose \( \mathcal{C} \) is a set of compact subsets of \( X \) such that (1) and (3) hold and such that for some real number \( c > 0 \),

\(^{\text{(3)}}\) For a general discussion of \( \mu \)-Vitali coverings together with more general coverings, in which condition (2) does not appear and (1) is negated, see Hahn-Rosenthal [2, Chapter V, §17]. Also, in the same regard, see Morse [5; 6].

\(^{\text{(3)}}\) For the definition of sigma-finite see Halmos [3, p. 31].
\[ \mu(C) > c[diam(C)]^q \quad \text{for each } C \in \mathcal{C}. \]

Then \( \mathcal{C} \) is a \( \mu \)-Vitali covering. For the statement of a more general theorem for \( R^q \) and pertinent bibliography, see Hahn-Rosenthal [2, §17, Part 5]. It follows, e.g., that the set of all closed cubes is a \( \mu \)-Vitali covering if \( X = R^q \). For the existence of Vitali coverings in a separable metric space, cf. Morse [5; 6].

Informally, what we require of the class \( \mathcal{C} \) is that it be such that each sub-class which, at each point \( t \), contains sets with arbitrarily small diameter which in turn contain \( t \), satisfies the conclusion of the Vitali Covering Theorem with respect to \( \mu \).

Now let \( \mathcal{C} \) be a \( \mu \)-Vitali covering in \( X \), and let \( \lambda \) be a non-negative, extended-real-valued-function whose domain includes \( \mathcal{C} \) and which is such that \( \lambda(\emptyset) = 0 \).

**Definition 2.** The upper \( \mathcal{C} \)-derivative \( D^o(t, \lambda) \) of \( \lambda \) at a point \( t \in X \) (with respect to \( \bar{\mu} \)) is defined by

\[
D^o(t, \lambda) = \limsup_{t \in C \in \mathcal{C}; \ diam(C) \to 0} \frac{\lambda(C)}{\mu(C)}. 
\]

The lower \( \mathcal{C} \)-derivative \( D_o(t, \lambda) \) of \( \lambda \) at a point \( t \in X \) (with respect to \( \bar{\mu} \)) is defined by

\[
D_o(t, \lambda) = \liminf_{t \in C \in \mathcal{C}; \ diam(C) \to 0} \frac{\lambda(C)}{\mu(C)}. 
\]

If for some \( t \in X \), \( D^o(t, \lambda) = D_o(t, \lambda) \), we say that \( \lambda \) is \( \mathcal{C} \)-differentiable at \( t \) (with respect to \( \bar{\mu} \)) and we denote the derivative by \( D(t, \lambda) \).

An extended-real-valued function \( f \) on \( X \) is \( \bar{\mu} \)-measurable if and only if \( f^{-1}(U) \subseteq \mathcal{B}^\infty \) for each open set \( U \) of real numbers.

**Theorem 1**. \( D^o(\cdot, \lambda) \) and \( D_o(\cdot, \lambda) \) are \( \bar{\mu} \)-measurable. If each member of \( \mathcal{C} \) is open, then \( D^o(\cdot, \lambda) \) and \( D_o(\cdot, \lambda) \) are Baire functions.

**Proof.** Suppose \( a \) is a real number. Let

\[ A = \{ t \in X \mid D^o(t, \lambda) \leq a \}. \]

For \( k = 1, 2, \ldots \), there is a sequence \( \{ C_j(k) \}_{j=1}^\infty \) of members of \( \mathcal{C}_k \) such that

\[
\lambda(C_j(k))/\bar{\mu}(C_j(k)) < a + (1/k) \quad (j = 1, 2, \ldots)
\]

and

\[
\bar{\mu} \left( A - \bigcup_{j=1}^\infty C_j(k) \right) = 0.
\]

(\(^{(*)}\) Essentially the first part of this theorem is found in Hahn-Rosenthal [2, Theorem 17.2.2, p. 247]. A proof will be indicated here for completeness.)
Let $B = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} C_j(k)$. Then $\bar{\mu}(A - B) = 0$. If $t \in B$, then $t \in \bigcap_{k=1}^{\infty} C_j(k)$ for some $j_k \ (k = 1, 2, \ldots)$. 

$$D_0(t, \lambda) \leq \lim \inf \frac{\lambda(C_{j_k}(k))}{\bar{\mu}(C_{j_k}(k))} \leq a,$$

and $t \in A$. Since $B \in \mathcal{B}^*$, $B \subset A$, and $\bar{\mu}(A - B) = 0$, we have $A \in \mathcal{B}^*$. Thus $D_0(, \lambda)$ is $\bar{\mu}$-measurable, and similarly $D_0^*(, \lambda)$ is $\bar{\mu}$-measurable. If each member of $\mathcal{C}$ is open, replace $\bigcup_{j=1}^{\infty} C_j(k)$ in the preceding argument by the union of all $C \in \mathcal{C}$ for which $\lambda(C)/\bar{\mu}(C) < a + (1/k)$. Then, modified accordingly, $B$ is a Borel set and $B = A$.

**Definition 3.** $\lambda$ is absolutely continuous with respect to $\bar{\mu}$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that if \{ $C_1, \ldots, C_n$ \} is any finite collection of members of $\mathcal{C}$ for which $\bar{\mu}(C_n \cap C_k) = 0$ for $k \neq m$, then $\sum_{k=1}^{\infty} \frac{\lambda(C_k)}{\bar{\mu}(C_k)} < \delta$ implies that $\sum_{k=1}^{\infty} \lambda(C_k) < \epsilon$.

**Definition 4.** $\lambda$ is $\bar{\mu}$-Lipschitzian (on $\mathcal{C}$) if there is a non-negative constant $K$ such that $\lambda(C) \leq K \bar{\mu}(C)$ for each $C \in \mathcal{C}$.

**Remark 3.** If $\lambda$ is $\bar{\mu}$-Lipschitzian, then $\lambda$ is absolutely continuous with respect to $\bar{\mu}$.

**Definition 5.** The strong upper $\mathcal{C}$-derivative $D_0^*(, \lambda)$ of $\lambda$ at a point $t \in X$ (with respect to $\bar{\mu}$) is defined by

$$D_0^*(t, \lambda) = D_0^*(t, \lambda; \mathcal{C}, \bar{\mu}) = \lim \sup_{\phi \in \mathcal{C}; \text{diam}(\phi) \to 0} \frac{\lambda(\phi)}{\bar{\mu}(\phi)}.$$

The strong lower $\mathcal{C}$-derivative $D_0^*(, \lambda)$ of $\lambda$ at a point $t \in X$ (with respect to $\bar{\mu}$) is defined by

$$D_0^*(t, \lambda) = D_0^*(t, \lambda; \mathcal{C}, \bar{\mu}) = \lim \inf_{\phi \in \mathcal{C}; \text{diam}(\phi) \to 0} \frac{\lambda(\phi)}{\bar{\mu}(\phi)}.$$

If $t \in X$ and $D_0^*(t, \lambda) = D_0^*(t, \lambda)$, we say that $\lambda$ is strongly $\mathcal{C}$-differentiable at $t$ (with respect to $\bar{\mu}$), and $D_0^*(t, \lambda) = D_0^*(t, \lambda) = D_0^*(t, \lambda)$ is the strong $\mathcal{C}$-derivative of $\lambda$ at $t$ (with respect to $\bar{\mu}$).

**Remark 4.** It is easily seen that for each $t \in X$: 

$$D_0^*(t, \lambda) = \max\{D_0(t, \lambda), \lim \sup_{s \to t} D_0(s, \lambda)\};$$

$$D_0^*(t, \lambda) = \min\{D_0(t, \lambda), \lim \inf_{s \to t} D_0(s, \lambda)\}.$$

Hence $D_0^*(, \lambda)$ is the smallest upper semi-continuous extended-real-valued function on $X$ which is $\geq D_0(, \lambda)$ and $D_0^*(, \lambda)$ is the largest lower semi-continuous extended-real-valued function on $X$ which is $\leq D_0(, \lambda)$. Thus $D_0^*(, \lambda)$ is continuous at each point of its domain. $D_0^*(, \lambda)$ and $D_0^*(, \lambda)$ are Borel functions, and hence the domain of $D_0^*(, \lambda)$ is a Borel set.

3. **Construction of the lower measure $m_\lambda$.** In this section M. E. Munroe's Method II (cf. Munroe [9, p. 105]) for the construction of metric outer measures will be sketched for later use.
Let there be given some subclass \( \mathcal{C} \) of \( \mathcal{P}(X) \) such that for each \( n \) a countable collection of members of \( \mathcal{C}_n \) (defined as in (3) of Definition 1 of §1) covers \( X \). We define \( \text{diam}(\emptyset) = 0 \).

Suppose now that we have a function \( \lambda \) whose domain includes \( \mathcal{C} \), whose image is contained in the set of non-negative, extended-real-numbers, and which is such that \( \lambda(\emptyset) = 0 \). For each \( A \in \mathcal{P}(X) \), let

\[
m_n^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \lambda(C_i) \mid C_i \in \mathcal{C}_n \text{ for each } i, \ A \subset \bigcup_{i=1}^{\infty} C_i \right\}.
\]

Since \( \mathcal{C}_{n+1} \subset \mathcal{C}_n \), we have \( m_n^*(A) \leq m_{n+1}^*(A) \) for each \( A \in \mathcal{P}(X) \). Hence, as \( n \to \infty \), \( \{ m_n^*(A) \}_{n=1}^{\infty} \) approaches a limit, finite or infinite, and we define

\[
m_0^*(A) = \lim_{n \to \infty} m_n^*(A) = \sup_n m_n^*(A) \quad (A \in \mathcal{P}(X)).
\]

We then have

**Theorem 2.** The set function \( m_0^* \) is a metric outer measure\(^5\). In particular, each Borel set of \( X \) is \( m_0^* \)-measurable\(^6\).

**Definition 6.** The outer measure \( m_0^* \) will be called the lower outer measure induced by \( \lambda \) and \( \mathcal{C} \). The restriction of \( m_0^* \) to the class of all \( m_0^* \)-measurable sets will be denoted by \( m_0 \) and is called the lower measure generated by \( \lambda \) and \( \mathcal{C} \).

**Remark 5.** The domain of \( m_0 \) contains \( \emptyset \) by Theorem 2.

**4. Construction of the upper measure \( m^0 \).** In this section \( \mathcal{C} \) is a subset of \( \mathcal{P}(X) \) and such that \( \emptyset \in \mathcal{C} \), and \( \lambda \) is a non-negative and extended-real-valued function whose domain includes \( \mathcal{C} \) and is such that \( \lambda(\emptyset) = 0 \).

**Definition 7.** For each non-void \( \mathcal{U} \subset \mathcal{P}(X) \), let \( \lambda^0(\mathcal{U}) = \sup \left\{ \sum_{j=1}^{\infty} \lambda(C_j) \mid C_j \in \mathcal{C} \text{ for } j = 1, 2, \ldots ; \ \mu(C_j \cap C_k) = 0 \text{ if } j \neq k \text{ for } j = 1, 2, \ldots , C_j \subset U \text{ for some } U \in \mathcal{U} \right\} \).

**Remark 6.** \( \lambda^0(\emptyset) \geq 0 \) for each \( \emptyset \subset \mathcal{P}(X) \).

**Definition 8.** If \( \mathcal{U} \) and \( \mathcal{V} \) are nonvoid sets contained in \( \mathcal{P}(X) \), then \( \mathcal{V} \) refines \( \mathcal{U} \) if and only if each member of \( \mathcal{V} \) is a subset of a member of \( \mathcal{U} \).

**Remark 7.** Observe that if \( \mathcal{V} \) refines \( \mathcal{U} \), then \( \lambda^0(\mathcal{V}) \leq \lambda^0(\mathcal{U}) \).

**Definition 9.** For each \( A \in \mathcal{P}(X) \) let \( \text{cov}(A) \) be the directed system of nonvoid coverings of \( A \) by open members of \( \mathcal{P}(X) \), with \( \text{cov}(A) \) directed by refinement. (Then \( \{ \emptyset \} \in \text{cov}(\emptyset) \), but \( \emptyset \notin \text{cov}(\emptyset) \).

It follows that \( \lambda^0 \) is nonincreasing on \( \text{cov}(A) \) and converges to a limit. We then define

\[
m_0^*(A) = \lim_{\mathcal{U} \in \text{cov}(A)} \lambda^0(\mathcal{U}) = \inf_{\mathcal{U} \in \text{cov}(A)} \lambda^0(\mathcal{U}) \quad (A \in \mathcal{P}(X)).
\]

\(^{(5)}\) For the definition of *metric outer measure*, cf. Munroe [7, pp. 85, 101].

\(^{(6)}\) For the definition of *\( m_0^* \)-measurable* and the standard theorem used here, cf. Munroe [7, pp. 86, 104].
**Lemma 1.** If \( U_i \) is a nonvoid subset of \( \mathcal{P}(X) \) for \( j = 1, 2, \ldots \), then

\[
\lambda^\circ \left( \bigcup_{j=1}^{\infty} U_i \right) \leq \sum_{j=1}^{\infty} \lambda^\circ(U_i).
\]

**Proof.** Let \( A_0 = \bigcup_{j=1}^{\infty} U_i \) and let \( \{ C_j \}_{j=1}^{\infty} \) be one of those systems used in the definition of \( \lambda^\circ(A_0) \). Then (where \( N_i \) is a non-negative integer or \( \infty \))

\[
\sum_{j=1}^{\infty} \lambda(C_j) \leq \sum_{i=1}^{N_i} \sum_{j=1}^{\infty} \lambda(C_{j,i})
\]

where \( \{ C_{j,i} \}_{j=1}^{N_i} \) is the set of those members of \( \{ C_j \}_{j=1}^{\infty} \) such that \( C_{j,i} \subset U \in U_i \). Now clearly

\[
\sum_{j=1}^{\infty} \lambda(C_j) \leq \lambda^\circ(U_i),
\]

hence

\[
\sum_{j=1}^{\infty} \lambda(C_j) \leq \sum_{i=1}^{\infty} \lambda^\circ(U_i),
\]

and so

\[
\lambda^\circ \left( \bigcup_{j=1}^{\infty} U_i \right) = \lambda^\circ(A_0) = \sup \left\{ \sum_{j=1}^{\infty} \lambda(C_j) \right\} \leq \sum_{i=1}^{\infty} \lambda^\circ(U_i).
\]

**Theorem 3.** \( m^\circ \) is a metric outer measure on \( \mathcal{P}(X) \). The domain of its restriction \( m^\circ \) to the class of \( m^\circ \)-measurable sets contains \( \emptyset \).

**Proof.** We first show that \( m^\circ \) is an outer measure and then that it is a metric outer measure. Since

\[
0 \leq m^\circ(\emptyset) \leq \lambda^\circ(\{ \emptyset \}) = 0, \quad m^\circ(\emptyset) = 0.
\]

Suppose \( A \subset \bigcup_{j=1}^{\infty} A_j \in \mathcal{P}(X) \), and suppose \( \epsilon > 0 \). For \( j = 1, 2, \ldots \), there is a \( U_j \in \text{cov}(A_j) \) such that \( \lambda^\circ(U_j) \leq m^\circ(A_j) + 2^{-j} \cdot \epsilon \). Then \( \bigcup_{j=1}^{\infty} U_j \in \text{cov}(A) \) and

\[
m^\circ(A) \leq \lambda^\circ \left( \bigcup_{j=1}^{\infty} U_j \right) \leq \sum_{j=1}^{\infty} \lambda^\circ(U_j)
\]

\[
\leq \sum_{j=1}^{\infty} \left[ m^\circ(A_j) + 2^{-j} \cdot \epsilon \right] = \sum_{j=1}^{\infty} m^\circ(A_j) + \epsilon.
\]

Thus since \( \epsilon \) is arbitrary

\[(?) \text{ See Theorem 2.}\]
In particular, $\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} A_j$; so $m^\circ(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} m^\circ(A_j)$. Hence $m^\circ$ is an outer measure.

Suppose now that $A, B \in \mathcal{F}(X)$, and $d(A, B) > 0$. For some real $\delta > 0$, we have that $d(A, B) > 2\delta$. Let $\epsilon > 0$ be given. Then there is a $\mathcal{U} \in \text{cov}(A \cup B)$ such that $\text{diam} (U) < \delta$ and $U \cap (A \cup B) \neq \emptyset$ for each nonvoid $U \in \mathcal{U}$ and such that

$$m^\circ(A \cup B) + \epsilon > \lambda^\circ(\mathcal{U}).$$

But if $\text{diam}(U) < \delta$ for each $U \in \mathcal{U}$, then $\mathcal{U}$ decomposes into two parts $\mathcal{U}_A$ and $\mathcal{U}_B$ so that

$$\mathcal{U} = \mathcal{U}_A \cup \mathcal{U}_B$$

where

$$\mathcal{U}_A \in \text{cov}(A), \quad \mathcal{U}_B \in \text{cov}(B)$$

and

$$U \cap V = \emptyset \quad \text{for all} \quad U \in \mathcal{U}_A \quad \text{and} \quad V \in \mathcal{U}_B.$$ 

Hence

$$m^\circ(A \cup B) + \epsilon > \lambda^\circ(\mathcal{U}_A) + \lambda^\circ(\mathcal{U}_B) \geq m^\circ(A) + m^\circ(B).$$

Since $\epsilon > 0$ is arbitrary, we have

$$m^\circ(A \cup B) \geq m^\circ(A) + m^\circ(B).$$

Hence $m^\circ$ is a metric outer measure.

The proof of the following theorem is routine.

**Theorem 4.** Suppose $\mu$ is regular (**), and suppose that $\lambda \mid \mathcal{C}$ is absolutely continuous with respect to $\mu$. Then each member of $\mathcal{B}^*$ is $m^\circ$-measurable, and $m^\circ \mid \mathcal{B}^*$ is absolutely continuous with respect to $\mu$.

In the remainder of this section $\mathcal{C}$ and $\lambda$ will be as stipulated in §2. The metric outer measure $m _\sigma^\circ$ of §3 will be shown to be the same as an outer measure about to be defined by means of coverings.

**Definition 10.** For each $A \subseteq X$ and each nonvoid $\mathcal{U} \subseteq \text{cov}(A)$, let

$$\lambda_0(\mathcal{U}; A) = \inf \left\{ \sum_{j=1}^{\infty} \lambda(C_j) \mid C_j \in \mathcal{C} \text{ for } j = 1, 2, \ldots; A \subseteq \bigcup_{j=1}^{\infty} C_j; \text{ for } j = 1, 2, \ldots, C_j \subseteq U \text{ for some } U \in \mathcal{U} \right\}.$$ 

Then $\lambda_0(\mathcal{U}; A)$ is nondecreasing under refinement. For each $A \subseteq X$, let

$$m^\circ_\sigma(A) = \lim_{\mathcal{U} \in \text{cov}(A)} \lambda_0(\mathcal{U}; A) = \sup_{\mathcal{U} \in \text{cov}(A)} \lambda_0(\mathcal{U}; A).$$

(*) Cf. §5, footnote (§).
The proof of the following lemma is straightforward. The details will be omitted.

**Lemma 2.** Suppose \( A_n \subseteq X \) and \( u_n \in \text{cov}(A_n) \) for \( n = 1, 2, \ldots \). Let \( A \subseteq \bigcup_{n=1}^{\infty} A_n \), and let \( u \in \text{cov}(A) \) be such that \( u_n \) refines \( u \) for \( n = 1, 2, \ldots \). Then \( \lambda_0(u; A) \leq \sum_{n=1}^{\infty} \lambda_0(u_n; A_n) \).

The preceding lemma specializes to the finite case via \( A_n = \emptyset \) and \( u_n = \{ \emptyset \} \) for all sufficiently large \( n \).

If \( u_n \) is the set of all open subsets of \( X \) of diameter \( \leq 1/n \), then, for each \( A \subseteq X \),

\[
m_n^*(A) \leq \lambda_0(u_n; A) \leq m_{n+1}^*(A)
\]

for \( n = 1, 2, \ldots \), and

\[
m_0^*(A) = \lim_{n \to \infty} \lambda_0(u_n; A) \leq \lim_{u \in \text{cov}(A)} \lambda_0(u; A) = m_0^*(A).
\]

The following theorem will be proved by showing that \( m_0^*(A) \leq m_0^*(A) \).

**Theorem 5.** \( m_0^* = m_0^* \).

**Proof.** Suppose \( A \subseteq X \) and \( u \in \text{cov}(A) \). For \( n = 1, 2, \ldots \), let \( U_n = \{ t \in X \mid \text{for some } r > 1/n, \text{the open sphere about } t \text{ of radius } r \text{ is a subset of a member of } u \} \).

Then \( U_n \) is open and \( U_n \subseteq U_{n+1} \) for \( n = 1, 2, \ldots \), and

\[
A \subseteq \bigcup_{n=1}^{\infty} U_n = \bigcup_{u \in u} U.
\]

Let \( U_0 = \emptyset \). For \( n = 1, 2, \ldots \), let \( u_n \) be the set of all open subsets of \( X \) of diameter \( \leq 1/n \) which meet \( U_n \), and let \( u_n = u_n \cup \{ \emptyset \} \). Then \( u_n \in \text{cov}(A \cap (U_n - U_{n+1})) \) refines \( u \), and

\[
\lambda_0(u_n; A \cap (U_n - U_{n-1})) \leq m_{n+1}^*(A \cap (U_n - U_{n-1}))
\]

for \( n = 1, 2, \ldots \). By Lemma 2,

\[
\lambda_0(u; A) \leq \sum_{n=1}^{\infty} \lambda_0(u_n; A \cap (U_n - U_{n-1}))
\]

\[
\leq \sum_{n=1}^{\infty} m_{n+1}^*(A \cap (U_n - U_{n-1}))
\]

\[
= \sum_{n=1}^{\infty} m_0^*(A \cap (U_n - U_{n-1}))
\]

\[
= m_0^*(A).
\]
Hence
\[ \overline{m}_o(A) = \sup_{U \in \text{cov}(A)} \lambda_0(U; A) \leq m_o(A). \]

The proof is complete.

**Remark 8.** It follows from the preceding theorem that \( \overline{m}_o \) is a metric outer measure. This can be proved directly without reduction to the corresponding result for \( m_o \). The direct proof would follow the pattern of the proof of the corresponding result for \( m_o \).

5. **Integral representation of \( m_o \).** In this section \( \mathcal{C} \) is a \( \mu \)-Vitali covering of \( X \), and \( \lambda \) is a non-negative extended-real-valued function whose domain includes \( \mathcal{C} \) and which is such that \( \lambda(\emptyset) = 0 \). Then the conditions placed on \( \mathcal{C} \) and \( \lambda \) in \$2 and \$3 hold, and \( D^\circ(\cdot, \lambda) \) and \( D_\circ(\cdot, \lambda) \) are defined as in \$2. It will be assumed also that \( \mu \) is regular\(^9\).

**Remark 9.** Suppose \( \nu \) is a measure defined on the set of Borel sets of a separable metric space \( S \). Then \( \nu \) is regular if and only if each \( s \in S \) has a neighborhood \( N \) such that \( \nu(N - \{s\}) \) is finite\(^{10}\). \( \nu \) is called point-finite if \( \nu(\{s\}) \) is finite for each \( s \in S \) and is called locally finite if each \( s \in S \) has a neighborhood \( N \) such that \( \nu(N) \) is infinite. Thus, if \( \nu \) is point-finite, then \( \nu \) is regular if and only if \( \nu \) is locally finite. Since the existence of a \( \mu \)-Vitali covering implies that \( \mu \) is point-finite and \( X \) separable, the assumption that \( \mu \) is regular is equivalent (under the previous assumption that \( \mathcal{C} \) is a \( \mu \)-Vitali covering of \( X \)) with the condition that \( \mu \) is locally finite.

**Lemma 3.** Let \( B \in \mathcal{B}^* \), and suppose \( K \) is a non-negative real number such that \( D_\circ(t, \lambda) < K \) for all \( t \in B \). If \( \nu: \mathcal{B} \to \{t| 0 \leq t \leq \infty\} \) is a measure which is continuous\(^{11}\) with respect to \( \mu \) and \( \nu(A) \leq m_o(A) \) for all \( A \in \mathcal{B} \), then for the completion \( \bar{\nu} \) of \( \nu \), we have \( \bar{\nu}(B) \leq K \bar{p}(B) \). (The domain of the completion \( \bar{\nu} \) contains \( \mathcal{B}^* \) by continuity.)

**Proof.** Suppose \( \epsilon > 0 \) is given. Since \( \bar{\mu} \) is regular, there is an open set \( U \) such that \( B \subseteq U \) and \( \bar{\mu}(U) < \bar{\mu}(B) + \epsilon \). For each \( t \in B \), there is a sequence \( \{C_n(t)\}_{n=1}^{\infty} \) in \( \mathcal{C} \) such that
\[ \lim_{n \to \infty} \text{diam}(C_n(t)) = 0, \]
and such that
\[ \lambda(C_n(t)) < K \frac{\bar{\mu}(C_n(t))}{\bar{\mu}(C_n(t))} \]
for \( n = 1, 2, \ldots \).

\(^9\) \( \mu \) is regular if and only if for each set \( B \) in the domain of \( \mu \) and each real \( \epsilon > 0 \) there is an open set \( U \subseteq X \) such that \( B \subseteq U \) and \( \mu(U - B) < \epsilon \). If \( \mu \) is regular, so is \( \bar{\mu} \).

\(^{10}\) To prove this, apply the proofs in Halmos [3, p. 52] within open sets of finite measure, and use second countability in the obvious way.

\(^{11}\) We say that \( \nu \) is continuous with respect to \( \mu \) if for each \( A \) in the domain of \( \mu \) for which \( \mu(A) = 0 \), \( A \) is in the domain of \( \nu \) and \( \nu(A) = 0 \).
From the definition of $C$, for $k = 1, 2, \ldots$, there are sequences $\{D_j(k)\}_{j=1}^{\infty}$ such that each nonvoid $D_j(k)$ is one of the sets $C_n(t)$, such that $D_j(k) \cap D_h(k) = \emptyset$ for $j \neq h$, such that $D_j(k) \subseteq \mathcal{C}_n$ for all $j$ and $k$, and such that $\mu(B - \bigcup_{j=1}^{\infty} D_j(k)) = 0$. Let $B_0 = \bigcap_{j=1}^{\infty} \bigcup_{j=1}^{\infty} D_j(k)$. Then $\mu(B - B_0) = 0$ and hence $\tilde{\nu}(B - B_0) = 0$. We then have that

$$\tilde{\nu}(B) \leq \tilde{\nu}(B_0) + \tilde{\nu}(B - B_0) = \tilde{\nu}(B_0) \leq m_0(B_0)$$

$$\leq \liminf_{k \to \infty} \sum_{j=1}^{\infty} \lambda(D_j(k)) \leq \liminf_{k \to \infty} \sum_{j=1}^{\infty} K\mu(D_j(k))$$

$$\leq \liminf_{k \to \infty} K\mu(U) \leq K(\bar{\mu}(B) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, $\tilde{\nu}(B) \leq K\bar{\mu}(B)$.

**Lemma 4.** Let $B \in \mathfrak{B}$. Let $\bar{\mu}(B) < +\infty$, and let $K$ be a non-negative real number such that $D(t, \lambda) < K$ for all $t \in B$. If $\nu: \mathfrak{B} \to \{t \mid 0 \leq t \leq \infty\}$ is a measure which is continuous with respect to $\mu$ and $\nu(A) \leq m_0(A)$ for all $A \in \mathfrak{B}$, then ($\tilde{\nu}$ being the completion of $\nu$)

$$\tilde{\nu}(B) \leq \int_B D_0(, \lambda) d\bar{\mu}.$$

**Proof.** For each positive integer $k$ and $j = 1, 2, \ldots, k$, let

$$B(j, k) = \{t \mid t \in B, (j - 1)K/k \leq D_0(t, \lambda) < jK/k\}.$$

For $k = 1, 2, \ldots,$ let

$$f_k = \sum_{j=1}^{k} \frac{jK}{k} C_{B(j, k)}$$

where $C_{B(j, k)}$ is the characteristic function of $B(j, k)$. Then $f_k$ is $\bar{\mu}$-measurable for each $k$, and $\{f_k\}_{k=1}^{\infty}$ converges uniformly to $D_0(, \lambda)$ on $B$ as $k \to \infty$. Hence

$$\lim_{k \to \infty} \int_B f_k d\bar{\mu} = \int_B D_0(, \lambda) d\bar{\mu}.$$

Using the definition of $B(j, k)$ and applying Lemma 3 to each of the $B(j, k)$ we obtain

$$\tilde{\nu}(B) = \sum_{j=1}^{k} \tilde{\nu}(B(j, k)) \leq \sum_{j=1}^{k} \frac{jK}{k} \bar{\mu}(B(j, k))$$

$$= \int_B f_k d\bar{\mu}$$

for $k = 1, 2, \ldots$.

Hence
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\[ \bar{v}(B) \leq \lim_{k \to \infty} \int_B f_k d\bar{\mu} = \int_B D_\alpha(\cdot, \lambda)d\bar{\mu}. \]

**Theorem 6.** If \( \nu: \mathcal{B} \to [0, \infty] \) is a measure which is continuous with respect to \( \mu \) and \( \nu(B) \leq m_\alpha(B) \) for each \( B \in \mathcal{B} \), then (\( \bar{v} \) being the completion of \( v \))

\[ \bar{v}(B) \leq \int_B D_\alpha(\cdot, \lambda)d\bar{\mu} \quad \text{for all } B \in \mathcal{B}^*. \]

**Proof.** Suppose \( B \in \mathcal{B}^* \). We may presume \( \int_B D_\alpha(\cdot, \lambda)d\bar{\mu} < +\infty \), for otherwise the inequality of Theorem 6 is trivial. For \( n = 1, 2, \ldots \), let

\[ A_n = \left\{ t \mid t \in B, \ \frac{1}{n+1} \leq D_\alpha(t, \lambda) < \frac{1}{n} \right\}. \]

\[ B_0 = \left\{ t \mid t \in B, 0 < D_\alpha(t, \lambda) < +\infty \right\}. \]

\[ B_n = \left\{ t \mid t \in B, n \leq D_\alpha(t, \lambda) < n+1 \right\}. \]

\[ B' = \left\{ t \mid t \in B, D_\alpha(t, \lambda) = 0 \right\}. \]

\[ B'' = \left\{ t \mid t \in B, D_\alpha(t, \lambda) = +\infty \right\}. \]

Then, for \( n = 1, 2, \ldots \),

\[ \bar{\mu}(A_n) \leq (n+1) \int_{A_n} D_\alpha(\cdot, \lambda)d\bar{\mu} < +\infty, \]

\[ \bar{\mu}(B_n) \leq \frac{1}{n} \int_{B_n} D_\alpha(\cdot, \lambda)d\bar{\mu} < +\infty. \]

Since \( \mu \) is sigma-finite, Lemma 4 may be applied countably often to yield

\[ \bar{v}(B') \leq \int_{B'} D_\alpha(\cdot, \lambda)d\bar{\mu} = 0. \]

Also \( B_0 \) is the disjoint union of the \( A_n \)'s and the \( B_n \)'s with \( n > 0 \). Hence, by Lemma 4,

\[ \bar{v}(B_0) = \sum_{n=1}^{\infty} \left\{ \bar{v}(A_n) + \bar{v}(B_n) \right\} \]

\[ \leq \sum_{n=1}^{\infty} \left\{ \int_{A_n} D_\alpha(\cdot, \lambda)d\bar{\mu} + \int_B D_\alpha(\cdot, \lambda)d\bar{\mu} \right\} \]

\[ = \int_B D_\alpha(\cdot, \lambda)d\bar{\mu}. \]

If \( \bar{\mu}(B'') = 0 \), then
\begin{align*}
\tilde{v}(B_0) + \tilde{v}(B) + \tilde{v}(B'') &= \tilde{v}(B_0) \leq \int_B D_\omega(\lambda) d\mu \leq \int_B D_\omega(\lambda) d\mu.
\end{align*}

If \( \tilde{v}(B'') > 0 \), then
\[
\tilde{v}(B) \leq \int_B D_\omega(\lambda) d\mu = + \infty.
\]

**Theorem 7.** \( \int_B D_\omega(\lambda) d\mu \leq m_\omega(B) \) for all \( B \in \mathcal{B}(12) \).

**Proof.** Suppose \( B \in \mathcal{B} \). For \( k = 1, 2, \ldots \), there is a sequence \( \{ C_n(k) \}_{n=1}^\infty \) in \( \mathcal{C}_k \) such that \( B \subset \bigcup_{n=1}^\infty C_n(k) \) and such that
\[
\sum_{n=1}^\infty \lambda(C_n(k)) \leq m_\omega(B) + 2^{-k}.
\]

For \( k = 1, 2, \ldots \), let
\[
f_k = \sum_{n=1}^\infty \frac{\lambda(C_n(k))}{\mu(C_n(k))} C_{C_n(k)}(t)
\]
where \( C_{C_n(k)} \) is the characteristic function of \( C_n(k) \). Then
\[
D_\omega(t, \lambda) \leq \liminf_{k \to \infty} f_k(t) \quad \text{for each } t \in B.
\]

Hence, by Fatou's Lemma
\[
\int_B D_\omega(\lambda) d\mu \leq \int_B \liminf_{k \to \infty} f_k d\mu
\]
\[
\leq \liminf_{k \to \infty} \int_B f_k d\mu = \liminf_{k \to \infty} \sum_{n=1}^\infty \frac{\lambda(C_n(k))}{\mu(C_n(k))} \mu(C_n(k))
\]
\[
= \liminf_{k \to \infty} \sum_{n=1}^\infty \lambda(C_n(k)) \leq \liminf_{k \to \infty} (m_\omega(B) + 2^{-k}) = m_\omega(B).
\]

**Theorem 8.** Suppose that \( D_\omega(\lambda) \) is \( \mu \)-integrable over \( X \). Then there is a maximum measure \( \nu \) on \( \mathcal{B} \) such that \( \nu \) is absolutely continuous with respect to \( \mu \) and \( \nu(B) \leq m_\omega(B) \) for each \( B \in \mathcal{B} \). It is the measure \( m_\omega \) on \( \mathcal{B} \) given by
\[
m_\omega(B) = \int_B D_\omega(\lambda) d\mu \quad \text{for all } B \in \mathcal{B}.
\]

---

\(^{(1)}\) In this theorem and its proof the assumption that \( \mu \) is regular may be dispensed with.

\(^{(2)}\) Regard \( \lambda(C_n(k))/\mu(C_n(k)) = 0 \) if \( C_n(k) = \emptyset \), i.e., if \( \mu(C_n(k)) = 0 \).

\(^{(3)}\) See Munroe [7, Corollary 27.1.1, p. 191].

\(^{(4)}\) We say that a measure \( \nu \) is absolutely continuous with respect to a measure \( \mu \) if the domain of \( \nu \) is part of the domain of \( \mu \) and if \( \epsilon > 0 \) implies the existence of a \( \delta > 0 \) such that if \( \nu(A) \) is defined, then \( \nu(A) < \epsilon \) whenever \( \mu(A) < \delta \).
Proof. $m_{o0}$ is absolutely continuous with respect to $\mu$, and $m_{o0}(B) \leq m_o(B)$ by Theorem 7. The maximum property of $m_{o0}$ follows from Theorem 6.

Lemma 5. Suppose $\nu$ is a sigma-finite measure on the sigma-algebra $S$ of certain subsets of $S \subseteq S$. Suppose $f$ is an extended-real-valued $\nu$-measurable function on $S$. Then $f$ is $\nu$-sigma-integrable$^{(16)}$ if and only if $f$ is finite almost everywhere (mod $\nu$).

Proof. $S = \bigcup_{n=1}^{\infty} S_n$ with $S_n \subseteq S$ and $\nu(S_n) < +\infty$ for $n = 1, 2, \cdots$. Suppose $f$ is finite almost everywhere (mod $\nu$). Let

$$A_0 = \{ x \in S \mid f(x) = +\infty \}.$$ 

Then $\nu(A_0) = 0$. Let

$$A_n = \{ x \in S \mid f(x) < n \}$$

for $n = 1, 2, \cdots$. Then

$$S = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} (S_n \cap A_k),$$

and $f$ is $\nu$-integrable over $S_n \cap A_k$ for $n = 1, 2, \cdots$, for $k = 0, 1, \cdots$. Thus $f$ is $\nu$-sigma-integrable. One part of the lemma follows; the other part is trivial.

Theorem 9. Suppose $D_0(\mu, \lambda)$ is finite almost everywhere (mod $\mu$). Then there is a maximum measure $\nu$ on $\mathcal{B}$ such that $\nu$ is continuous with respect to $\mu$ and $\nu(B) \leq m_o(B)$ for each $B \subseteq \mathcal{B}$. It is the measure $m_{o0}$ on $\mathcal{B}$ given by $m_{o0}(B) = \int_B D_0(\mu, \lambda) d\mu$ for all $B \subseteq \mathcal{B}$.

Proof. It follows from Lemma 5 that $m_{o0}$ is continuous with respect to $\mu$, and $m_{o0} \leq m_o$ by Theorem 7. The maximum property of $m_{o0}$ follows from

Remark 10. In Theorems 8 and 9, $m_{o0}$ may be called the $\mu$-nonsingular component$^{(17)}$ of $m_o$.

Theorem 10. If $m_o$ is continuous with respect to $\mu$, then $m_o(B) = \int_B D_0(\mu, \lambda) d\mu$ for all $B \subseteq \mathcal{B}^*$.

Proof. Apply Theorems 6 and 7 with $\nu = m_o$.

6. Integral representation of $m^o$. In this section $\mathcal{C}$ and $\lambda$ are assumed to be given as at the beginning of §5, but $\mu$ will not be assumed to be regular.

$^{(16)}$ $f$ is $\nu$-sigma-integrable if and only if there is a sequence $\{ S_n \}_{n=1}^\infty$ of members of $S$ such that $S = \bigcup_{n=1}^\infty S_n$ and $\int_S |f| d\nu < +\infty$ for $n = 1, 2, \cdots$.

$^{(17)}$ If $m_o$ is sigma-finite, then $m_{o0}$ is the nonsingular component in a Lebesgue decomposition of $m_o$ with respect to $\mu$, which exists (uniquely) by a standard theorem (cf., e.g., Halmos [3, p. 134]). The standard theorem on the existence of Lebesgue decomposition does not give Theorems 8 and 9 since $m_o$ is not assumed to be sigma-finite.
Lemma 6. Let $B \in \mathcal{B}$, and suppose $K$ is a real number such that $D^\circ(t, \lambda) > K$ for all $t \in B$. Then $m^\circ(B) \geq K \mu(B)$.

Proof. Suppose $\mathcal{U} \in \text{cov}(B)$. For each $t \in B$, there is a sequence $\{C_n(t)\}_{n=1}^\infty$ in $\mathcal{C}$ such that $\lim_{n \to \infty} [\text{diam}(C_n(t))] = 0$, $t \in C_n(t) \subseteq U$ for some $U \in \mathcal{U}$ and

$$\frac{\lambda(C_n(t))}{\mu(C_n(t))} > K$$

for $n = 1, 2, \ldots$.

Because $\mathcal{C}$ is a $\mu$-Vitali covering, there is a sequence $\{D_j\}_{j=1}^\infty$ such that each nonvoid $D_j$ is one of the sets $C_n(t)$, such that $D_j \cap D_h = \emptyset$ for $j \neq h$, and such that $\bar{\mu}(B - B_0) = 0$ where $B_0 = \bigcup_{j=1}^\infty D_j$. Then

$$\lambda^\circ(\mathcal{U}) \geq \sum_{j=1}^\infty \lambda(D_j) > \sum_{j=1}^\infty K \mu(D_j)$$

$$\geq K \mu(B_0) = K \mu(B_0) + K \mu(B - B_0)$$

$$= K \mu(B) = K \mu(B).$$

Finally

$$m^\circ(B) = \lim_{\mathcal{U} \in \text{cov}(B)} \lambda^\circ(\mathcal{U}) \geq K \mu(B).$$

Lemma 7. Let $B \in \mathcal{B}$, let $\mu(B) < +\infty$, and let $K$ be a real number such that $D^\circ(t, \lambda) < K$ for all $t \in B$. Then

$$m^\circ(B) \geq \int_B D^\circ(\cdot, \lambda) d\bar{\mu}.$$

Proof. For each positive integer $k$ and $j = 1, 2, \ldots, k$, let

$$B(j, k) = \{t \mid t \in B, (j - 1)K/k \leq D^\circ(t, \lambda) < jK/k\}.$$

For $k = 1, 2, \ldots$, let

$$f_k = \sum_{j=1}^k \frac{(j - 1)K}{k} C_{B(j, k)}$$

where $C_{B(j, k)}$ is the characteristic function of $B(j, k)$. Then $f_k$ is $\bar{\mu}$-measurable for each $k$ and converges uniformly to $D^\circ(\cdot, \lambda)$ on $B$ as $k \to \infty$. Thus

$$\int_B D^\circ(\cdot, \lambda) d\bar{\mu} = \lim_{k \to \infty} \int_B f_k d\bar{\mu}.$$

By Lemma 6 applied to $B(j, k)$, and from the definition of $B(j, k),$

$$m^\circ(B) \geq \sum_{j=1}^k m^\circ(B(j, k)) \geq \sum_{j=1}^k \frac{(j - 1)K}{k} \bar{\mu}(B(j, k)) = \int_B f_k d\bar{\mu}.$$
for $k = 1, 2, \cdots$. Hence

$$m^\circ(B) \geq \lim_{k \to \infty} \int_B f_k d\tilde{\mu} = \int_B D^\circ(\cdot, \lambda) d\tilde{\mu}.$$  

**Theorem 11.** For each $B \in \mathfrak{B}$,

$$m^\circ(B) \geq \int_B D^\circ(\cdot, \lambda) d\tilde{\mu}.$$  

**Proof.** Since $\mu$ is sigma-finite, it suffices to prove the inequality for $\mu(B)$ finite.

The proof from here on is the same, mutatis mutandis, as the proof of Theorem 8 and uses Lemma 7.

**Lemma 8.** Let $B \in \mathfrak{B}$ and let $\{f_n\}_{n=1}^\infty$ be a sequence of non-negative, $\tilde{\mu}$-measurable, real-valued functions defined on $B$ such that

$$\int_B \sup_{n=1}^\infty f_n d\tilde{\mu} < +\infty.$$  

Then

$$\int_B \limsup_{n \to \infty} f_n d\tilde{\mu} \leq \limsup_{n \to \infty} \int_B f_n d\tilde{\mu}.$$  

**Proof.** Let $g = \sup_{n=1}^\infty f_n$, let $g_j = g - f_j$ for $j = 1, 2, \cdots$, and apply Fatou's Lemma to $\{g_j\}_{j=1}^\infty$.

**Theorem 12.** Suppose $B \in \mathfrak{B}^*$, and suppose $D^{\circ*}(\cdot, \lambda)$ is finite almost everywhere on $B$ (mod $\tilde{\mu}$). Then, if $\mu$ is regular,

$$m^\circ(B) = \int_B D^\circ(\cdot, \lambda) d\tilde{\mu}.$$  

**Proof.** It may be supposed that $D^{\circ*}(\cdot, \lambda)$ is finite everywhere on $B$. For $k = 1, 2, \cdots$, let

$$B_k = \{t \in B \mid k - 1 \leq D^{\circ*}(t, \lambda) < k\}.$$  

Then $\{B_k\}_{k=1}^\infty$ is a partitioning of $B$ by members of $\mathfrak{B}^*$. It suffices to show that for $k = 1, 2, \cdots$,

$$m^\circ(B_k) = \int_{B_k} D^\circ(\cdot, \lambda) d\tilde{\mu}.$$  

Thus we may assume that for some real number $r > 0$, $D^{\circ*}(t, \lambda) < r$ for all $t \in B$. Moreover, by standard arguments we may assume $\tilde{\mu}(B)$ finite (since $\tilde{\mu}$ is sigma-finite). Since $\tilde{\mu}$ is regular, there is an open set $V \subset X$ such that
Let $B \subseteq V$ and $\mu(V)$ is finite. For each $t \in B$ there is an open neighborhood $U_t \subseteq V$ of $t$ in $X$ such that for each $C \subseteq E$ for which $\emptyset \neq C \subseteq U_t$, $\lambda(C)/\mu(C) < r$. Let $\mathcal{U} = \{U \mid U = U_t \text{ for some } t \in B\}$. Then $\mathcal{U} \subseteq \text{cov}(B)$. For $k = 1, 2, \cdots$, there is some $\mathcal{U}_k \subseteq \text{cov}(B)$ such that $\mathcal{U}_k$ refines $\mathcal{U}$, such that each member of $\mathcal{U}_k$ has diameter $< 1/k$, and such that

$$\lambda^o(\mathcal{U}_k) < m^o(B) + \frac{1}{k}.$$  

For $k = 1, 2, \cdots$, there is a sequence $\{C_n(k)\}_{n=1}^\infty$ such that $C_n(k) \subseteq \mathcal{C}_k$ for each $n$, such that each $C_n(k)$ is a subset of a member of $\mathcal{U}_k$, such that $\mu(C_p(k) \cap C_n(k)) = 0$ if $p \neq n$, and such that

$$\sum_{n=1}^\infty \lambda(C_n(k)) > \lambda^o(\mathcal{U}_k) - 1/k \geq m^o(B) - 1/k.$$  

For $k = 1, 2, \cdots$, let

$$f_k = \sum_{n=1}^\infty \frac{\lambda(C_n(k))}{\mu(C_n(k))} C_{C_n(k)}$$

where $C_{C_n(k)}$ is the characteristic function of $C_n(k)$ on $X$. Then for $k = 1, 2, \cdots$ and almost every $t \in X \pmod{\mu}$, $f_k(t) \leq r$. By Lemma 8, since $D^o(t, \lambda) \geq \limsup_{k \to \infty} f_k(t)$ for almost all $t \in B \pmod{\mu}$,

$$\int_B D^o(\ , \lambda) d\mu \geq \int_B \left( \limsup_{k \to \infty} f_k \right) d\mu$$

$$\geq \limsup_{k \to \infty} \int_B f_k d\mu$$

$$= \limsup_{k \to \infty} \sum_{n=1}^\infty \lambda(C_n(k)) \geq m^o(B).$$

By Theorem 11,

$$m^o(B) \geq \int_B D^o(\ , \lambda) d\mu.$$  

Thus

$$m^o(B) = \int_B D^o(\ , \lambda) d\mu.$$  

**Corollary 12.1.** If $D^o(\ , \lambda)$ is finite almost everywhere on $X \pmod{\mu}$, then $m^o(B) = \int_B D^o(\ , \lambda) d\mu$ for all $B \subseteq \mathcal{B}^*$, and $m^o$ is continuous with respect to $\mu$. If $D^o(\ , \lambda)$ is finite almost everywhere on $X \pmod{\mu}$ and $D^o(\ , \lambda)$ is $\mu$-integrable on $X$, then $m^o$ is absolutely continuous with respect to $\mu$. 

7. Relationship between \( m_0 \) and \( m^o \). Throughout this section, as in §3, \( \mathcal{C} \) is a \( \mu \)-Vitali covering of \( X \), and \( \lambda \) is a non-negative extended-real-valued function whose domain includes \( \mathcal{C} \) and which is such that \( \lambda(\emptyset) = 0 \). It will be assumed also that \( \mu \) is regular. Thus the theorems of §5 and §6 will apply.

Since \( D_0(\cdot, \lambda) \) and \( D^o(\cdot, \lambda) \) are non-negative, we may define measures \( \mu_0 \) and \( \mu^o \) on \( \mathcal{B}^* \) by \( \mu_0(B) = \int_B D_0(\cdot, \lambda) \, d\mu, \mu^o(B) = \int_B D^o(\cdot, \lambda) \, d\mu \) for all \( B \in \mathcal{B}^* \). Then \( \mu_0 \) (resp., \( \mu^o \)) is continuous with respect to \( \mu \) if and only if \( D_0(\cdot, \lambda) \) (resp., \( D^o(\cdot, \lambda) \)) is finite almost everywhere \( \pmod{\mu} \), in which case \( \mu_0 \) (resp., \( \mu^o \)) is also sigma-finite by Lemma 5. If \( D_0(\cdot, \lambda) \) is finite almost everywhere \( \pmod{\mu} \), then (cf. Theorem 9 and Remark 10), \( \mu_0 \) is the \( \mu \)-non-singular component of \( m_0 \).

By Theorems 7 and 11, \( \mu_0 \leq \mu^o \leq m^o \). In particular, if \( m_0 \) is continuous with respect to \( \mu \), then, by Theorem 10, \( m_0 = m_0 \leq m^o \). The inequality \( m_0 \leq m^o \) in case \( m_0 \) is continuous with respect to \( \mu \) also is a corollary of Theorem 13 below. Finally, \( \lambda \) is differentiable with respect to \( \mu \) almost everywhere \( \pmod{\mu} \) \( (+\infty \text{ allowed as a value of the derivative}) \) if and only if \( m_0 = m^o \).

The relation between \( \mu^o \) and \( \mu^* \) might be clarified by the study of a more general situation. In the following definition and three lemmas, \( \nu, \nu_1, \) and \( \nu_2 \) are measures on a sigma-ring \( \mathcal{S} \) of subsets of a set \( S \in \mathcal{S} \), and \( \nu^\#_1 \) and \( \nu^\#_2 \) are outer measures on the set of all subsets of \( S \) (no relation between \( \nu^\#_1 \) and \( \nu^\#_2 \) need be assumed). Also, \( \bar{\nu} \) with domain \( \bar{\mathcal{S}} \) will be the completion of \( \nu \).

Definition 11. \( \nu_2 \) (resp., \( \nu^\#_2 \)) will be said to \( \nu \)-dominate \( \nu_1 \) (resp., \( \nu^\#_1 \)) if and only if for each \( A \in \mathcal{S} \) (resp., \( A \subset S \)) there is some \( B \in \mathcal{S} \) such that \( \nu(A - B) = 0 \) and \( \nu_1(B) \leq \nu_2(A) \) (resp., \( \nu^\#_1(B) \leq \nu^\#_2(A) \)).

Lemma 9. Suppose \( \nu \) is sigma-finite. Then there is a set \( H \in \mathcal{S} \) such that (i) and (ii) below hold.

(i) \( \nu_1 \) (resp., \( \nu^\#_1 \) or \( \nu^\#_1 \mid \mathcal{S} \)) is sigma-finite on \( H \);
(ii) if \( A \in \mathcal{S} \) is such that \( \nu_1 \) (resp., \( \nu^\#_1 \) or \( \nu^\#_1 \mid \mathcal{S} \)) is sigma-finite on \( A \), then \( \nu(A - H) = 0 \). (Briefly, there is a \( \nu \)-maximal set \( H \in \mathcal{S} \) on which \( \nu_1 \) (resp., \( \nu^\#_1 \) or \( \nu^\#_1 \mid \mathcal{S} \)) is sigma-finite.)

Proof. Standard arguments reduce the lemma to the case in which \( \nu(S) \) is finite. So suppose \( \nu(S) \) is finite. (The proof for \( \nu^\#_1 \) and \( \nu^\#_1 \mid \mathcal{S} \) are sufficiently similar to that for \( \nu_1 \) that they will be omitted.) Let

\[
\mathcal{C} = \sup\{\nu(A) \mid A \in \mathcal{S} \text{ and } \nu_1 \text{ is sigma-finite on } A\}.
\]

For \( n = 1, 2, \ldots \), there is a set \( H_n \in \mathcal{S} \) such that \( \nu_1 \) is sigma-finite on \( A \) and \( \nu(H_n) > \mathcal{C} - (1/n) \). Let \( H = \bigcup_{n=1}^\infty H_n \). Then (i) and (ii) may be verified by standard arguments.

Lemma 10. Suppose \( \nu \) is sigma-finite, and suppose \( \nu_1 \) or \( \nu_2 \) is sigma-finite (resp., \( \nu^\#_1 \mid \mathcal{S} \) or \( \nu^\#_2 \mid \mathcal{S} \) is sigma-finite and \( \nu^\#_1 \mid \mathcal{S} \) is additive). Suppose \( \nu_2 \) (resp., \( \nu^\#_2 \)) \( \nu \)-dominates \( \nu_1 \) (resp., \( \nu^\#_1 \)). Then there is a set \( K \in \mathcal{S} \) such that \( \nu(S - K) = 0 \) and
such that \( v_2(Z) \leq v_2^*(Z) \) (resp., \( v_2^*(Z) \leq v_2^*(Z) \)) for each \( Z \in S \) with \( Z \subseteq K \) (resp., for each \( Z \subseteq K \)).

**Proof.** The proof for \( v_1 \) and \( v_2 \) is sufficiently similar to that for \( v_1^* \) and \( v_2^* \) that the former proof will be omitted. Standard arguments reduce the lemma to the case in which \( v_1^*(S) \) or \( v_2^*(S) \) is finite, and then \( \nu \)-domination reduces the lemma to the case in which \( v_1^*(S) \) is finite. So suppose \( v_1^*(S) \) is finite. Let

\[
\begin{align*}
c &= \sup \{ v_1^*(A) \mid A \in S \text{ and } \nu(A) = 0 \}. \end{align*}
\]

Then \( c < +\infty \). For \( n = 1, 2, \ldots \), there is a set \( D_n \in S \) such that \( \nu(D_n) = 0 \) and \( v_1^*(D_n) > c - (1/n) \). Let \( D = \bigcup_{n=1}^{\infty} D_n \) and \( K = S - D \). Then \( \nu(S - K) = \nu(D) = 0 \). Suppose \( Z \subseteq K \). By \( \nu \)-domination there is \( B \in S \) such that \( \nu(Z - B) = 0 \) and \( v_1^*(B) \leq v_2^*(Z) \). There is an \( F \in S \) such that \( Z - B \subseteq F \) and \( \nu(F) = 0 \). Let \( G = (K - B) \cap F \). Then \( G \in S \), \( Z - B \subseteq G \subseteq K \), \( G \cap B = \emptyset \), \( G \cap D = \emptyset \), and \( \nu(G) = 0 \). Since \( v_1^* \mid S \) is additive, for \( n = 1, 2, \ldots \),

\[
\begin{align*}
c &\geq v_1^*(D_n \cup G) = v_1^*(D_n) + v_1^*(G) \geq [c - (1/n)] + v_1^*(G). \end{align*}
\]

Hence \( v_1^*(G) = 0 \), and

\[
\begin{align*}
v_1^*(Z) &\leq v_1^*(B) + v_1^*(Z - B) \leq v_1^*(B) + v_1^*(G) \leq v_2^*(Z). \end{align*}
\]

The lemma is established.

**Lemma 11.** Suppose \( \nu \) is sigma-finite (and \( v^*_1 \mid S \) is a measure). Suppose \( \nu_2 \) (resp., \( v_2^* \)) \( \nu \)-dominates \( v_1 \) (resp., \( v_1^* \)). Then there is a set \( K \in S \) such that for each \( A \in S \) (resp., for each \( A \subseteq S \)), (i) and (ii) below hold.

(i) \( v_1(A \cap K) \leq v_2(A \cap K) \) (resp., \( v_1^*(A \cap K) \leq v_2^*(A \cap K) \));

(ii) neither \( v_1 \) nor \( v_2 \) (resp., if \( A \in \emptyset \), neither \( v_1^* \mid S \) nor \( v_2^* \mid S \)) is sigma-finite on \( A - K \) unless \( \nu(A - K) = 0 \).

**Proof.** Let \( H \) be given by Lemma 9 (resp., Lemma 9 with the parentheses). Apply Lemma 10 with \( S \) and \( \emptyset \) replaced by \( H \) and \( \{ T \mid T \in S, T \subseteq H \} \) respectively to obtain \( K \subseteq H \). Then (i) and (ii) for each \( A \in S \) (resp., \( A \subseteq S \)) may be verified.

**Theorem 13**(18). \( m^0 \mid \emptyset \) (resp., \( m^0 \mid \emptyset \)) \( \mu \)-dominates \( m_0 \mid \emptyset \) (resp., \( m^0 \mid \emptyset \)).

**Proof.** It suffices to prove: if \( A \subseteq X \), then there is a Borel set \( B_0 \) such that \( \mu(A - B_0) = 0 \) and \( m_0(B_0) \leq m^0(A) \).

So suppose \( A \subseteq X \). There are \( \mathcal{U}_n \subseteq \text{cov}(A) \) \( (n = 1, 2, \ldots) \) such that \( m^0(A) = \lim_{n \to \infty} \lambda^0(\mathcal{U}_n) \) and such that each member of \( \mathcal{U}_n \) has diameter \( \leq 1/n \). For \( n = 1, 2, \ldots \) there are sets

\[
C_j^n \subseteq \emptyset \quad (j = 1, 2, \ldots)
\]

(18) In this theorem and its proof the assumption that \( \mu \) is regular may be dispensed with. A similar statement holds for Theorems 14, 16 (in parts i-iv), 17, 19, Lemma 12, and Corollary 20.1.
such that $C^*_j \cap C^*_k = \emptyset$ for $j \neq k$, such that $C^*_j$ is a subset of a member of $\mathcal{U}_n$ for $j = 1, 2, \ldots$, and such that $\bar{\mu}(A - \cup_{n=1}^{\infty} C^*_n) = 0$. For $j, n = 1, 2, \ldots$, there is a set $B^*_j \in \mathcal{B}$ such that $B^*_j \subset C^*_j$ and $\bar{\mu}(C^*_j - B^*_j) = 0$. Let

$$B_0 = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} B^*_j.$$ 

Then $\bar{\mu}(A - B_0) = 0$. For $n = 1, 2, \ldots$,

$$m^*(B_0) \leq \sum_{j=1}^{\infty} \lambda(C^*_j) \leq \lambda(\mathcal{U}_n).$$

Hence

$$m_0(B_0) = \lim_{n \to \infty} m^*(B_0) \leq \lim_{n \to \infty} \lambda(\mathcal{U}_n) = m^*(A).$$

**Theorem 14.** There is a Borel set $K \subset X$ such that for each $A \in \mathcal{B}$ (resp., for each $A \in X$):

(i) $m_0(A \cap K) \leq m_0(A \cap K)$ (resp., $m^*_0(A \cap K) \leq m^*_0(A \cap K)$);

(ii) neither $m_0|\mathcal{B}$ nor $m^*_0|\mathcal{B}$ (if $A \in \mathcal{B}$) is sigma-finite on $A - K$ unless $\bar{\mu}(A - K) = 0$.

**Proof.** Apply Theorem 14 and Lemma 11.

In the following theorem the possibility $D_0(t, \lambda) = D^*(t, \lambda) = D(t, \lambda) = +\infty$ for all $t$ in some set of positive $\bar{\mu}$-measure is not excluded. A measure $\nu$ on $\mathcal{B}$ is $\mu$-Lipschitzian if and only if $\nu \leq c \mu$ for some real $c$ (cf. Definition 4).

**Theorem 15.** Suppose $m^*_0|\mathcal{B} \geq \nu$ for every $\mu$-Lipschitzian measure $\nu$ on $\mathcal{B}$ such that $m_0|\mathcal{B} \geq \nu$ (e.g., suppose $m_0|\mathcal{B} \geq m^*_0|\mathcal{B}$). Then $\lambda$ is differentiable almost everywhere (mod $\mu$).

**Proof.** Suppose the theorem is false. There is a set $B_0 \in \mathcal{B}$ such that $\mu(B_0) > 0$ and $D_0(t, \lambda) < D^*(t, \lambda)$ for all $t \in B_0$. There are a set $A \in \mathcal{B}$ and real numbers $r > 0$ and $\delta > 0$ such that $A \subset B_0$, $0 < \mu(A) < +\infty$, and $D_0(t, \lambda) \leq r$ and $D^*(t, \lambda) > D_0(t, \lambda) + \delta$ for all $t \in A$. Define

$$\nu(B) = \int_{A \cap B} [D_0(\cdot, \lambda) + \delta]d\bar{\mu}$$

for each $B \in \mathcal{B}$. Then $\nu$ is a measure on $\mathcal{B}$, $\nu \leq (r + \delta)\mu$, and

$$\nu(B) \leq \int_{A \cap B} D^*(\cdot, \lambda)d\bar{\mu} \leq m^*(A \cap B) \leq m^*(B)$$

for each $B \in \mathcal{B}$ by Theorem 11. Since $\nu \leq m^*_0|\mathcal{B}$, $\nu \leq m_0|\mathcal{B}$. Hence, by Theorem 6,
\[ \nu(A) \leq \int_A D_\infty(\lambda, \lambda) d\mu + \int_A D_\infty(\lambda, \lambda) d\mu + \delta \mu(A) \]
\[ = \int_A [D_\infty(\lambda, \lambda) + \delta] d\mu = \nu(A) \]

(for \( \int_A D_\infty(\lambda, \lambda) d\mu \leq r \mu(A) < +\infty \)), a contradiction. Thus the theorem is established.

**Corollary 15.1.** Suppose \( D^\infty(t, \lambda) < +\infty \) for almost every \( t \in X \) (mod \( \mu \)). Then \( \lambda \) is differentiable almost everywhere (mod \( \mu \)) with respect to \( \lambda \) and \( \mathfrak{C} \) if and only if \( m_0 \mid \mathfrak{B} \geq m^0 \mid \mathfrak{B} \).

**Proof.** If \( \lambda \) is differentiable almost everywhere (mod \( \mu \)), then \( m_0 \mid \mathfrak{B} \geq m^0 \mid \mathfrak{B} \) by Theorem 9 and Corollary 12.1. For the converse, apply Theorem 15.

**Lemma 12.** Suppose \( \nu \) is a measure on a sigma-ring \( \mathfrak{S} \) with \( \mathfrak{B} \cup \mathfrak{C} \subset \mathfrak{S} \subset \mathfrak{B}^* \) such that \( \nu \) is continuous with respect to \( \mu \mid \mathfrak{C} \) and such that \( \nu \mid \mathfrak{C} \leq \lambda \). Then \( \nu \leq m^\infty \mid \mathfrak{S} \).

**Proof.** Suppose \( B \in \mathfrak{S} \) and \( \mathfrak{C} \subseteq \text{cov}(B) \). There are sets \( C_n \in \mathfrak{C} \) \( (n = 1, 2, \ldots) \) such that each \( C_n \) is a subset of a member of \( \mathfrak{C} \), such that \( C_j \cap C_h = \emptyset \) if \( j \neq h \), and such that

\[ \mu \left( B - \bigcup_{n=1}^\infty C_n \right) = 0. \]

Then

\[ \nu(B) \leq \nu \left( \bigcup_{n=1}^\infty C_n \right) = \sum_{n=1}^\infty \nu(C_n) \leq \sum_{n=1}^\infty \lambda(C_n) \leq \lambda^o(\mathfrak{C}). \]

Hence, for each \( B \in \mathfrak{S} \),

\[ \nu(B) \leq \inf_{\mathfrak{C} \subseteq \text{cov}(B)} \lambda^o(\mathfrak{C}) = m^\infty(B). \]

The theory of upper and lower measures induced by a measure is done in the following theorem; (v)-(vii) are variants of known results (cf., e.g., Hahn-Rosenthal [2] and Morse [5; 6]). Differentiability is taken with respect to \( \mathfrak{C} \) and \( \mu \).

**Theorem 16.** Suppose \( \lambda \) is a measure on a sigma-ring \( \mathfrak{S} \) with \( \mathfrak{B} \cup \mathfrak{C} \subset \mathfrak{S} \subset \mathfrak{B}^* \). Then (i)-(vii) below hold.

(i) \( \lambda \leq m^\infty \mid \mathfrak{S} \).

(ii) If \( \lambda \) is continuous with respect to \( \mu \mid \mathfrak{S} \), then \( \lambda \leq m^\infty \mid \mathfrak{S} \).
(iii) If \( \lambda \) is regular, then \( m^{\circ*} S \leq \lambda \), and each measure \( \nu \) on \( S \) which is \( \leq \lambda \) and continuous with respect to \( \bar{\mu} | S \) is \( \leq m^{\circ*} S \).

(iv) If \( \lambda \) is regular and continuous with respect to \( \bar{\mu} | S \), then \( m^{\circ*} S = \lambda \).

(v) \( \lambda \) is differentiable at almost every \( t \in X \mod \mu \) for which \( \lambda(N - \{ t \}) \) is finite for some neighborhood \( N \) of \( t \) in \( X \).

(vi) If \( \lambda \) is regular, \( \lambda \) is differentiable almost everywhere \( \mod \mu \).

(vii) If, where \( C^o \) is the interior of \( C \), \( \bar{\mu}(C^o) = \bar{\mu}(C) \) for each \( C \in \mathcal{C} \), then \( \lambda \) is differentiable almost everywhere \( \mod \mu \).

**Proof.** (i) follows trivially from the definition of \( m^{\circ*} \). (ii) follows from Lemma 12 with \( \nu = \lambda \).

To prove (iii), suppose \( \lambda \) is regular. Suppose \( S \in \mathcal{S} \), and suppose \( \epsilon > 0 \). There is an open set \( V \subset X \) such that \( S \subset V \) and \( \lambda(V) < \lambda(S) + \epsilon \). Then \( \{ V \} \in \text{cov}(S) \), and it is easily verified that \( \lambda^o(\{ V \}) \leq \lambda(V) \). Hence

\[
m^{\circ*}(S) = \inf_{U \in \text{cov}(S)} \lambda^o(U) \leq \lambda^o(V) \leq \lambda(V) \leq \lambda(S) + \epsilon.
\]

Hence \( m^{\circ*}(S) \leq \lambda(S) \) for each \( S \in \mathcal{S} \). Part of (iii) is established. The other part follows from Lemma 12.

(iv) follows from (ii) and (iii).

To prove (v), let \( X_{[1]} \) be the set of all \( t \in X \) for which \( \lambda(N - \{ t \}) \) is finite for some neighborhood \( N \) of \( t \) in \( X \). Then \( X_{[1]} \) is open. Let \( \lambda_{[1]} = \lambda | X_{[1]} \) where \( s_{[1]} = \{ S | S \in \mathcal{S}, S \subset X_{[1]} \} \), let \( \mu_{[1]} = \mu | B_{[1]} \) where \( B_{[1]} \) is the set of all Borel sets of \( X_{[1]} \), and let \( \mathcal{C}_{[1]} = \{ C | C \in \mathcal{C}, C \subset X_{[1]} \} \). Then \( \mathcal{C}_{[1]} \) is a \( \mu_{[1]} \)-Vitali covering of \( X_{[1]} \) with \( \lambda_{[1]} \) and \( \mu_{[1]} \) regular measures on \( X_{[1]} \) (cf. Remark 9, §5). Let \( m_{\circ[1]} \) and \( m_{[1]} \) be the lower and upper measures induced by \( \lambda_{[1]} \) and \( \mu_{[1]} \). Then \( m_{\circ[1]} \leq \lambda_{[1]} \leq m_{[1]} \) on \( \mathcal{C}_{[1]} \) by (iii) and (i). Hence \( \lambda_{[1]} \) is differentiable with respect to \( \mathcal{C}_{[1]} \) and \( \mu_{[1]} \) almost everywhere \( \mod \mu_{[1]} \) in \( X_{[1]} \) by Theorem 15. Hence \( \lambda \) is differentiable with respect to \( \mathcal{C} \) and \( \mu \) almost everywhere \( \mod \mu \) in \( X \). Thus (v) is established. (vi) follows from (v).

To prove (vii) let

\[
\mathcal{D}_n = \{ C | C \in \mathcal{C}_n, \lambda(C) < + \infty \} \quad (n = 1, 2, \ldots),
\]

\[
D_0 = \bigcap_{n=1}^{\infty} \bigcup_{C \in \mathcal{D}_n} C.
\]

There is a sequence \( \{ C_j \}_{j=1}^{\infty} \) of members of \( \bigcup_{n=1}^{\infty} \mathcal{D}_n \) such that \( C_j \cap C_h = \emptyset \) for \( j \neq h \) and

\[
\bar{\mu} \left( D_0 - \bigcup_{n=1}^{\infty} C_n \right) = 0.
\]

Suppose \( \mu(C^o) = \bar{\mu}(C) \) for each \( C \in \mathcal{C} \). Since \( \mu(C_n^o) < + \infty \) for \( n = 1, 2, \ldots \), \( \lambda \) is differentiable almost everywhere in \( \bigcup_{n=1}^{\infty} C_n \) by (v), hence almost every-
where in $U_{n=1}^\infty C_n$, and hence almost everywhere in $D_0$. Trivially, $D(t, \lambda) = + \infty$ for each $t \in X - D_0$. Thus (vii) is established.

The theorem is proved.

Many classical examples of $\mu$-Vitali coverings have the following property $P$.

**Definition 12.** We shall say that $\mathcal{C}$ has property $P$ if and only if for all $A \subseteq X$ and $\mathcal{U} \subseteq \text{cov}(A)$, there is a sequence $\{C_j\}_{j=1}^\infty$ of members of $\mathcal{C}$ such that $\mu(C_j \cap C_k) = \emptyset$ for $j \neq k$, such that each $C_j$ is a subset of a member of $\mathcal{U}$, and such that $A \subseteq \bigcup_{j=1}^\infty C_j$.

**Theorem 17.** Suppose $\mathcal{C}$ has property $P$. Then $m_0^* \leq m^*$. 

**Proof.** Given $A$, $\mathcal{U}$, and $\{C_j\}_{j=1}^\infty$ as in Definition 12, we have

$$\lambda_\nu(\mathcal{U}; A) \leq \sum_{j=1}^\infty \lambda(C_j) \leq \lambda^\nu(\mathcal{U}).$$

Hence

$$m_0^*(A) = m_0^*(A) = \lim_{\mathcal{U} \in \text{cov}(A)} \lambda_\nu(\mathcal{U}; A) \leq \lim_{\mathcal{U} \in \text{cov}(A)} \lambda^\nu(\mathcal{U}) = m^*(A).$$

**Definition 13.** If $\lambda$ is such that $m_0|\mathcal{B} = m^0|\mathcal{B}$, we say that $m = m_0|\mathcal{B} = m^0|\mathcal{B}$ is the $\lambda$-determined measure on $\mathcal{B}$.

**Theorem 18.** Suppose $\mathcal{C}$ has property $P$ and $D^\nu(t, \lambda) < + \infty$ for almost every $t \in X$ (mod $\mu$). Then $\lambda$ is differentiable almost everywhere (mod $\mu$) with respect to $\mu$ and $\mathcal{C}$ if and only if $m_0|\mathcal{B} = m^0|\mathcal{B}$, i.e., if and only if there is a $\lambda$-determined measure on $\mathcal{B}$.

**Proof.** Apply Theorem 17 and Corollary 15.1.

**Theorem 19.** Suppose $\mathcal{C}$ has property $P$, and suppose $\lambda$ is a regular measure on a sigma-ring $\mathcal{S}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{B}^*$. Then $m_0^*|\mathcal{S} = m^0|\mathcal{S} = \lambda$, and $\lambda|\mathcal{B}$ is the $\lambda$-determined measure on $\mathcal{B}$.

**Proof.** Apply Theorem 17 and (iii) and (i) of Theorem 16 to obtain $m_0^*|\mathcal{S} \leq m^0|\mathcal{S} \leq \lambda \leq m_0^*|\mathcal{S}$.

The following theorem follows from the definitions of $m^0*$ and $m_0^* = m_0^*$ and is of interest mainly if $\mathcal{C}$ has property $P$.

**Theorem 20.** Suppose there is a $\lambda$-determined measure $m$ on $\mathcal{B}$. Then for each $B \subseteq \mathcal{B}$ with $m(B)$ finite and each real $\varepsilon > 0$ there is some $\mathcal{U} \subseteq \text{cov}(B)$ such that: if $C_j \subseteq \mathcal{C}$ is a subset of a member of $\mathcal{U}$ for $j = 1, 2, \ldots$, $\mu(C_j \cap C_k) = 0$ for $j \neq k$, and $B \subseteq \bigcup_{j=1}^\infty C_j$, then
Corollary 20.1. Suppose there is a \( \lambda \)-determined measure \( m \) on \( \mathcal{B} \). Then for each compact \( B \in \mathcal{B} \) with \( m(B) \) finite and each real \( \epsilon > 0 \) there is some real \( \delta > 0 \) such that: if \( C_j \in \mathcal{C} \) meets \( B \) and has diameter \( < \delta \) for \( j = 1, 2, \ldots, p(C_j \cap C_k) = 0 \) for \( j \neq k \), and \( B \subseteq \bigcup_{j=1}^{\infty} C_j \), then

\[
\left| m(B) - \sum_{j=1}^{\infty} \lambda(C_j) \right| < \epsilon.
\]

Proof. Let \( \mathcal{U} \) be given by Theorem 20. By standard methods\(^{(19)}\) there is a real \( \delta > 0 \) such that each subset of \( X \) meeting \( B \) and having diameter \( < \delta \) is a subset of a member of \( \mathcal{U} \). The conclusion of Corollary 20.1 follows.

8. Specialization to Lebesgue measure in \( \mathbb{R}^q \). Lebesgue measure in \( \mathbb{R}^q \) will be written \( L_q \). All intervals in \( \mathbb{R}^1 \) will be considered to be bounded and to contain more than one point. A closed interval in \( \mathbb{R}^q \) is a cartesian product \( \prod_{j=1}^{q} J_j \) of closed intervals \( J_j \) in \( \mathbb{R} \); if \( J_j \) has length \( a \) for \( j = 1, \ldots, q \), then \( \prod_{j=1}^{q} J_j \) is a closed cube of edge-length \( a \). A closed \( j \)-simplex in \( \mathbb{R}^q \) is the convex hull of any set of \( j+1 \) affinely independent points of \( \mathbb{R}^q \), the vertices of the simplex.

Definition 14. Suppose \( A \subseteq \mathbb{R}^q \) is bounded, Lebesgue measurable, and contains more than one point. The regularity of \( A \) is defined to be

\[
r(A) = \frac{L_q(A)}{[\text{diam}(A)]^q}.
\]

Suppose \( S \) is a closed \( q \)-simplex in \( \mathbb{R}^q \), and suppose \( J \subseteq \mathbb{R}^q \) is a closed interval. It is well-known that there is a "decomposition" \( \mathcal{D} \) of \( S \times J \subseteq \mathbb{R}^{q+1} \) such that \( \mathcal{D} \) is a set of closed \( (q+1) \)-simplices for which (1)-(4) below hold\(^{(20)}\).

1. If \( A \in \mathcal{D} \), \( B \in \mathcal{D} \), and \( A \neq B \), then \( L_{q+1}(A \cap B) = 0 \);
2. \( S \times J = \bigcup_{A \in \mathcal{D}} A \);
3. \( \mathcal{D} \) has exactly \( q+1 \) members;
4. if \( A \in \mathcal{D} \), then

\[
L_{q+1}(A) = \frac{1}{q+1} \cdot \frac{L_1(J)}{L_q(S)} \cdot L_{q+1}(S \times J).
\]

This may be used to prove the following well-known lemma by induction on the dimension.

Lemma 13. Suppose \( H \) is a closed interval in \( \mathbb{R}^q \). Then there is a set \( \mathcal{M} \) of \( q \)-simplices such that (1)-(4) below hold.

1. If \( A \in \mathcal{M} \), \( B \in \mathcal{M} \), and \( A \neq B \), then \( L_q(A \cap B) = 0 \).

\(^{(19)}\) Cf. Eilenberg and Steenrod \([1, \text{p. 65}]\) for the method of proof.
\(^{(20)}\) Cf. Eilenberg and Steenrod \([1, \text{p. 70}]\) for a description.
(2) $H = \bigcup_{A \in \mathcal{M}} A$.
(3) $\mathcal{M}$ has exactly $q!$ members.
(4) If $A \subseteq \mathcal{M}$, then $L_q(A) = L_q(H) / q!$ and $r(A) \geq r(H) / q!$

If $H$ is a cube in $\mathbb{R}^q$ of edge-length $a$, then trivially

$$r(H) = a^q / ((qa^2)^{1/2})^q = q^{-q/2}.$$ 

**Definition 15.** A *sigma-interval* in $\mathbb{R}^q$ is any subset of $\mathbb{R}^q$ which is the union of countably many closed intervals.

Any closed interval in $\mathbb{R}^q$ is a sigma-interval. Any open subset of $\mathbb{R}^q$ is a sigma-interval.

**Theorem 21.** Suppose $X$ is a sigma-interval in $\mathbb{R}^q$, $\mathcal{B}$ is the set of all Borel sets of $X$, and $\mu = L_q|_{\mathcal{B}}$. Suppose $c$ is a real number such that $0 < c < q^{-q/2}$.

Suppose (a) or (b) or (c) below holds.

(a) $G$ is the set consisting of $\emptyset$ and all closed intervals of $X$ of regularity $> c$.
(b) $C$ is the set consisting of $\emptyset$ and all closed $q$-simplices of $X$ of regularity $> c / q$.
(c) $\mathcal{E}$ is the set consisting of $\emptyset$ and all compact subsets of $X$ of regularity $> c / q$.

Then $\mathcal{E}$ is a $\mu$-Vitali covering of $X$ having property $P$. Hence, if $X$ is a non-negative extended-real-valued function whose domain includes $G$ such that $X(0) = 0$, then all the theorems of §§2-7 are valid (for the conditions placed on $X$, $\mathcal{B}$, $\mu$, and $\mathcal{E}$ within §§2-7 are valid).

**Proof.** We first prove (1)–(3) of Definition 1. (1) and (2) are trivial.

To prove (3) we write $X = \bigcup_{n=1}^{\infty} J_n$, each $J_n$ being a closed interval. Consider a positive integer $n$. Each $J_n$ is the union of finitely many cubes of diameter $< 1/n$ and regularity $> c$, and each of these cubes is (by Lemma 13) the union of finitely many simplices of regularity $> c$. Thus (3) of Definition 1 holds.

By Remark 2 of §2, $\mathcal{E}$ is a $\mu$-Vitali covering of $X$.

It remains to prove that $\mathcal{E}$ has property $P$.

Suppose $A \subseteq X$ and $\mathcal{U} \subseteq \text{cov}(A)$. There is a sequence $\{J_n\}_{n=1}^{\infty}$ of closed intervals in $\mathbb{R}^q$ such that $X = \bigcup_{n=1}^{\infty} J_n$. For $n = 1, 2, \cdots$, we will arrive at a set $\mathcal{J}(n)$ of closed intervals in $\mathbb{R}^q$ such that (a)–(d) below hold where $U_0 = \bigcup_{U \in \mathcal{U}} U$.

(a) $\bigcup_{J \in \mathcal{J}(n)} J = J_n \cap (U_0 - \bigcup_{j < n} J_j)$.
(b) If $J \subseteq \mathcal{J}(n)$, $J' \subseteq \mathcal{J}(n)$, and $J \neq J'$, then $\mu(J \cap J') = 0$.
(c) Each member of $\mathcal{J}(n)$ is a subset of a member of $\mathcal{U}$.
(d) If $J \subseteq \mathcal{J}(n)$, $r(J) > c$.

Since $U_0 - \bigcup_{j < n} J_j$ is open in $X$, by standard methods there are sets $\mathcal{J}'(n)$ of closed intervals such that (a) and (b) hold for $\mathcal{J}(n) = \mathcal{J}'(n)$. For each

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(a) Cf. Munroe [7, p. 126] for the method of proof (there applied to half-open intervals).

There is a number $\delta(J) > 0$ such that each subset of $J$ of diameter $< \delta(J)$ is a subset of a member of $\mathcal{U}$. By elementary means one may establish the existence for each $J \in \mathcal{G}'(n)$ of a finite set $\mathcal{G}(J)$ of closed intervals of diameter $< \delta(J)$ and regularity $> c$ such that $J = \bigcup_{H \in \mathcal{G}(J)} H$ and such that $\mu(H \cap H') = 0$ if $H, H' \in \mathcal{G}(J)$ and $H \neq H'$. Let

$$\mathcal{G}(n) = \bigcup_{J \in \mathcal{G}'(n)} \mathcal{G}(J).$$

Then (a)–(d) may be verified. Let $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{G}(n)$. By Lemma 13 for each $H \in \mathcal{K}$ there is a finite set $\mathcal{M}(H)$ of closed simplices satisfying (1)–(4) of Lemma 13 with $\mathcal{M} = \mathcal{M}(H)$. Let $\mathcal{K} = \bigcup_{H \in \mathcal{K}} \mathcal{M}(H)$. Then $\mathcal{K}$ and $\mathcal{K}$ are countable and refine $\mathcal{U}$, $A \subset \bigcup_{H \in \mathcal{K}} H = \bigcup_{K \in \mathcal{K}} K$, the intersection of any two distinct members of $\mathcal{K}$ has $\mu$-measure 0, the intersection of any two distinct members of $\mathcal{K}$ has $\mu$-measure 0, and either $\mathcal{K} \subset \mathcal{C}$ or $\mathcal{K} \subset \mathcal{E}$.

Hence $\mathcal{C}$ has property $P$.

References


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(2) Cf. Eilenberg and Steenrod [1, Lemma 7.5, p. 65].