ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF
ORDINARY, NONLINEAR, SECOND ORDER
DIFFERENTIAL EQUATIONS(1)

BY

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This paper treats various problems connected with systems of differential
equations of the form

\[ x'' = f(t, x, x') \]

for a vector \( x \). The first part (§§1–5) deals with a priori bounds for \( |x'| \) for
a solution \( x = x(t) \). The next part (§§8–9) gives existence theorems for non-
singular boundary value problems

\[ x(0) = x_0 \quad \text{and} \quad x(T) = x_T. \]

§§10–11 give existence theorems for solutions of singular boundary value
problems, that is, solutions which exist for all \( t \geq 0 \) and satisfy

\[ x(0) = x_0 \quad \text{and} \quad |x| \leq R \quad \text{for} \quad t \geq 0 \quad \text{or} \quad x \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \]

In §§12–13, there are obtained uniqueness and continuity theorems for the
solutions satisfying (2) or (3).

The results are applied in §§14–15 to obtain existence theorems for peri-
odic and almost periodic solutions. This application was suggested by a lec-
ture of G. Seifert.

Finally, §§16–17 deal with existence of solutions of

\[ x'' = X(t, x, x', z), \quad z' = Z(t, x, x', z) \]
satisfying

\[ x(0) = x_0, \quad x(T) = x_T \quad \text{and} \quad z(0) = z_0, \]

where \( x \) and \( z \) are vectors (not necessarily of the same dimension).

1. Below, if \( x, f \) are vectors, \( |x| \) denotes the Euclidean length of \( x \) and
\( x \cdot f \) the scalar product of \( x \) and \( f \).

In §§1–5, there will be obtained a priori bounds for the first derivatives of
\( n \)-vector functions \( x(t) \) subject to second order differential inequalities. In
this direction, our results are new only for \( n > 1 \). For reference and compar-
isition, the following is stated for the case \( n = 1 \).

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States Government.
Lemma 1 (Nagumo [8]). Let $\phi(s)$, where $0 \leq s < \infty$, be a positive continuous function satisfying

\[(1.1) \quad \int^{s} s ds/\phi(s) = \infty.\]

Let $R > 0$ and $T \geq S > 0$. Then there exists a constant $M$, depending only on $\phi$ and $R, S$, with the following property: If $x = x(t)$ is a real-valued function of class $C^2$ for $0 \leq t \leq T$ satisfying

\[(1.2) \quad |x| \leq R, \quad |x''| \leq \phi(|x'|), \]

then $|x'| \leq M$ for $0 \leq t \leq T$.

For example, $M$ can be chosen to be the solution of the equation

\[(1.3) \quad \int_{2R/S}^{M} s ds/\phi(s) = 2R.\]

(Actually, the condition (1.1) on $\phi$ can be relaxed to the assumption that equation (1.3) have a solution $M$.)

Functions $\phi(s)$ satisfying the conditions of Lemma 1 will be called Nagumo functions. (For example, $\phi(s) = \gamma s^2 + C > 0$, where $\gamma, C$ are constants, is a Nagumo function.)

2. An example of Heinz [5] given in connection with partial differential inequalities shows that Lemma 1 is false if $x = x(t)$ is a vector-valued function. His example is the binary vector $x(t) = (\cos pt, \sin pt)$ which satisfies $|x| = 1$, $|x'| = |p|$ and $|x''| = |x'|^2$. Thus (1.2) holds with $R = 1$ and $\phi(s) = s^2 + 1$, but there is no a priori bound for arbitrary $p$.

Heinz's arguments suggest the consideration of auxiliary inequalities, different in form from (1.2), say

\[(2.1) \quad |x| \leq R, \quad |x''| \leq \rho'', \]

where $\rho(t)$ is a (scalar) function of class $C^2$ on $0 \leq t \leq T$.

The desired a priori bounds for $|x'|$, in case of a vector $x$, are given by the following lemma and its consequences.

Lemma 2. Let $\phi(s)$, $0 \leq s < \infty$, be a Nagumo function. Let $\rho = \rho(t)$ be a (scalar) function of class $C^2$ and $0 \leq \rho(t) \leq K_1$ on $0 \leq t \leq T$. Let $R > 0$, $T \geq S > 0$. Then there exists a constant $M$, depending only on $\phi(s)$, $K_1$, $R$ and $S$, with the following property: If $x = x(t)$ is a (vector-valued) function of class $C^2$ on $0 \leq t \leq T$ satisfying (1.2) and (2.1), then $|x'| \leq M$ for $0 \leq t \leq T$.

Heinz's example above shows that condition (2.1) in $x(t)$ cannot be omitted. It will be seen below that condition (1.2) cannot be omitted either. If, however, (1.2) is omitted (and (2.1) retained), one obtains a priori bounds for $|x'|$ on all subintervals $\mu \leq t \leq T - \mu$ of $0 \leq t \leq T$, where $0 < \mu < T$; cf. §3.
Remark. It will be shown (§4) that condition (1.2) on \( x(t) \) can be omitted in Lemma 2 if \( \rho(t) \) satisfies

\[
(2.2) \quad |\rho'| \leq \theta |x'| + C_1
\]

for some constants \( \theta, C_1 \) with \( 0 < \theta < 1 \).

If one chooses \( \phi, \rho \) to be \( \gamma s^2 + C, \alpha |x|^2 + K \), respectively, then Lemma 2 implies the following:

**Lemma 3.** Let \( \alpha, \gamma, R, S, T, C, K \) be non-negative constants and \( T \geq S > 0 \).
Then there exists a constant \( M = M(\alpha, \gamma, R, S, C, K) \) with the following property: If \( x = x(t) \) is of class \( C^2 \) on \( 0 \leq t \leq T \) satisfying

\[
(2.3) \quad |x| \leq R, \quad |x''| \leq \gamma |x'|^2 + C,
\]

\[
(2.4) \quad |x| \leq R, \quad |x''| \leq \alpha r'' + K, \quad \text{where} \quad r = |x|^2,
\]

then \( |x'| \leq M \) on \( 0 \leq t \leq T \).

Heinz's example of the binary vector \( x = (\cos pt, \sin pt) \) shows that condition (2.4) cannot be omitted. It is easy to give an example of a family of (scalar) functions \( x(t) \) satisfying inequalities of the form (2.4) but not of the form (2.3) and for which there is no a priori bound for \( |x'| \). To this end, let \( \epsilon, \rho > 0 \) and let \( x(t) = x(t; \rho, \epsilon) \) be the scalar function which is \( 1 + \epsilon \rho^4(t - 1/\rho)^4 \) or 1 according as \( 0 \leq t \leq 1/\rho \) or \( t > 1/\rho \). Then \( x' = 4 \epsilon \rho^4(t - 1/\rho)^3 \) or 0 and \( x'' \) is \( 12 \epsilon \rho^4(t - 1/\rho)^2 \) or 0 according as \( 0 \leq t \leq 1/\rho \) or \( t > 1/\rho \). Since \( 1 \leq x \leq 1 + \epsilon \) and \( x'' \geq 0 \), it is clear that (2.4) holds with \( R = 1 + \epsilon, \alpha = 1/2 \) and \( K = 0 \). As \( x'(0) = -4 \epsilon \rho \), there is no a priori bound for all \( \rho > 0 \) (and \( \epsilon > 0 \) fixed).

Note that if \( \gamma R < 1 \), then (2.3) implies (2.4) with

\[
(2.5) \quad \alpha = \gamma/2(1 - \gamma R) \quad \text{and} \quad K = C/(1 - \gamma R).
\]

For since

\[
(2.6) \quad r'' = 2(x \cdot x'' + |x'|^2),
\]

(2.3) shows that \( r'' \geq 2(1 - \gamma R)|x'|^2 - 2CR \) and another application of (2.3) gives \( r'' \geq 2(1 - \gamma R)(|x''| - C)/\gamma - 2CR \). This inequality is equivalent to (2.4)--(2.5). Conversely, if \( 2Ra < 1 \), then (2.4) implies (2.3) with

\[
(2.7) \quad \gamma = 2\alpha/(1 - 2Ra) \quad \text{and} \quad C = K/(1 - 2Ra).
\]

It can also be remarked that if (2.3) holds and, in addition,

\[
(2.8) \quad x \cdot x'' \geq 0,
\]

then (2.4) holds with \( \alpha = \gamma/2 \) and \( K = C \).

In view of the remark concerning (2.5), Lemma 3 has the following consequence:

**Corollary 1.** Let \( \gamma, R, S, T, C \) be non-negative constants subject to \( \gamma R < 1 \)
and \( T \geq S > 0 \). Then the analogue of Lemma 3 holds with an \( M = M(\gamma, R, S, C) \) if condition (2.4) on \( x(t) \) is omitted.

In view of Heinz's example \( x = (\cos pt, \sin pt) \), \( \gamma R < 1 \) cannot be relaxed to \( \gamma R \leq 1 \) in this assertion. (Heinz's results on partial differential inequalities involve the condition \( \gamma R < 1/2 \).)

The remark concerning (2.7) and Lemma 3 (or the remark concerning (2.2) and Lemma 2) imply

**Corollary 2.** Let \( \alpha, R, S, T, K \) be non-negative constants subject to \( 2R\alpha < 1 \) and \( T \geq S > 0 \). Then the analogue of Lemma 3 holds with an \( M = M(\alpha, R, S, K) \) if condition (2.3) on \( x(t) \) is omitted.

For the family of functions \( x = x(t; \rho, \epsilon) \), mentioned after Lemma 3, with \( 2R\alpha = 1 + \epsilon \). This shows that 1 in the inequality \( 2R\alpha < 1 \) cannot be replaced by a larger constant.

3. **A priori bound on \([\mu, T - \mu] \).** It will first be shown that the inequality (2.1) implies an a priori bound for \(|x'| \) on \( \mu \leq t \leq T - \mu \), where \( 0 < \mu < T \).

Let \( 0 < \mu < T \) and \( 0 \leq t \leq T - \mu \). The relations

\[
(3.1) \quad x(t + \mu) - x(t) - \mu x'(t) = \int_{t}^{t+\mu} (t + \mu - s)x''(s)ds,
\]

\( t + \mu - s \geq 0 \), and (2.1) imply that

\[
\mu \left| x'(t) \right| \leq 2R + \rho(t + \mu) - \rho(t) - \mu \rho'(t).
\]

Hence

\[
(3.2) \quad \left| x'(t) \right| \leq (2R + K_1)/\mu - \rho'(t) \quad \text{for } 0 \leq t \leq T - \mu,
\]

where \( 0 \leq \rho \leq K_1 \) for \( |x| \leq R \).

Replacing (3.1), for \( \mu \leq t \leq T \), by

\[
x(i) - x(t - \mu) - \mu x'(t) = -\int_{t-\mu}^{t} (t - \mu - s)x''(s)ds
\]

leads to

\[
(3.3) \quad \left| x'(t) \right| \leq (2R + K_1)/\mu + \rho'(t) \quad \text{for } \mu \leq t \leq T.
\]

Adding (3.2) and (3.3) gives

\[
(3.4) \quad \left| x'(t) \right| \leq (2R + K_1)/\mu \quad \text{for } \mu \leq t \leq T - \mu.
\]

4. **On (2.2).** If, in addition to (2.1), the inequality (2.2) holds, then the choice \( \mu = T/2 \) in (3.2) and (3.3) shows that, for \( 0 \leq t \leq T \),

\[
(4.1) \quad \left| x'(t) \right| \leq (M_1 + C_1)/(1 - \theta), \quad \text{where } M_1 = 2(2R + K_1)/T.
\]

Let \( M \) denote the value of \( (M_1 + C_1)/(1 - \theta) \), when \( T = S \). Since \( x(t) \) is given
on an interval of length \( T \geq S \), it follows by applying the inequality \( |x'| \leq M \) on every subinterval of length \( S \), that \( M \) is the desired a priori bound.

5. Proof of Lemma 2. Let \( \mu = T/2 \) in (3.2) and (3.3) and let \( M_1 = M_1(T) \) be the constant defined in (4.1). Then (1.2) and (3.2)–(3.3) imply that

\[
\begin{align*}
| x' \cdot x'' | / \phi(|x'|) & \leq |x'| \leq M_1 \pm \rho',
\end{align*}
\]

where \( \pm \) is required according as \( t \geq T/2 \) or \( t \leq T/2 \).

Defined \( \Phi(s) \) by

\[
\Phi(s) = \int_0^s \frac{u}{u} \phi(u).
\]

Then

\[
\begin{align*}
|\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| &= \left| \int x' \cdot x'' \, dt / \phi(|x'|) \right|
\end{align*}
\]

where the integral is taken over the \( t \)-interval with endpoints \( t \) and \( T/2 \). It follows, therefore, from (5.1) that the expression on the left of (5.3) is majorized by \( 2^{-1} M_1 T + K_1 \). Since \( |x'(T/2)| \leq M_1 \) by the case \( \mu = T/2 \) of (3.4), it follows from (5.3) that

\[
| x'(t) | \leq M_2 \quad \text{on} \quad 0 \leq t \leq T,
\]

where \( M_2 = M_2(T) \) is defined by

\[
M_2 = \Phi^{-1} \left( \frac{1}{2} M_1 T + K_1 + \Phi(M_1) \right),
\]

in terms of the inverse function \( \Phi^{-1} \) of the increasing function \( \Phi \). Clearly \( M = M_2(S) \) is the desired a priori bound.

6. Below there will also be needed the following:

**Lemma 4 (Scorza-Dragoni [10]).** Let \( g(t, x, x') \) be a continuous and bounded (vector-valued) function for \( 0 \leq t \leq T \) and arbitrary \( (x, x') \). Then, for arbitrary \( x_0 \) and \( x_T \), the system of differential equations

\[
\begin{align*}
x'' &= g(t, x, x')
\end{align*}
\]

has at least one solution \( x = x(t) \) satisfying

\[
\begin{align*}
x(0) &= x_0 \quad \text{and} \quad x(T) = x_T.
\end{align*}
\]

It has been pointed out by Bass [2] that this lemma is easily derived from Schauder’s fixed point theorem if one considers (6.1) as an inhomogeneous form of the linear homogeneous equation \( x'' = 0 \).

7. In the remainder of the paper, the function \( \rho(t) \) in Lemma 2 will be taken to be \( \rho(t) = \alpha |x|^2 + K \); \( \phi(s) \) will be a Nagumo function that is, a function \( \phi \) satisfying the conditions of Lemma 1.
In order to be able to apply Lemma 4 below, it will be convenient to have the following remark: Let \( f(t, x, x') \) be a continuous (vector-valued) function on a set

\[
D(R, T): \quad 0 \leq t \leq T, \quad |x| \leq R, \quad x' \text{ arbitrary},
\]

and let \( f \) have one or more of the following properties:

\[
\begin{align*}
(7.2) \quad & x \cdot f + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| > 0, \\
(7.3) \quad & x \cdot f + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R, \\
(7.4) \quad & f \leq 2a(x \cdot f + |x'|^2) + K, \\
(7.5) \quad & f \leq \phi(|x'|).
\end{align*}
\]

Let \( M > 0 \). Then there exists a continuous, bounded function \( g(t, x, x') \) defined for \( 0 \leq t \leq T \) and arbitrary \((x, x')\) with the corresponding set of properties among the following:

\[
\begin{align*}
(7.2') \quad & x \cdot g + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| > 0, \\
(7.3') \quad & x \cdot g + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R, \\
(7.4') \quad & |g| \leq 2a(x \cdot g + |x'|^2) + K, \\
(7.5') \quad & |g| \leq \phi(|x'|),
\end{align*}
\]

and, at the same time, satisfying

\[
(7.6) \quad g(t, x, x') = f(t, x, x') \quad \text{for} \quad 0 \leq t \leq T, \quad |x| \leq R, \quad |x'| \leq M.
\]

In fact, one obtains such a \( g \) as follows: Let \( \delta(s) \), where \( 0 \leq s < \infty \), be a scalar continuous function which satisfies

\[
\delta = 1, \quad 0 < \delta < 1, \quad \delta = 0 \quad \text{according as} \quad s \leq M, \quad M < s < 2M, \quad s \geq 2M.
\]

Put

\[
\begin{align*}
g(t, x, x') &= \delta(|x'|) f(t, x, x') \quad \text{on} \quad D(T, R), \\
g(t, x, x') &= (R/|x|)g(t, Rx/|x|, x') \quad \text{for} \quad |x| > R.
\end{align*}
\]

On \( D(T, R) \), the relation

\[
x \cdot g + |x'|^2 = \delta(|x'|)(x \cdot f + |x'|^2) + (1 - \delta(|x'|)) |x'|^2
\]

makes it clear that \( g \) has the desired properties on \( D(T, R) \). Furthermore, the validity of the properties for \( |x| = R \) implies their validity for \( |x| > R \).

Note that \((7.5), (7.4), \text{respectively, imply that a solution} \ x = x(t) \text{of} \ x'' = f(t, x, x') \text{satisfies} (1.2), (2.1) \text{with} \ \rho = \alpha |x|^2 + K.\)

8. The next desired result in the following theorem dealing with the existence of solutions of nonlinear, nonsingular, boundary value problems (under conditions more general than those in Lemma 4) for a system
\( (8.1) \quad x'' = f(t, x, x'). \)

**Theorem 1.** Let \( f(t, x, x') \) be a continuous function on \( D(T, R) \) in (7.1) satisfying
\[
(8.2) \quad x \cdot f + |x'|^2 \geq 0 \quad \text{if} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R.
\]
In addition, let \( f \) satisfy (7.4) and (7.5), where \( \alpha, K \) are non-negative constants and \( \phi(s) \) is a Nagumo function. Let \( |x_0|, |x_T| \leq R \). Then the system (8.1) has at least one solution \( x = x(t) \) satisfying \( x(0) = x_0, x(T) = x_T \).

In the case \( x \) is scalar, condition (7.4) can be omitted; [8].

In Theorem 1 and the assertions below, *(7.5) can be omitted if \( 2Ra < 1 \). Also, (7.4) can be omitted, if (7.5) is replaced by
\[
(8.3) \quad |f| \leq \gamma |x'|^2 + C,
\]
where \( \gamma, C \) are non-negative constants and \( \gamma R < 1 \). Cf. the remarks concerning (2.5) and (2.7) above.

**Proof of Theorem 1.** The proof will be given first for the case that \( f \) satisfies (7.3) instead of (8.2).

Let \( M \) be the constant (with \( T = S \)) occurring in Lemma 2 (where \( \rho = \alpha |x|^2 + K \)). Let \( g(t, x, x') \) be a continuous bounded function for \( 0 \leq t \leq T \) and arbitrary \( (x, x') \) satisfying (7.3'), (7.6) and, correspondingly, (7.4') (7.5'). By Lemma 4, (6.1) has a solution \( x = x(t) \) satisfying the boundary conditions (6.2).

Condition (7.3') means that \( r = |x(t)|^2 \) satisfies \( r'' > 0 \) if \( r' = 0 \) and \( r \geq R^2 \); cf. (2.6). Hence \( r(t) \) cannot have a maximum value \( \geq R^2 \) in the interval \( 0 < t < T \). Since \( r(0) = |x_0|^2, r(T) = |x_T|^2 \) satisfy \( r(0), r(T) \leq R^2 \), it follows that \( r(t) \leq R^2 \) (that is, \( |x| \leq R \)) on \( 0 \leq t \leq T \). In view of (7.4')--(7.5'), Lemma 2 is applicable to \( x(t) \). Hence, \( |x'(t)| \leq M \) for \( 0 \leq t \leq T \).

By virtue of (7.6), it follows that \( x = x(t) \) is a solution of (8.1). Hence Theorem 1 is proved provided that (7.3), rather than (8.1), is assumed.

In order to remove this proviso, note that if \( \epsilon > 0 \), the function \( f(t, x, x') + \epsilon x \) satisfies the conditions of Theorem 1 as well as condition (7.3). It is only necessary to replace \( K, \phi \) in (7.4), (7.5) by \( K + \epsilon R, \phi + \epsilon R \), respectively. Hence, by what has been proved,
\[
(8.4) \quad x'' = f(t, x, x') + \epsilon x
\]
has a solution \( x = x_\epsilon(t) \) satisfying the boundary conditions. It is clear that \( |x_\epsilon(t)| \leq R \) and that, for a suitable \( N \) independent of \( \epsilon \) \((< 1)\), \( |x_\epsilon'(t)| \leq N \) for \( 0 \leq t \leq T \). Ascoli's selection theorem shows that there exists a sequence \( \epsilon_1, \epsilon_2, \cdots \) such that \( 0 < \epsilon_n \to 0 \) as \( n \to \infty \) and that \( x(t) = \lim x_\epsilon(t) \), as \( \epsilon = \epsilon_n \to 0 \), exists and is a solution of (8.1) and (6.2). This completes the proof of Theorem 1.

**Remark.** In Theorem 1, let (8.2) be strengthened to
(8.4) \[ x \cdot f + |x'|^2 \geq 0 \quad \text{if} \quad x \cdot x' = 0, \]
and let
(8.5) \[ x_T = 0. \]
Then, for the solution \( x = x(t) \) just obtained, \( r = |x(t)|^2 \) satisfies
(8.6) \[ r \geq 0, \quad r' \leq 0. \]

For if (8.4) is first replaced by (7.2), it is seen that \( r(t) \) has no maximum on \( 0 < t < T \). Hence \( r(t) \leq \max(r(0), r(T)) = r(0) \). Since \( r(T) = 0 \), the same argument applies if \( 0 < t < T \) is replaced by any subinterval \( t_0 < t < T \). This gives (8.6) if (7.2) holds. If (8.4) holds, the proof of Theorem 1 shows that \( r = |x_t(t)|^2 \) satisfies (8.6). But these inequalities are not lost during the limit process \( \epsilon = \epsilon_n \to 0 \).

9. In this section, there will be proved a theorem analogous to Theorem 1, but the assumption (8.2) will be replaced by conditions on the magnitude of \( |x_0|, |x_T| \) and \( T \).

**Theorem 2.** Let \( f \) satisfy the conditions of Theorem 1 except that (8.2) need not hold. Let \( x_0, x_T, R \) and \( T \) be such that
(9.1) \[ \beta = \max(|x_0|, |x_T|) \]
satisfies
(9.2) \[ \alpha \beta^2 + \beta + KT^2/8 \leq R. \]
Then (8.1) has at least one solution \( x = x(t) \) satisfying \( x(0) = x_0, x(T) = x_T \).

One can obtain the following assertion:

**Corollary.** Let \( f \) be defined and continuous on \( D(T, R) \) and satisfy (8.3) for some non-negative constants \( \gamma, C \) such that \( \gamma R < 1 \). Let \( \beta \) in (9.1) and \( T \) satisfy
(9.3) \[ \gamma \beta^2 + 2(1 - \gamma R)\beta + CT^2/4 \leq 2R(1 - \gamma R). \]
Then (8.1) has at least one solution \( x = x(t) \) satisfying \( x(0) = x_0, x(T) = x_T \).

Theorems 1 and 2 can be considered to be the ordinary (vector) analogue of Nagumo's results [9] for a partial (scalar) differential equation.

**Proof of Theorem 2.** Let \( M \) be the constant supplied by Lemma 2 (with \( T = S \) and \( \rho = \alpha |x|^2 + K \)) and let \( g(t, x, x') \) be the function supplied by §7.

By Lemma 4, (6.1) has a solution satisfying (6.2). Let \( y = y(t) \) be the linear (vector) function satisfying
(9.4) \[ y(0) = x_0 \quad \text{and} \quad y(T) = x_T, \]
so that
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\[ x(t) = y(t) - \int_0^T G(t, s)x''(s)ds, \]

where \( TG(t, s) \) is \((T-t)s \) or \( t(T-s) \) according as \( 0 \leq s \leq t \leq T \) or \( 0 \leq t \leq s \leq T \). By (6.1) and (7.4'), \( x \) satisfies the differential inequality in (2.4). Thus \( G \geq 0 \) implies

\[ |x(t)| \leq |y(t)| + \int_0^T G(t, s)(ar''(s) + K)ds. \]

In this inequality, \( r'' \) can be replaced by \((r-u)''\), where \( u = u(t) \) is the linear function determined by

\[ u(0) = r(0) = |x_0|^2, \quad u(T) = r(T) = |x_T|^2. \]

Thus, by (9.5),

\[ |x(t)| \leq |y(t)| + \alpha(u(t) - r(t)) + 2^{-1}K(T-t)t. \]

Since \( |y(t)| \leq \max(|x_0|, |x_T|) = \beta \), \( u(t) \leq \max(|x_0|^2, |x_T|^2) = \beta^2 \) and \( r \geq 0 \),

\[ |x(t)| \leq \beta + \alpha \beta^2 + KT^2/8. \]

By condition (9.3), \( |x(t)| \leq R \) for \( 0 \leq t \leq T \). Also, \( |x'(t)| \leq M \) by Lemma 2. It follows from (7.6) that \( x = x(t) \) is a solution of (8.1). This proves Theorem 2.

10. In some of the theorems to follow, the bounded interval \( 0 \leq t \leq T \) is replaced by \( 0 \leq t < \infty \).

**Theorem 3.** Let \( f(t, x, x') \) be defined and continuous on

\[ D(R): 0 \leq t < \infty, \quad |x| \leq R < \infty, \quad x' \text{ arbitrary.} \]

For every \( T > 0 \), let \( f \) satisfy the conditions of Theorem 1 on \( D(T, R) \), where the constants \( \alpha, K \) and Nagumo function \( \phi(s) \) which occur can depend on \( T \). Then, for every \( x_0 \) in the sphere \( |x_0| \leq R \), there is at least one solution \( x = x(t) \) of (8.1) which satisfies \( x(0) = x_0 \) and exists for \( t \geq 0 \).

**Remark.** If, in addition (8.4) is assumed in Theorem 3, then \( r = |x(t)|^2 \) satisfies (8.6). Also, if

\[ x \cdot f + |x'|^2 \geq 0, \]

then

\[ r \geq 0, \quad r' \leq 0, \quad r'' \geq 0. \]

For a scalar equation in which \( f \) does not depend on \( x' \), this type of theorem goes back to A. Kneser [6]; cf. [7]. For the scalar analogue of Theorem 3 in which the conditions (7.4), (7.5) on \( D(T, R) \) are replaced by a Nagumo condition (7.5) alone, see [3]. For an \( f \) linear in \( x \) and independent of \( x' \),
see [12]; for the general linear case, [4]. For a nonlinear system, see [2], where \( f \) is subject to a majorant linear in \( |x'| \),

\[
|f| \leq \gamma |x'| + C,
\]
on \( D(T, R) \). In contrast to (10.4), Theorem 3 implies that it is sufficient to require on each \( D(T, R) \) an inequality of the form (8.3) if \( \gamma R < 1 \). There is, of course, no limitation on \( R \) if \( \gamma \) can be chosen arbitrarily small, that is, if

\[
f(t, x, x')/ |x'|^2 \to 0 \quad \text{as} \quad |x'| \to \infty
\]
uniformly for bounded \( t \) and \( |x| \leq R \).

**Remark.** In some of the papers just mentioned, it is assumed that

\[
x \cdot f(t, x, x') \geq 0.
\]

In this case, the conditions (7.4), (7.5) on \( f \) on \( D(T, R) \) are satisfied if, for example, (8.3) holds on \( D(T, R) \) with arbitrary constants \( \gamma = \gamma(T) > 0, C = C(T) > 0 \). (A restriction of the type \( \gamma R < 1 \) is not needed.)

**Proof of Theorem 3.** Let \( m = 1, 2, \ldots \). By Theorem 1, (8.1) has a solution \( x = x_m(t) \) on \( 0 \leq t \leq m \) satisfying \( x(0) = x_0, x(m) = 0 \). Let \( m \geq T \). Then, by Lemma 2, there is a constant \( M = M_T \) such that \( |x_m(t)| \leq M \) for \( 0 \leq t \leq T \). Hence the sequences \( \{x_m(t)\}, \{x'_m(t)\}, \{x''_m(t)\} \), where \( m \geq T \), are uniformly bounded and equicontinuous on \( 0 \leq t \leq T \). Theorem 3 follows from Ascoli's selection theorem applied to a sequence of intervals \( 0 \leq t \leq T \), where \( T \to \infty \).

The assertion concerning (8.6) follows from the Remark at the end of §8 and that concerning (10.3) follows from (2.6).

11. The next theorem gives a sufficient condition for the solutions \( x = x(t) \) of (8.1) given by Theorem 3 to satisfy

\[
x(t) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Theorem 4.** Let \( f(t, x, x') \) be defined on \( D(R) \). For every number \( m, 0 < m < R \), let there exist a non-negative function \( \sigma(t) = \sigma(t, m) \) for large \( t \) satisfying

\[
x \cdot f(t, x, x') \geq \sigma(t) \geq 0 \quad \text{for large} \ t, 0 < m \leq |x| \leq R, x' \text{ arbitrary},
\]

\[
\int_0^\infty \sigma(t) dt = \infty.
\]

Let \( x = x(t) \) be a solution of (8.1) on \( t \geq 0 \). Then (11.1) holds.

This is an analogue of (IV) in [3] dealing with scalar equations.

**Remark.** Let \( f(t, x, x') \) satisfy the conditions of Theorem 3 and, in addition, let the constants \( \alpha, K \) and the function \( \phi(s) \) be independent of \( T \). Let \( x = x(t) \) be a solution (8.1) satisfying (11.1). Then

\[
x'(t) \to 0 \quad \text{as} \quad t \to \infty
\]

For, by Lemma 2, \( x'(t) \) is bounded for large \( t \) and, by (7.5), \( x''(t) \) is bounded.
for large $t$. The relation (11.4) then follows from the simplest Tauberian theorem (Hadamard) which states that $M_1 \leq \text{Const}(M_0 M_2)^{1/2}$ if $M_0$, $M_1$, $M_2$ are the least upper bounds for the moduli of a $C^2$ function and its first and second derivatives on $T \leq t < \infty$, respectively.

Proof of Theorem 4. Let $r(t) = |x(t)|^2$. Since (11.2) holds for large $t$, $r$ satisfies $r'' \geq 0$ for large $t$. Suppose, if possible, that (11.1) fails to hold. Then there exists a constant $m$ such that $0 < m \leq r(t) \leq R$ for large $t$. Let $\sigma(t)$ be the function belonging to the number $m$. Then

$$q(t) = 2(x \cdot f(t, x, x') + |x'|^2)/r, \quad \text{where} \quad x = x(t) \quad \text{and} \quad r = |x(t)|^2,$$

satisfies

$$(11.5) \quad q(t) \geq 2\sigma(t)/m \quad \text{for large} \quad t.$$

Note that $r = r(t)$ satisfies the linear equation

$$(11.6) \quad r'' - q(t)r = 0;$$

cf. (2.6). But the boundedness of $r(t)$ and (11.3), (11.5), (11.6) imply that

$$(11.7) \quad r(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

(Weyl; cf., e.g., [13, pp. 601–602]). This contradicts $r \geq m \geq 0$ for large $t$. Hence Theorem 4 is proved.

12. This section deals with the uniqueness of solutions of (8.1) given by Theorems 1–3. In order to obtain a uniqueness criterion, consider the linear system of differential equations

$$(12.1) \quad y'' = A(t)y + B(t)y',$$

where $A(t)$, $B(t)$ are real matrices and $y$ is a vector. It is easily verified that, by virtue of (12.1),

$$y \cdot y' + |y'|^2 = |y' + \frac{1}{2} B^* y|^2 + y \cdot A y - B^* y \cdot B y^2/4,$$

where $B^*$ is the transpose of $B$. Thus a sufficient condition (cf. [4]) for every solution $y = y(t)$ of (12.1) to satisfy

$$(12.2) \quad |y(t)| \geq 0, \quad (|y(t)|^2)'' \geq 0,$$

is that

$$(12.3) \quad 4A - BB^* \geq 0,$$

where "$Q \geq 0$" for a matrix $Q$ means that "$y \cdot Q y \geq 0$ (or, equivalently, $y \cdot (Q + Q^*) y \geq 0$) for all real vectors $y". The fact that (12.3) implies (12.2) leads to the following uniqueness theorem:

Theorem 5. Let $f(t, x, x')$ be defined on $D(T, R)$ [or $D(R)$] and possess
continuous partial derivatives with respect to the components of $x$ and $x'$. Let $F(t, x, x')$ and $G(t, x, x')$ denote the Jacobian matrices

$$\begin{align*}
F(t, x, x') &= (\partial f/\partial x), \\
G(t, x, x') &= (\partial f/\partial x'),
\end{align*}$$

and suppose that

$$4F - GG^* \geq 0. \tag{12.5}$$

Then (8.1) has, at most, one solution which satisfies $x(0) = x_0$ and $x(T) = x_T$ [or which satisfies $x(0) = x_0$ and exists for $t \geq 0$].

Of course, if an a priori bound $|x'| \leq M$ is known for the possible solutions involved, then (12.5) is only required for $0 \leq t \leq T$ [or $t \geq 0$], $|x| \leq R$, $|x'| \leq M$.

In the case that $x$ is a scalar and $f$ does not depend on $x'$, the proof of Theorem 5 will show that the conditions on $f$ can be replaced by the assumption that $f(t, x)$ is nondecreasing in $x$ for fixed $t$.

**Proof of Theorem 5.** Note that if there are two such solutions $x = x_1(t)$ and $x = x_2(t)$, the difference $y = x_2 - x_1$ satisfies a linear equation (12.1), where

$$\begin{align*}
A(t) &= \int_0^1 F ds, \\
B(t) &= \int_0^1 G ds,
\end{align*}$$

and the argument of $F, G$ in these integrals is

$$\begin{align*}
(t, sx_2(t) + (1 - s)x_1(t), sx'_2(t) + (1 - s)x'_1(t)).
\end{align*}$$

For any (constant) vector $y$, Schwarz’s inequality (applied to each component of $B^*(t)y$) gives

$$|B^*(t)y|^2 \leq \int_0^1 |G^*y|^2 ds,$$

where the argument of $G^*$ is (12.6). Hence

$$y \cdot (4A - BB^*)y \geq y \cdot \int_0^1 (4F - GG^*) ds y;$$

that is, (12.3) follows from (12.5). Consequently, $y(t) = x_2(t) - x_1(t)$ satisfies (12.2).

If either $y(0) = y(T) = 0$ or $y(0) = 0$ and $y(t)$ exists and is bounded for $t \geq 0$, then (12.2) implies $y(t) \equiv 0$. Hence Theorem 5 is proved.

13. This section gives a “continuity” theorem for the solutions furnished by Theorems 1–3; namely:

**Lemma 5.** Let $f(t, x, x')$ and $f_1(t, x, x')$, $f_2(t, x, x')$, … be continuous functions defined on $D(T, R)$ [or on $D(R)$] such that

$$f_n(t, x, x') \to f(t, x, x') \quad \text{as} \quad n \to \infty \tag{13.1}$$
uniformly on compact subsets of \( D(T, R) \) [or of \( D(R) \)]. On \( D(T, R) \) [or on every \( D(T, R) \)], let \( f \) satisfy the conditions of Theorem 1 [with constants \( \alpha, K \) and Nagumo function \( \phi(s) \) depending on \( T \)]. Let \( |x_0|, |x_T| \leq R \). Finally, let

\[
x'' = f_n(t, x, x')
\]

possess a solution \( x = x_n(t) \) on \( 0 \leq t \leq T \) satisfying \( x(0) = x_0, x(T) = x_T \) [or on \( 0 \leq t < \infty \) and satisfying \( x(0) = x_0 \)]. Then there exists a sequence of positive integers \( n_1 < n_2 < \cdots \) such that

\[
\lim_{k \to \infty} x_n(t) = x(t), \quad \text{where} \quad n = n_k,
\]

exists uniformly on \( 0 \leq t \leq T \) [or compact subsets of \( 0 \leq t \leq T \)] and is a solution of \( (8.1) \) satisfying \( x(0) = x_0, x(T) = x_T \) [or \( x(0) = x_0 \)].

In order to see this, consider only the case of the nonsingular boundary value problem \( x(0) = x_0, x(T) = x_T \). The considerations in the singular case are similar.

Lemma 5 is an immediate consequence of Lemma 2. In fact, the inequality

\[
|f_n - f| \leq 1
\]

together with the inequalities for \( f \) in (7.4) and (7.5) imply that

\[
|f_n| \leq 2\alpha|x| + |x'|^2 + K + 1 + 2Ra, \quad |f_n| \leq \phi(|x'|) + 1.
\]

Let \( M \) denote the constant furnished by Lemma 2, where \( \rho(t), \phi(s) \) are replaced by \( \alpha|x|^2 + K + 1 + 2Ra, \phi(s) + 1 \), respectively.

In view of assumption (13.1), the inequality (13.4) holds for \( 0 \leq t \leq T, |x| \leq R, |x'| \leq M \) if \( n \) is sufficiently large. It follows from Lemma 2 that \( |x'| \leq M \) for \( 0 \leq t \leq T \) and large \( n \). Hence there exists a sequence of positive integers \( n_1 < n_2 < \cdots \) such that \( \lim x_n'(0) \) exists as \( n = n_k \to \infty \). Lemma 5 now follows from standard theorems.

14. Existence of periodic or almost periodic solutions is usually proved under conditions which assure that all solutions exist for \( t \geq 0 \). Recently, Seifert [11] has given an existence theorem for almost periodic solutions in which this is not the case. Theorem 7 of the next section can be considered an analogue of his result for systems (even though the scalar case of Theorem 7, without modification, does not yield Seifert's theorem which involves a differential equation of a rather special form). In this section, there will be obtained a similar theorem for the existence of periodic solutions.

**Theorem 6.** Let \( f(t, x, x') \) be defined for \( -\infty < t < \infty, |x| \leq R, x' \) arbitrary with the properties: (i) \( f \) is continuous and periodic in \( t \) of period 1 for fixed \( (x, x') \); (ii) the Jacobian matrices (12.4) exist and are continuous; (iii) \( f \) satisfies (8.2); (iv) \( f \) satisfies (7.4) and (7.5) with non-negative constants \( \alpha, K \) and Nagumo function \( \phi(s) \) independent of \( t, x \); finally, (v) if \( M \) is the constant sup-
plied by Lemma 2 (with some fixed $S>0$ and $\rho(t) = \alpha |x|^2 + K$), then (12.5)
holds for $-\infty < t < \infty$, $|x| \leq R$, $|x'| \leq M$.

Then (8.1) has at least one periodic solution of period 1. If $x = x_1(t)$ and $x = x_2(t)$ are bounded solutions for $-\infty < t < \infty$, then $|x_1(t) - x_2(t)| \equiv \text{const}$ (and const $= 0$ if “$\geq 0$” in (12.5) is replaced by “$> 0$”). If $x = x_1(t)$ and $x = x_2(t)$ are bounded solutions for $t \geq 0$ or $t \leq 0$, then $|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)|$.

Remark. When $x$ is a scalar, one can improve Theorem 6 slightly: condition (7.4) can be omitted in (iv); in which case, $M$ in (v) should be replaced by the $M$ supplied by Lemma 1. If, in addition, $f(t, x, x') = f(t, x)$ does not depend on $x'$, the differentiability assumptions (ii) and (v) can be replaced by the condition that $f$ is nondecreasing in $x$ for fixed $t$ (and uniqueness results if $f$ is increasing in $x$).

Proof of Theorem 6. By Theorems 3 and 5, the equation (8.2) has a unique solution on $t \geq 0$ satisfying $x(0) = x_0$ for any $x_0$ in the sphere $|x_0| \leq R$. Let this solution be denoted by $x = x(t, x_0)$. Define a map $x_0 \rightarrow x_1$ of the sphere $|x_0| \leq R$ into itself by putting $x_1 = x(1, x_0)$. It is clear from $|x'(t, x_0)| \leq M$ and from the uniqueness of the solution $x = x(t, x_0)$ that the map $x_0 \rightarrow x_1$ is continuous. Hence, by Brouwer’s fixed point theorem, there exists a point $x_0 = x*$ such that $x(1, x*) = x*$.

The periodicity of $f$ implies that if $x = x(t)$ is a solution of (8.1), then $x = x(t+1)$ is also. In particular, $x(t+1, x_0) = x(t, x_1)$. For the fixed point $x_0 = x*$, we have $x(t+1, x*) = x(t, x*)$, i.e., periodicity of period 1. This gives the existence assertion of Theorem 6.

The “uniqueness” assertions have nothing to do with the periodicity of $f$. If $x = x_1(t)$, $x_2(t)$ are two solutions of (8.1) for $-\infty < t < \infty$, then the proof of Theorem 5 shows that $r = |x_1 - x_2|^2$ satisfies $r'' \geq 0$ for all $t$. But the boundedness of $r$, $0 \leq r \leq 4R^2$, implies therefore that $r(t)$ is a constant. (If “$> 0$” holds in (12.5) it is seen that $r'' > 0$ for some $t$ unless $r \equiv 0$.) The stability assertion concerning $|x_1 - x_2|^2$ for $t \geq 0$ or $t \leq 0$ follows similarly. This completes the proof of Theorem 6.

15. An analogous theorem for almost periodic solutions is the following:

Theorem 7. Let $f(t, x, x')$ be defined for $-\infty < t < \infty$, $|x| \leq R$, $x'$ arbitrary with the properties (i) $f(t, x, x')$ is uniformly continuous for $-\infty < t < \infty$, $|x| \leq R$, $x'$ bounded; and is uniformly almost periodic in $t$ for fixed $(x, x')$; (ii) the Jacobian matrices (12.4) exist and are uniformly bounded and uniformly continuous for $-\infty < t < \infty$, $|x| \leq R$, $x'$ bounded; and conditions (iii)--(v) of Theorem 6 hold.

Then (8.1) has at least one uniformly almost periodic solution.

The last parts of Theorem 6 concerning uniqueness on $-\infty < t < \infty$ and stability for $t \geq 0$ or $t \leq 0$ are valid here. Also, the Remark following Theorem 6 on the scalar case is applicable to Theorem 7.

Theorem 7 is a consequence of Theorem 3, the proof of Theorem 5 and
results of Amerio [1]; cf. [1] for references to Favard. By Theorem 3, (8.1) possesses solutions \( x = x(t) \) on \( 0 \leq t \leq \infty \). Also, by Lemma 2, any solution of (8.1) satisfies \( |x'(t)| \leq M \) if \( x(t) \) exists on an interval of length \( \geq S \). In particular, \( |x'(t)| \leq M \) on \( 0 \leq t \leq \infty \). The boundedness of \( x(t) \) and \( x'(t) \) on \( t \geq 0 \) for some solution implies, by [1], the existence of a solution \( x = x_1(t) \) for \( -\infty < t < \infty \).

If \( x = x_1(t), x_2(t) \) are two solutions of (8.1) for \( -\infty < t < \infty \), then, as in the proof of Theorem 6, \( |x_1(t) - x_2(t)| \equiv \text{const.} \)

Let \( f_1(t, x, x') \) belong to the closure of the set \( \{ f(t+s, x, x') : -\infty < s < \infty \} \) with respect to the sup norm for \( -\infty < t < \infty, |x| \leq R, |x'| \leq M \). It is clear that \( f_1 \) has properties (i)-(v) analogous to those of \( f \) (if, in (i) and (ii), “\( x' \) bounded” is replaced by “\( |x'| \leq M \)”). Thus, if \( x = x_1(t) \) and \( x = x_2(t) \) are two solutions of \( x'' = f_1(t, x, x') \) for \( -\infty < t < \infty \), then \( |x_1 - x_2| \equiv \text{const.} \)

It follows from [1] that if \( x = x_1(t) \) is any solution of (8.1) on \( -\infty < t < \infty \), then \( x_1(t) \) is uniformly almost periodic. This completes the proof of Theorem 7.

16. Let \( x, z \) be vectors, not necessarily of the same dimension, and let \( x' \) be a vector of the same dimension as \( x \). Let \( R, Q \) be positive constants and \( E \) the \((t, x, x', z)\)-set:

\[
(16.1) \quad E: 0 \leq t \leq T, \quad |x| \leq R, \quad x' \text{ arbitrary}, \quad |z| \leq 2Q.
\]

Let \( X, Z \) be continuous vector valued functions on \( E \) of the same dimension as \( x, z \), respectively. The system of differential equations

\[
(16.2) \quad x'' = X(t, x, x', z), \quad z' = Z(t, x, x', z),
\]

will now be considered.

The following conditions will be imposed on \( X \):

\[
(16.3) \quad x \cdot X(t, x, x', z) + |x'|^2 \geq 0 \quad \text{if} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R;
\]

there exist non-negative constants \( \alpha, K \) and a Nagumo function \( \phi(s) \) such that

\[
(16.4) \quad |X| \leq 2\alpha(x \cdot X + |x'|^2) + K, \quad |X| \leq \phi(|x'|);
\]

the Jacobian matrices

\[
(16.5) \quad F(t, x, x', z) = (\partial X/\partial x), \quad G(t, x, x', z) = (\partial X/\partial x')
\]

exist, are continuous and satisfy (12.5) on \( E \).

For \( Z \), it will be supposed that there exist continuous, positive functions \( \sigma(t), \tau(s) \) for \( 0 \leq t \leq T, Q^2 \leq s \leq (2Q)^2 \), respectively, satisfying

\[
(16.6) \quad 2 |z \cdot z| \leq \sigma(t)\tau(|z|^2) \quad \text{for} \quad 0 \leq t \leq T, Q \leq |z| \leq 2Q, (x, x') \text{ arbitrary},
\]

\[
(16.7) \quad \int_0^T \sigma(t)dt < \int_{Q^2}^{(2Q)^2} ds/\tau(s) < \infty,
\]
and that the Jacobian matrix \((\partial Z/\partial z)\) exists and is continuous on \(E\).

**Theorem 8.** Let \(|x_0|, |x_T| \leq R, |z_0| \leq Q\). The system (16.2) has at least one solution \(x = x(t), z = z(t)\) which satisfies

\begin{equation}
(16.8) \quad x(0) = x_0, \quad x(T) = x_T \quad \text{and} \quad z(0) = z_0.
\end{equation}

It is clear that the first inequality for \(X\) in (16.4) is redundant if the second is of the form \(|X| \leq \gamma |x'|^2 + C\) and \(\gamma R < 1\). It is also clear that Theorem 8 leads to an analogue of Theorem 3.

The proof of Theorem 8 depends on Lemma 5 and on Schauder’s fixed point theorem.

**17. Proof of Theorem 8.** Let \(H\) be the Banach space of vector functions \((x(t), z(t))\) on \(0 \leq t \leq T\) with the product topology arising from \(x(t) \in C^2, z(t) \in C^1\). Let \(M\) be the constant furnished by Lemma 2 (with \(S = T\) and \(\rho = \alpha |x|^2 + K\)) and let \(N\) be a bound for \(|X|, |Z|\) on the set

\begin{equation}
(17.1) \quad E_M: 0 \leq t \leq T, \quad |x| \leq R, \quad |x'| \leq M, \quad |z| \leq 2Q.
\end{equation}

Let \(\omega(\epsilon) = \omega_M(\epsilon)\) be defined by

\begin{equation}
(17.2) \quad \omega(\epsilon) = \max \sup_{j = x, z} |\Delta J|,
\end{equation}

where \(\Delta J = J(t, x, x', z) - J(t^{*}, x^{*}, x'^{*}, z^{*})\) and \(\sup\) refers to \((t, x, x', z), (t^{*}, x^{*}, x'^{*}, z^{*})\) in \(E_M\) and subject to \(|t - t^{*}| \leq \epsilon, |x - x^{*}| \leq M\epsilon, |x' - x'^{*}| \leq N\epsilon, |z - z^{*}| \leq N\epsilon\).

Let \(H_0\) be the subset of \(H\) consisting of vector functions \((x(t), z(t))\) which, for \(0 \leq t \leq T\), satisfy \(|x| \leq R, |x'| \leq M, |x''| \leq N, |z| \leq 2Q, |z'| \leq N\) and

\[|j(t) - j(t^{*})| \leq \omega(\epsilon)\] if \(0 \leq t, t^{*} \leq T, |t - t^{*}| \leq \epsilon\) and \(j = x', z'\),

and, in addition, satisfy the boundary condition (16.8). Clearly, \(H_0\) is a compact, convex subset of \(H\).

Define a map \(L: H_0 \to H_0\) as follows: if \((\bar{x}(t), \bar{z}(t)) \subseteq H_0\), let \(L(\bar{x}(t), \bar{z}(t)) = (x(t), z(t))\), where \(z(t)\) is defined as the unique solution of

\begin{equation}
(17.3) \quad z' = Z(t, x(t), z(t), \bar{z}), \quad z(0) = z_0,
\end{equation}

and \(x(t)\) is the unique solution of

\begin{equation}
(17.4) \quad x'' = X(t, x, x', z(t)), \quad x(0) = x_0 \quad \text{and} \quad x(T) = x_T.
\end{equation}

In order to see that \(L\) is well defined, note that (17.3) defines \(z(t)\) uniquely, at least for small \(t \geq 0\), since \((\partial Z/\partial z)\) exists and is continuous. Actually, \(z(t)\) is defined for \(0 \leq t \leq T\) (and satisfies \(|z(t)| < 2Q\)). For otherwise, there is a subinterval \(t_0 \leq t \leq t_1\) of \(0 \leq t \leq T\) on which \(Q \leq |z| \leq 2Q\) and \(|z(t_0)| = Q, |z(t_1)| = 2Q\). But if \(s(t) = |z(t)|^2\), then \(s' = 2z \cdot Z(t, x(t), \bar{z}, z)\) satisfies \(|s'| \leq \sigma(t)r(s)\). An integration of this differential inequality over the interval \(t_0 \leq t \leq t_1\) leads to

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\[ \int_{t_0}^{t_1} \sigma(t) \, dt \geq \int_{Q}^{(2Q)^2} ds / \tau(s), \]

which contradicts (16.7). Also, (17.4) has a unique solution \( x = x(t) \) by the existence Theorem 1 and the uniqueness Theorem 5. Finally, \( (x(t), \varphi(t)) \) is in \( H_0 \).

The mapping \( L: H_0 \to H_0 \) is continuous. In order to see this, it is sufficient to show that if \( (\bar{x}_n(t), \bar{z}_n(t)) \), \( n = 1, 2, \cdots \), is a sequence of elements of \( H_0 \) such that \( (\bar{x}_n, \bar{z}_n) \to (\bar{x}, \bar{z}) \) in \( H \), as \( n \to \infty \), and \( (x_n, z_n) = L(\bar{x}_n, \bar{z}_n) \), then \( (x_n, z_n) \to (x, z) \) in \( C^1(0, T) \) is clear from (17.3) and the uniqueness of the solution of (17.3). That \( x_n \to x \) in \( C^2(0, T) \) follows from (17.4) and Lemma 5.

Schauder's fixed point theorem implies that there is a point \( (x(t), \varphi(t)) \in H \) which is a fixed point of the map \( L \). The point \( (x(t), \varphi(t)) \) is a solution of (16.2) satisfying (16.8). This gives Theorem 8.

References


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