LOWER BOUNDS FOR THE DEGREE OF APPROXIMATION

BY
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1. Introduction. The problem which is discussed in this paper may be described as follows. Let $X$ be a Banach space of real or complex valued functions $f$ defined on a compact metric space $A$. Let $G = \{g_1, \ldots, g_n, \ldots \} \subset X$. Then

1. $E_n(f) = \inf_{a} \left\| f - \sum_{i=1}^{n} a_i g_i \right\|$

is the degree of approximation of $f$ by $G$, and

2. $E_n(\mathcal{A}) = \sup_{f \in \mathcal{A}} E_n(f)$

is the degree of approximation of a class $\mathcal{A} \subset X$. Finally,

3. $D_n(\mathcal{A}) = \inf_{\mathcal{A}} E_n(\mathcal{A})$

is the optimal degree of approximation of $\mathcal{A}$. In this paper we shall give simple methods which permit to find the order of magnitude of $D_n(\mathcal{A})$ for several important classes $\mathcal{A}$: for some classes of analytic functions ($\S 6$); for the unit ball $L^p+a$ of the space $C^p+a$ of functions with continuous derivatives of order $p$, which satisfy a Lipschitz condition with exponent $\alpha$, $0 < \alpha \leq 1$ ($\S 5$). We also consider approximation of continuous functions ($\S \S 2, 3$) and give results about condensation of singularities (Theorems 2, 7).

The main content of this paper consists of results which show that standard means (trigonometric approximation, series of orthogonal polynomials) give the best possible approximation, at least up to a bounded factor. Since estimates of $D_n(\mathcal{A})$ from above follow from classical results [1; 5], we are interested in estimating $D_n(\mathcal{A})$ from below. Clearly, results of this type are the better, the smaller the norm used in the definition (1). This is why most of our theorems are for the $L^1$-norm. Kolmogorov [2] (see also [6]) discussed $D_n(\mathcal{A})$ in the $L^2$-norm, and gave an asymptotic formula for it when $\mathcal{A}$ is the class of functions $f$ on an interval with $f^{(n)}$ bounded in the $L^2$-norm. Originally the present paper was written to improve (for linear approximation) the

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estimates of $D_n(\mathcal{A})$ given by Vituškin [7]. After the research announcement [4] had been accepted for publication, a book of Vituškin [8] on the theory of tabulation appeared with improved estimates. We shall compare his results with our Theorems 3, 5, 7. Vituškin considers only approximation of functions $f$ of a variable $x$ in the uniform norm, $||f|| = \sup_{x} |f(x)|$. His results are generally of the type $n \geq C\phi(\epsilon)$, where $\epsilon = D_n(\mathcal{A})$, $\phi(\epsilon)$ is a known decreasing function of $\epsilon$, and $C$ is an unknown bounded function of $n$. This leads to the same lower bound as that given in our Theorem 5 for the $L^1$-norm, but his lower estimates for the cases treated by Theorems 3 and 7 are not as good as ours. It should be noted, however, that he considers also approximation by expressions more general than linear forms $\sum_{i=1}^{n} a_i g_i(x)$. He replaces these by polynomials of degree $k$ in the $a_i$, $i = 1, \ldots, n$, with coefficients which are given functions of $x$, or by a quotient of two such polynomials. His results may be roughly described by saying that in his previous inequality, $n$ may now be replaced by $n \log ((K+1)/\epsilon)$, $K$ being the maximum of the degrees of the denominator and the numerator. Vituškin's methods are quite different from ours. Vituškin uses the metric entropy of Kolmogorov (compare [3]), while we prefer a direct approach.

2. Uniform approximation. Let $A$ be a compact metric space containing infinitely many points with the distance function $\rho$, and let $\mathcal{A}$ be a set of continuous real- or complex-valued functions on $A$. Throughout this paper, $||f||$ shall denote the uniform norm $||f|| = \sup_{x \in A} |f(x)|$ of $f$.

**Definition.** The characteristic (at the origin) of a compact family $\mathcal{A}$, $x_n(\mathcal{A})$, is the largest number $\delta \geq 0$ with the following property. There exist $n+1$ points of $A$, $x_0, x_1, \ldots, x_n$ such that for each distribution of real or complex signs $e_k$, $k = 0, \ldots, n$, there exists a function $f \in \mathcal{A}$ with

\[
\text{sign } f(x_k) = e_k, \quad k = 0, \ldots, n, \\
|f(x_k)| \geq \delta, \quad k = 0, \ldots, n.
\]

It is clear from the compactness of $A$ and $\mathcal{A}$ that a largest $\delta$ exists.

**Theorem 1.** For each compact family $\mathcal{A}$,

\[
D_n(\mathcal{A}) \geq x_n(\mathcal{A}).
\]

**Proof.** Let $\delta = x_n(\mathcal{A})$, and let $x_0, \ldots, x_n$ be the corresponding points of $A$. Consider arbitrary continuous functions $g_0, \ldots, g_n$ defined on $A$. We can select constants $c_k$, $k = 0, \ldots, n$ with $\sum_{k=0}^{n} |c_k| = 1$ so that

\[
\sum_{k=0}^{n} c_k g_i(x_k) = 0, \quad i = 1, \ldots, n.
\]

Put $L(f) = \sum_{k=0}^{n} c_k f(x_k)$. Then for arbitrary $a_i$,

\[
||f - \sum_{i=1}^{n} a_i g_i|| \geq L\left( f - \sum_{i=1}^{n} a_i g_i \right) = L(f).
\]
We select $f \in \mathcal{A}$ to satisfy (4) and (5) with $\varepsilon_k = \text{sign } c_k$ and $\delta = \chi_n(\mathcal{A})$. Then

\begin{equation}
E_n^g(f) \geq L(f) \geq \chi_n(\mathcal{A}).
\end{equation}

This proves (6), since $G$ is arbitrary.

Sometimes one can show that relations such as (7) hold for some $f \in \mathcal{A}$ and an infinity of $n$. In this direction we have:

**Theorem 2.** Let the family $\mathcal{A}$ contain 0 and be compact and convex. Then for each sequence $\varepsilon_n \to 0$ and each $G$ there is a function $f_0 \in \mathcal{A}$ such that for infinitely many $n$,

\begin{equation}
E_n^g(f_0) \geq \varepsilon_n \chi_n(\mathcal{A}).
\end{equation}

**Proof.** This is a special case of the following lemma. Let $A$ be a convex and closed subset of a Banach space $X$ and let

\[ \|f\| = E_0(f) \geq E_1(f) \geq \cdots \geq E_n(f) \geq \cdots \]

be a sequence of semi-norms on $X$. Let $\chi_n = \sup_{f \in \mathcal{A}} E_n(f) < +\infty$, $n = 0, 1, \cdots$. Then for each sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, there exists an $f_0 \in \mathcal{A}$ with

\begin{equation}
E_n(f_0) \geq \varepsilon_n \chi_n
\end{equation}

for infinitely many $n$.

Assume on the contrary that for each $f \in \mathcal{A}$

\begin{equation}
E_n(f) < \varepsilon_n \chi_n
\end{equation}

for all sufficiently large $n$. We select $f_n \in \mathcal{A}$ such that $E_n(f_n) \geq \chi_n / 2$. By induction we construct a sequence $b_k > 0$, $k = 1, 2, \cdots$, $\sum_k b_k \leq 1$ and a sequence of natural numbers $n_1 < n_2 < \cdots < n_k < \cdots$ as follows. If the $b_i$, $n_i$, $i \leq k - 1$ are known, $\sum_1^{k-1} b_i < 1$, we first take $b_k > 0$ so small that $b_k < b_{k-1} / 2$, $\sum_i b_i < 1$, and

\begin{equation}
b_k < \frac{1}{2} \varepsilon_{n_2-1} \chi_{n_2-1}.
\end{equation}

We then take $n_k$ so large that

\begin{equation}
b_k \leq \epsilon_{n_k} < b_k,
\end{equation}

\begin{equation}
E_n^g \left( \sum_1^k b_i f_{n_i} \right) < \epsilon_{n_k} \chi_{n_k}.
\end{equation}

Let $f_0 = \sum_1^\infty b_i f_{n_i}$. Then, assuming without loss of generality $\chi_0 = 1$, we have

\[ E_{n_k}(f_0) \geq E_{n_k}(b_i f_{n_i}) - E_{n_k} \left( \sum_1^{k-1} b_i f_{n_i} \right) - E_{n_k} \left( \sum_{k+1}^\infty b_i f_{n_i} \right), \]

\[ E_{n_k} \left( \sum_{k+1}^\infty b_i f_{n_i} \right) \leq \sum_{k+1}^\infty b_i E_0(f_{n_i}) \leq \sum_{k+1}^\infty b_i \leq 2b_{k+1} < \epsilon_{n_k} \chi_{n_k}. \]
by (11), hence by (12) and (13)

\[ E_n(f_0) \geq \frac{1}{2} b_kX_{n_k} - 2\epsilon_kX_{n_k} \geq \epsilon_kX_{n_k}, \quad k = 1, 2, \ldots \]

This is a contradiction with our assumption and proves the lemma.

In §5 we shall see that sometimes \( \epsilon_n \) may be dropped in the statement of Theorem 2.

3. Approximation of continuous functions. We consider a continuous increasing function \( \omega(h) \) defined for \( h \geq 0 \), vanishing at zero and subadditive: \( \omega(h_1 + h_2) \leq \omega(h_1) + \omega(h_2) \). We denote by \( \Lambda^\omega = \Lambda^\omega(A) \) the set of all continuous real functions on a compact metric space \( A \) with infinitely many points which satisfy \( |f(x)| \leq 1 \), \( x \in A \) and

\[ |f(x) - f(x')| \leq \omega(\rho(x, x')) \]  

**Theorem 3.** Let \( \sigma_n = \sigma_n(A) \) be the largest number \( \sigma > 0 \) such that \( A \) contains \( n + 1 \) points at mutual distances \( \leq 2\sigma \). Then for all large \( n \),

\[ D_n(\Lambda^\omega) \geq \frac{1}{2} \omega(\sigma_n); \]

**this result is essentially the best possible.**

**Proof.** Let \( x_0, x_1, \ldots, x_n \) be points of \( A \) at mutual distances \( \leq 2\sigma \), \( \sigma = \sigma_n \).

Let \( \epsilon_k = \pm 1, k = 0, \ldots, n \). Put

\[ f(x) = \begin{cases} \frac{1}{2} \epsilon_k\omega(\sigma - \rho(x, x_k)) & \text{if } \rho(x, x_k) \leq \sigma, \quad k = 0, \ldots, n, \\ 0 & \text{elsewhere.} \end{cases} \]

To estimate \( f(x) - f(x') \) we may assume that \( |f(x)| \geq |f(x')| \) and that \( x \) belongs to the ball \( \rho(x, x_k) \leq \sigma \). If also \( x' \) belongs to this ball,

\[ |f(x) - f(x')| = \frac{1}{2} \left\{ \omega(\sigma - \rho(x, x_k)) - \omega(\sigma - \rho(x', x_k)) \right\} \leq \frac{1}{2} \omega(\rho(x', x_k) - \rho(x, x_k)) \leq \frac{1}{2} \omega(\rho(x, x')); \]

and if \( x' \) does not belong to the ball,

\[ |f(x) - f(x')| \leq 2 |f(x)| = \omega(\sigma - \rho(x, x_k)) \leq \omega(\rho(x, x')). \]

Also, for large \( n \), \( |f(x)| \leq 2^{-1} \omega(\sigma_n) \leq 1 \), \( x \in A \). Thus \( f \in \Lambda^\omega \). From the definition of \( \chi_n \), we see that \( \chi_n(\Lambda^\omega) \geq 2^{-1} \omega(\sigma_n) \), and Theorem 1 gives (15).

This inequality cannot be improved in the following sense. For a given \( n \), let \( \sigma_n' = \sigma' > 0 \) be chosen so that there exist \( n \) open balls \( U_1, \ldots, U_n \) with centers \( x_1, \ldots, x_n \) and radii \( \sigma' \) which cover \( A \). Then there exists a system of real continuous functions \( G = \{ g_1, \ldots, g_n \} \) with
A reasonably chosen $\sigma'$ is usually close to $\sigma_n$; for example, one always has $\sigma_{n+1} < \sigma'_n$ [3, p. 8]. To prove (17), consider a decomposition of unity $1 = g_1(x) + \cdots + g_n(x), x \in A$, where $g_k(x)$ are continuous functions on $A$ with $0 \leq g_k(x) \leq 1, x \in A, g_k(x) = 0$ for $x \in U_k, k = 1, \ldots, n$. If $f \in \Lambda_n$ and $P = \sum_{k=1}^{n} f(x_k)g_k$, then

$$|f(x) - P(x)| \leq \sum_{k=1}^{n} |f(x) - f(x_k)| g_k(x) \leq \omega(\sigma').$$

4. Approximation in the $L^1$ norm. Let $A$ be a compact metric space and $\mu$ a positive regular Borel measure on $A$. Let $\mathfrak{A}$ be a subset of $L^1(A; \mu)$. We prove a theorem similar to Theorem 1 of §2. For this purpose we need the following:

**Definition.** For two natural numbers $n, p$, $D_n^p(\mathfrak{A})$ denotes the supremum of numbers $\delta \geq 0$ with the following property. There exist disjoint measurable subsets $A_k, k = 1, \ldots, n+p$ of $A$ such that for any $p$ different sets $A_k, j = 1, \ldots, p$ and any distribution of real or complex signs $\varepsilon_k, k = 1, \ldots, n+p$, there exists a function $f_0 \in \mathfrak{A}$ with

$$\text{sign} \int_{A_k} f_0 d\mu = \varepsilon_k, \quad k = 1, \ldots, n+p,$$

$$|\int_{A_{kj}} f_0 d\mu| \geq \delta, \quad j = 1, \ldots, p.$$ 

**Theorem 4.** For each subset $\mathfrak{A} \subset L^1(A; \mu)$ one has in the $L^1$-norm

$$D_n(\mathfrak{A}) \geq pD_n^p(\mathfrak{A}).$$

**Proof.** Let $g_i, i = 1, \ldots, n$ be arbitrary functions from $L^1(A; \mu)$. The equations

$$\sum_{k=1}^{n+p} c_k \int_{A_k} g_i d\mu = 0, \quad i = 1, \ldots, n$$

have at least $p$ linearly independent solutions $c = \{c_k\}_{k=1}^{n+p}$. We use the following:

**Lemma.** The vector space $X$ spanned by $p$ linearly independent vectors $c = \{c_k\}_{k=1}^{n+p}$ contains at least one vector $c = \{c_k\}$ for which

$$|c_k| = \|c\| = \sup_{l=1, \ldots, n+p} |c_l| = 1$$

holds for at least $p$ values of $k$.

Using this we select constants $c_k$ satisfying (21) and (22) and put for $f \in L^1$,

$$L(f) = \sum_{k=1}^{n+p} c_k \int_{A_k} f d\mu.$$
This is a continuous linear functional of norm one and \( L(g_i) = 0, i = 1, \cdots, n \).

As in §2 we see that \( E_2^0(f) \geq L(f) \). Hence the theorem will be proved if we can find an \( f_0 \in \mathcal{A} \) with \( L(f_0) \geq p \delta \) for each \( \delta < \delta_\alpha^0(\mathcal{A}) \). If we define the signs \( \epsilon_k \) by \( \epsilon_k = \text{sign } c_k \) and take for \( k_1, \cdots, k_p \) some \( p \) values of \( k \) satisfying (22), the inequality \( L(f_0) \geq p \delta \) follows from the definition of \( \delta_\alpha^0(\mathcal{A}) \) for a properly chosen \( f_0 \).

**Proof of the lemma.** Let \( j, 1 \leq j \leq p \) denote the largest integer for which

\[
(23) \quad |x_k| = \|x\| = \sup_{i=1, \ldots, n+p} |x_i| = 1
\]

holding for \( j \) different values of \( k \). We shall show that the assumption \( j < p \) leads to a contradiction. Let \( x \) be a vector in \( X \) for which (23) holds for \( k = k_1, \cdots, k_j \); and consider the \( j' \leq j \) dimensional space \( Y \) spanned by the vectors \( d = \{c_k\}_{k=1}^{j} \). We can find \( j' \) of the vectors \( c \) for which the corresponding \( d \) span \( Y \). Subtracting proper linear combinations of these \( c \) from other vectors \( d \), we obtain vectors \( c' \), not all zero, with coordinates \( c'_k, i = 0, 1, \cdots, j \).

For one of such \( c' \) and the \( x \) chosen above put \( y = x + \lambda c' \), with real \( \lambda \). The coordinates of \( y \) change continuously with \( \lambda \), and \( |y_{k_i}| = |x_{k_i}| = 1 \). It is easy to see that one can find a value of \( \lambda \) for which \( y \) will satisfy (23) for more than \( j \) values of \( k \). This is a contradiction.

As an application of Theorem 4, we can obtain for the class \( \mathcal{A} \) of analytic functions of §6, for the \( L^1 \)-norm and the Lebesgue measure, \( D_n(\mathcal{A}) \leq C n^{-p+\alpha} \).

We omit the details.

5. \( L^1 \) approximation of smooth functions. Let \( A_0 \) be a fixed parallelepiped of the \( q \)-dimensional euclidean space, and let \( A \) be a subset of \( A_0 \), which is a union of finitely many parallelepipeds. A function \( f(x) \), \( x = (x^{(1)}, \cdots, x^{(\mu)}) \), defined on \( A_0 \), belongs to \( C^p+\alpha(A_0) \), where \( p = 0, 1, \cdots, 0 < \alpha \leq 1 \), if it has continuous partial derivatives of order \( p \) on \( A_0 \), each of which satisfies a Lipschitz condition

\[
(24) \quad |f^{(p)}(x) - f^{(p)}(x')| \leq M_p(x, x')^\alpha.
\]

If \( M_0 \) is the smallest possible value of \( M \) in (24), we put \( \|f\|_{p+\alpha} = \max(\|f\|, M_0) \). Finally, \( \Lambda^{p+\alpha} \) will be the set of all functions \( f \) on \( A_0 \) with \( \|f\|_{p+\alpha} \leq 1 \).

The following theorem states essentially that \( D_n(\Lambda^{p+\alpha}) \leq \text{const. } n^{-(p+\alpha)/\mu} \); details will be needed for a "gliding hump" argument in Theorem 6. In this section, \( \mu \) will be the Lebesgue measure.

**Theorem 5.** There exists a constant \( B > 0 \) depending only on \( p+\alpha, q, m > 0 \) such that for each \( A \) with \( \mu A \geq m \) and each \( \epsilon > 0 \), for each sufficiently large \( n, n \geq n(A, \epsilon, p+\alpha, q) \), and each system \( G = \{g_1, \cdots, g_n, \cdots\} \) of integrable functions on \( A_0 \), there exists a function \( f_0 \in \Lambda^{p+\alpha} \) defined on \( A_0 \) and vanishing outside of \( A \) such that for its degree of approximation in the \( L^1 \)-norm on \( A_0 \),
\begin{align*}
\text{(25)} & \quad E_n^G(f_0) \geq Bn^{-(p+\alpha)/q}; \\
\text{moreover for all partial derivatives } f_0^{(k)} \text{ of orders } k = 0, 1, \cdots, p, \\
\text{(26)} & \quad \|f_0^{(k)}\| < \varepsilon.
\end{align*}

This \( f_0 \) belongs to \( C^{p+1}(A') \), where \( A' \) is obtained from \( A \) by removing a finite number of open parallelepipeds of arbitrary small total measure.

\textbf{Proof.} We use the construction of Theorem 4 to obtain our \( f_0 \). Let \( F(u) \) denote a fixed increasing continuous function for \( 0 \leq u \leq 1 \), given by \( F(u) = u^{\alpha} \) for \( 0 \leq u \leq 1/2 \), with continuous derivatives of all orders up to \( p+1 \) for \( 0 < u \leq 1 \), which all vanish at \( u = 1 \). Clearly, \( F^{(p)}(u) \) will satisfy on \([0, 1]\) a Lipschitz condition \( |F^{(p)}(u) - F^{(p)}(u')| \leq B_1 |u - u'|^p \). (The constants \( B_1, B_2, \cdots \) will depend only on \( p+\alpha, q, m \).)

For all sufficiently large \( n \), \( A \) contains \( 2^n \) points \( x_1, \cdots, x_{2^n} \) at mutual distances from each other \( \geq 2B_2n^{-1/q} \), such that the closed balls \( A_k \) with center \( x_k \) and radius \( r = B_2n^{-1/q} \) are contained in \( A \). Let \( \epsilon_k = \pm 1 \) be an arbitrary distribution of signs. We put

\[ f_1(x) = \begin{cases} 
\epsilon_k r^{\alpha} F(1 - r^{-1} \rho(x, x_k)), & x \in A_k, \\
0 & \text{elsewhere on } A_0.
\end{cases} \]

Elementary computations show that \( f_1 \in C^{p+\alpha} \) with \( \|f_1\|_{p+\alpha} \leq B_2 \). On the other hand, \( |f_1(x)| \geq B_2 r^{p+\alpha} \) for \( \rho(x, x_k) \leq r/2 \), hence

\[ \left| \int_{A_k} f_1 d\mu \right| \geq B_2 r^{p+\alpha} \geq B_2 n^{-(p+\alpha)/q} = \delta. \]

If we put \( f_0 = f_1/B_2 \), then \( f_0 \) belongs to \( \Lambda^{p+\alpha} \) and satisfies conditions (18) and (19) of \( \S 4 \), with \( n \) replaced by \( 2^n \), and \( p \) by \( n \), hence \( E_n^G(f_0) \geq n \delta \) with \( \delta \) given by (27). This proves (25), and the other conditions are also satisfied for each large \( n \).

For \( 0 < \alpha < 1 \) we can prove more.

\textbf{Theorem 6.} Let \( A \) be a closed \( q \)-dimensional parallelepiped and \( 0 < \alpha < 1 \).

There exists a constant \( B \) such that for each system \( G = \{g_n\} \) of integrable functions on \( A \), there exists an \( f_0 \in \Lambda^{p+\alpha} \) for which (for the \( L^1 \)-approximation)

\[ E_n^G(f_0) \geq Bn^{-(p+\alpha)/q} \]

holds for an infinity of values of \( n \).

\textbf{Proof.} We shall have \( B = B_2/6 \), where \( B_2 \) is the constant of Theorem 5 corresponding to \( m = \mu A/2 \). Let the sequence \( G \) be given. For the construction we shall need two sequences of sets, \( A_k, A'_k \) consisting each of finitely many closed parallelepipeds and satisfying

\[ A = A_1 \supset A'_1 \supset \cdots \supset A_k \supset A'_k \supset \cdots ; \mu A_k > m, \]
functions $f_k$ defined on $A$ and vanishing outside of $A_k$, integers $1 \leq n_j < \cdots < n_k < \cdots$, and two sequences of numbers $e_k > 0$, $\delta_k > 0$ with $\sum_{i=1}^k e_i < 1$; $\delta_k$ will be the distance from $A_k'$ to the boundary of $A_k$. We put $M_k = \sum_{i=1}^k e_i$, $F_k = \sum f_i^2$, and denote by $N_k$ the $C^{p+1}$-norm of $F_k$ on $A_k + 1$.

Let all these elements with subscripts not exceeding $k - 1$ be defined; we shall define all elements with subscripts $k$. We can assume that for all sufficiently large $n$,

$$E_n(F_{k-1}) < \frac{1}{3} B_1 n^{-\rho}, \quad \rho = \frac{p + \alpha}{q},$$

for otherwise Theorem 6 would follow immediately.

We first select $A_k, A', A_{k-1} \supset A_k \supset A_{k-1}$ and $\mu A_k' > m$ such that $F_{k-1}$ belongs to the class $C^{p+1}$ on $A_k$, and that the distance $\delta_k$ of $A_k'$ to the boundary of $A_k$ satisfies

$$0 < \delta_k < (N_{k-1} e_{k-1})^{(1-(1-\alpha))}, \quad \delta_k < 1.$$

Then we take $n_k > n_{k-1}$ so large that (29) holds with $n = n_k$. Next $e_k > 0$ is defined with $\sum_{i=1}^k e_i < 1$ and

$$\epsilon_k < \frac{1}{3} B_1 n_k^{-\rho} \mu A,$$

$$\epsilon_k < \frac{1}{2} \epsilon_{k-1} \delta_k.$$

Thus $\epsilon_k < \epsilon_{k-1}/2$. Finally we take $f_k$ to be the function $f_0$ of Theorem 5, with $n, A_0, A, \epsilon$ now replaced by $n_k, A_k, A_k'$ and $e_k$, respectively. Put

$$f = \sum_{i=1}^\infty f_i.$$

As in Theorem 2, we obtain for the degree of approximation by the $g_i, g_k, \cdots$ in the $L^1$ norm on $A$,

$$E_n(f) \geq E_n(f_k) - E_n(F_{k-1}) - \sum_{i=k+1}^\infty \|f_i\| \mu A$$

$$\geq B_1 n_k^{-\rho} - \frac{1}{3} B_1 n_k^{-\rho} - \sum_{i=k+1}^\infty e_i \mu A$$

$$\geq \frac{2}{3} B_1 n_k^{-\rho} - e_k \mu A \geq \frac{1}{3} B_1 n_k^{-\rho}.$$

We now have to show that $f \in A^{p+\alpha}$. We have $\|f\| \leq \sum \|f_i\| \leq \sum e_i < 1$. The derivatives of $f$ of orders $k = 1, \cdots, p$ are obtained by termwise differentia-
tion of (33); in particular, each partial derivative $f^{(p)}$ is by (26) and the choice of $\varepsilon_k$ equal to $f^{(p)}(x) = \lim_{k \to +\infty} F_k^{(p)}(x)$. We show by induction that

$$
| F_k^{(p)}(x) - F_k^{(p)}(x') | \leq (1 + M_{k-1}) h^a, \quad h = \rho(x, x').
$$

We may assume that at least one of the points $x, x'$ belongs to $A_k'$, for otherwise $F_k = F_{k-1}$ at these points. Let $x \in A_k'$. We consider two cases. If $h \leq \delta_k$, then $x' \in A_k$ and by (30), $N_{k-1} h^{a-1} \leq \varepsilon_{k-1}$. Therefore

$$
| F_k^{(p)}(x) - F_k^{(p)}(x') | = | F_k^{(p)}(x) - F_k^{(p)}(x') | + | f^{(p)}(x) - f^{(p)}(x') |
\leq N_{k-1} h + h^a \leq (\varepsilon_{k-1} + 1) h^a
\leq (1 + M_{k-1}) h^a.
$$

If on the other hand $h > \delta_k$, then by (32), $\varepsilon_{k-1} h^a > 2\varepsilon_k$, hence

$$
| F_k^{(p)}(x) - F_k^{(p)}(x') | \leq (1 + M_{k-2}) h^a + 2|f^{(p)}|,
\leq (1 + M_{k-2}) h^a + 2\varepsilon_k
\leq (1 + M_{k-1}) h^a.
$$

If we put $f_0 = f/2$, we see that $f_0 \in \Lambda^{a+\alpha}$ and that $E_n(f_0) \geq 6^{-1} B_1 n_k^{-\rho}$, $k = 1, 2, \ldots$.

6. $L^1$-approximation of analytic functions. Let $\mathcal{A}$ be the family of the complex valued analytic functions on $[-1, +1]$, which have analytic continuation bounded by one into the ellipse $E_\rho$ with foci $-1, +1$ and the sum of the half axes $\rho > 1$. We consider the measure given by $d\mu = (1-x^2)^{-1/2} dx$ on $[-1, +1]$.

**Theorem 7.** For the family $\mathcal{A}$ in the $L^1(\mu)$ norm,

$$
D_n(\mathcal{A}) \geq C \rho^{-n},
$$

$$
C = 1 - \rho^{-1}.
$$

For the uniform approximation, (35) may be derived from Theorem 1. One can take for the function $f$ with properties (4), (5) the interpolating polynomial with values $\varepsilon_k \delta$ assumed at the $n+1$ zeros $x_k$ of the Chebyshev polynomial $T_{n+1}$.

Turning to the proof of Theorem 7, we note that $|T_k(x)| \leq \rho^k$ in $E_\rho$ [5], hence the functions $t_k = \rho^{-k} T_k$ belong to $\mathcal{A}$.

Let $g_1, \ldots, g_n$ be arbitrary functions from $L^1(\mu)$; there exist linear combinations $h_k$ of the $g_i$ such that

$$
\|t_k - h_k\|_{L^1} \leq E_{n}(\mathcal{A}),
$$

$$
k = 0, 1, \ldots, n.
$$

As a set of $n+1$ vectors in an $n$-dimensional space is always linearly dependent, there exist constants $c_0, \ldots, c_n$ such that
\[(37) \quad \sum_{k=0}^{n} c_k h_k(x) = 0,\]

\[(38) \quad \sum_{k=0}^{n} |c_k| = 1.\]

The function \( f = \sum_{k=0}^{n} c_k h_k \) belongs to \( \mathcal{A} \) by (38). By (36) and (37), \( \|f\|_{L^1} \leq E_n^\alpha(\mathcal{A}) \). We also have

\[\|f\|_{L^1} = \left| \int_{-1}^{1} f(x) T_k(x) d\mu(x) \right| = c_k \rho^{-k}.\]

Hence

\[\rho^k E_n^\alpha(\mathcal{A}) \geq |c_k|, \quad k = 0, 1, \ldots, n.\]

Summing over \( k \) we get

\[(\rho^{n+1} - 1) E_n^\alpha(\mathcal{A}) \geq \rho^{-1}.\]

From this the inequality (35) follows. Classical estimates in the uniform norm give \( D_n(\mathcal{A}) \leq C_1 \rho^{-n} \), \( C_1 = 2(\rho - 1)^{-1} \).

We conclude by remarking that similar estimates can be given for the class \( \mathcal{A}' \) of functions analytic on \( |z| < 1 \), which have analytic continuation bounded by one into the circle \( |z| < \rho, \rho > 1 \).

REFERENCES