PERIODICITY MODULO m AND DIVISIBILITY PROPERTIES OF THE PARTITION FUNCTION(1)

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Introduction. The subject matter of this paper arose in considering the distribution of the values of the unrestricted partition function $p(n)$ modulo m, m an integer $\geqq 2$. Here $p(n)$ is defined by

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}.$$

The Ramanujan congruences

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7},$$
$$p(11n + 6) \equiv 0 \pmod{11},$$

show that 5, 7, 11 divide $p(n)$ for infinitely many values of n, and Watson [9] has shown that the same is true for all powers of 5 and 7. In addition Lehner [3; 4] has shown that 11^2 and 11^3 also have this property. It is natural to conjecture therefore that $p(n)$ fills *all* residue classes modulo m infinitely often; that is, that if r is any integer such that $0 \leqq r \leqq m-1$, then the congruence

$$p(n) \equiv r \pmod{m}$$

has infinitely many solutions in non-negative integers n.

This conjecture seems difficult and I have only scattered results. In §2 of this paper it will be shown that the conjecture is true for $m=5$ and 13 by means of congruences derived from the elliptic modular functions, and similar theorems will be proved; for example that $p(5n+4)/5$ and $p(7n+5)/7$ fill all residue classes modulo 5 and 7 respectively, infinitely often. In §1 it will be shown in an elementary way that the conjecture is also true for $m=2$. Thus $p(n)$ is odd infinitely often and even infinitely often. In §1 we will also consider the question of the periodicity of a sequence modulo m in some generality.

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1. We first state and prove a theorem which leads immediately to a proof of the fact that $p(n)$ is odd infinitely often and even infinitely often.

THEOREM 1. *Let*

$$f(x) = \sum_{n=0}^{\infty} c_n x^{e_n}, \qquad 0 \leqq e_0 < e_1 < e_2 < \cdots,$$

be a power series with integral coefficients and exponents such that

$$e_{n+1} - e_n \to \infty \text{ as } n \to \infty, \tag{1}$$

$$(c_n, c_{n+1}, \cdots) = 1, \qquad n \geqq 0. \tag{2}$$

Then there do not exist polynomials $\alpha(x)$, $\beta(x)$ with integral coefficients, $\alpha(0) = 1$, such that

$$f(x) \equiv \beta(x)/\alpha(x) \pmod{m}. \tag{3}$$

In congruence (3) it is understood that the coefficients of corresponding powers of x are congruent.

Proof. We note that if

$$\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} c_n x^{e_n} = \sum_{n=0}^{\infty} b_n x^n$$

then

$$\sum_{e_k \leq n} c_k a_{n-e_k} = b_n, \qquad n \geqq 0.$$

Put

$$\alpha(x) = \sum_{n=0}^{r} a_n x^n, \qquad a_0 = 1,$$

$$\beta(x) = \sum_{n=0}^{s} b_n x^n.$$

Then

$$\alpha(x) f(x) \equiv \beta(x) \pmod{m}$$

implies that

$$\sum_{e_k \leq n} c_k a_{n-e_k} \equiv b_n \pmod{m};$$

and replacing n by e_n,

$$\sum_{k=1}^{n} c_k a_{e_n - e_k} = c_n + \sum_{k=0}^{n-1} c_k a_{e_n - e_k} \equiv b_{e_n} \pmod{m}. \tag{4}$$

By (1) we can choose n_0 so large that for all $n \geqq n_0$, $e_n - e_{n-1} > r$, $e_n > s$. Then (4) implies that for all $n \geqq n_0$,

$$c_n \equiv 0 \pmod{m}.$$

But this contradicts (2), since $m > 1$. The theorem is thus proved.

Theorem 1 now implies

THEOREM 2. *The sequence $\{p(n)\}$ is never ultimately periodic modulo m and consequently $p(n)$ fills at least two different residue classes modulo m infinitely often. Thus $p(n)$ is odd infinitely often and even infinitely often.*

Proof. The Euler product

$$\phi(x) = \prod_{n=0}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{(3n^2+n)/2}$$

satisfies the requirements (1), (2). Thus $\phi(x)$ (and so $1/\phi(x)$) is never congruent modulo m to a quotient $\beta(x)/\alpha(x)$. But it is easy to show that a sequence of integers $\{t_n\}$ is ultimately periodic modulo m if and only if $\sum t_n x^n$ is congruent modulo m to a quotient $\beta(x)/\alpha(x)$. (This is proved below.) Theorem 2 is therefore proved.

We now look at these questions somewhat more generally. Let R_m be the totality of formal power series in x which are congruent modulo m to a quotient of polynomials with integral coefficients, the denominator having constant term 1; and let $S_m \subset R_m$ be defined by the additional requirement that the numerator have constant term 1 also. Then R_m is closed with respect to addition, multiplication and differentiation, and S_m is closed with respect to multiplication and division. Further, we have the theorem

THEOREM 3. *The sequence of integers $T = \{t_n\}$, $n \geqq 0$ is ultimately periodic modulo m if and only if the formal power series*

$$f(x) = \sum_{n=0}^{\infty} t_n x^n$$

is in R_m.

Proof. If T is ultimately periodic modulo m, then integral polynomials $\alpha(x)$, $\beta(x)$ and a positive integer q exist such that

$$f(x) \equiv \alpha(x) + \beta(x) + x^q\beta(x) + x^{2q}\beta(x) + \cdots \pmod{m}$$

Hence

$$f(x) \equiv \frac{\alpha(x)(1 - x^q) + \beta(x)}{1 - x^q} \equiv \frac{\gamma(x)}{1 - x^q} \pmod{m},$$

where $\gamma(x)$ is an integral polynomial. Thus $f(x)$ is in R_m.

If $f(x)$ is in R_m, then integral polynomials $\alpha(x)$, $\beta(x)$ exist, $\alpha(0)=1$, such that

$$f(x) \equiv \beta(x)/\alpha(x) \pmod{m}.$$

This implies that fixed integers $a_1, a_2, \cdots, a_r$ exist such that for all $n \geqq n_0$,

$$t_n \equiv a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_r t_{n-r} \pmod{m}. \tag{5}$$

Now the total number of vectors

$$(t_{n-1}, t_{n-2}, \cdots, t_{n-r}) = \tau_n$$

is finite modulo m (m^r is an upper bound) and so there are integers $n_2 > n_1 \geqq n_0$ such that

$$\tau_{n_2} \equiv \tau_{n_1} \pmod{m}.$$

But then (5) implies that for all $k \geqq -r$,

$$t_{n_2+k} \equiv t_{n_1+k} \pmod{m},$$

and so T is ultimately periodic modulo m, completing the proof of the theorem.

An easy deduction from the closure properties of the sets R_m, S_m is

LEMMA 1. *If r is a rational number in lowest terms whose numerator is prime to m and $f(x)^r$ is in S_m, then $f'(x)/f(x)$ is in R_m.*

Using this lemma, we can prove

LEMMA 2. *Suppose that $T=\{t_n\}$, $Q=\{q_n\}$ are two sequences of integers defined for $n \geqq 0$ such that $t_0=0$, $q_0=1$ and*

$$nq_n = \sum_{k=1}^{n} t_k q_{n-k}, \qquad n \geqq 1. \tag{6}$$

Then if T is not ultimately periodic modulo m neither is Q.

Proof. Set $t(x)=\sum_{n=1}^{\infty} t_n x^n$, $q(x)=\sum_{n=0}^{\infty} q_n x^n$. Then (6) implies that

$$xq'(x)/q(x) = t(x).$$

Suppose Q is ultimately periodic modulo m. Then $q(x)$ is in R_m (Theorem 3) and since $q_0=1$, $q(x)$ is also in S_m. Thus $q'(x)/q(x)$ is in R_m (Lemma 1), and therefore so is $t(x)$. Thus T is also ultimately periodic modulo m (Theorem 3) and the lemma is proved.

If n is a positive integer, let $\sigma(n)$ denote the sum of the divisors of n, and define $\sigma(n)$ as 0 otherwise. Then we have

LEMMA 3. *Let $r_1, e_1, r_2, e_2, \cdots, r_s, e_s$ be integers such that $0<e_1<e_2< \cdots <e_s$. Set*

$$t_n = -\sum_{k=1}^{s} r_k e_k \sigma(n/e_k), \qquad n \geqq 1. \tag{7}$$

Let p be a prime such that $(r_1e_1, p) = 1$. Then the sequence $T = \{t_n\}$ is not ultimately periodic modulo p.

Proof. Suppose the contrary. Then there is a positive integer d such that for all $n \geqq n_0$ the numbers

$$t_{e_1(nd+1)}$$

are all in the same residue class modulo p. Choose $n_1 \geqq n_0$ so that $q = n_1d+1$ is prime and larger than p and e_s (Dirichlet's theorem). Since

$$q^2 = (n_1^2d + 2n_1)d + 1 > n_1d + 1$$

we have that $t_{e_1q} \equiv t_{e_1q^2} \pmod{p}$. Then (7) implies that

$$-r_1e_1\sigma(q) \equiv -r_1e_1\sigma(q^2) \pmod{p}$$

and since $(r_1e_1, p) = 1$ and q is prime this implies that $q^2 \equiv 0 \pmod{p}$, a contradiction since $q \not\equiv p$. The lemma is therefore proved.

If we notice that with the values of t_n as given in (7) the associated function $q(x)$ of Lemma 2 becomes

$$\phi(x^{e_1})^{r_1}\phi(x^{e_2})^{r_2} \cdots \phi(x^{e_s})^{r_s}, \tag{8}$$

where $\phi(x) = \prod_{n=1}^{\infty}(1-x^n)$ is the Euler product, then we find from Lemmas 2 and 3

LEMMA 4. *Suppose that m is divisible by a prime not dividing r_1e_1. Then the coefficients of the function* (8) *are not ultimately periodic modulo m.*

We remark that the restriction that m be divisible by a prime not dividing r_1e_1 is sometimes necessary, since for example if p is a prime

$$\phi(x)^p\phi(x^p)^{-1} \equiv 1 \pmod{p}.$$

Many of the functions common in analytic number theory are of the type (8). $\phi(x)^{-1}$ enumerates $p(n)$ and has already been discussed. Some others are

$\phi(x)^{-1}\phi(x^2)$, which enumerates $q(n)$, the number of partitions of n into distinct parts (or into odd parts);

$\phi(x)^{-1}\phi(x^2)^2\phi(x^4)^{-1}$, which enumerates $q_0(n)$, the number of partitions of n into distinct odd parts;

$\phi(x)^{-2s}\phi(x^2)^{5s}\phi(x^4)^{-2s}$, which enumerates $r_s(n)$, the number of representations of n as the sum of s squares.

Then Lemma 4 implies

THEOREM 4. *The sequences $\{q(n)\}$, $\{q_0(n)\}$ are not ultimately periodic* *modulo m. If m contains a prime factor not dividing* $2s$, *the sequence* $\{r_s(n)\}$ *is not ultimately periodic modulo m.*

We can draw a similar conclusion for the function $p_r(n)$ defined by

$$\sum_{n=0}^{\infty} p_r(n)x^n = \phi(x)^r = \prod_{n=1}^{\infty} (1 - x^n)^r;$$

but in this case a somewhat stronger result holds. We first prove the following strengthened version of Lemma 1:

LEMMA 5. *Let* $r \neq 0$ *be an integer and suppose that* $f(x)^r \in S_m$. *Then if* p *is any prime dividing* m, $f'(x)/f(x) \in R_p$.

Proof. If $(p, r) = 1$ the conclusion is evident from Lemma 1. Suppose $p \mid r$, and put $r = pr_0$. Let $\alpha(x)$ be the polynomial of least degree with integral coefficients and constant term 1 such that

$$f(x)^r \equiv \beta(x)/\alpha(x) \pmod{p}$$

where $\beta(x)$ is also a polynomial with integral coefficients and constant term 1. Then

$$\alpha(x)f(x)^r \equiv \beta(x) \pmod{p}$$

and differentiating both sides,

$$\alpha'(x)f(x)^r \equiv \beta'(x) \pmod{p}$$

since $p \mid r$. But this implies that

$$\alpha'(x) \equiv 0 \pmod{p} \tag{9}$$

(since $\alpha(x)$ was of least degree) and so also

$$\beta'(x) \equiv 0 \pmod{p}. \tag{10}$$

But (9) and (10) now imply that

$$\alpha(x) \equiv \alpha_0(x^p) \pmod{p},$$
$$\beta(x) \equiv \beta_0(x^p) \pmod{p},$$

where $\alpha_0(u)$, $\beta_0(u)$ are again integral polynomials with constant terms 1. Since

$$f(x)^r = f(x)^{pr_0} \equiv f(x^p)^{r_0} \pmod{p},$$

it follows that

$$f(x)^{r_0} \equiv \beta_0(x)/\alpha_0(x) \pmod{p}$$

and the argument can be repeated until an exponent r_1 is reached such that $f(x)^{r_1} \in S_p$ and $(r_1, p) = 1$. The conclusion then follows from Lemma 1.

We now obtain from Lemma 5

THEOREM 5. *The sequence* $\{p_r(n)\}$, *where* r *is a nonzero integer, is not ultimately periodic modulo* m.

2. In this section we leave the elementary(²) and make use of some deep identities and congruences from the theory of the elliptic modular functions. We first prove the following lemma which will be needed later:

LEMMA 6. *Let* p *be a prime not dividing the positive integer* c. *Let* e *be the exponent of* c *modulo* p. *Then if* a *is not divisible by* p *the solutions of*

$$xc^x \equiv a \pmod{p} \tag{11}$$

fall into e *classes modulo* p *which are given by*

$$x = (ey + r)p - ac^{e-r}(p - 1), \qquad 0 \leqq r \leqq e - 1. \tag{12}$$

If $p|a$ *then* (11) *has just one class of solutions modulo* p *given by*

$$x = yp.$$

In either case (11) *is satisfied by infinitely many positive integers* x.

Proof. We can assume that $(a, p) = 1$, the latter part of the lemma being trivial. Set $x = et + r$, $0 \leqq r \leqq e - 1$. Then (11) becomes

$$(et + r)c^r \equiv a \pmod{p}$$

so that

$$t \equiv \frac{p-1}{e}(r - ac^{e-r}) \pmod{p}.$$

Thus $t = yp + ((p-1)/e)(r - ac^{e-r})$ and so x is of the form given in (12).

Two solutions with different r's are in different classes modulo p, since for

$$x_1 = (ey_1 + r_1)p - ac^{e-r_1}(p - 1),$$

$$x_2 = (ey_2 + r_2)p - ac^{e-r_2}(p - 1),$$

we have $x_1 \equiv ac^{e-r_1} \pmod{p}$, $x_2 \equiv ac^{e-r_2} \pmod{p}$, so that $x_1 \equiv x_2 \pmod{p}$ if and only if $c^{r_1 - r_2} \equiv 1 \pmod{p}$, which implies that $r_1 = r_2$.

We now collect together some congruences for easy reference.

LEMMA 7. *The following congruences are valid*:

$$p(5n + 1) \equiv p_{23}(5n) \pmod{5} \quad (\textit{Kolberg } [2]), \tag{13}$$

(²) It might be argued that Euler's pentagonal number theorem $\prod_{n=1}^{\infty}(1 - x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{(3n^2+n)/2}$ used in §1, Theorem 2 is not elementary, but the fact that a combinatorial proof of this result has been given by F. Franklin (see [1, pp. 83–85]) justifies its occurrence in §1. Of course it is possible that the same fate awaits some of the identities used in §2 which are presently not considered elementary.

(14) $p(5n+2) \equiv p_{23}(5n+1) \pmod 5$ (*Kolberg* [2]),

(15) $\frac{1}{5}\, p(5n+4) \equiv p_{19}(n) \pmod 5$ (*Ramanujan* [1, pp. 89–90]),

(16) $\frac{1}{25}\, p(25n+24) \equiv 3p_{23}(n) \pmod 5$ (*Zuckerman* [10]),

(17) $\frac{1}{7}\, p(7n+5) \equiv p_{17}(n) \pmod 7$ (*Ramanujan* [1, pp. 89–90]),

(18) $\frac{1}{49}\, p(49n+47) \equiv 5p_{23}(n) \pmod 7$ (*Zuckerman* [10]),

(19) $p\left(84n^2 - \frac{1}{24}(n^2-1)\right) \equiv 0 \pmod{13}$, $(n, 6) = 1$ (*Newman* [6]),

(20) $p((24t+11)\Delta_n + 13t + 6) \equiv 6^n p(13t+6) \pmod{13}$,

$$\Delta_n = \frac{13}{24}(13^{2n} - 1) \text{ (Newman [8])},$$

(21) $c(13n) \equiv -p_{24}(n-1) \equiv -\tau(n) \pmod{13}$,

where

$$\sum_{n=-1}^{\infty} c(n)x^n = j(\tau) = 12^3 J(\tau), \qquad x = \exp 2\pi i\tau,$$

$J(\tau)$ the complete modular invariant (*Newman* [1]).

The author has shown in [5] that if r is even, $0 \leqq r \leqq 24$, and p is a prime such that $\delta = r(p-1)/24 \equiv 0 \pmod 1$ then for all integral n

$$p_r(np+\delta) = p_r(n)p_r(\delta) - p^{(r/2)-1}p_r\left(\frac{n-\delta}{p}\right). \tag{22}$$

Set $a_n = r(p^n-1)/24$, $p_r(a_n) = t_n$. Then $t_0 = 1$, $t_1 = p_r(\delta)$, and replacing n by a_n in (22),

$$t_{n+1} = p_r(\delta)t_n - p^{(r/2)-1}t_{n-1}, \qquad n \geqq 1. \tag{23}$$

Define

$$\Delta = p_r(\delta)^2 - 4p^{(r/2)-1}.$$

Then it is easy to prove from (23) that

$$t_n = 2^{-n} \sum_{0 \leq k \leq n/2} \binom{n+1}{2k+1} p_r(\delta)^{n-2k}\Delta^k. \tag{24}$$

Thus if q is any odd prime divisor of Δ,

$$t_n \equiv (n+1)\left(\frac{p_r(\delta)}{2}\right)^n \pmod{q}. \tag{25}$$

Lemma 6 and (25) together now imply

THEOREM 6. *Suppose that r is even, $0 \leqq r \leqq 24$, and p is a prime such that $\delta = r(p-1)/24$ is an integer. Let q be any odd prime divisor of $\Delta = p_r(\delta)^2 - 4p^{(r/2)-1}$ which is different from p. Then the sequence $\{p_r(n)\}$ fills all residue classes modulo q infinitely often.*

An interesting application arises by choosing $r=24$, $p=79$ or 163. These primes are the only ones <200 such that

$$\tau(p)^2 = p_{24}(p-1)^2 \equiv 4p^{11} \pmod{13}.$$

Then Theorem 6 and (21) imply

THEOREM 7. *The sequence $\{c(13n)\}$ fills all residue classes modulo 13 infinitely often.*

The author has shown in [6] that if r is odd, $1 \leqq r \leqq 23$, and p is a prime such that $r\nu = r(p^2-1)/24 \equiv 0 \pmod 1$ then for all integral n,

$$p_r(np^2 + r\nu) - \gamma_n p_r(n) + p^{r-2} p_r\left(\frac{n - r\nu}{p^2}\right) = 0, \tag{26}$$

where

$$\gamma_n = c - \left(\frac{r\nu - n}{p}\right) p^{(r-3)/2} a, \qquad c = p_r(r\nu) + \left(\frac{r\nu}{p}\right) p^{(r-3)/2} a.$$

Here $((r\nu - n)/p)$ is the Legendre-Jacobi symbol of quadratic reciprocity and

$$a = (-1)^{(p-1)(p-1-2r)/8}.$$

Let a_0 be arbitrary, and set

$$a_n = p^{2n} a_0 + r(p^{2n} - 1)/24,$$
$$t_n = p_r(a_n).$$

Then a_n satisfies

$$a_{n+1} = p^2 a_n + r\nu$$

and replacing n by a_n in (26) we find that $t_0 = p_r(a_0)$, $t_1 = p_r(p^2 a_0 + r\nu)$,

$$t_{n+1} - c t_n + p^{r-2} t_{n-1} = 0, \qquad n \geqq 1. \tag{27}$$

We are going to apply Lemma 7 to the recurrence (27). The general procedure is to find values of r and p which make the desired divisibility properties of the sequences under consideration evident. This entails much numerical work and involves a knowledge of the coefficients $p_r(n)$, some of which are tabulated in [6].

We notice first that congruences (19) and (20) settle the conjecture mentioned in the introduction for $m=13$. Thus (19) shows that the zero class occurs infinitely often; and if we notice in (20) that 6 is a primitive root of 13 and t may be chosen so that $(p(13t+6), 13)=1$ then also all other classes must occur infinitely often. Thus

THEOREM 8. *The sequence* $\{p(n)\}$ *fills all residue classes modulo* 13 *infinitely often.*

We now consider congruences (13) and (14). For this purpose we choose $r=23$, and $p\equiv\pm 1 \pmod 5$. Then $a_n\equiv a_0 \pmod 5$ and we will choose a_0 so that either $a_0\equiv 0 \pmod 5$ or $a_0\equiv 1 \pmod 5$. Then (13) and (14) imply that

$$p(1 + a_n) \equiv t_n \pmod 5.$$

Since $p^2\equiv 1 \pmod 5$ (27) implies that

$$p(1 + a_{n+1}) - cp(1 + a_n) + p\cdot p(1 + a_{n-1}) \equiv 0 \pmod 5, \qquad n \geqq 1. \tag{28}$$

Here c satisfies

$$c \equiv p_{23}(23\nu) + \left(\frac{-69}{p}\right) \pmod 5.$$

We make the choice $p=11$. Then from the tables given in [6] we find that $c\equiv 2 \pmod 5$, and (28) becomes

$$p(1 + a_{n+1}) - 2p(1 + a_n) + p(1 + a_{n-1}) = 0 \pmod 5)$$

which implies that

$$p(1 + a_n) \equiv (t_1 - t_0)n + t_0 \pmod 5. \tag{29}$$

Everything depends on the initial values t_0, t_1. We have for $r=23$, $p=11$ from (26) that

$$\begin{gathered} p_{23}(121a_0 + 115) - \gamma_{a_0}p_{23}(a_0) + 11^{21}p_{23}\left(\frac{a_0 - 115}{121}\right) = 0, \\ \gamma_{a_0} = c - \left(\frac{115 - a_0}{11}\right)11^{10}. \end{gathered} \tag{30}$$

Choose a_0 so that

$$a_0 \not\equiv 115 \pmod{121}.$$

Then (30) implies that

$$t_1 - t_0 \equiv \left(1 + \left(\frac{5 - a_0}{11}\right)\right)t_0 \pmod 5.$$

Since we do not want $t_1-t_0\equiv 0 \pmod 5$, we choose a_0 so that

$$\left(\frac{5-a_0}{11}\right) \not\equiv -1.$$

Then

$$a_0 \equiv 0, 1, 2, 4, 5, 7 \pmod{11}.$$

Thus a_0 must satisfy

$$\begin{aligned} &a_0 \equiv 0, 1 \pmod 5, \\ &a_0 \equiv 0, 1, 2, 4, 5, 7 \pmod{11}, \\ &a_0 \not\equiv 115 \pmod{121}. \end{aligned} \tag{31}$$

After a_0 is chosen to satisfy (31), we must still verify that $t_0\not\equiv 0 \pmod 5$. The first few a_0's satisfying (31) and the associated t_0's and t_1-t_0's follow:

(32)

a_0	0	1	5	11	15	16	26	35	40	45	46	51	55	56	60
$t_0 \pmod 5$	1	2	1	2	1	2	0	2	3	3	4	4	3	4	0
$t_1-t_0 \pmod 5$	2	4	1	4	2	2	0	4	1	1	3	3	1	3	0

Summarizing, we have proved

THEOREM 9. *Let* $a_n=121^na_0+(23/24)(121^n-1)$. *Then for all* $n\geqq 0$,

$$p(1+a_n) \equiv \begin{cases} 2n+1 \pmod 5, & a_0 = 0, 15, \cdots \\ 4n+2 \pmod 5, & a_0 = 1, 11, 35, \cdots \\ n+1 \pmod 5, & a_0 = 5, \cdots \\ 2n+2 \pmod 5, & a_0 = 16, \cdots \\ n+3 \pmod 5, & a_0 = 40, 45, 55, \cdots \\ 3n+4 \pmod 5, & a_0 = 46, 51, 56, \cdots \end{cases} \tag{33}$$

Thus the sequence $\{p(n)\}$ *fills all residue classes modulo* 5 *infinitely often.*

We make the remark that the choice $p=11$ was one of expediency only. Other primes would do as well.

Since $p(125n+24)/25\equiv 3p(5n+1) \pmod 5$ (by (13) and (16)) Theorem 9 implies

THEOREM 10. *The sequence* $\{p(125n+24)/25\}$ *fills all residue classes modulo* 5 *infinitely often.*

Precisely the same procedure can be applied to the remaining congruences of Lemma 7. Some typical results are

$$\frac{1}{5}\,p\left(\frac{95\cdot 13^{2n}+1}{24}\right) \equiv 2^n \pmod 5, \tag{34}$$

$$\frac{1}{7}\,p\left(\frac{119\cdot 11^{2n}+1}{24}\right)\equiv(-1)^n(4n+1)\pmod 7, \tag{35}$$

$$\frac{1}{49}\,p\left(\frac{1127\cdot 11^{2n}+1}{24}\right)\equiv(-1)^{n-1}(n+2)\pmod 7. \tag{36}$$

Since $p(5n+4)/5$ certainly fills the zero class modulo 5 infinitely often, and 2 is a primitive root of 5, we can conclude with the aid of Lemma 6.

THEOREM 11. *The sequences* $\{p(5n+4)/5\}$, $\{p(7n+5)/7\}$, $\{p(49n+47)/49\}$ *fill all residue classes infinitely often modulo* 5, 7, 7 *respectively.*

We mention in conclusion that similar results modulo 10, 26, 65 may be derived.

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