CARTAN DECOMPOSITIONS FOR L* ALGEBRAS

BY

JOHN R. SCHUE

1. Introduction. In the discussion of L* algebras given in [5] a classification theory was obtained for the separable simple algebras under the assumption of the existence of a Cartan decomposition relative to some Cartan subalgebra. The main result of this paper is a proof that any semi-simple L* algebra of arbitrary dimension has such a decomposition relative to any Cartan subalgebra.

In the process of proving this several additional results of interest in themselves are obtained, among them one concerning representations of finite-dimensional semi-simple Lie algebras which seems to be new. This is stated in detail in the second corollary of 4.5. The conclusion obtained in 4.5 also adds a new result to the theory of commutators of operators on a Hilbert space.

2. Continuous decompositions.

Definitions and notation. An L* algebra is defined as a Lie algebra L over the complex numbers whose underlying vector space is a Hilbert space and such that for each x in L there exists an element x* with ([x, y], z) = (y, [x*, z]) for all y and z. For an x in L, X (occasionally Dx) will denote the linear operator defined by Xy = [x, y] for all y and we will assume that the norm on L is chosen such that \|X\| \leq \|x\| . An L* algebra is semi-simple if and only if the mapping x \rightarrow x* is one-one. For the remainder of this paper L will denote an arbitrary (but fixed) semi-simple L* algebra unless further restrictions are explicitly stated. As shown in [5] this implies the mapping x \rightarrow x* is a Hilbert space conjugation and anti-multiplicative, Dx* is the adjoint of D_x, and that L is a direct sum of simple L* ideals. A Cartan subalgebra of L is defined as a maximal abelian self-adjoint subalgebra of L. For subsets M, N of L, Sp(M) will represent the smallest closed linear subspace of L containing M and \([M, N] = \text{Sp} \{ [m, n]: m \in M, n \in N \} \). For subspaces S_1, S_2 the notation S_1 + S_2 will be used only when S_1 is orthogonal to S_2.

Suppose A is a bounded self-adjoint operator on L. For \lambda \ real and \epsilon > 0 let \( V(\lambda, \epsilon) = \{ x: \| (A - \lambda)x \| \leq \epsilon \| x \| , n = 1, 2, \ldots \} \). For a Borel set M of the real numbers let \( V(M, \epsilon) = \text{Sp} \{ V(\lambda, \epsilon): \lambda \in M \} \) and \( V(M) = \cap_{\epsilon > 0} V(M, \epsilon) \). It is proved in [1, pp. 66–69] that \( V(\lambda, \epsilon) \) is a closed subspace and equal to the set of x such that the sequence \( \{ (e^{-1}(A - \lambda))nx \} \) is bounded. Furthermore, if \( E \) is the real spectral measure such that \( A = \int \lambda dE \) then the range of \( E(M) \) is equal to \( V(M) \) for \( M \) compact. For any Borel set \( M \) the range of \( E(M) \) will be denoted by \( S(M) \). Finally, for Borel sets \( M \) and \( N \) let \( M + N \)

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$= \{m+n: m \in M, n \in N\}$ and $-M = \{-m: m \in M\}$. Then $M+N$ and $-M$ are also Borel sets.

2.1. Suppose $A$ is a bounded self-adjoint derivation on $L$ and $M$, $N$ are Borel sets of the real line. Then $[S(M), S(N)] \subset S(M+N)$ and $S(M)^* = S(-M)$.

Proof. Suppose first that $M$ and $N$ are compact and $\epsilon > 0$. Let $M = \bigcup M_k$ and $N = \bigcup N_k$ where $\{M_k\}$, $\{N_k\}$ are sequences of disjoint Borel sets, each of diameter less than $(1/2)\epsilon$. Let $x \in S(M)$ and $y \in S(N)$. Then $x = \sum x_j$, $y = \sum y_k$ where $x_j \in S(M_j)$, $y_k \in S(N_k)$ and $[x, y] = \sum [x_j, y_k]$. Suppose $\lambda_j \in M_j$, $\mu_k \in N_k$. Then it follows from the spectral theorem that $\|(A - \lambda_j)x_j\| \leq (2^{-n}\epsilon^n)\|x_j\|$ and $\|(A - \mu_k)y_k\| \leq (2^{-n}\epsilon^n)\|y_k\|$ for each positive integer $n$. Since $A$ is a derivation it follows by induction on $n$ that $\|(A - (\lambda_j + \mu_k))^{n}[x_j, y_k]\| = \|\sum C_m [(A - \lambda_j)^{n-1}x_j, (A - \mu_k)^{n-1}y_k]\| \leq \sum C_m [(A - \lambda_j)^{n-1}x_j, (A - \mu_k)^{n-1}y_k] \leq \sum C_m 2^{-m\epsilon^n}\|x_j\|\|y_k\| = \epsilon^n\|x_j\|\|y_k\|$. Hence the sequence

$\{(e^{-1}(A - (\lambda_j + \mu_k))^n[x_j, y_k]\}$

is bounded and this implies $[x_j, y_k] \in V(\lambda_j + \mu_k, \epsilon)$. Thus $[x, y] \in V(M+N, \epsilon)$. The compactness of $M$ and $N$ implies $M+N$ is also and hence $[x, y] \in S(M+N)$. Thus $[S(M), S(N)]$ is a subset of $S(M+N)$ for $M$ and $N$ compact.

It is proved in [1] that $E$ is regular, i.e., for any Borel sets $M$ and $N$, $E(M) = \sup\{E(C): C \subset M, C$ compact $\}$ and similarly for $E(N)$. For $C \subset M$ and $D \subset N$ with $C$ and $D$ compact we have $[S(C), S(D)] \subset S(C + D) \subset S(M+N)$. Letting $C$ vary gives $[S(M), S(D)] \subset S(M+N)$.

Notation. Suppose $\mathcal{C}$ is a closed self-adjoint abelian subalgebra of $L$. Let $\mathfrak{g} = \mathfrak{g}(\mathcal{C})$ be the commutative $C^*$ algebra of bounded operators generated by $\{H: h \in \mathcal{C}\}$. Since each $H$ is zero on $\mathcal{C}$ then the identity operator is not in $\mathfrak{g}$. Let $\Delta = \Delta(\mathcal{C})$ be the set of all homomorphisms of $\mathfrak{g}$ into the complex numbers. For $A \in \mathcal{C}$ let $\hat{A}$ be the function on $\Delta$ defined by $\hat{A}(\alpha) = \alpha(A)$. If $\Delta$ is given the weakest topology making these functions continuous then $\Delta$ is a compact Hausdorff space and the theory of $C^*$ algebras shows that the mapping $A \mapsto \hat{A}$ is an algebraic isomorphism mapping $\mathfrak{g}$ isometrically onto the set of all continuous functions on $\Delta$ vanishing at the zero homomorphism with $A^*$ corresponding to the complex conjugate of $\hat{A}$. The set $\Delta$ with its topology will be called the spectrum of $\mathcal{C}$. It is also known that the spectrum of an operator $A \in \mathfrak{g}$ is the range of the function $\hat{A}$.

2.2. For each $\alpha \in \Delta$ there is a unique $x_\alpha \in \mathcal{C}$ such that $\alpha(H) = (h, x_\alpha)$ for all $h \in \mathcal{C}$. Also $\|x_\alpha\| \leq 1$ and $x_\alpha^* = x_\alpha$. If $\{x_\alpha\}$ is given the induced weak topology of $\mathcal{C}$ then $\Delta$ is homeomorphic to $\{x_\alpha\}$ under the mapping $\alpha \mapsto x_\alpha$. 

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Proof. All except the last statement were proved in [5]. The family 
\{ \hat{A}_h : h \in \mathcal{A} \} separates points of \( \Delta \) and all members vanish at infinity (zero). By Theorem 5G of [3], the topology of \( \Delta \) is that generated by the family. But this is clearly equivalent to the weak topology on \{ \xi_\alpha \}, finishing the proof.

There is a unique spectral measure \( E \) on the Borel sets of \( \Delta \) such that 
\[(Ax, y) = \int \hat{A}(\alpha) d(E(\alpha)x, y) \] for all \( x, y \) in \( L \). Also it is easily seen that the range of \( E(0) \) is \{ \( x : [\mathcal{A}, x] = 0 \} \) where \{ \( 0 \} \) denotes the Borel set consisting of the zero homomorphism. For an arbitrary Borel set \( M \) of \( \Delta \) let \( S(M) \) denote the range of \( E(M) \).

Suppose \( \alpha, \beta \in \Delta \). If \( \xi_\alpha + \xi_\beta = \xi_\gamma \) for some \( \gamma \in \Delta \), let \( \gamma \) be denoted by \( \alpha + \beta \). If \( \xi_\alpha = -\xi_\gamma \) for some \( \gamma \in \Delta \), let \( \gamma \) be denoted by \( -\alpha \). Using this notation we have the following theorem which is the continuous version of the desired composition for \( L \) relative to \( \mathcal{A} \).

2.3. Suppose \( M, N \) are Borel sets of \( \Delta \) and \( M + N = \{ m + n : m \in M, n \in N \} \). Then \( M + N \) is a Borel set and \( \{ S(M), S(N) \} \subset S(M + N) \). If \( -M = \{ -m : m \in M \} \) then \( S(-M) = S(M)^* \).

Proof. Choose a set \{ \( x_i : i \in I \} \) of elements of \( \mathcal{A} \) such that the set spans \( \mathcal{A} \) and \( x_i^* = x_i \) for each \( i \). Let \( \sigma_i \) be the spectrum of \( x_i \). Then \( \sigma_i \) is compact so that \( P = \prod \sigma_i \) is compact. For \( \alpha \in \Delta \) let \( f(\alpha) \) be the element of \( P \) whose \( i \)th coordinate is \( (x_i, x_\alpha) \). Then \( f \) is a homeomorphism of \( \Delta \) onto a compact subset of \( P \). If addition is defined in \( P \) (whenever possible) in the obvious coordinate-wise fashion then \( f \) preserves the algebraic structure of \( \Delta \) as well as the topology. The spectral measure \( E \) can be defined directly on \( P \) (hence on \( f(\Delta) \) and \( \Delta \)) by constructing the product measure obtained from the \( E_i \)'s where \( X_i = \int f_\alpha dE_i \). The measure-theoretic details will be omitted here but a discussion of this type of problem may be found in [2].

A subset \( M \) of \( P \) will be called a rectangle if and only if \( M = \prod M_i \) where \( M_i \) is a Borel set of \( \sigma_i \) and \( M_i = \sigma_i \) for all but a finite number of indices. Then the Borel sets of \( P \) will coincide with the \( \sigma \)-algebra generated by the rectangles. In fact, Chapter 7 of [2] shows that this \( \sigma \)-algebra is obtained as the smallest monotone class containing all finite unions of disjoint rectangles. 2.1 can be used to prove 2.3 for the case when \( M, N \) are finite unions of disjoint rectangles. The collection of all sets for which 2.3 holds is clearly a monotone class, hence contains all Borel sets.

2.4. Corollary. Suppose \( L \) is separable and \( \mathcal{A} \) is a closed self-adjoint abelian subalgebra of \( L \) such that \{ \( \| \xi_\alpha \| : \alpha \in \Delta, \xi_\alpha \neq 0 \} \) is bounded away from zero. Then there exist disjoint Borel sets \( M_k \) of \( \Delta \), \( k = 0, 1, \cdots \) such that \( M_0 = \{ 0 \} \), \( L = \sum S(M_k) \), and \( \{ S(M_k), S(M_k)^* \} \subset \{ x : [\mathcal{A}, x] = 0 \} \).

Proof. \( L \) separable implies \{ \( x_\alpha : \alpha \in \Delta \} \) is a separable metric space in the norm topology, hence contains a countable dense subset. Suppose \( \| \xi_\alpha \| > c > 0 \) for all \( \alpha \neq 0 \). Let \( \alpha_0 = 0 \) and let \{ \( x_{\alpha_k} : k \geq 1 \} \) be a countable dense subset of
\[ \{ x_\alpha \} - \{ 0 \}. \] If \( N_k = \{ \beta : \| x_\beta - x_\alpha \| \leq 3^{-k} \} \) then \( N_0 = \{ 0 \} \) and \( \Delta = \bigcup N_k \). Furthermore, since \( \{ x_\alpha \} \) is weakly compact, \( N_k \) is weakly closed hence a Borel set. If \( \alpha, \beta \in N_k \) the triangle inequality implies \( \| x_\alpha - x_\beta \| < c \) so that either \( \alpha = \beta \) or \( \alpha - \beta \in \Delta \). Hence \( N_k + (-N_k) = \{ 0 \} \). In the usual way it is possible to choose Borel sets \( M_k \) such that \( M_k \subset N_k \) and \( \Delta \) is the disjoint union of the \( M_k \)'s. Then \( [S(M_k), S(M_k)^*] \subset S(M_k + (-M_k)) \subset S(\{ 0 \}) \) and the remaining statements follow easily from the spectral theory.


3.1. There exists a nonzero element \( a \in L \) such that \( (a, a^*) = 0 \) and \( A^3 = 0 \).

**Proof.** Let \( x \) be a self-adjoint element of \( L \) with \( \| x \| = 1 \) and let \( X = \bigwedge dE \).

If \( M \) is the real interval \((2/3, 1]\) and \( V \) is the range of \( E(M) \) then \( V \neq 0 \) and, since \( V^* \) is the range of \( E(-M) \), \( (V, V^*) = 0 \). Using 2.1, \( \{ [V, [V, [F, L]]] \} = 0 \). Thus \( a \) may be chosen as any element of \( V \) different from zero.

**Notation.** For this section we choose a fixed \( a \) having the properties listed in 3.1 and let \( b = a^* \), \( c = [a, b] \). Then \( C = C^* \) and \( A^3 = B^3 = 0 \). This section is devoted to an analysis of the \( L^* \) subalgebra generated by \( a \), culminating in 3.8. This turns out to be a key result in the general existence proof for Cartan decompositions.

3.2. For any \( x \) in \( L \), \( A^2XA = AXA^2 \) and \( B^2XB = BXB^2 \).

**Proof.** \([A, [A, [A, X]]]] = 0 \) and \( A^3 = 0 \) together imply \( -3A^2XA + 3AXA^2 = 0 \). The second equation follows from the first by taking adjoints.

3.3. Suppose \( x \) is any element in the closure of the range of \( A^2 \). Then \( X^3 = 0 \).

**Proof.** By continuity of the adjoint representation it is sufficient to prove this for the case \( x = A^2z \) for some \( z \) in \( L \). Then \( X = [A, [A, Z]] = A^2Z - 2AZA + ZA^2 \). Using 3.2, \( A^2ZA = AZA^2 \) so that \( A^2ZA^2 = 0 \). A direct computation then shows that \( X^2 = A^2Z^2A^2 \) and \( X^3 = 0 \).

3.4. Suppose \( n \) is any positive integer. Then

(a) \( C^na = -A^2(BA)^{n-1}b = (AB)^na \),

(b) \( C^nb = (-1)^n(BA)^{n-1}a = (-1)^n(BA)^nb \).

**Proof.** Since \( C = C^* \), \( (C^nx)^* = (-1)^nC^nx^* \) for any \( x \). Thus (b) follows from (a) by using adjoints.

The first equation of (a) is proved by induction. For \( n = 1 \), \( Ca = [[a, b], a] = -A^2b \). Assuming the result for \( n \), \( C^{n+1}a = C(-A^2(BA)^{n-1}b) = -(AB - BA)(BA)^{n-1}b = -AB(A^2BA)^{n-1}b = -A^4(BA)^nb \), after using 3.2.

For the second, repeated application of 3.2 gives \( C^na = - (AB)^{n-1}A^2b = (AB)^na \).

**Corollary.** For all integers \( n \geq 0 \),

(a) \( A^2C^nb = BC^nb = 0 \),

(b) \( ABC^na = C^{n+1}a, BAC^nb = -C^{n+1}b \),

(c) \( B^2C^na = (-1)^nC^{n+1}b, A^2C^nb = (-1)^nC^{n+1}a \).

3.5. Let \( S_0 = Sp\{ a, b \}, S_0 = Sp\{ D_1 \cdots D_n : s = a, b; D_i = A, B \} \) for \( n = 1, 2, \cdots \). Let \( S = Sp \{ S_n : n = 0, 1, \cdots \} \).
(a) $S$ is the $L^*$ subalgebra generated by $a$.
(b) $S_{2n} = \text{Sp}\{C^n a, C^n b\}, \quad n = 0, 1, \ldots$
(c) $S_{2n+1} = \text{Sp}\{BC^n a, AC^n b\}, \quad n = 0, 1, \ldots$
(d) $(S_{2n}, S_{2n+1}) = 0, m, n = 0, 1, \ldots$

**Proof.** (a) It is clear that $S_n = S_n^*$ and hence $S = S^*$. Since $AS_n \subseteq S_{n+1}$, $BS_n \subseteq S_{n+1}$, then $S$ is invariant under $A$ and $B$. Hence $S$ is invariant under $X$ for any $x$ in the $L^*$ subalgebra $S'$ generated by $a$, i.e., $[S', S] \subseteq S$. But clearly $S \subseteq S'$. Hence $[S, S] \subseteq S$ and therefore $S = S'$.

(b) and (c) are true for $n = 0$. Suppose they hold for some $n$. Then

$$S_{2n+2} = \text{Sp}\{A S_{2n+1}, B S_{2n+1}\} = \text{Sp}\{ABC^n a, A^n C^n b, B^n C^n a, BAC^n b\} = \text{Sp}\{C^{n+1} a, C^{n+1} b\},$$

using the corollary of 3.4. Hence

$$S_{2n+4} = \text{Sp}\{A S_{2n+3}, B S_{2n+3}\} = \text{Sp}\{AC^{n+2} b, BC^{n+2} a\}$$

since $AC^{n+2} a = BC^{n+2} b = 0$. Thus, by induction on $n$, (b) and (c) are true for all $n$.

(d) $(C^n a, BC^n a) = (AC^n a, C^n a) = 0$ and

$$(C^n a, AC^n b) = (-1)^n(C^n a, A(BA)^n b) = (-1)^n(C^n a, (AB)^n a b) = (-1)^{n+1}(C^{n+1} a, B a) = (-1)^{n+1}(AC^{n+1} a, a) = 0.$$ Similarly $(C^n b, AC^n b) = (C^n b, BC^n a) = 0$, completing the proof of (d).

3.6. Letting $n$ range over the non-negative integers, let $V_0 = \text{Sp}\{S_{2n+1}\}$, $V_1 = \text{Sp}\{C^n a\}$, $V_2 = \text{Sp}\{C^n b\}$.

(a) $S = V_0 + V_1 + V_2$.
(b) $V_0 = V_0^*, V_1^* = V_2, V_2^* = V_1$.
(c) $[V_1, V_1] = [V_2, V_2] = 0$.
(d) $[V_1, V_1^*] = [V_2, V_2^*] = V_0$.

**Proof.** (a) It only remains to prove that $V_1$ is orthogonal to $V_2$. Now $(a, b) = 0$ and, if either $n$ or $m$ is nonzero, then $(C^n a, C^n b) = - (A^n (BA)^{n-1} b, b) = - ((BA)^{n-1} b, B^n b) = 0$. Hence $(V_1, V_2) = 0$.

(b) $V_1^* = \text{Sp}\{C^n a\} = \text{Sp}\{C^n b\} = V_2$. Similarly $V_2^* = V_1$. Since $S^* = S$, $V_0^* = V_0$.

(c) It is sufficient to prove $[V_1, V_1] = 0$ or $[C^n a, C^n a] = 0$ for all $m$ and $n$. This is done by induction on $n$. The case $n = 0$ is given by the corollary of 3.4. Suppose $[C^{n-1} a, C^n a] = 0$ for all $m$. Then $C^n [a, C^n a] = C^n 0 = 0$. By Leibniz's rule, $0 = [C^n a, C^n a] + \text{terms of the form } [C^n a, C^n a]$ where $p < n$. Each of these latter terms is zero by the induction hypothesis.

(d) $V_0 = \text{Sp}\{[a, C^n b], [b, C^n a]\}$ implies $V_0 \subseteq [V_1, V_1^*]$. But $(V_1, [V_1, V_1^*]) = ([V_1, V_1], V_1) = 0$ implies $[V_1, V_1^*]$ is orthogonal to $V_1$. Similarly, since $[V_1, V_1^*] = [V_2, V_2^*]$, $[V_1, V_1^*]$ is also orthogonal to $V_2$, hence must be a subset of $V_0$.

3.7. $[c, V_0] = 0$. 

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Proof. By using adjoints it is sufficient to prove $CAC^nb = 0$ for all $n$. Since $C$ is self-adjoint it is sufficient to prove $C^2AC^nb = 0$. Using the appropriate cases of the corollary of 3.4,

$$CAC^nb = (AB - BA)AC^nb = A(BAC^nb) - B(A^2C^nb)$$

$$= - AC^{n+1}b + (-1)^nBC^{n+1}a.$$ 

Hence

$$C^2AC^nb = - A(BAC^{n+1}b) + B(A^2C^{n+1}b) + (-1)^nAB^2C^{n+1}a + (-1)^{n+1}BABC^{n+1}a$$

$$= AC^{n+2}b + (-1)^nBC^{n+2}a + (-1)^{n+1}AC^{n+2}b + (-1)^{n+1}BC^{n+2}a = 0.$$ 

**Corollary 1.** $[C^na, C^mb] = (-1)^n[a, C^{n+m}b]$ for all $n, m$.

**Proof.** $[C^na, C^mb] \in V_0$ by 3.6. Also we may assume $n$ is positive. Then

$$0 = C^[0,la, C^mb] = [C^na, C^mb] + [C^na, C^{m+1}b]$$

so that $[C^na, C^mb] = - [C^{n-1}a, C^{n+1}b]$. Applying this repeatedly gives the result in general.

**Corollary 2.** $[F^0, V_i] = V_i$ for $i = 1, 2$.

**Proof.** It is sufficient to prove this for $i = 1$. Now $[V_0, C^na] = C^a [V_0, a]$ and $V_1$ is invariant under $C$ so it is enough to prove $[V_0, a] \subseteq V_1$ in order to prove $[V_0, V_1] \subseteq V_1$. But $A V_0 = Sp\{ A^2C^nb, ABC^ma \} \subseteq V_1$ by the corollary of 3.4.

For the reverse inclusion, suppose $x \in V_1$ and $(x, [V_0, V_1]) = 0$. Then $([x, x^*], V_0) = 0$ so that $[x, x^*] = 0$ and $X$ is normal. To prove $x$ must be zero it is sufficient to show that $X$ is nilpotent. In fact, for future reference, we will prove that $X^2 = 0$ for all $x \in V_1$. Since $V_1 = Sp\{ C^na : n \geq 0 \}$, it is clear that $Sp\{ C^na : n \geq 1 \}$ is either all of $V_1$ or a hyperplane in $V_1$. In the first case $V_1$ is contained in the closure of the range of $A^2$ (see 3.4) and the assertion follows from 3.3. In the other case an element $x \in V_1$ must be of the form $x = \mu a + y$ where $\mu$ is a scalar and $y$ is in the closure of the range of $A^2$. A proof like that of 3.3 then shows that $A^2 Y = A^2 = 0$. Since $A^2 = Y^2 = 0$, the binomial theorem gives $X^2 = 0$.

**Corollary 3.** $[V_0, F^0] = 0$.

**Proof.** Suppose $x \in V_0$ and $(x, AC^nb) = 0$ for all $n$. Then $(b, x) = C^a(b, x) = 0$ for all $n$ because $[b, x] \in V_2$. But then $0 = C^aXb = XC^nb$ and this implies $X$ is zero on $V_0$ so that $(x, V_0) = (x, [V_0, V_0^*]) = 0$ and hence $x$ is zero. Thus $V_0 = Sp\{ AC^nb \}$. Now $0 = [C^nb, C^nb]$ implies $A^2[C^nb, C^nb] = 0$ so that $[A^2C^nb, C^nb] + 2[A^2b, A^2C^nb] + C^nb, A^2C^nb] = 0$. The sum of the first and last terms on the left side is $(-1)^{n+1}[C^{n+1}a, C^nb] + (-1)^{n+1}[C^nb, C^{n+1}a]$ which is zero by Corollary 1. Thus $[AC^nb, AC^nb] = 0$ and since $V_0 = Sp\{ AC^nb \}$, $V_0$ isabelian.

**Corollary 4.** $S$ is semi-simple with $V_0$ as a Cartan subalgebra.
Proof. Suppose \( x \in S \) and \([x, V_0] = 0\). Then \( 0 = ([x, V_0], V_i) = (x, V_i) \) for \( i = 1, 2 \) implies \( x \in V_0 \) and \( V_0 \) is maximal abelian. To show that \( S \) is semi-simple suppose that \( x \in S \) and \([x, S] = 0\). Then \( x \in V_0 \) and \([x, b] = 0\). The proof of Corollary 3 shows that \( x \) must be zero.

3.8. Suppose \( S \) is a semi-simple \( L^* \) algebra and \( S = V_0 + V_1 + V_2 \) with \( V_0 \) as a Cartan subalgebra and that the relations of 3.6 hold. Then \( S \) is a direct sum of three-dimensional ideals \( I_i \) where \( I_i = \text{Sp} \{ e_j, e_j^*, [e_j, e_j^*] \} \) for some nonzero \( e_j \in V_1 \).

Proof. The decomposition theorem of [5] for semi-simple algebras shows that \( S \) can be written as the direct sum of simple ideals \( I_i \) where \( I_i = H_i + [H_i, S] \) for some closed self-adjoint subspace \( H_i \) of \( V_0 \). We choose a fixed \( I_i \) and let \( U_0 = H_i \), \( U_1 = [H_i, V_1] \), \( U_2 = [H_i, V_2] \). Then \( I_i = U_0 + U_1 + U_2 \) and it is clear that \([U_i, U_i] = 0\) for each \( i \), \([U_0, U_i] = U_i \) for \( i = 1, 2 \), while \([U_i, U_i^*] = U_0 \) for \( i = 1, 2 \).

Suppose \( U_1 = P + Q \) where \( P \) and \( Q \) are closed subspaces invariant under \( U_0 \). Then \( ([U_1, U_1^*], P), 0 = 0 \) implies \( ([U_1, U_1^*], [Q, P^*]) = 0 \) so that \([Q, P^*] = 0\). Since \([Q, P] \) is also zero, it follows that \( U_0 = [P + Q, P^* + Q^*] = [P, P^*] + [Q, Q^*] \) and, furthermore, that \( XY = 0 \) (on \( S \)) for all \( x \in [P, P^*] \) and \( y \in [Q, Q^*] \). Referring to the proof in [5] of the decomposition theorem and using the simplicity of \( I_i \) we must have \([P, P^*] = 0 \) or \([Q, Q^*] = 0\). But every element of \( U_1 \) is nilpotent on \( S \) so that necessarily either \( P = 0 \) or \( Q = 0 \). Hence \( U_1 \) contains no nontrivial closed subspaces invariant under \( U_0 \). By the spectral theorem \( U_1 \) must be one-dimensional and this completes the proof.

Corollary 1. The \( L^* \) algebra generated by \( a \) is a direct sum of three-dimensional ideals.

Corollary 2. There exists a nonzero element \( x \in L \) such that \( X^3 = 0 \) and \([x, x^*], x] = \lambda x \) with \( \lambda \) positive. In fact \( \lambda \| x \|^2 = \| [x, x^*] \|^2 \).

Proof. Let \( I_i \) be a simple ideal of \( S \) as above and let \( x = e_j \). Then \([x, x^*], x] = \lambda x \) for some \( \lambda \). Hence \( \lambda \| x \|^2 = \| [x, x^*] \|^2 \). Since \( x \in V_1 \), the proof of Corollary 2 of 3.7 shows that \( X^3 = 0 \). Thus \([x, x^*] \neq 0 \) and \( \lambda \) must be different from zero.

4. A commutator equation. For this section, \( A \) will denote a fixed nonzero bounded operator on a Hilbert space such that \([A, A^*], A] = \lambda A \) for some \( \lambda \neq 0 \). From this considerable information about the spectra of \( AA^*, A^*A \), and \([A, A^*] \) can be obtained. Also there are some interesting consequences for representations of Lie algebras as bounded operators on a Hilbert space. By assuming 4.1 and Corollary 1 of 4.2 much of what is done here is valid for elements \( A, A^* \) of an arbitrary algebra with identity over a field of characteristic zero.

4.1. \( A \) is nilpotent.

Proof. The mapping \( B \rightarrow [[A, A^*], B] \) is a derivation on the set of all
bounded operators and has norm not exceeding \(2\|A, A^*\|\). Hence
\[\|[A, A^*], A^n\| = n\lambda A^n\text{ for } n \text{ a positive integer.} \]
Then \(n\lambda A^n = [[A, A^*], A^n]\) implies \(n\|\|A^n\|\leq 2\|\|A, A^*\|\|\|A^n\|\) so that \(A^*\) is zero for some \(n\).

4.2. \([A^*, A^n] = nA^n - (\lambda/2)(n)(n-1)A^{n-1}\) for \(n = 1, 2, \ldots\).

**Proof.** The case \(n = 1\) is trivial. Assuming the equation for \(n\) gives
\[\begin{align*}
[A^*, A^{n+1}] &= [A^*, A^n] + [A^*, A^n]A \\
&= A^n[A*, A] + nA^n - (\lambda/2)(n)(n-1)A^n \\
\end{align*}\]

**Corollary 1.** \(\lambda\) is real and positive and \([A^*, A], A^*\] = \(\lambda A^*\).

**Proof.** Choose \(n\) such that \(A^n \neq 0\) but \(A^{n+1} = 0\). Then \(0 = [A^*, A^{n+1}] = (n+1)A^n[A^*, A] - (\lambda/2)(n)(n+1)A^n\) implies \(A^n(A^nA - n(\lambda/2)) = 0\). Since \(A^nA - n(\lambda/2)\) does not have a bounded inverse, hence \(n(\lambda/2)\) is in the spectrum of the positive operator \(A^nA\) and \(\lambda\) must be positive. Since \(\lambda\) is real, taking adjoints of both sides of the equation \([[A, A^*], A] = \lambda A^*\) gives the second assertion.

**Corollary 2.** \([A^n, A] = nA^{n-1}[A^*, A] + (\lambda/2)(n)(n-1)A^{n-1}\).

**Proof.** Taking adjoints of both sides of the equation in 4.2 gives
\[\begin{align*}
[A^n, A] &= n[A^*, A]A^{n-1} - (\lambda/2)(n)(n-1)A^{n-1} \\
&= nA^{n-1}[A^*, A] + \lambda n(n-1)A^{n-1} - (\lambda/2)(n)(n-1)A^{n-1}.
\end{align*}\]

**Corollary 3.** \(AA^*\) commutes with \(A^*A\).

**Proof.** Since \([A, A^*] = AA^* - A^*A\), it is sufficient to prove \(AA^*\) commutes with \([A, A^*]\). But
\[\|[A, A^*], AA^*\| = \|[A, A^*], A\]A^n + A[[A, A^*], AA^*] = \lambda AA^* - \lambda A^* = 0.\]

4.3. For each non-negative integer \(n\) let \(B_n = A^nAA^*\) and \(D_n = A^nA^n\).

(a) \(B_nAA^* = (1/n + 1)B_{n+1} + (n/n + 1)B_nA^nA^n + (\lambda/2)B_n\).

(b) \(D_nAA^* = (1/n + 1)D_{n+1} + (n/n + 1)D_nA^nA^n + (\lambda/2)D_n\).

(c) For all \(n, m \geq 0\), \(B_n\) and \(D_n\) commute with \(B_m\) and \(D_m\).

**Proof.** (a)
\[B_nAA^* = A^nAA^*A^nAA^* = A^nAA^*A^nAA^* + A^n[A^nA^n, A]AA^* = B_{n+1} + A^n(nA^{n-1}[A^*, A] + (\lambda/2)(n)(n-1)A^{n-1})AA^* = B_{n+1} + (\lambda/2)(n)(n-1)B_n + nA^nA^{n-1}[A^*, A]AA^* = B_{n+1} + (\lambda/2)(n)(n-1)B_n + n\lambda A^nA^n + nA^nA^n[A^*, A] = B_{n+1} + (\lambda/2)(n)(n+1)B_n + nB_nA^nA^* - nB_nAA^*.

Solving for \(B_nAA^*\) gives the assertion in (a).
(b) Because of the symmetry between $A$ and $A^*$ the proof is like that for (a).

(c) If either $n$ or $m$ is zero the result is immediate. By Corollary 3 of 4.2, $B_1$ commutes with $D_1$. Using equations (a) and (b) an induction on $n$ shows that $B_n$ and $D_n$ are polynomials in $B_1$ and $D_1$ and this gives (c).

4.4. Let $p$, $q$ be non-negative integers and $n=p+q$. Then

\[ B_p D_q AA^* = (q + 1/n + 1)B_{p+1}D_q + (p/n + 1)B_p D_{q+1} + (\lambda/2)(p)(q + 1)B_p D_q. \]

**Proof.** The case $q=0$ is given by equation (a) of 4.3 and the case $p=0$ reduces to $D_n AA^* = AA^* D_n$. Thus we may assume both $p$ and $q$ are positive. Now $B_p D_q AA^* = (B_p AA^*) D_q = (1/p + 1)B_{p+1} + (p/p + 1)B_p A^* A + p(\lambda/2)B_p) D_q$ implies

\[ B_p D_q AA^* = (1/p + 1)B_{p+1}D_q + (p/p + 1)B_p D_q + (\lambda/2)(p)(q + 1)B_p D_q. \]

But, using equation (b) of 4.3,

\[ B_p(D_q AA^*) = B_p((q + 1/q)D_q A^* A - (1/q)D_q + (p/p + 1)B_p D_q A^* A. \]

which gives

\[ B_p D_q AA^* = (1/q)B_p D_q - (\lambda/2)(q + 1)B_p D_q + (q + 1/q)B_p D_q A^* A. \]

Using (1) and (2) to eliminate the term $B_p D_q A^* A$ gives the conclusion.

**Corollary.** For $n$ a positive integer $(AA^*)^n$ is a linear combination of the $B_p D_q$ where $1 \leq p \leq n, 0 \leq q \leq n$.

**Proof.** For $n = 1$, $AA^* = B_1 D_0$. 4.4 and an induction on $n$ gives the result in general.

4.5. Let $n$ be the greatest integer such that $A^n \neq 0$. Then

\[ (AA^*)^n \prod_k (AA^* - (\lambda/2)(p)(q + 1)) = 0 \]

where the product is taken over all pairs $p$, $q$ with $1 \leq p \leq n, 0 \leq q \leq n$.

**Proof.** $(AA^*)^n$ is a linear combination of terms of the form $B_p D_q$ with $1 \leq p \leq n, 0 \leq q \leq n$. Thus it is sufficient to show that for each such pair $p$, $q$, $B_p D_q \prod_k (AA^* - (\lambda/2)(p)(q + 1)) = 0$ where the product is taken over all pairs $k, m$ with $p \leq k \leq n, q \leq m \leq n$. If we define the degree of $B_p D_q$ as $p+q$ then $B_p D_q (AA^* - (\lambda/2)(p)(q + 1)) = (q + 1/p + q + 1)B_{p+1}D_q + (p/p + q + 1)B_p D_{q+1}$ (by 4.4) and hence is a sum of terms of degree greater than that of $B_p D_q$. If the degree of $B_p D_q$ is $n$ each of the terms on the right is zero since $A^{n+1} = A^{n+1} = 0$. In general an induction on the terms of higher degree will yield the conclusion.

**Corollary.** $AA^*$ and $A^* A$ have finite spectra contained in the set $\{k(\lambda/2): k = 0, 1, \ldots, n(n+1)\}$. $[A, A^*]$ has spectrum contained in this set and its negatives.

**Proof.** Let $\alpha$ be the commutative $C^*$ algebra generated by $AA^*$ and $A^* A$. 
For any homomorphism $\alpha$ of $\mathfrak{G}$ onto the complex numbers the value of $\alpha$ at $AA^*$ must satisfy the same polynomial relation as $AA^*$, proving the first part. Because of the symmetry between $A$ and $A^*$, $A^*A$ must also satisfy a polynomial identity like that of 4.5. Since $[A, A^*] = AA^* - A^*A$, a similar argument applies here.

**Corollary.** Suppose $L$ is a finite-dimensional semi-simple complex Lie algebra with $\mathfrak{G}$ as a Cartan subalgebra and \{ $h_\alpha, e_\alpha$: $\alpha$ a root \} is a Weyl basis of $L$ relative to $\mathfrak{G}$. Let $\sigma$ be the associated involution and $x^* = -\sigma(x)$ for all $x$ in $L$. Suppose $\phi$ is a representation of $L$ as bounded operators on a Hilbert space with $\phi(x^*) = \phi(x)^*$ for all $x$.

(a) If $[x, x^*] = 0$ then $\phi(x)$ is diagonalizable with finite spectrum.

(b) The eigenvalues of $\phi(h_\alpha)$ are integer multiples of $(1/2)\alpha(h_\alpha)$.

(c) $\phi(\mathfrak{G})$ is diagonalizable.

(d) $\phi(e_\alpha)$ is nilpotent.

**Proof.** For each $\alpha$, $h_\alpha = [e_\alpha, e_\alpha^*]$ and $[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha$ together with the first corollary give (b). (d) is a consequence of this and 4.1. (c) is true since $\phi(\mathfrak{G})$ is spanned by the finitely many diagonalizable operators $\phi(h_\alpha)$ which are mutually commutative. If $[x, x^*] = 0$ then $x$ is contained in some Cartan subalgebra of the $L^*$ algebra $L$ and this subalgebra is spanned by elements of the form $[f_\beta, f_\beta^*]$ where $f_\beta$ is a root vector relative to it so that the arguments used in (a) and (b) can be used to prove $\phi(x)$ is diagonalizable.

A slightly improved version of 4.5 for a special case will be needed later and this is proved below.

4.6. Suppose $A$ is equal to $D_a$ for some $a$ in $L$ and $A^3 = 0$. Then the spectrum of $[A, A^*]$ lies in the set \{ $k(\sqrt{2})$: $k = 0, 1, 2, -1, -2$ \}.

**Proof.** By 3.2, $A^2A^*A = AA^*A^2$ and, using the argument in the proof of the first corollary of 4.2, each of these is equal to $\lambda A^2$. Because of symmetry similar relations hold with $A$ and $A^*$ interchanged.

Now $(AA^*)^2 = A^2A^* + A[A^*, A]A^* = A^2A^* + AA^*[A^*, A] + \lambda AA^*$ which implies $2(AA^*)^2 = A^2A^* + AA^*A + \lambda^2 AA^*$. From these two relations a direct computation shows that $2(AA^*)^3 - 3\lambda (AA^*)^2 + \lambda^2 AA^* = 0$ and (by symmetry) that the same relation holds for $A^*A$. Then an argument like that used in proving the corollary of 4.5 will finish the proof.

5. **Reduction to the separable case.**

**Definition.** Let $\mathfrak{G}$ be a Cartan subalgebra of $L$. If $L'$ is a semi-simple subalgebra of $L$, $L'$ will be called regular (with respect to $\mathfrak{G}$) if and only if $L'$ is separable and $\mathfrak{G}' = \mathfrak{G} \cap L'$ is a Cartan subalgebra of $L'$. It will be proved here that if each regular $L'$ has a Cartan decomposition with respect to $\mathfrak{G}$ then $L$ has a decomposition with respect to $\mathfrak{G}$. For $x$ in $L$ let $M(x)$ denote the smallest closed subspace of $L$ containing $x$ and invariant under $\mathfrak{G}$. Then $M(x) = \operatorname{Sp} \{ V_n: n = 0, 1, \cdots \}$ where $V_0 = \operatorname{Sp} \{ x \}$, $V_n = [\mathfrak{G}, V_{n-1}]$ for $n \geq 1$.

5.1. Let $x$ be fixed and let $B$ be the bounded operator on $\mathfrak{G}$ defined by
\((Bh, h') = (Hx, H'x)\) for \(h, h'\) in \(\mathcal{C}\). Then \(B\) is self-adjoint and completely continuous.

**Proof.** \((Bh, h') \geq 0\) implies \(B = B^*\). Let \(E\) be the spectral measure on the spectrum of \(\mathcal{C}\) such that \((Hy, z) = \int (h, x_\alpha) d(E(\alpha)y, z)\) for \(h\) in \(\mathcal{C}\) and \(y, z\) in \(L\).

Then \((Bh, h') = \int (h, x_\alpha)(x_\alpha, h') d(E(\alpha)x, x)\). If \(\{h_n\}\) converges weakly to \(h\) and \(\{h_n^\prime\}\) to \(h'\) then both sequences are bounded and the Lebesgue dominated convergence theorem implies \((Bh_n, h_n^\prime)\) converges to \((Bh, h')\). By [4, Definition 2, p. 206], \(B\) is completely continuous.

5.2. For \(x\) in \(L\) let \(\mathcal{C}'(x) = \{h : h \in \mathcal{C}, [h, x] = 0\}\) and let \(\mathcal{C}(x)\) be the orthogonal complement of \(\mathcal{C}'(x)\) in \(\mathcal{C}\). If \(x\) is self-adjoint then so are \(\mathcal{C}(x)\) and \(M(x)\) and both are separable.

**Proof.** It is clear that \(\mathcal{C}'(x)\) is self-adjoint and hence the same is true of \(\mathcal{C}(x)\). Since \(V_0\) is self-adjoint, induction on \(n\) proves that each \(V_n\) is also and hence \(M(x)\) is.

Let \(h' \in \mathcal{C}\). By the definition of the operator \(B\) in 5.1, \((Bh, h') = 0\) for all \(h\) if and only if \(H'x = 0\). Hence \(\mathcal{C}'(x)\) is the null-space of \(B\) and, since \(B\) is self-adjoint, \(\mathcal{C}(x)\) is the closure of the range of \(B\). Since \(B\) is completely continuous, the reference in [4] shows that \(\mathcal{C}(x)\) must be separable. Then an induction on \(n\) proves that \(V_n = [H(x), V_{n-1}]\) and that each \(V_n\) is separable so that \(M(x)\) is separable.

5.3. Suppose \(x\) is self-adjoint, nonzero, and orthogonal to \(\mathcal{C}\). Let \(L'\) be the \(L^*\) algebra generated by \(\mathcal{C}(x) + M(x)\). Then \(\mathcal{C}(x) = \mathcal{C} \cap L'\), \(L'\) is regular, \([\mathcal{C}'(x), L'] = 0\), and \((\mathcal{C}'(x), L') = 0\).

**Proof.** Since the orthogonal complement of \(\mathcal{C}\) is invariant under \(\mathcal{C}\) and contains \(x\) it also contains \(M(x)\) so that the indicated sum is actually direct. Since \(\mathcal{C}(x)\) and \(M(x)\) are separable and self-adjoint it is possible to choose a countable (or finite) orthogonal basis of the space \(\mathcal{C}(x) + M(x)\), say \(\{e_n\}\), such that each \(e_n\) is self-adjoint. Then a proof like that for 3.5 (a) shows that \(L'\) is spanned by products of the form \(E_{e_1} \cdots E_{e_n} e_n\) and, since the set of these is countable, \(L'\) is separable.

For \(h' \in \mathcal{C}'(x)\), \(H'\) is zero on \(\mathcal{C}(x) + M(x)\), hence on a set of generators of \(L'\). Since \(H'\) is a derivation, \(H'\) is zero on \(L'\), proving \([\mathcal{C}'(x), L'] = 0\). Now \(\mathcal{C}'(x), e_n = 0\) for each \(n\) and, since \([\mathcal{C}'(x), L'] = 0\), it follows readily that \(\mathcal{C}'(x)\) is orthogonal to each of the finite products of generators, hence is orthogonal to \(L'\). This implies \(\mathcal{C}(x) = \mathcal{C} \cap L'\).

Finally, if \(y \in L'\) and \([y, \mathcal{C}(x)] = 0\) then \([y, \mathcal{C}] = 0\) so that \(y \in \mathcal{C}\) and hence \(y \in \mathcal{C}(x)\). Thus \(\mathcal{C}(x)\) is a maximal abelian subalgebra of \(L'\). If \(y \in L'\) and \([y, L'] = 0\) then \(y \in \mathcal{C}(x) \cap \mathcal{C}'(x)\) implies \(y\) is zero. Thus \(L'\) is semi-simple and \(\mathcal{C}(x)\) is a Cartan subalgebra of \(L'\).

5.4. Suppose every regular subalgebra \(L'\) of \(L\) has a Cartan decomposition with respect to \(\mathcal{C}' = \mathcal{C} \cap L'\). Then \(L\) has a decomposition with respect to \(\mathcal{C}\).

**Proof.** Let \(K\) be the \(L^*\) subalgebra of \(L\) obtained by letting \(K = \mathcal{C} + V\) where \(V\) is the span of all the nonzero root vectors of \(L\) relative to \(\mathcal{C}\). It is sufficient to prove \(K = L\). Now \(K\) is invariant under \(\mathcal{C}\) so that \(K'\), the orthog-
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Onal complement of $K$ in $L$, is also invariant. Furthermore, $K'^* = K'$ and if $K' \neq 0$ there is a nonzero self-adjoint element $x \in K'$. Then $M(x) \subset K'$. Let $L'$ be the $L^*$ subalgebra generated by $3\mathcal{C}(x) + M(x)$. By 5.3 $L'$ is regular so that the hypothesis here implies $L'$ has a Cartan decomposition with respect to $3\mathcal{C}(x)$. Since $M(x)$ is invariant under $3\mathcal{C}(x)$ it will be spanned by root vectors of $3\mathcal{C}(x)$ and hence there is a nonzero $v$ in $M(x)$ which is a common eigenvector for all $H, h \in 3\mathcal{C}(x)$. But if $h' \in 3\mathcal{C}'(x)$ then $H'v = 0$ so that it follows immediately that $v$ is a common eigenvector for $3\mathcal{C}$. Since $v \in M(x) \subset K'$ this gives the desired contradiction.


Remark. It will be proved here that if $L$ is simple and separable there is a Cartan subalgebra $\mathcal{C}$ of $L$ such that $L$ as a decomposition with respect to $\mathcal{C}$. Hence $L$ must be one of the five types $A, A', B, C, D$ obtained in [5]. Since each of these is a Lie subalgebra of an $H^*$ algebra, Theorem 2 of [5] shows that $L$ has a decomposition with respect to any Cartan subalgebra.

From this it is clear that any separable semi-simple $L^*$ algebra has a decomposition with respect to any Cartan subalgebra. Finally, 5.4 shows that this is true with no restriction on the dimension of $L$.

6.1. Suppose $a_1, a_2$ are self-adjoint elements of $L$ and $A_1A_2 = 0$. Then either $a_1 = 0$ or $a_2 = 0$.

Proof. Since $A_1$ is self-adjoint, $A_2A_1$ is also zero. Let $C_i$ be the null-space of $A_i$ and $R_i$ the closure of the range of $A_i, i = 1, 2$. Then $L = C_1 + R_1$ and both $C_i$ and $R_i$ are self-adjoint. Let $I_i = \text{Sp}\{R_i, [R_i, R_i]\}$. Since $A_1A_2 = 0$ then $(R_i, R_j) = 0$. Also $[[a_1, L], [a_2, L]] = [[[a_1, L], a_2], L] + [[a_2, [a_1, L]], L] = [a_2, [a_1, L], L] \subset R_2$. Similarly $[[a_1, L], [a_2, L]] \subset R_1$. From this we have $(R_1, R_2) \subset R_1 \cap R_2 = 0$. Hence the Jacobi identity gives $[I_1, I_2] = 0$. From the above it is easy to see that $I_1$ is orthogonal to $I_2$. Let $W$ be the orthogonal complement of $I_1 + I_2$. Then $(W, R_i) = 0$ implies $W \subset C_1 \cap C_2$. This in turn implies that $R_i$ is invariant under $W$. But $(W, R_i) = (W, [R_i, R_i]) = 0$ so that $[W, R_i] = 0$ and hence $[W, I_i] = 0$. Since $L = W + I_1 + I_2$ it follows immediately that $I_i$ is an ideal of $L$. By the simplicity of $L$ either $I_1$ or $I_2$ must be zero. Now $A_1L \subset R_i \subset I_i$ so that either $A_1$ or $A_2$ is zero.

Notation. By Corollary 2 of 3.8 there exists an element $a$ of $L$ such that $A^2 = 0, ||a|| = 1,$ and $[[A, a^*], a] = \lambda a$ where $\lambda = ||[a, a^*]||^2 \neq 0$. Thus $[[A, A^*], A] = \lambda A$ and 4.6 implies $L = V_0 + V_{\lambda^2} + V_{-\lambda^2} + V_\lambda + V_{-\lambda}$ where $V_\mu$ is the eigenspace for $[A, A^*]$ with the indicated subscript as eigenvalue. The usual relations hold between these subspaces, i.e., $[V_\mu, V_\nu] \subset V_{\mu^\star + \nu}$ and $V_\mu^\star = V_{-\mu}$ for each $\mu$. In particular, $X^\star = 0$ for all $x \in V_\lambda$.

6.2. Let $S = [V_\lambda, V_\lambda^*] + V_\lambda + V_\lambda^*$. Then $S$ is a semi-simple $L^*$ algebra.

Proof. Clearly $S^* = S$ and $[V_\lambda, V_\lambda] = [V_\lambda^*, V_\lambda^*] = 0$. Since $V_\lambda, V_\lambda^*$ are both invariant under $[V_\lambda, V_\lambda^*]$, so is $[V_\lambda, V_\lambda^*]$. From this it follows that $S$ is a subalgebra. If $x \in S$ and $[x, S] = 0$ then $[A, A^*]x = 0$ implies $x \in [V_\lambda, V_\lambda^*]$. But $(x, [V_\lambda, V_\lambda^*]) = 0$ since $[x, V_\lambda] = 0$. Thus $x$ is zero and $S$ is semi-simple.

Definition. By Zorn's Lemma it is possible to choose a subset $F$ of $V$
which is maximal with respect to the following properties:
(i) $b \in \mathcal{F}$ implies $\|b\| = 1$ and $[\|b, b^*\|, b] = \lambda b$ ($\lambda = \|b, b^*\|^2$).
(ii) $b, c \in \mathcal{F}$ implies $[\|b, b^*\|, \|c, c^*\|] = 0$.

Necessarily $a$ is in $\mathcal{F}$ since $[V_\alpha, V_\alpha^*] \subset V_\alpha$. Let $M = \text{Sp}\{b, b^* : b \in \mathcal{F}\}$. Then $M$ is self-adjoint and abelian. Let $C(M) = \{x : x \in S, [x, M] = 0\}$.

6.3. Let $\Delta$ be the spectrum of $M$ (acting on $S$) and suppose $\alpha \in \Delta$ with $\alpha$ nonzero. Then $\|x_\alpha\| \geq (1/2)\lambda^1/2$.

**Proof.** For any $b$ in $\mathcal{F}$, $\langle [b, b^*], [a, a^*] \rangle = \lambda$ implies $\lambda \leq \|a, a^*\| \|b, b^*\| = \lambda^1/2\lambda^1/2$ so that $\lambda \leq \lambda b$. Since $\alpha$ is not zero there is a $b \in \mathcal{F}$ such that $\langle x_\alpha, [b, b^*] \rangle \neq 0$. By the first corollary of 4.5 the spectrum of $[B, B^*]$ consists of integer multiples of $(1/2)\lambda$. Thus we must have $\|x_\alpha\| \|b, b^*\| \geq (1/2)\lambda b$ which gives $\|x_\alpha\| \geq (1/2)\lambda^1/2$.

**Corollary.** There exist subspaces $V_k$ of $S$, invariant under $C(M)$, such that $S = C(M) + \sum V_k$ and $[V_k, V^*_k] \subset C(M)$.

**Proof.** The existence of the $V_k$'s is implied by 2.4. Since they are spectral subspaces they are invariant under all operators commuting with $\{X : x \in M\}$, hence invariant under $C(M)$.

6.4. $M = C(M)$ and $M$ is a Cartan subalgebra of $S$.

**Proof.** For $x$ in $C(M)$, $[A, A^*]x = 0$ implies $x \in V_\alpha$, hence $x \in [V_\alpha, V^*_\alpha]$. Thus $V_\alpha$ is invariant under $C(M)$. Then if $W_k = V_\alpha \cap V_k$ we have $W_k$ is invariant under $C(M)$ for any $x$ commuting with $M$, $V_\alpha = \sum W_k$, and $[W_k, W^*_k] \subset C(M)$.

Suppose $c \in W_k$. Then $W_k$ is invariant under $[C, C^*]$. Hence $\text{Sp}\{X^c : X = [C, C^*], n = 0, 1, \ldots \} \subset W_k$. Since $C^* = 0$, the proof of 3.8 shows that there exists an orthonormal set $\{e_i\} \subset W_k$ such that $[e_i, e_i^*], e_i = \lambda e_i, [e_i, e_j^*] = [e_i, e_j^*] = 0$ for $i \neq j$, and $c = \sum c_i e_i$. Then $[c, [c, c^*]] = \sum c_i [e_i, e_i^*]$. By the maximality of $\mathcal{F}$, each $e_i \in \mathcal{F}$ since $[e_i, e_i^*] \subset C(M)$. Hence $[e_i, e_i^*] \in M$ so that $[c, c^*] \in M$.

For future reference we will now prove that if $x$ is any element of $L$ with $[x, M] = 0$ and $(x, M) = 0$ then $[x, V_\alpha] = 0$. To see this note first that $x \in V_\alpha$ (since $[a, a^*] \in M$) and this implies $V_\alpha$ and $V_k$ are invariant under $X$ so that $W_k$ is also invariant. If $c \in W_k$ then $[c, [c, c^*]] \in M$ so that $0 = (x, [c, c^*]) = (Xc, c)$. Since the operator $X$ on $W_k$ is completely determined by the quadratic form $(Xc, c)$ this gives $X$ is zero on $W_k$ for each $k$ so that $X$ is zero on $V_\alpha$.

The preceding paragraph shows that if $x \in C(M)$ and $(x, M) = 0$ then $[x, V_\alpha] = 0$. Now $M$ is self-adjoint which implies the same for $C(M)$ and thus $x^* \in C(M), (x^*, M) = 0$ so that we also have $[x^*, V_\alpha] = 0$ and this gives $[x, V^*_\alpha] = 0$. But then $[x, [V_\alpha, V^*_\alpha]]$ is also zero and this implies $X$ is zero on $S$ so that $x = 0$. Hence $M = C(M)$ so that $M$ is maximal abelian in $S$, hence a Cartan subalgebra of $S$.

6.5. $S$ has a Cartan decomposition with respect to $M$.

**Proof.** Using the notation of 6.4 we now have $V_\alpha = \sum W_k$ with $[W_k, W^*_k]$
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CM. Also \([W_k, W_k^*] = 0\). Let \(k\) be fixed and let \(S_1 = [W_k, W_k^*] + W_k + W_k^*\). Then it is easily seen that \(S_1\) is an \(L^*\) subalgebra since \(W_k\) is invariant under \(M\). Let \(P\) be the projection of \(S\) onto \(S_1\) and \(a_0 = P[a, a^*]\). Then for \(z \in S_1\), 
\[\begin{align*}
[a_0, z] &= [P[a, a^*], z] = P[[a, a^*], z] = [[a, a^*], z] \text{ since } Z \text{ and } Z^* \text{ leave } S_1 \\
&\text{invariant. Hence if } [z, S_1] = 0 \text{ then } [z, [a, a^*]] = 0 \text{ so that } z \in [W_k, W_k^*].
\end{align*}\]
But \([z, [W_k, W_k^*]] = 0\) since \([z, W_k] = 0\), hence \(z\) is zero and \(S_1\) is semi-simple. Now \([W_k, W_k^*] \subset M\) implies \([W_k, W_k^*]\) is abelian and the proof above shows that it is maximal abelian in \(S_1\), thus a Cartan subalgebra. By 3.8, \(S_1\) is a direct sum of simple ideals \(I_j\) where \(I_j = \text{Sp}\{e_i, e_i^*, [e_i, e_i^*]\}\) for some \(e_i\) in \(W_k\). If \(x \in M\) then \([x, S_1] \subset S_1\) and \(X\) is zero on \(S_1\) if and only if it is zero on \(W_k\) which is equivalent to \((x, [W_k, W_k^*]) = 0\). From this it follows readily that each \(e_i\) is a common eigenvector for all \(X, x \in M\). Thus each \(W_k\), and therefore \(V_x\), is spanned by root vectors for \(M\). By symmetry the same is true of \(V_x^*\). Since \(S\) is generated by \(V_x\) and \(V_x^*\), and since each finite product of eigenvectors for \(M\) is again an eigenvector for \(M\), \(S\) is spanned by eigenvectors and this completes the proof.

**Definition.** Let \(\mathfrak{S}_1\) be a set in \(V_{1/2}\) which is maximal with respect to the following properties:

(i) \(b \in \mathfrak{S}_1\) implies \(\|b\| = 1\), \([[b, b^*], b] = \lambda_b b\),
(ii) \(b, c \in \mathfrak{S}_1\) implies \([[b, b^*], [c, c^*]] = 0\),
(iii) \(\varepsilon \in \mathfrak{S}_1\) implies \([[b, b^*], M] = 0\).

Let \(\mathfrak{S}_0 = \text{Sp}\{b, b^*: b \in \mathfrak{S}_1 \cup \mathfrak{S}\}\). Then \(M \subset \mathfrak{S}_0\) and \(\mathfrak{S}_0\) is abelian and self-adjoint. Let \(\mathfrak{S} = \{x: [x, \mathfrak{S}_0] = 0\}\). Since \([a, a^*] \subset \mathfrak{S}_0\), \(\mathfrak{S} \subset V_0\). We will show that \(\mathfrak{S}\) is the desired Cartan subalgebra. Note that \(V_\mu\) is invariant under \(\mathfrak{S}\) for each eigenspace \(V_\mu\) of \([A, A^*]\).

6.6. Let \(\Delta\) be the spectrum of \(\mathfrak{S}_0\) and suppose \(\alpha\) is a nonzero element of \(\Delta\). Then \(\|x_\alpha\| \geq (1/4)\alpha^{1/2}\). Hence \(L = \mathfrak{S} + \sum V_\lambda\) where \([V_\lambda, V_\lambda^*] \subset \mathfrak{S}\) and \(V_\lambda\) is invariant under \(\mathfrak{S}\).

**Proof.** For \(b \in \mathfrak{S}_1\), \(([b, b^*], [\alpha, \alpha^*]) = (1/2)\alpha\) implies \(\lambda_b^{1/2}\lambda_\alpha^{1/2} \geq (1/2)\lambda\) so that \(\lambda_b \geq (1/4)\lambda\). The remainder of the argument is like that of 6.3 and the corollary.

6.7. \(\mathfrak{S}\) is abelian and is a Cartan subalgebra of \(L\).

**Proof.** If \(\mathfrak{S}\) is abelian it is necessarily maximal abelian so that we need only prove the first assertion. Let \(W_k = V_{1/2} \cap V_k\). Then \(W_k\) is invariant under \(\mathfrak{S}\) and \(V_{1/2} = \sum W_k\). Also \([W_k, W_k^*] \subset \mathfrak{S}\). Suppose \(\mathfrak{S}\) is not abelian. Then \([\mathfrak{S}, \mathfrak{S}]\) is a semi-simple \(L^*\) subalgebra and \(([\mathfrak{S}_0, \mathfrak{S}, \mathfrak{S}]) = 0\). Thus there exists a \(w \in [\mathfrak{S}, \mathfrak{S}]\) with \(\|w\| = 1\) and \([[w, w^*], w] = \mu w\). Also \((w, w^*], \mathfrak{S}_0) = 0\).

Now \([W, W^*]\) has spectrum contained in the set \(\{r\mu\}\) where \(r\) is a half-integer (by the first corollary of 4.5). Let \(T_r\) be the eigenspace associated with the value \(r\mu\). Choose a fixed \(W_k\). Since \(W_k\) is invariant under \([W, W^*]\), \(W_k = \sum Z_{r\mu}\) where \(Z_{r\mu}\) is the intersection of \(W_k\) with \(T_r\).

Suppose \(r \neq 0\). Then \([Z_{r\mu}, Z_{r\mu}] \subset T_{2r\mu} \subset \text{Range} [W, W^*]\). But \([Z_{r\mu}, Z_{r\mu}]\)
$\subset V_\lambda$. Since $([w, w^*], M) = 0$, the remark made in the proof of 6.4 shows that $[W, W^*]$ is zero on $V_\lambda$ and thus $V_\lambda$ is contained in the null-space of $[W, W^*]$. Hence $[Z_{r\mu}, Z_{r\mu}] = 0$.

Now suppose some $Z_{r\mu \neq 0}$ for $r \neq 0$. For this $r$ let $S_1 = [Z_{r\mu}, Z_{r\mu}^*] + Z_{r\mu} + Z_{r\mu}^*$. Since $[Z_{r\mu}, Z_{r\mu}^*] \subset \mathfrak{C}\cap \text{null-space } [W, W^*]$ then $Z_{r\mu} = W_\lambda \cap T_{r\mu}$ is invariant under $[Z_{r\mu}, Z_{r\mu}^*]$ so it is easy to see that $S_1$ is an $L^*$ subalgebra. Using the technique of projecting $[w, w^*]$ onto $S_1$, the proof of 6.5 can be used to show that $S_1$ is semi-simple. Let $c \in S_1$. Then $C^2 = 0$ on $S_1$. Hence it follows as in 6.4 that $c = \sum c_i e_i$ where $\{e_i\}$ is an orthonormal set in $Z_{r\mu} \subset W_\lambda$ with $[[e_i, e_i^*], e_i] = \lambda_i e_i$ and $[c, c^*] = \sum |c_i|^2 [e_i, e_i^*]$. Now $[e_i, e_i^*] \in [W_\lambda, W_\lambda^*] \subset \mathfrak{C}$. By the maximality of $S_1$ this implies $[e_i, e_i^*] \in \mathfrak{C}_0$. Thus $[c, c^*] \in \mathfrak{C}_0$ so that $0 = ([w, w^*], [c, c^*]) = ([w, w^*], [c, c])$. Since $Z_{r\mu}$ is invariant under the operator $[W, W^*]$ we must have $[W, W^*]$ is zero on $Z_{r\mu}$. But this implies $Z_{r\mu}$ is zero for all $r \neq 0$ and hence $[W, W^*]$ is zero on $V_{\lambda/2}$ as well as on $V_\lambda$. Using the self-adjointness of $[W, W^*]$ this implies $[W, W^*]$ is zero on $V_{\lambda\mu}$ and $V_\lambda$ so that $[W, W^*] [A, A^*] = 0$. By 6.1 we must have $[W, W^*] = 0$ and, since $w$ is an eigenvector for $[W, W^*]$, this gives $w = 0$, finishing the proof.

6.8. $\mathfrak{C}$ has at least one nonzero root.

Proof. $[\mathfrak{C}, V_\lambda] \subset V_\lambda$ and $M \subset \mathfrak{C}$. In the remark in the proof of 6.4 shows that $X$ is zero on $V_\lambda$. Now, using 6.5, $V_\lambda$ is spanned by eigenvectors for $M$ and it is immediate that each of these will be a root vector for $\mathfrak{C}$, corresponding to a nonzero root.

6.9. Suppose $L'$ is one of the simple algebras $A, A', B, C, D$ discussed in §3 of [5]. If $e_\alpha$ is a normalized root vector corresponding to some root $\alpha$ then $E_{\alpha} = 0$. There exists an $e_\alpha$ in $L'$ such that the eigenspace corresponding to the maximal eigenvalue of $[E_\alpha, E_\alpha^*]$ is finite-dimensional.

Proof. It is easily seen that $\beta + 3\alpha$ is never a root for any root $\beta$ in the same root set as $\alpha$ and this implies $E_{\beta} = 0$. Hence the spectrum of $[E_\alpha, E_\alpha^*]$ is contained in the set $\{r(\alpha, \alpha) : r = 0, 1/2, -1/2, 1, -1\}$. Now the eigenspace in question is spanned by root vectors $e_\beta$ with $(\alpha, \beta) = (\alpha, \alpha)$. Using the notation of [5], the roots are obtained from the set $\{\lambda_i - \lambda_j, \lambda_i + \lambda_j, 2\lambda_i, \lambda_i, i, j \text{ integers}\}$ with $(\lambda_i, \lambda_i) = \delta_{i,j}$. Furthermore, $L'$ must contain a root of the form $\lambda_i - \lambda_j$. Taking $\alpha = \lambda_i - \lambda_j$, $(\alpha, \beta) = (\alpha, \alpha)$ implies $\beta$ is contained in the finite set $\{\lambda_i - \lambda_j, 2\lambda_i\}$.

6.10. $L$ has a Cartan decomposition with respect to $\mathfrak{C}$.

Proof. Let $R$ be the (nonempty) set of all nonzero roots of $L$ relative to $\mathfrak{C}$ and choose a normalized root vector $e_\alpha$ for each $\alpha \in R$. Let $V' = \text{Sp} \{e_\alpha : \alpha \in R\}, \mathfrak{C}' = \text{Sp} \{e_\alpha, e_\alpha^* : \alpha \in R\}$, and $S' = \mathfrak{C}' + V'$. Then $S'$ is semi-simple, $\mathfrak{C}'$ is a Cartan subalgebra of $S'$, and $S$ has a Cartan decomposition with respect to $\mathfrak{C}'$ with $R$ as a complete set of roots. If $x \in \mathfrak{C}'$ and $(x, \mathfrak{C}') = 0$ then $[x, e_\alpha] = 0$ for all $\alpha$ so that $[x, S'] = 0$. Let $W$ be the orthogonal complement of $\text{Sp} \{\mathfrak{C}, S'\}$. Then $W$ is invariant under both $S'$ and $\mathfrak{C}$. If $[S', W] = 0$ then it is immediate that $[S', L] \subset S'$ so that $S'$ is a nonzero ideal, $S' = L$, and the existence theorem is proved.
By the remarks above it is enough to prove $[S', W] = 0$. Now $W$ is invariant under $\mathcal{C}$ so that $W = \sum W_k$ where $W_k$ is invariant under $\mathcal{C}$ and $[W_k, W_*] \subseteq \mathcal{C}$.

Let $S''$ be any simple ideal of $S'$. By the classification theory of [5] and 6.9, there is an $e_a$ in $S''$ such that the eigenspace associated with the maximal eigenvalue of $[E_a, E_a^*]$ (restricted to $S''$) is finite-dimensional. We will show that $[E_a, E_a^*]$ is zero on $W$.

Since $[[e_a, e_a^*], e_a] = \mu e_a$, $L$ is spanned by subspaces $T_r$, where $T_r$ is the eigenspace for $[E_a, E_a^*]$ associated with the value $r\mu$, $r$ being a half-integer. Suppose $W \cap T_r$ contains a nonzero subspace $U$ invariant under $X$ with $[U, U] = 0$ for some $k$. If $S_1 = [U, U^*] + U + U^*$ it is easy to see that $S_1$ is a semi-simple $L^*$ subalgebra (see the proof of 6.5) with $[U, U^*]$ as a Cartan subalgebra. By 3.8, $S_1$ is a direct sum of three-dimensional ideals $I_j$ with $I_j = \text{Sp} \{ e_j, e_j^*, [e_j, e_j^*] \}$ for some $e_j$ in $U$. Thus each $e_j$ is a root vector for $[U, U^*] \subseteq \mathcal{C}$. Since $S_1$ is invariant under $\mathcal{C}$ it follows as in the proof of 6.8 that each $e_j$ is a root vector for $\mathcal{C}$, a contradiction of the definition of $S'$.

Thus if $r\mu$ is the maximal eigenvalue for $[E_a, E_a^*]$, since $T_\alpha$ is abelian, we must have $W \cap T_\alpha = 0$. But then $(T_\alpha, W) = 0$ and this implies $T_\alpha$ is entirely contained in $S''$. Since $E_a^* = 0$ on $S'$, $r_\alpha$ must equal one. By the choice of $\alpha$, $T_1$ is finite-dimensional, say of dimension $m$.

Suppose $[E_a, E_a^*] \neq 0$ on $W$. Since $W$ is self-adjoint we must have $Z = T_1 \cap W_k \neq 0$ for some $k$. Then $Z$ is invariant under $\mathcal{C}$. Suppose $Z = Z_1 + \cdots + Z_{m+1}$ with each $Z_i$ invariant under $\mathcal{C}$. Then $([Z_i, Z_j^*], \mathcal{C}) = (Z_i, [\mathcal{C}, Z_j]) = 0$ for $i \neq j$. Hence $[Z_i, Z_j^*] = 0$ for $i \neq j$ and this implies $([Z_i, Z_i], [Z_j, Z_j]) = 0$. Since $[Z_i, Z_i] \subseteq T_i$ for each $i$, at least one $Z_i$ must be abelian, so that the remarks above imply that $Z$ contains no more than $m$ mutually orthogonal subspaces invariant under $\mathcal{C}$. From this it is a simple consequence of the spectral theorem that $Z$ must contain a root vector for $\mathcal{C}$; a contradiction. Hence $[E_a, E_a^*]$ must be zero on $W$. Using the simplicity of $S''$, this implies the representation of $S'$ on $W$ obtained by restricting the adjoint representation is trivial. Since $S'$ was an arbitrary simple component of $S'$ this implies $[S', W] = 0$.

**Bibliography**