BANACH ALGEBRAS WITH SCATTERED STRUCTURE SPACES

BY

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1. For a commutative semisimple Banach algebra $B$, let $\mathfrak{m}_B$ denote its space of nonzero multiplicative linear functionals, and $x \mapsto \hat{x}$ its Gelfand representation. A well-known theorem of Šilov [1; 15] shows that for any compact-open subset $U$ of $\mathfrak{m}_B$ there is an $x$ in $B$ with $\hat{x}$ the characteristic function $\phi_U$ of $U$. An immediate consequence is the fact that $B$ is regular if $\mathfrak{m}_B$ is totally disconnected. The present note is devoted to a similar application of Šilov’s result which has apparently escaped notice.

Normally when $A$ is a subalgebra of $B$ (closed or not), $\mathfrak{m}_A$ may contain functionals other than those provided by the restrictions of the elements of $\mathfrak{m}_B$. But at least when $A$ is closed the Šilov boundary $\partial_A$ of $A$ is produced by the elements of $\partial_B$. In any case, if we assume $\partial_A$ is produced by $\partial_B$, and that $\partial_B$ is scattered (i.e., contains no nonvoid perfect subset), then Šilov’s theorem shows not only that $B$ is regular but indeed that $A$ is regular as well so that all of $\mathfrak{m}_A = \partial_A$ arises from $\partial_B = \mathfrak{m}_B$. When $\partial_B$ is discrete the same is true of $\partial_A$, and we can trivially identify the smallest hull-less ideal $\mathfrak{a}(\infty)$ of $A$; it is precisely the span of the idempotents in $A$. Consequently $A$ is tauberian if and only if it is the closed span of its idempotents.

As an application we can easily determine all closed tauberian subalgebras of $L_1(G)$ and $L_2(G)$ when $G$ is a compact abelian group. In the $L_2$ case every closed subalgebra $A$ is tauberian, and is determined by just the sets of constancy of the Fourier transforms $A^\sim$; the same prescription applies to the tauberian closed subalgebras $A$ of $L_1(G)$, but whether nontauberian subalgebras exist seems to be a difficult problem. Borrowing from the $L_2$ case we can easily see that $A$ is tauberian if (and of course only if) $A \cap L_2$ is dense in $A$. (Some application can also be made to closed commutative semisimple subalgebras of $L_1(G)$ and $L_2(G)$ when $G$ is compact nonabelian, cf. §4.)

The notation used below is essentially standard, as in [9], with the exception of our use of “scattered,” as defined above. We denote the hull of an ideal $I$ by $hI$, and by $kF$ the kernel of a subset $F$ of $\mathfrak{m}_A$, while $j_A(\infty)$ is the set of $a$ in $A$ for which $\hat{a}$ has compact support. $A$ is called tauberian when every hull-less ideal is dense in $A$; when $A$ is regular this amounts to the density of $j_A(\infty)$ [9]. All algebras will be assumed commutative semisimple. Although $\sim$ may be used for any Gelfand representation which arises, it will always be clear from the context which is intended. Finally, $A_a$ will be used to

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denote the closed subalgebra of $A$ generated by an element $a$, $\sigma_A(a)$ the spectrum of $a$ in $A$, and $\text{bdry} F$ the ordinary boundary of a plane set $F$.

It is a pleasure to record my indebtedness to Edwin Hewitt for raising a question which led to this investigation, and to Walter Rudin for suggesting the use of scattered sets.

2. The main result. Let $A$ be a Banach algebra which forms, algebraically, a subalgebra of a commutative semisimple Banach algebra $B$. If the functionals in $\partial_A$ are included in the restrictions to $A$ of the elements of $\partial_B$ we shall simply say $\partial_A$ arises from, or is produced by, $\partial_B$. As is well known this occurs\(^{(2)}\) if $A$ is a closed subalgebra of $B$, and as a consequence, $\text{bdry} \sigma_A(a) \subseteq \partial(\partial_A)$. Indeed, $\sigma_{\Lambda_\alpha}(a)$ can be identified with $\mathcal{M}_\Lambda_{\alpha}$ in such a way that $\text{bdry} \sigma_{\Lambda_\alpha}(a)$ is precisely $\partial_{\Lambda_\alpha}$ (which arises from $\partial_A$).

Theorem 2.1. Let $A$ be a Banach algebra which is algebraically a subalgebra of a commutative semisimple Banach algebra $B$. Suppose $\partial_A$ arises from $\partial_B$, while $\partial_B$ is scattered. Then $A$ is regular, and consequently $\mathcal{M}_A = \partial_A$ arises from $\partial_B = \mathcal{M}_B$.

Proof. Rudin [12, Theorem 1] has shown that a continuous image of a compact scattered space is scattered. Since the one-point compactification $\partial_B \cup \{0\}$ of $\partial_B$ is clearly scattered, and $\text{bdry} \sigma_{\Lambda_\alpha}(a) \subseteq \partial(\partial_A) \subseteq \partial(\partial_B)$, we conclude that $\text{bdry} \sigma_{\Lambda_\alpha}(a)$ is scattered. But nontrivial closed connected plane sets cannot be scattered, so $\text{bdry} \sigma_{\Lambda_\alpha}(a)$ is totally disconnected; consequently $\sigma_{\Lambda_\alpha}(a)$, and its subset $\sigma_A(a)$, are all boundary. Thus $\partial(\mathcal{M}_A) = \sigma_A(a)$ is totally disconnected, for each $a$ in $A$, so $\mathcal{M}_A$ is totally disconnected.

By Silov's theorem [1; 15] for each compact-open $U$ in $\mathcal{M}_A$ there is an $a$ in $A$ with $\partial = \phi_U$, so that $A$ is regular, and of course $\mathcal{M}_A = \partial_A$. Since in particular we may take $A = B$, $B$ is regular and $\mathcal{M}_B = \partial_B$, completing the proof.

Clearly $A^\sim$ contains sufficiently many real valued functions to apply Stone-Weierstrass, yielding a result of Rudin [12].

Corollary 2.2. $A^\sim$ is dense in $C_0(\mathcal{M}_A)$.

Corollary 2.3. Let $A$ and $B$ be as in 2.1, while $\partial_B$ is discrete. Then $\mathcal{M}_A$ is discrete and, for each $M \in \mathcal{M}_A$, $\phi_{\{M\}} \in A^\sim$.

We need only verify discreteness. But since $\mathcal{M}_A$ arises from $\partial_B$, and $\partial$, considered as a function on $\partial_B$, has $|\partial(M)| \geq \varepsilon > 0$ for only finitely many $M$ in $\partial_B$, $|\partial(M)| \geq \varepsilon$ for only finitely many $M$ in $\mathcal{M}_A$. Clearly then the compact subsets of $\mathcal{M}_A$ are finite, and thus $\mathcal{M}_A$ is discrete.

Corollary 2.4. Let $A$ be a commutative semisimple Banach algebra, and $E$\(^{(2)}\) By the Beurling-Gelfand formula, which shows $a \in A$ has the same spectral norm in $A$ and in $B$. Thus under the dual to the injection $A \rightarrow B$, $\partial_B$ maps onto a closed set on which $|\partial|$ maximizes.
a scattered hull-kernel closed subset of \( \mathcal{M}_A \). Then the relative topology on \( E \) is the hull-kernel topology.

For the semisimple algebra \( B = A/kE \) has \( E \) in its relative topology as \( \mathcal{M}_B \), whether the usual \((w^*)\) or hull-kernel topologies are used \([9, 20G]\); since \( B \) is regular by 2.1, these two topologies coincide on \( \mathcal{M}_B \), so that the relative topology on \( E \) is the same whether \( \mathcal{M}_A \) is taken in its usual topology or the hull-kernel topology.

There is, more or less, a converse to the fact (in 2.1) that \( \partial_A \) is scattered if \( \partial_B \) is. Indeed

**Corollary 2.5.** Let \( A \) and \( B \) be commutative semisimple Banach algebras and \( \tau \) a homomorphism of \( A \) onto a dense subalgebra of \( B \). Then if \( \partial_A \) is scattered, so is \( \partial_B \).

**Proof.** By semisimplicity \( \tau \) is continuous, with closed kernel \( I \). Let \( C \) be the semisimple algebra \( A/I \), and \( \tilde{\tau} \) the induced isomorphism of \( C \) into \( B \). Since we can identify \( \mathcal{M}_C \) with the scattered set \( hI \), and \( C \) is regular, by a result of Rickart \([11, \text{Theorem } 1]\) the dual map to \( \tilde{\tau} \) takes \( \mathcal{M}_B \) onto \( \partial_C = \mathcal{M}_C \), and, since \( \tilde{\tau}C \) is dense, is a homeomorphism. Consequently \( \mathcal{M}_B \) is scattered, yielding the result.

Actually fuller use of Rickart's result can be made if \( \tau \) is an algebraic isomorphism, since then we do not require the continuity of \( \tau \) produced by the semisimplicity of \( B \).

**Corollary 2.6.** Let \( \tau \) be an algebraic isomorphism of the commutative semisimple Banach algebra \( A \) onto a dense subalgebra of the non-semisimple Banach algebra \( B \), and suppose \( \partial_A = \mathcal{M}_A \) is scattered. Then \( \mathcal{M}_B \) is scattered.

For the dual to \( \tau \) maps \( \mathcal{M}_B \) onto \( \partial_A = \mathcal{M}_A \), and again is a homeomorphism.

2.7. **Remarks.** Since countable locally compact spaces are scattered the results of \([2; 3]\) can be obtained from ours. As a more novel application, we note the following. Let \( M_0(T^1) \) denote the subalgebra of the algebra of measures on the circle group \( T^1 \) consisting of those \( \mu \) with Fourier-Stieltjes transforms in \( C_0(Z) \) (\( Z \) the integers). It is known that \( \mathcal{M}_{M_0(T^1)} \) contains \( Z \) properly \([14, 2.5 (i)]\); as a consequence we conclude that \( Z \) does not contain \( \partial_{M_0(T^1)} \) (by 2.1). (And in general, a scattered closed proper subset of \( \mathcal{M}_A \) cannot contain \( \partial_A \).) In passing, we may as well note that the Fourier-Stieltjes transformation, as a map of \( M_0(T^1) \) into \( C_0(Z) \), provides us with an example of a nontopological isomorphism \( A \rightarrow B \) with \( \mathcal{M}_B \) scattered and \( \mathcal{M}_A \) not scattered. Indeed with \( A = M_0(T^1) \), \( Z \) is a closed subset of \( \mathcal{M}_A \), as is easily seen, and thus a closed proper subset on which no nonzero representative function can vanish (by the 1-1 nature of the Fourier-Stieltjes transformation); consequently \( A \) cannot be regular.

Finally, in 2.1, if \( \partial_B \) is not scattered but each \( \delta(\partial_B) \) is, then \( A \) is regular and \( \mathcal{M}_A = \partial_A \) arises from \( \partial_B \) by the same argument.
3. Tauberian algebras with discrete boundaries. Let \( \partial_A \) be discrete. Since \( A \) is regular it contains a smallest hull-less ideal, \( j_A(\infty) \), consisting of all \( a \) for which \( a \) vanishes off a compact (i.e., finite) subset of the discrete space \( \mathcal{M}_A = \partial_A \). Moreover since we know \( A \) contains \( \phi_M \) for each \( M \) in \( \mathcal{M}_A \), \( j_A(\infty) \) is just the span of the corresponding idempotent elements of \( A \). Recalling that a semisimple algebra is called tauberian if it has no proper closed hull-less ideals (which, for regular algebras, says exactly that \( j_A(\infty) \) is dense), we have

**Theorem 3.1.** Let \( \partial_A \) be discrete. Then \( A \) is tauberian if and only if it is the closed span of its idempotent elements.

As a consequence we obtain a result due to Rudin [13](\(^\dagger\)).

**Theorem 3.2.** Let \( A \) be a closed subalgebra of a tauberian algebra \( B \), and suppose \( \mathcal{M}_B \) is discrete. Then if \( A \) separates the elements of \( \mathcal{M}_B \cup \{0\} \), \( A = B \).

**Proof.** By 2.1 and 2.3 and our hypothesis of separation, we have \( \mathcal{M}_A = \mathcal{M}_B \) and \( \phi_M \in A \) for every \( M \) in \( \mathcal{M}_B \). Thus \( j_B(\infty) \subset A \), and \( A = B \).

**Corollary 3.3.** Let \( A \) be tauberian and \( \mathcal{M}_A \) discrete. The following are equivalent:

1°. \( A \) is separable.
2°. \( \mathcal{M}_A \) is countable.
3°. \( A \) is singly generated.

**Proof.** If 1° holds then \( C_0(\mathcal{M}_A) \) is separable by 2.2, and \( \mathcal{M}_A \), being discrete, is then clearly countable. But if \( \mathcal{M}_A = \{ M_1, M_2, \ldots \} \), let \( a_n = \phi_M \), and choose a sequence \( \lambda_1, \lambda_2, \ldots \) of distinct nonzero numbers satisfying \( \sum |\lambda_n| \cdot ||a_n|| < \infty \). Then \( a = \sum \lambda_n a_n \) is an element of \( A \) for which \( a \) separates \( \mathcal{M}_A \cup \{0\} \), so that \( A_a = A \) by 3.2. Finally that 3° implies 1° is clear.

It is trivial to identify the maximal closed subalgebras of a tauberian \( B \) with \( \mathcal{M}_B \) discrete, by virtue of 3.2.

**Theorem 3.4.** Let \( B \) be a tauberian algebra with \( \mathcal{M}_B \) discrete, and let \( A \) be a maximal closed subalgebra which is not a maximal regular ideal. Then(\(^\ddagger\)) \( A = \{ x: x \in B, \hat{x}(M_1) = \hat{x}(M_2) \} \) for some \( M_1 \neq M_2 \) in \( \mathcal{M}_B \).

3.5. If \( \mathcal{M}_A \) is discrete one can easily determine all closed tauberian subalgebras of \( A \) (an example is given in §4). But closed subalgebras need not

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\(^\dagger\) Rudin's proof (which makes no use of Silov's theorem) appears in mimeographed notes of a Symposium on Harmonic Analysis and Related Integral Transforms held in summer 1956 at Cornell University. (It was applied there only to the special case \( B = L_1(G) \), \( G \) a compact abelian group.) We might note that in 3.2 we could alternatively assume \( \partial_B \) is scattered and \( B \) spanned by idempotents, obtaining a result of Katznelson and Rudin [8, Theorem 3] (for once \( \mathcal{M}_A = \mathcal{M}_B \) we have \( \epsilon = \epsilon B \) by Silov's theorem).

\(^\ddagger\) The special case \( B = L_1(G) \), \( G \) a compact abelian group, strengthens the final remark of [3].
be tauberian even if \( A \) is; Mirkil [10] gives an example of such a tauberian \( A \) containing a closed ideal \( I \neq khl \), so that spectral synthesis fails, and \( khl \) provides a nontauberian subalgebra since \( I \) is always a hull-less ideal in \( khl \).

Indeed for just this reason it is apparent that for a given algebra \( A \), all closed subalgebras are tauberian if and only if all admit spectral synthesis. (For a tauberian \( A \) with \( \mathcal{M}_A \) discrete, spectral synthesis is equivalent to \( x \in (Ax)^{-} \), all \( x \in A \) [10].)

3.6. Coddington [4] has given an example of a tauberian \( A \) with \( \mathcal{M}_A \) discrete which is not self-adjoint.

3.7. Our next results yield a class of algebras \( A \) with \( \mathcal{M}_A \) discrete. Let \( a \to L_a \) denote the regular representation of \( A : L_ax = ax, x \in A \). Let \( \mathcal{B}(X) \) denote the algebra of all bounded linear maps of a Banach space \( X \) into itself; we shall say \( T \in \mathcal{B}(X) \) has an essentially simple spectrum if \( \sigma_{\mathcal{B}(X)}(T) \setminus \{0\} \) consists only of eigenvalues of finite multiplicity, having only 0 as a point of accumulation.

**Theorem 3.8.** Let \( A \) be a commutative semisimple Banach algebra. If each \( L_a \) has an essentially simple spectrum, \( \mathcal{M}_A \) is discrete.

When each \( L_a \) is actually a compact operator this is a special case of a result of Kaplansky [7, 5.1]; our tauberian \( A \)'s with \( \mathcal{M}_A \) discrete all fall into this category since \( f_a(\infty) \) provides a uniformly dense set of compact \( L_a \). But even with all \( L_a \) compact \( A \) need not be tauberian(5).

For the proof of 3.8 we require the following well-known fact.

**Lemma 3.9.** Let \( A \) be a commutative Banach algebra, and let \( A^- \) be the uniform closure of \( \{L_a : a \in A\} \) in \( \mathcal{B}(A) \). Then we can identify the spaces \( \mathcal{M}_{A^-} \) and \( \mathcal{M}_A \) in such a way that \( L_a = L_{\hat{a}} \).

**Proof.** Each \( M \) in \( \mathcal{M}_{A^-} \) of course produces a nonzero multiplicative linear functional on \( A \) since \( a \to L_a \) is multiplicative and \( \{L_a : a \in A\} \) is dense. Conversely if \( M \in \mathcal{M}_A \) choose a \( u \in A \) with \( M(u) = 1 \), and set \( M^*(T) = M(Tu) \), \( T \in A^- \). Clearly \( M^* \) is linear; if \( L_{a_n} \to T \) and \( L_{b_n} \to S \) in \( A^- \) then \( M^*(TS) = M(TSu) \) if \( M(L_{a_n}L_{b_n}u) = \lim M(b_nu)M(a_nu) = M(Tu)M(Su) = M^*(T)M^*(S) \), and \( M^* \in \mathcal{M}_{A^-} \). Finally since \( M^*(L_a) = M(a) \) and \( \{L_a : a \in A\} \) is dense in \( A^- \) our correspondence clearly preserves topology and yields \( L_{\hat{a}} = \hat{a} \) on the identified space.

**Proof of 3.8.** Since the spectrum \( \sigma_{\mathcal{B}(A)}(L_a) \) of the operator \( L_a \) has at most 0 as a point of accumulation, the same is true of \( \text{bdry } \sigma_{\mathcal{B}(A)}(L_a) \subset \sigma_{\mathcal{B}(A)}(L_a) \).

Thus \( \sigma_{\mathcal{B}(A)}(L_a) = L_a(\mathcal{M}_A^-) = \hat{a}(\mathcal{M}_A) \) has the same property, and \( \mathcal{M}_A \) is totally disconnected and \( A \) regular, as in 2.1.

Now if \( \mathcal{M}_A \) is not discrete it contains some compact infinite subset, and

(5) For example Mirkil's algebra [10], and its closed subalgebras, satisfy the hypothesis of 3.8.
thus for some $a$ in $A$, $\delta > 0$, and sequence $\{M_n\}$ of distinct elements of $\mathfrak{M}_A$ we have $|\phi(M_n)| \geq \delta$. In view of the nature of $\phi(M_n)$ we may as well assume $\phi(M_n) = 1$ for all $n$. But again for the same reason $\phi = 1$ on a neighborhood $V_n$ of $M_n$, which can of course be chosen so that $M_m \in V_n$ for $m < n$. Let $a_n \in A$ be chosen so that $\phi_n(M_n) = 1$ while $\phi_n$ vanishes off $V_n$, so $\phi_n(M_m) = 0$ for $m < n$. Clearly the $a_n$ are linearly independent, while $\phi a_n = a_n$ implies $L_\phi a_n = a_n$. Since the operator $L_\phi$ can only have finitely many linearly independent eigenvectors corresponding to a single eigenvalue, we have obtained the desired contradiction, completing the proof.

As a consequence of 3.8 we can say something about some commutative semisimple subalgebras of $L_1(G)$ when $G$ is noncommutative (below).

Neither 3.8 nor our next result contains the other, although both are variants of the same theme.

**Theorem 3.10.** Let $A$ be a commutative, semisimple, and uniformly closed algebra of operators with essentially simple spectra on a Banach space $X$. Then $\mathfrak{M}_A$ is discrete.

**Proof.** The relation bndry $\sigma_A(a) \subset \sigma_{G(X)}(a)$ yields the fact that $\sigma_A(a) = \phi(\mathfrak{M}_A)$ has at most 0 as a point of accumulation; thus $\mathfrak{M}_A$ is totally disconnected. Again if $\mathfrak{M}_A$ is not discrete we may assume $\phi(M_n) = 1$ for a sequence of distinct $M_n$ in $\mathfrak{M}_A$, with $\delta = 1$ on a neighborhood $V_n$ of $M_n$, and with the $V_n$ now chosen so that $V_n \cap V_m = \emptyset$, $n \neq m$. By Silov's theorem we have a nonzero idempotent $a_n$ in $A$ with $\phi_n$ vanishing off $V_n$, and thus $a a_n = a_n$, while $a_m a_n = 0$, $n \neq m$.

Now if $x_n$ is any nonzero element of the range of $a_n$ we have $x_n = a_n x_n$, so $a x_n = a_n a_n x_n = a_n x_n = x_n$, $a x_n = x_n$; on the other hand, for $n \neq m$, $a_m x_n = a_m a_n x_m = 0$, so the $x_n$ are surely linearly independent, contradicting the spectral property of $a$ and completing the proof.

4. **Applications to group algebras.** Let $G$ be a compact abelian group. Trivially every idempotent in $L_1(G)$ or $L_2(G)$ is a finite sum of characters, and thus if $A$ is a closed subalgebra of either of these, $\mathfrak{M}_A(\infty)$ is just the span of an appropriate set of such finite sums. Moreover it is quite trivial to identify the basic set of generating idempotents $\{e_M: M \in \mathfrak{M}_A\}$, where $\phi_M = \phi(M)$, since $\mathfrak{M}_A$ arises from $G^\wedge$: given $M$, for all $a$ in $A$ we have $\phi(M) = \phi(\hat{g})$, for $\hat{g}$ in a certain subset $G_M$ of $G^\wedge$, and clearly such characters $\hat{g}$ are just those (finitely many) for which $\hat{g}(M) = 1 = \hat{g}(M)$. Since the Fourier transform $\hat{g}(M) = \hat{g}(M)$ must vanish elsewhere, $e_M$ is precisely the sum of these characters.

Consequently $A$ (or rather, the map of $G^\wedge \to \mathfrak{M}_A \cup \{0\}$) provides a subdivision of $G^\wedge$ into certain finite “sets of constancy” $\{F_M\}$, on each of which all the Fourier transforms in $A^\wedge$ are constant (plus a possibly infinite set which we ignore for the moment, the “hull” of $A$, on which all these transforms vanish). Conversely the subdivision $\{F_M\}$ determines $j_A(\infty)$ at least, and thus determines $A$ if $A$ is tauberian. In case $A$ is a closed subalgebra of...
$L_2$, A is tauberian; indeed it is simply a matter of rearranging terms in the (in $L_2$) unconditionally convergent Fourier series expansion of $a \in A$ to write $a = \sum_M \delta(g_M)v_M$, where $g_M$ is any element of $F_M$.(a). In case A is a closed subalgebra of $L_1$ it is not at all clear that A must be tauberian; but we can fall back upon the $L_2$ case at least to assert that A is tauberian if (and obviously only if) $A \cap L_2$ is dense in A. Indeed approximation of $a \in A \cap L_2$ by an element of $j_A(\infty)$ in the $L_2$ norm improves when we pass to the $L_1$ norm so that $A \cap L_2 \subseteq j_A(\infty)^{-}$ and $A = j_A(\infty)^{-}$.

Whether all closed subalgebras of $L_1(G)$ are tauberian seems a rather difficult question. But of course under various assumptions about the “sets of constancy” an algebra must be tauberian; as the simplest example suppose each $F_M$ consists of a single character. Then $j_A(\infty)^{-}$ is clearly an ideal in $L_1(G)$ whose hull is $G^\sim \setminus (UF_M)$, i.e., the “hull” of A. By spectral synthesis for $L_1(G)$ it coincides with the kernel of its hull, which of course contains A. (Thus the closed ideals of $L_1(G)$ are characterized as those closed subalgebras with degenerate (i.e., single element) sets of constancy.) As a second example, A is tauberian if the union of the nondegenerate sets of constancy forms a lacunary subset E of $G^\sim$ in the sense of [6, 9.2] (which includes the classical case when $G^\sim$ is the group $Z$ of integers). For let $(f, h) = \int f(h)dg, f*(g) = f(g^{-1})$. By the Hahn-Banach Theorem it suffices to show $h \in L_m(G)$ with $(\epsilon_M, h) = 0$ for all $M$ satisfies $(a, h) = 0, a \in A$ ; alternatively that the continuous function $a \ast h^*$ vanishes at the identity 1 of G. Since $0 = (\epsilon_M, h) = \sum_{\tilde{g} \in \pi_M} \tilde{h}^*(\tilde{g})$ and $\tilde{a}$ is constant on $F_M$, the formal series

\[(4.01) \sum (a \ast h^*)^-(\tilde{g}) = \sum \tilde{a}(\tilde{g})\tilde{h}^-(\tilde{g})\]

for $a \ast h^*(1)$ can be rearranged and grouped in blocks with each block having sum zero. Thus A will be tauberian if we can guarantee that (4.01), as regrouped, sums to $a \ast h^*(1)$. In our special case the fact that $\tilde{h}$ vanishes on each degenerate $F_M$ shows $(a \ast h^*)^-$ vanishes off the lacunary set E, thus (4.01) is absolutely convergent by [6, 9.2, 8.5], hence converges unconditionally to $a \ast h^*(1) = 0$. (The approach through (4.01) yields many special cases when $G^\sim = Z$.)

Finally we should note that if $UF_M$ forms a lacunary subset of $G^\sim$ in the somewhat different sense of [5] or [6, 8.6], A is tauberian since it is actually a subset of $L_2(G)$.

If G is an arbitrary compact group, any commutative semisimple closed subalgebra A of $L_1(G)$ or $L_2(G)$ satisfies the hypotheses of 3.8, and thus $M_A$ is discrete and A regular. The form of idempotents in $L_1$ or $L_2$ is of course easily obtained, but now even in the $L_2$ case it is not at all clear that distinct idempotents $e_{M_1}, e_{M_2}$, are orthogonal in $L_2$ if G is nonabelian. Consequently

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(a) The argument applies more generally: if $M_B$ is discrete and each $b$ in B can be expressed by an unconditionally convergent series $\sum b_M e_M$, then all closed subalgebras are tauberian.
we shall restrict our attention to subalgebras $A$ consisting of normal elements $a: a * a^* = a^* * a$.

**Theorem 4.1.** Let $G$ be a compact group, $A$ a closed commutative subalgebra of $L_1(G)$ or $L_2(G)$ consisting of normal elements, and suppose $A \cap L_2$ is dense in $A$. Then $A$ is semisimple and self-adjoint, and is the closed span of ($L_2$- and ring theoretically-) orthogonal self-adjoint idempotent elements of $C(G)$ which provide, in a natural fashion, the discrete space $\mathcal{M}_A$.

**Proof.** $A$ is semisimple since the nonzero compact normal operator $f \rightarrow a * f$ on $L_2$ must have a nonzero eigenvalue, which, as is easily seen, must lie in $\sigma_A(a)$. Each idempotent $e_M$ produces a compact normal idempotent operator on $L_2$ in the same way, so $f \rightarrow e_M * f$ is an orthogonal projection onto a finite dimensional subspace of $L_2$. Since this projection must reduce to zero on all but finitely many minimal 2-sided ideals, we easily identify $e_M$ as an element of their span, and thus a continuous function. Of course the self-adjointness of $f \rightarrow e_M * f$ yields $e_M = e_M^*$ (since $f \rightarrow e_M^* * f$ is the adjoint); consequently if $M_1 \neq M_2$, $(e_{M_1}, e_{M_2}) = e_{M_1} * e_{M_2}(1) = e_{M_1} * e_{M_2}(1) = 0$ (since $e_{M_1} * e_{M_2} = 0$), and the $e_M$ are orthogonal.

Since $(a, e_M) = a * e_M^*(1) = a * e_M(1) = a(M)e_M(1)$, and $e_M(1) = e_M * e_M^*(1) = (e_M, e_M) > 0$, any $a$ in $A \cap L_2$ which is orthogonal to $j_A(\infty)$ must be zero by semisimplicity. Thus $A \cap L_2$ lies in the closed $L_2$-span of the $e_M$, and each $a$ in $A \cap L_2$ can be appropriately approximated in $L_2$ norm; since the approximation improves in passing to the $L_1$ norm and $A \cap L_2$ is dense in $A$, even if $A$ is a subalgebra of $L_1$ the desired approximation is available. Finally since our involution is an isometry and maps a dense subset onto itself, $A = A^*$ clearly, completing the proof.

Much the same argument yields

**Theorem 4.2.** Let $H$ be an $H^*$ algebra which has only finite dimensional minimal 2-sided ideals. Let $A$ be a commutative closed subalgebra consisting of normal elements. Then $A$ is semisimple, self-adjoint, and spanned by a set of orthogonal self-adjoint idempotents; properly renormed, $A$ is an $H^*$ algebra.

(The first hypothesis of course guarantees that the operators $L_a$ are compact.)

**References**


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