HOMOGENEOUS EXTENSIONS OF POSITIVE LINEAR OPERATORS

BY

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1. Introduction. A positive linear operator is (roughly speaking) a countably additive, order preserving, \( \sigma \)-finite linear mapping \( \phi \) from one function space, \( F \), to another, \( F' \).\(^1\) (For precise definitions, see §2 below. We assume in particular that \( F \) and \( F' \) satisfy the countable chain condition.) It has been shown in \([4]\) that a normal form representation can be given for \( \phi \): if the function space \( F' \) consists of the "measurable" functions (modulo "null" functions) on a space \( X \), then \( F \) is isomorphic to a subspace of the space \( F^* \) of "measurable" functions on \( X \times Y \), where \( Y \) is an ordinary numerical measure space, and \( \phi \) can be extended to a positive linear operator \( \phi^* \) from \( F^* \) to \( F' \), in such a way that (to within the isomorphism mentioned) \( \phi^* f = g \), where \( g(x) = \int_X f(x, y) \, dy \).

We are concerned here with the case in which \( F' = F \). It is now of importance (for instance in ergodic theory) to consider the iterates of \( \phi \); and the normal form representation just mentioned has now the drawback that the isomorphism imbedding \( F \) in \( F^* \) interferes with the description of these iterates. The present paper takes a first step towards obtaining a more satisfactory representation of \( \phi \) and its iterates.

Given a positive linear operator \( \phi \) from \( F \) to \( F \), we shall show (Theorem 1, 4.1) that the function space \( F \) can be imbedded in a larger space \( F^* \), and the operator \( \phi \) extended to a positive linear operator \( \phi^* \) from \( F^* \) to \( F^* \), in such a way that the extended operator has the following property, which we call "full homogeneity": For each characteristic function \( \chi \in F^* \), and each function \( g \in F^* \) such that \( 0 \leq g \leq \phi^* \chi \), there exists a characteristic function \( \chi' \in F^* \) such that \( \chi' \leq \chi \) and \( \phi^* \chi' = g \).\(^2\) It follows that the iterates of the extended operator \( \phi^* \) will also be fully homogeneous, and therefore \( \sigma \)-finite. (Even in simple cases, the iterates of \( \phi \) itself may fail to be \( \sigma \)-finite\(^3\).) A routine application of the results of \([4]\) would then lead easily to a simultaneous representation theorem for \( \phi \) and its iterates; however, a sharper theorem can be

\(^1\) Or from one conditionally complete vector \( \sigma \)-lattice, satisfying the countable chain condition, to another; cf. \([4, p. 156]\).

\(^2\) To improve the legibility of formulae, we often omit brackets, writing (as here) \( \phi^* \chi \) for \( \phi^*(\chi) \), and later \( Rx \) for \( F(x) \), etc.

\(^3\) For example, let \( F \) be the space of measurable functions \( f(x, y) \) modulo null functions on the plane (with ordinary measure), and let \( \phi f = g \) where \( g(x, y) = \int_x f(x, t) \, dt \) (independent of \( y \)). Then \( \phi \) is an \( L^1 \)-integral on \( F \) (and in particular is \( \sigma \)-finite), but if \( f \in F^* \) and \( \phi f \) is finite, then \( f = 0 \), whence \( \phi^* \) is not \( \sigma \)-finite.
deduced with more trouble, so we leave this application for a later paper. Meanwhile we show (Theorem 3, §7) that when $F$ arises from a genuine numerical measure (that is, $F$ is the space of measurable functions modulo null functions on a measure space) then the extended space $F^*$ in Theorem 1 can also be taken to arise from a numerical measure. The deduction of Theorem 3 depends on a property of measure algebras (Theorem 2, §6) which may be of independent interest: Given a $\sigma$-subalgebra $A$ of a measure algebra $(E, \mu)$, and given a $\sigma$-finite measure $\nu$ on $A$ which is equivalent to $\mu$ on $A$ (that is, $\nu$ vanishes only for the zero element of $A$), there exists a $\sigma$-finite measure $\nu^*$ on $E$ which extends $\nu$ on $A$ and is equivalent to $\mu$ on $E$.

The technique employed for the proofs makes considerable use of representation spaces and of continuous functions on them; thus, after giving the notation and some preliminary results in §2, we collect some results on the representation spaces of an algebra and a complete subalgebra of it in §3. Theorem 1 and its proof occupy §§4 and 5, Theorem 2 is dealt with in §6, and Theorem 3 in §7. The background material and a few specific results are quoted without proof from [2–5]; apart from this the present paper is largely self-contained.

2. Notation and preliminaries.

2.1 Algebras and representation spaces. In general we use the same notations as in [2; 4], an acquaintance with which is assumed. The term "algebra" always means "Boolean algebra"; if $E$ is an algebra, the symbols $o$ and $e$ denote the zero and unit elements of $E$ respectively, and $-x$ denotes the complement of $x \in E$; and the symmetric difference of $x, y \in E$ (written $x+y$ in [2; 4]) is here written as $x+y$.

The representation space of an algebra $E$ is the space $R$ of ultrafilters on $E$; to each $x \in E$ corresponds the set $Rx \subseteq R$ consisting of those ultrafilters which contain $x$ (thus $Ro=\phi, Re=R$), and $T$ is topologised by taking the sets $Rx$ as a basis; $R$ is compact Hausdorff, and each $Rx$ is both open and closed. The correspondence $x \leftrightarrow Rx$ is a finite isomorphism between $E$ and the algebra of all open-closed subsets of $R$. We write $B(R)$ for the family of Borel subsets of $R$, $B_0 R$ for the family of the family of "restricted Borel sets" (the Borel field generated by the open-closed sets of $R$), and $\kappa R$ for the family of first category subsets of $R$. We have:

(1) If $X \subseteq B_0 R$, $X = G + H$ where $G$ is open and $H \subseteq \kappa R$.

(Here again $+$ denotes symmetric difference.) As the method of proof of (1) ("Borel induction") will be used frequently in what follows, we sketch it: The family of all sets of the form $G + H$, where $G$ is open and $H \subseteq \kappa R$, is closed under complementation and under countable unions; hence it is a Borel field containing all open sets, and so it contains $B_0 R$.

Similarly, if $E$ is a $\sigma$-algebra, we have

(2) If $X \subseteq B_0 R$, $X = Rx + H$ where $x \in E$ and $H \subseteq \kappa R$.

It follows that the $\sigma$-algebra $E$ is isomorphic to $B_0 R / \kappa R$; if further $E$ is a

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complete algebra (as it will be if it satisfies the countable chain condition) then this $\sigma$-isomorphism is necessarily complete. If $E$ is complete, (2) applies to all sets in $\mathcal{R}^*\mathcal{R}$, so that $E$ is (completely) isomorphic to $\mathcal{R}^*E/\mathcal{K}R$.

2.2 Subalgebras. Let $A$ be a subalgebra of an arbitrary algebra $E$, and let $S$, $R$ be their respective representation spaces. There is a natural mapping $\xi$ of $R$ onto $S$, obtained as follows: each point of $R$ is an ultrafilter $\mathcal{U}$ on $E$, and its trace $\mathcal{U} \cap A$ on $A$ is an ultrafilter on $A$; we take $\xi(\mathcal{U}) = \mathcal{U} \cap A$. It is easily verified that

(1) $\xi^{-1}(S_y) = R_y$, 

so that $\xi$ is continuous. If further $B$ is a subalgebra of $A$, with $T$ as its representation space, and $\xi'$, $\xi''$ are the corresponding mappings from $R$ to $T$, $S$ to $T$, then clearly

(2) $\xi' = \xi'' \xi$.

Let $\mathcal{B}_a$ be the Borel field (of subsets of $R$) generated by the sets $R_y$, $y \in B$; a “Borel induction” argument shows that

(3) $\mathcal{B}_a \supset \mathcal{B}_R = \xi^{-1}\mathcal{B}S$.

2.3 Complete subalgebras. A subalgebra $A$ of an arbitrary algebra $E$ will be called a complete subalgebra of $E$ if, for every $H \subset A$, the supremum $\bigvee H = \bigvee \{h | h \in H\}$ of $H$ in $E$ exists and belongs to $A$; thus $A$ (but not necessarily $E$) is then itself a complete algebra. An isomorphism $\theta$ of an algebra $B$ onto a subalgebra $A$ of $E$ is “complete with respect to $E$” if $A$ is a complete subalgebra of $E$; this implies that $B$ is itself a complete algebra and that, for every $H \subset B$, $\theta(\bigvee H) = \bigvee (\theta H)$ (the supremum in $A$ or $E$).

Now suppose $A$ is a complete subalgebra of an arbitrary algebra $E$, and let $S$, $R$, be their respective representation spaces, and $\xi$ the mapping of $R$ onto $S$ introduced in 2.2. For each $x \in E$, write $x^* = \bigwedge \{y | y \in A, y \geq x\}$ (the infimum referring to $E$, but $x^* \in A$). It is now easily verified that

(1) $\xi Rx = Sx^*$,

so that $\xi$ is now open as well as continuous (and so closed, as $R$, $S$ are compact Hausdorff). It follows that

(2) $\xi^{-1}\mathcal{K}S \subset \mathcal{K}R$.

We deduce:

(3) If $H \in \mathcal{B}S$ and $\xi^{-1}H \in \mathcal{K}R$, then $H \in \mathcal{K}S$.

For, by 2.1(2), $H = G + Z$ where $G$ is open and $Z \in \mathcal{K}S$. Hence $\xi^{-1}H = \xi^{-1}G + \xi^{-1}Z$, showing that the open set $\xi^{-1}G$ is (from (2)) of first category in the compact space $R$; hence $\xi^{-1}G = \emptyset$, giving $H = Z$ as required.

These results, together with 2.2(3), give

(4) $\mathcal{K}R \cap \mathcal{B}_a = \xi^{-1}(\mathcal{K}S \cap \mathcal{B}S)$.
2.4 Function spaces. Let \( S \) be any set, \( \mathcal{B} \) any Borel field of subsets of \( S \), and \( \mathcal{I} \) any \( \sigma \)-ideal of subsets of \( S \). By a “function” \( f \) on \( S \) we mean an extended-real function (so that, for each \( s \in S \), \( f(s) \) is real or \( +\infty \) or \( -\infty \)). We make the convention that \( 0 \cdot -\infty = 0 \), but \( -\infty - -\infty \) is not defined. \( \mathcal{F}(\mathcal{B}) \) denotes the collection of “\( \mathcal{B} \)-measurable” functions on \( S \), that is, of functions \( f \) such that, for each real \( \rho \), \( \{ s \mid s \in S, f(s) > \rho \} \in \mathcal{B} \). The set of non-negative \( \mathcal{B} \)-measurable functions is denoted by \( \mathcal{F}^+(\mathcal{B}) \); generally, if \( \mathcal{A} \) is any set of functions, \( \mathcal{A}^+ \) denotes the set of non-negative functions in \( \mathcal{A} \). The support \( [f] \) (called “locus” in [4; 5]) of a function \( f \) on \( S \) is the set \( \{ s \mid f(s) \neq 0 \} \). A function \( f \) is “null” or \( \mathcal{I} \)-negligible if \( [f] \subseteq \mathcal{I} \); the set of \( \mathcal{I} \)-negligible functions is denoted by \( \mathcal{I}(\mathcal{I}) \). The equivalence class modulo \( \mathcal{I}(\mathcal{I}) \) of a function \( f \) is written \( f \equiv \mathcal{I} \) (and not \( \{ f \} \), as in [2; 4]). We write \( \tilde{f} \ll \tilde{g} \) to mean \( f(s) < g(s) \) for all \( s \in S \setminus N \) where \( N \in \mathcal{I} \), and \( \tilde{f} \leq \tilde{g} \) to mean \( f(s) \leq g(s), s \in S \setminus N \); \( \tilde{f} < \tilde{g} \) means \( \tilde{f} \leq \tilde{g} \) and \( \tilde{f} \neq \tilde{g} \). The set \( \{ [f] \mid f \in \mathcal{F}(\mathcal{B}) \} \) is written \( \mathcal{F}(\mathcal{B})/\mathcal{I}(\mathcal{I}) \) or \( \mathcal{F}(S, \mathcal{B}, \mathcal{I}) \).

If \( E \) is the \( \sigma \)-algebra \( \mathcal{B}/\mathcal{I} \), \( E \) determines \( \mathcal{F}(S, \mathcal{B}, \mathcal{I}) \) to within “strict” isomorphism (a 1-1 correspondence preserving \( \leq \) and (pointwise) sums and products) (see [4, p. 159]), and we write the strict isomorphism class of \( \mathcal{F}(S, \mathcal{B}, \mathcal{I}) \) as \( \mathcal{F}(E) \); we also use \( \mathcal{F}(E) \) to stand for any one member of this class. \( \mathcal{F}^+(E) \) of course denotes the subset of \( \mathcal{F}(E) \) corresponding to the non-negative functions. We say that \( \mathcal{F}(S, \mathcal{B}, \mathcal{I}) \) is a realisation of \( \mathcal{F}(E) \).

For every \( \sigma \)-algebra \( E \), the function space \( \mathcal{F}(E) \) exists and has the representation space realisation \( \mathcal{F}(R, \mathcal{B}, \mathcal{I}) \) where \( R \) = representation space of \( E \), \( \mathcal{B} = \mathcal{B} \) (or \( \mathcal{B} \) if \( E \) is complete), \( \mathcal{I} = \mathcal{I}R \). If \( E \) is complete, each \( f \in \mathcal{F}(E) \) now has a unique continuous representative \( f \in \mathcal{F}(R, \mathcal{B}, \mathcal{I}) \), as follows from [2, p. 285].

The support \( [f] \) of \( f \) in \( \mathcal{F}(E) \) is the element \( [f] + \mathcal{I} \) of \( E \). Dually, if \( x \in E \), its characteristic function \( \chi_x \) or \( \chi x \) is the element \( \chi(X) \) of \( \mathcal{F}(E) \), where \( \chi(X) \) is the characteristic function of any set \( X \in \mathcal{B} \) for which the equivalence class \( X + \mathcal{I} = x \). The value of \( \chi(X) \) at \( s \in S \) is denoted by \( \chi(X, s) \).

2.5 Cylinder mappings. Let \( A \) be a \( \sigma \)-subalgebra of a \( \sigma \)-algebra \( E \). There is then a natural strict isomorphism \( c \) of \( \mathcal{F}(A) \) in \( \mathcal{F}(E) \), which we call the “cylinder mapping” (by analogy with the case in which \( E \) is the product of \( A \) with another factor); it can be described as follows. Each \( \tilde{g} \in \mathcal{F}(A) \) determines a “spectrum” (cf. [6; 4, p. 159]) on \( A \); in terms of any realisation \( \mathcal{F}(S, \mathcal{B}, \mathcal{I}) \) of \( \mathcal{F}(A) \), the spectrum consists of the equivalence classes modulo \( \mathcal{I} \) of the sets \( \{ s \mid g(s) < \rho \} \) where \( \rho \) is rational. Conversely, each spectrum on \( A \) determines a unique \( \tilde{g} \in \mathcal{F}(A) \). The imbedding of \( A \) in \( E \) turns the spectrum of \( \tilde{g} \) into the spectrum of a unique \( \tilde{f} \in \mathcal{F}(E) \), and we take \( c(\tilde{g}) = \tilde{f} \). In particular, if \( \mathcal{F}(S', \mathcal{B}', \mathcal{I}') \) is any realisation of \( \mathcal{F}(E) \), the sets of \( \mathcal{B}' \) which correspond to elements in \( A \) form a Borel field \( \mathcal{B}'' \subseteq \mathcal{B}' \) such that \( \mathcal{F}(S', \mathcal{B}'', \mathcal{I}') \) is a realisation of \( \mathcal{F}(A) \); the cylinder mapping of \( \mathcal{F}(A) \) in \( \mathcal{F}(E) \) is now that induced by the identity mapping on \( S' \).

(1) It is an “extended vector \( \sigma \)-lattice with a unit,” in the sense that the classes of the finite functions form a vector \( \sigma \)-lattice with a unit, and conversely every vector \( \sigma \)-lattice with a unit arises in this way.
We shall later require the form taken by the cylinder mapping in terms of the representation space realisations. Let $R, S$ be the representation spaces of the $\sigma$-algebras $E, A$ where $A$ is a complete subalgebra of $E$; let $\xi$ be the corresponding mapping of $R$ onto $S$ (cf. 2.2), and let $\mathcal{C}, \mathcal{D}$ be the families of continuous functions on $R, S$. As $\xi$ is continuous, $\xi$ induces a mapping $\xi^*$ of $\mathcal{D}$ in $\mathcal{C}$ by the rule $(\xi^*g)p = g(\xi p)$, $p \in R$, $g \in \mathcal{D}$. Each $\bar{g} \in F(A)$ is the equivalence class of a unique $g \in \mathcal{D}$ (cf. end of 2.4), and we have

$$c\bar{g} = \xi^*g,$$

as follows from 2.2(1) and 2.3(2) applied to the spectrum of $\bar{g}$.

2.6 $F'$-integrals. Let $E, E'$ be $\sigma$-algebras satisfying the countable chain condition, and write $F = F(E), F' = F(E')$. A mapping $\phi$ of a subset $G$ of $F$ in $F'$ is called a positive linear operator from $F$ to $F'$, or an $F'$-integral on $F$ (cf. [4, p. 161; 5, p. 232]) if $G \supset F^+$ and:

1. If $f \in F^+$, $\phi f = 0$.
2. If $f^+ \in F^+$ (n = 1, 2, \ldots), then $\phi f = \sum \phi f^+$.
3. There exist $\bar{g}_1, \bar{g}_2, \ldots \in F^+$ such that $\sum g_n \gg 0$ and $\phi g_n \ll \infty$.
4. $G = \{ f \mid \phi f \wedge f^- \ll \infty \}$, and if $\bar{f} \in G$ then $\phi f = \phi(f^+) - \phi f^-$. 

As $\phi$ is determined by its values on $F^+$, we shall usually regard $\phi$ as a mapping of $F^+$ in $F'^+$ satisfying (2) and (3); for every such mapping can be extended to a suitable $G$ [4, pp. 161, 162]. The extended mapping $\phi$ is necessarily linear on $G$.

If further $\phi$ satisfies
(i) $\phi f > 0$, if $f > 0$,
(ii) $\phi f \gg 0$,
$\phi$ is called strict. (In [4; 5], a strict $F'$-integral on $F$ was called simply an $F'$-integral; the present $F'$-integral was called "relaxed.") Every $F'$-integral $\phi$ on $F$ determines in a natural way a strict $F_1'$-integral $\phi_1$ on $F_1$, where $F_1 = F(E_1)$, $F_1' = F(E_1')$, and $E_1, E_1'$ are suitable principal ideals of $E, E'$ (see [5, p. 238]); $\phi_1$ is called the "strict form" of $\phi$.

For any $F'$-integral $\phi$ on $F$, we write $\lambda x = \phi(x)(x \in E)$; $\lambda$ is the induced "$F'$-measure" on $E$; it is countably additive and $\sigma$-finite [5, p. 233 (a) and (b)] and determines $\phi$ uniquely.

An $F'$-integral $\phi$ on $F$ is fully homogeneous if the corresponding $F$-measure $\lambda$ is "full-valued" in the sense of [4, p. 166]; that is, given $x \in E$ and $g \in F'^+$ such that $\bar{g} \leq \lambda x$, there exists $y \leq x$ such that $\lambda y = \bar{g}$. A fully homogeneous $\phi$ is itself full-valued [4, p. 174]; that is, given $\bar{f} \in F^+$ and $\bar{g} \in F'^+$ such that $\bar{g} \leq \phi \bar{f}$, there exists $h \in F^+$ such that $\bar{h} \leq \bar{f}$ and $\phi h = \bar{g}$.

Let $E, E', E''$ be $\sigma$-algebras, and $F, F', F''$ their function spaces (that is, $F' = F(E'),$ etc.). Suppose $\phi$ is an $F'$-integral on $F$, and $\psi$ an $F''$-integral on $F'$. In general, $\psi \phi$ need not be an $F''$-integral on $F$, as the $\sigma$-finiteness requirement (3) may fail. But:

5. If $\phi$ and $\psi$ are fully homogeneous, $\psi \phi$ is a fully homogeneous $F''$-
integral on \( F \), provided that \( E \) satisfies the countable chain condition.

There is no difficulty in seeing that \( \psi \phi \) satisfies conditions (1) and (2). Suppose \( 0 \leq g \leq \psi \phi x \) where \( x \in E, g \in F' \). Put \( h = \lambda x = \phi x \); then \( 0 \leq g \leq \psi h \), so (as \( \psi \) is full-valued) there exists \( k \in F^+ \) such that \( k \leq h \) and \( \psi k = g \). Hence there exists \( y \leq x \) in \( E \) such that \( \lambda y = k \); and \( \psi \phi y = g \), proving that \( \psi \phi \) is fully homogeneous. Put \( \psi \phi = \theta \); condition (3) now follows in this way. Put \( h_1 = 1 \setminus \theta 1 \); there exists \( x_1 \in E \) such that \( \theta x_1 = h_1 \). If \( \alpha \) is any countable ordinal, and disjoint elements \( x_\alpha \in E \) have been defined for all \( \beta < \alpha \) so that \( \theta x_\alpha \leq 1 \), we put \( h_\alpha = 1 \setminus \theta x_\alpha \leq 1 \). If \( h_\alpha \neq 0 \), we take \( x_\alpha \leq V x_\beta \) so that \( \theta x_\alpha = h_\alpha \leq 1 \); if \( h_\alpha = 0 \), we put \( x_\alpha = - V x_\beta \) and terminate the process. Because of the countable chain condition, this process terminates for some countable \( \alpha \). Renumbering the elements \( x_\alpha (\beta < \alpha) \) into a simple sequence \( x_1, \ldots, x_\alpha, \ldots \), we put \( \theta g_n = x_\alpha \) and have \( \sum \theta g_n = 1 \gg 0 \) and \( \theta g_n \leq 1 \ll \infty \) as required.

For any \( F' \)-integral \( \phi \) on \( F \) we have:

(6) If \( f, g \in F' \) and \( [f] = [g] \), then \( [\phi f] = [\phi g] \).

For \( \phi f = \phi g \); hence \( [\phi f] = [\phi g] \).

Now let \( E_i, E'_i \) be \( \sigma \)-subalgebras of \( \sigma \)-algebras \( E_i, E'_i \) satisfying the countable chain condition; write \( F_i, F'_i \) for \( F(E_i), F(E'_i) \) respectively \((i = 1, 2)\), and let \( c, c' \) be the respective cylinder mappings of \( F_i \) in \( F_2, F'_i \) in \( F'_2 \). If \( \phi_i (i = 1, 2) \) is an \( F'_i \)-integral on \( F_i \) we say that \( \phi_2 \) is a cylinder extension of \( \phi_1 \) if, for each \( f \in F_i \), \( c'(\phi_i f) = \phi_2(c f) \). As remarked in 2.5, we can choose realisations of \( F_i \) and \( F'_i \) for which \( c \) and \( c' \) are induced by identity mappings, and then \( \phi_2 \) is a cylinder extension of \( \phi_1 \) if and only if \( \phi_1 \) is (in an obvious sense) the restriction of \( \phi_2 \) to \( F_i \).

The following result is basic for the construction in the present paper. It is proved (in a slightly different formulation) in [4, Theorem 6] for the case in which \( \phi \) is strict; the general result follows easily on considering the "strict form" of \( \phi \). (For details see the beginning of §7.)

(7) If \( \phi \) is an \( F' \)-integral on \( F \), where \( F = F(E) \) and \( F' = F(E') \), there exists an algebra \( E^* \), of which \( E \) is a complete subalgebra, and an \( F' \)-integral \( \phi^* \) on \( F(E) \), such that \( \phi^* \) is a fully homogeneous cylinder extension of \( \phi \).

Except where the contrary is stated, all algebras in what follows are assumed to be \( \sigma \)-algebras satisfying the countable chain condition. Further, the term "subalgebra" is understood to mean a \( \sigma \)-subalgebra, and hence a complete subalgebra.

3. Functions on representation spaces.

3.1 Let \( A \) be a (complete) subalgebra of an arbitrary algebra \( E \); \( A \) is of course assumed to be a \( \sigma \)-algebra satisfying the countable chain condition, but \( E \) is not. We derive for later use some properties of the representation spaces \( R, S \) of \( E, A \), and of various families of functions on them. We write \( \mathfrak{N}_0 = \mathfrak{B}_0 \setminus \mathfrak{K} R \), where \( \mathfrak{B}_0 \) is the Borel field generated by the sets \( R y, y \in A \), and \( \mathfrak{N}_0 \) for the family of all subsets \( N \) of \( R \) which are subsets of sets in \( \mathfrak{N}_0 \). From 2.2 and 2.3(4), \( \mathfrak{S}(R, \mathfrak{B}_0, \mathfrak{N}_0) \) and \( \mathfrak{S}(R, \mathfrak{B}_0, \mathfrak{K} R) \) are realisations of \( F(A) \); if \( E \)
is a \( \sigma \)-algebra, the corresponding cylinder mappings of \( F(A) \) in \( F(E) \) are induced by the identity mapping. As in 2.2 we write \( \xi \) for the natural mapping of \( R \) onto \( S \), and \( \mathcal{C}, \mathcal{D} \) for the families of continuous functions on \( R, S \).

3.2 Lemma. If \( H \in \mathcal{K}S \), there exist \( x_{mn} \in A \) (\( m = 1, 2, \ldots \)) such that

\[
x_{m1} \geq x_{m2} \geq \cdots \land A \land x_{mn} = 0 \text{ for each } m, \text{ and } H \subset \bigcup_m \cap_n S x_{mn}.
\]

For let \( F \) be closed and nowhere dense in \( S \); the open set \( S - F \) can be written as \( \cup S y_a \) for suitable \( y_a \in A \), and (because \( F \) is nowhere dense) \( \forall y_a = e \).

There is therefore a sequence of values of \( \lambda \), which we denote by 1, 2, \ldots, \( i, \ldots \), such that \( \forall y_i = e \). Put \( x_n = - (y_1 \lor y_2 \lor \cdots \lor y_n) \); then \( x_1 \geq x_2 \geq \cdots \land A \land x_n = 0 \), and \( S x_n = S - (S y_1 \cup \cdots \cup S y_n) \supset F \). Now we have \( H \subset \bigcup F_m \quad (m = 1, 2, \ldots) \) where \( F_m \) is closed and nowhere dense; applying the foregoing to \( F_m \) instead of \( F \), we obtain the elements \( x_{mn} \) required.

Corollary. If \( H \in \mathcal{K}S \), there exists \( H^* \in \mathcal{K}S \cap \mathcal{O}S \) such that \( H \subset H^* \).

3.3 Definition. A function \( f \) on \( \mathcal{A} \) is \( 0 \)-continuous if, for each real (or, equivalently, rational) \( \rho, \{ p \in R, f(p) > \rho \} \) is a union of sets of the form \( R y \) where \( y \in A \). (This implies that \( f \) is continuous.) The set of \( 0 \)-continuous functions on \( \mathcal{A} \) is denoted by \( \mathcal{C}_0 \). We have

(1) \( f \in \mathcal{S}(\mathcal{O}0) \) if and only if there exists \( h \in \mathcal{S}(\mathcal{O}(S)) \) such that \( f = h \xi \).

The nontrivial implication here can be seen by considering the spectrum of \( f \), or by observing that (by the argument in \cite[4, p. 157]{4}) each \( f \in \mathcal{S}(\mathcal{O}0) \) is expressible as \( \sum_{\alpha} \chi X_n \), where \( X_n \in \mathcal{A} \) and \( \alpha_n \) is real. We have \( X_n = \xi^{-1} Y_n \) where \( Y_n \in \mathcal{O}S \), and then, if we set \( h = \sum_{\alpha} \chi Y_n \), we have \( f = h \xi \).

We deduce:

(2) If \( f \in \mathcal{S}(\mathcal{O}0) \), there exists \( h \in \mathcal{A}_0 \) such that \( f = h \mod \mathcal{A}_0 \).

By (1), \( f = g \xi \) where \( g \in \mathcal{S}(\mathcal{O}(S)) \). There exists a continuous function \( g \) on \( S \) such that \( g(s) = g_i(s) \) for \( s \in S - H \), where \( H \in \mathcal{K}S \); and by 3.2, Corollary, we may assume \( H \in \mathcal{O}S \) also. Put \( h = g \xi \). Using 2.2(1) we see that \( h \) is \( 0 \)-continuous; and \( f(p) = h(p) \) for \( p \in R - N \) where \( N = \xi^{-1} H \in \mathcal{A}_0 \) by 2.3(4).

Conversely:

(3) If \( f \in \mathcal{A}_0 \), there exists \( g \in \mathcal{S}(\mathcal{O}0) \) such that \( f = g \mod \mathcal{A}_0 \).

For write \( X_n = \{ p \in R, f(p) < \rho \} \); by hypothesis, this is of the form \( \cup S y_a \) for suitable elements \( y_a \in A \). Let \( z_a = V y_a \); then \( R z_a = X_n \cup H_a \), where \( H_a = \xi^{-1}(S z_a - \cup S y_a) \). As \( S z_a - \cup S y_a \) is closed and nowhere dense in \( S \), it is contained in a set \( K_a \in \mathcal{K}S \cap \mathcal{O}S \) (3.2, Corollary); hence \( H_a \subset \xi^{-1} K_a \in \mathcal{A}_0 \). Put \( N = \bigcup \{ \xi^{-1} K_a \mid \rho \text{ rational} \}, \quad g = f \chi(R - N) \). Then \( N \in \mathcal{A}_0 \), and \( f(p) = g(p) \) for \( p \in R - N \). Let \( Y_p = \{ p \in R, g(p) < \rho \} \); one verifies that, if \( \rho \) is rational, \( Y_p = R z_a \cap (R - N) \) for \( \rho \leq 0 \), \( Y_p = R z_a \cup N \) for \( \rho > 0 \). Hence \( Y_p \in \mathcal{O}0 \) for all rational \( \rho \), and hence for all \( \rho \), proving \( g \in \mathcal{S}(\mathcal{O}0) \).

Next we deduce:

(4) \( f \in \mathcal{A}_0 \) if and only if \( f = h \xi \) for some \( h \in \mathcal{O} \).

The "if" is trivial from 2.2(1). Conversely, given \( f \in \mathcal{A}_0 \), apply (3) and (1).
to obtain \( g \in \mathcal{F}(\mathcal{B}(S)) \) such that \( f = g \xi \) modulo \( \mathcal{M}_0 \). There exists a continuous \( h \) on \( S \) such that \( h(s) = g(s) \) for \( s \in S - H \), where \( H \in \mathcal{K}S \). Thus \( f(p) = h \xi(p) \) for all \( p \in R - N \) where \( N \in \mathcal{K}R \). As \( R - N \) is dense in \( R \), and \( f \), \( h \xi \) are both continuous, it follows that \( f(p) = h \xi(p) \) for all \( p \in R \).

3.4 Let \( \mathcal{M}_0 \) denote \( \mathcal{B}_0 + \mathcal{M}_0 \) (that is, the family of all sets of the form \( B + N \) where \( B \in \mathcal{B}_0 \), \( N \in \mathcal{M}_0 \)). Equivalently (from 2.1(2), 2.2(3), 2.3(4)) \( \mathcal{M}_0 \) consists of all sets of the form \( R_y + N \) where \( y \in A \), \( N \in \mathcal{M}_0 \). Clearly \( \mathcal{M}_0 \) is a Borel field. We say that a function \( f \) on \( R \) is 0-measurable if all the sets \( \{ p \mid f(p) < \rho \} \) are in \( \mathcal{M}_0 \). If \( f \in \mathcal{Z}(\mathcal{M}_0) \) (that is, if \( [f] \in \mathcal{M}_0 \)) we say that \( f \) is 0-negligible. Every 0-continuous function is 0-measurable (cf. end of 2.1). Conversely,

(1) Given an 0-measurable function \( f \), there exists a unique 0-continuous function \( g \) such that \( f = g \) mod \( \mathcal{M}_0 \).

By [4, p. 157] we have \( f = \sum \alpha_n \chi_{X_n} \) for suitable real numbers \( \alpha_n \) and sets \( X_n \in \mathcal{M}_0 \) \((n = 1, 2, \ldots)\); and each \( X_n \) has the form \( R_{x_n} + H_n \) where \( H_n \in \mathcal{M}_0 \) and \( x_n \in A \). Consider the function \( h = \sum \alpha_n \chi_{X_n} \) on \( S \); being \( \mathcal{B}_S \)-measurable, it differs from a continuous function \( k \) on \( S \) on a first category set \( K \). Then \( g = k \xi \) is 0-continuous, and we have \( f(p) = k \xi(p) \) if \( p \in R - \{ x^{-1}K \cup \cup H_n \} \). The uniqueness of \( g \) is trivial.

Given \( g \in F(A) \), the class \( g \), in the representation space realisation of \( F(A) \), contains a unique continuous function \( g_0 \) [2, p. 287]. Then \( g_0 \xi \) is 0-continuous on \( R \); further, from 3.3(4), every 0-continuous function arises in this way. We write \( g_0 \xi = R_0 g \); \( R_0 \) induces a strict isomorphism (a 1-1 correspondence preserving \( \leq \) and finite sums and products) between \( F(A) \) and \( \mathcal{C}_0 \). It follows that every subset \( \{ f_a \} \) of \( \mathcal{C}_0 \) has a least upper bound \( f = \bigvee f_a \) in \( \mathcal{C}_0 \), and there is a countable subfamily \( \{ f_{a_n} \} \) of \( \{ f_a \} \) such that \( f = \bigvee f_{a_n} \). Moreover we have

(2) If \( f = \bigvee f_a \) in \( \mathcal{C}_0 \), \( f(p) = \sup f_a(p) \) for all \( p \in R - N \), where \( N \in \mathcal{M}_0 \).

For we have \( f_a = g_a \xi \) where \( g_a \) is continuous on \( S \), and if \( g = \bigvee g_a \) in \( F(A) \) then \( g = \bigvee g_a \). If \( g_0 \) is the continuous function on \( S \) which is in \( g \), then \( g_0(s) = \sup g_a(s) \) if \( s \in S - H \) where \( H \in \mathcal{K}S \), and we can assume \( (3.2) \) \( H \in \mathcal{B}_S \) also. Then \( f = g_0 \xi \), and \( f(p) = \sup f_a(p) \) if \( p \in R - \xi^{-1}H \) where \( \xi^{-1}H \in \mathcal{M}_0 \) by 2.3(4).

Note that, from 2.2(1),

(3) \[ R_0 \chi y = \chi_{R y}, \quad \text{if } y \in A. \]

If \( E \) is itself a \( \sigma \)-algebra, the cylinder mapping \( c \) of \( F(A) \) in \( F(E) \) is defined, and from 2.5(1) we have, for \( g \in F(E) \),

(4) \[ R_0 g = \text{the unique 0-continuous function in the class } c g. \text{(6)} \]

For later use, we deduce:

(6) Even when \( E \) is only finitely additive, it would be possible to define a "cylinder mapping" from \( F(A) \) to the finitely additive function space corresponding to \( F(E) \), whenever \( A \) is a \( \sigma \)-subalgebra of \( E \). This mapping would then be realised by \( R_0 \).
If $f \in \mathcal{E}^+$, there exist positive real numbers $\sigma_n$, elements $x_n \in A$ ($n = 1, 2, \ldots$), and a non-negative function $h$ such that

$$f = \sum \sigma_n \chi x_n + h.$$

We have $f = R_0 \bar{g}$ where $\bar{g} \in F(A)^+$, and a slight modification of the argument in [4, p. 157, Lemma 1] gives $\bar{g} = \sum \sigma_n \chi x_n$ where $x_n \in A$ and $\sigma_n > 0$. Hence, by (3), $f = \sum \sigma_n \chi x_n$ on $R - N$, where $N \in \mathcal{G}_0$. By continuity, $f \geq$ every finite sum of terms $\sigma_n \chi x_n$, so the difference $h$ between $f$ and $\sum \sigma_n \chi x_n$ is $\geq 0$.

4. The main theorem and its proof (first part).

4.1 We now state the main theorem of this paper. We recall that “algebra” means “Boolean $\sigma$-algebra satisfying the countable chain condition” except where the contrary is stated.

**Theorem 1.** Let $E_0$ be an algebra, $F_0 = F(E_0)$ its function space, and $\phi_0$ a positive linear operator from $E_0$ to itself (that is, an $F_0$-integral on $F_0$). There exist an algebra $E$, of which $E_0$ is a subalgebra, and a fully homogeneous $F$-integral $\phi$ on $F$, where $F = F(E)$, such that $\phi$ is a cylinder extension of $\phi_0$.

**Remark.** If $\phi_0$ is a strict $F_0$-integral on $E_0$, I do not know whether $\phi$ can always be taken to be a strict $F$-integral on $F$. If $\phi_0(1) \gg 0$, then automatically $\phi(1) \gg 0$ because $\phi$ is a cylinder extension.

Before proving the theorem, we note the following consequence of it.

**Corollary.** For each $n = 1, 2, \ldots$, $\phi^n$ is also a fully homogeneous $F$-integral on $F$, and is a cylinder extension of $\phi^n$. This follows from Theorem 1 by an easy induction, using 2.6(5).

4.2 The algebras $E_n$. The proof of Theorem 1 requires a number of steps. First we note that by successive applications of 2.6(7) we obtain a sequence of algebras $E_0, E_1, E_2, \ldots$, where $E_k$ is a subalgebra of $E_{k+1}$, and a sequence $\{\phi_k\}$ ($k = 0, 1, 2, \ldots$) where $\phi_k$ is an $E_k$-integral on $F_k$, $F_k$ denoting $F(E_k)$, such that

1. $\phi_{k+1}$ is a cylinder extension of $\phi_k$; that is, $\phi_{k+1} c_{k,k+1} = c_{k,k+1} \phi_k$, where $c_{k,k+1}$ is the cylinder mapping of $F_k$ in $F_{k+1}$,
2. $\phi_{k+1}(F_{k+1}^+) \subset c_{k,k+1}(F_k^+)$, and further
3. $(c_{k,k+1})^{-1} \phi_{k+1}$ is a fully homogeneous $F_k$-integral on $F_{k+1}$.

(We merely put $\phi_{k+1} = c_{k,k+1} \phi^*$ where $\phi^*$ is the extension provided by 2.6(7).)

We write $c_{n,n+k}$ for the cylinder mapping of $F_n$ into $F_{n+k}$ ($n, k \geq 0$), noting that $c_{n,n+k}$ is 1-1, that $c_{nn}$ is the identity mapping, and that $c_{n,n+k} c_{n+k,n} = c_{n+k,n+k} c_{n,n+k}$. To simplify printing, we write the inverse mapping $(c_{n,n+k})^{-1}$ as $c_{n+k,n}$. By induction, first over $k$ for $m = 1$ and then over $m$, we obtain

$$\phi_{n+k} = c_{n,n+k} \phi_n$$ on $F_n^+$ (m = 1, 2, \ldots).

Restated in terms of the inverse cylinder mappings, this is
In fact, if $r \geq 1$, $c_{n+r,n+r-1}\phi_{n+r}$ is a fully homogeneous $F_{n+r-1}$-integral on $F_{n+r}$, by (3). Now, on $F_{n+k}$, put

$$
\psi = (c_{n+1,n}\phi_{n+1})(c_{n+2,n+1}\phi_{n+2}) \cdots (c_{n+k,n+k-1}\phi_{n+k}).
$$

Then $\psi$ is a fully homogeneous $F_n$-integral on $F_{n+k}$, by 2.6(5). But, in view of (5),

$$
\psi = c_{n+1,n}(\phi_{n+1}c_{n+2,n+1}) \cdots (\phi_{n+k-1}c_{n+k,n+k-1})\phi_{n+k}
$$

$$
= c_{n+1,n}c_{n+2,n+1}\phi_{n+2}\phi_{n+3,n+2} \cdots
$$

$$
= c_{n+2,n}(\phi_{n+2})^2c_{n+3,n+2}\phi_{n+3} \cdots
$$

$$
= c_{n+2,n}c_{n+3,n+2}(\phi_{n+3})^2 \cdots = c_{n+k,n}(\phi_{n+k})^k
$$

finally. This also shows $c_{n+k,n}(\phi_{n+k})^k$ is defined for all $f \in F_{n+k}$, giving the first part of the assertion.

**Remark.** It follows from (6) that $(\phi_n)^m$ is an $F_n$-integral on $F_n$ provided $m \leq n$; compare footnote 3.

4.3 The algebra $E'$. Let $E' = \bigcup E_n$, where $E_0$, $E_1$, $E_2$, \cdots is the sequence of algebras obtained in 4.2. For $x$, $y \in E'$, define $x \leq y$ to mean that $x \leq y$ in some $E_n$ (and so in $E_m$ for all $m \geq n$). It is easily verified that $E'$ becomes a finitely additive Boolean algebra satisfying the countable chain condition, and that each $E_n$ is a complete subalgebra of $E'$. Let $\mathcal{R}$ be the representation space of $E'$. The required extended function space $F$ of Theorem 1 (4.1) will be defined by a certain realisation $\mathcal{F}(\mathcal{R}, \mathcal{B}, \mathcal{M})$; but before we define $\mathcal{B}$ and $\mathcal{M}$ it is convenient to have the extended integral $\phi$ more or less available. This we achieve by defining an operator $\Phi$ on a suitable class $\mathcal{C}'$ of continuous functions on $R$ (4.5). By measure-theoretic considerations we are then able to extend a modified form of $\Phi$ to an operator $\Phi^*$ on the family $\mathcal{F}(\mathcal{M}')$ of $\mathcal{M}'$-measurable functions, where $\mathcal{M}'$ is a certain Borel field of subsets of $R$ (4.8); and all that remains is to define the ideal $\mathcal{M}$ of null sets—an operation of some delicacy since $\mathcal{M}$ must be large enough to produce the countable chain condition and not so large as to annihilate $\Phi^*$.

4.4 The function-class $\mathcal{C}'$. As in §2, we let $Rx$ denote the open-closed subset of $R$ corresponding to $x \in E'$. We write $\mathcal{E}' = \{ Rx \mid x \in E' \}$, $\mathcal{E}_k = \{ Rx \mid x \in E_k \}$, $\mathcal{B}' = \mathcal{A}R = \text{Borel field (of subsets of } R \text{) generated by } \mathcal{E}'$, $\mathcal{B}_k = \text{Borel field generated by } \mathcal{E}_k$. A set $N \subset R$ is called $k$-negligible if it is contained in some $Y \in \mathcal{B}_k \setminus \mathcal{A}R$; $N$ is negligible if it is of the form $\bigcup N_k$ ($k = 0, 1, \cdots$) where $N_k$ is $k$-negligible. The families of $k$-negligible and of negligible sets are written
\( \mathcal{M}_k, \mathcal{M}' \) respectively; they are \( \sigma \)-ideals. We have \( \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \subseteq \mathcal{B}' \), \( \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}' \subseteq \mathcal{N}R \). We say that a function on \( R \) is \( k \)-negligible or negligible if it is in \( Z(\mathcal{M}_k) \) or \( Z(\mathcal{M}') \), respectively.

We define \( \mathcal{M}_k = \mathcal{B}_k + \mathcal{N}_k \) (cf. 3.4), \( \mathcal{M}'_k = \mathcal{B}_k + \mathcal{N}' \), \( \mathcal{M}' = \mathcal{B} + \mathcal{N}' \); all these are Borel fields. By 3.3(2), \( \mathcal{B}_k \subseteq \mathcal{E}_k + \mathcal{N}_k \), so that \( \mathcal{M}_k = \mathcal{E}_k + \mathcal{N}_k \), and hence \( \mathcal{M}'_k = \mathcal{E}_k + \mathcal{N}' \). Clearly \( \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}_k \subseteq \mathcal{N}_k' \subseteq \cdots \subseteq \mathcal{N}_k + \mathcal{N}' \). If \( x \in \mathcal{E}_k \), the correspondence between \( x \) and \( (Rx) + \mathcal{N}_k \) is a (complete) isomorphism between \( E_k \) and \( \mathcal{M}_k / \mathcal{N}_k \); similarly, for \( x \in \mathcal{E}' \), the correspondence between \( x \) and \( (Rx) + \mathcal{N}' \) is a finite isomorphism between \( E' \) and a finitely additive subalgebra of \( \mathcal{N}' / \mathcal{N}' \). (Note that in general \( E' \) need not be a \( \sigma \)-algebra, and that the \( \sigma \)-algebra \( \mathcal{M}' / \mathcal{N}' \) need not satisfy the countable chain condition.) If we restrict \( x \) to \( E_k \) here, we obtain an isomorphism of \( E_k \) onto the (complete) subalgebra \( \mathcal{M}_k / \mathcal{N}_k \) of \( \mathcal{M}' / \mathcal{N}' \).

We call a function \( f \) on \( R \) "\( k \)-continuous" if for each real \( p \in \{ p \mid |f(p)| > \rho \} \) is a union of sets in \( \mathcal{E}_k \); that is, if \( f \) is "\( 0 \)-continuous" in the sense of 3.3, taking \( E = E' \), \( A = E_k \). We write \( \mathcal{E}_k \) for the family of all \( k \)-continuous functions on \( R \), and \( \mathcal{E}' \) for \( \mathcal{E}_0 \); note that \( \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \). A function is called "\( k \)-measurable" if it is in \( \mathcal{F}(\mathcal{M}_k) \) (this notation is consistent with that in 3.4), "\( k' \)-measurable" if it is in \( \mathcal{F}(\mathcal{M}'_k) \). The following assertions follow easily from 3.3(3) and 3.4(1):

1. Every \( k \)-continuous function is \( k \)-measurable, and hence \( k' \)-measurable.
2. If \( f \) is \( k \)-measurable, there is a unique \( k \)-continuous function \( g_k \) such that \( f = g_k \mod \mathcal{M}_k \); and \( g_k = g_{k+1} = \cdots \).

Let \( R_k \) be the isomorphism between \( F_k = F(E_k) \) and \( \mathcal{E}_k \) described in 3.4 (where we replace \( A \) by \( E_k \), \( B \) by \( E' \) and \( R_0 \) by \( R_k \)).\(^*\) If we use the realisation \( \mathcal{F}(R, \mathcal{M}_k, \mathcal{M}_k') \) of \( F_k \), a typical element \( f \) of \( F_k \) consists of all functions differing from a \( k' \)-measurable function \( f \) by a negligible function, the cylinder mapping \( c_{k,k+n} \) becomes the identity mapping, and \( R_k f \) is the unique \( k \)-continuous function in \( f \). It follows that, for arbitrary realisations,

\[
R_k f = R_{k+n} g_k \quad (k, n = 0, 1, 2, \cdots, f \in F_k).
\]

4.5 The operator \( \Phi \). Given \( f \in \mathcal{E}'^+ \), we have \( f \in \mathcal{E}_k \) for some \( k \); put \( g = R_k^{-1} f \in F_k \), and define \( \Phi f = R_k \phi_k g \in \mathcal{E}_k^\dagger \). This definition does not depend on the choice of \( k \), as follows from 4.4(3) and 4.2(4), so \( \Phi \) is a single-valued mapping of \( \mathcal{E}'^+ \) into itself. \( \Phi \) and its iterates have the following properties; in all of them, \( k = 0, 1, 2, \cdots, m \), \( n = 1, 2, \cdots, \), and, where no proofs are given, the proofs are straightforward inductions over \( m \).

1. If \( f = R_k g \), where \( g \in \mathcal{E}_k^+ \), then \( \Phi^n f = R_k \phi_k^n g \).
2. \( \Phi^n(C_{k+m}^+) \subseteq C_{k+m}^+ \).

The case \( m = 1 \) of (2) follows from the definition of \( \Phi \), together with 4.2(2) and 4.4(3); the general case then follows by induction over \( m \).

\(^*\) Our notation in this paragraph is not quite exact; some isomorphisms have been suppressed. Strictly speaking, \( R_k \) depends on which realisation of \( F_k \) is used, but this should not cause confusion here.
(3) If $f, g \in C^+$ and $a, b$ are non-negative real numbers,
\[ \Phi^m(af + bg) = a\Phi^m f + b\Phi^m g. \]

(4) Let $f_n, f \in C^+$, and write $\Phi^m f_n = g_n, \Phi^m f = g$. Suppose that $f_n(p) \rightarrow f(p)$ for each $p \in R - N$, where $N \subseteq \mathfrak{M}$, and that either (a) $f_1 \leq f_2 \leq \cdots$ or (b) $f_1 \equiv f_2 \equiv \cdots$ and $g_1(p) < \infty$ if $p \in R - N$. Then $g_n(p) \rightarrow g(p)$ for all $p \in R - N^*$ where $N^* \subseteq \mathfrak{M}$.

(5) If $X \in \mathfrak{M}_{k+m}, g \in C^+, g \leq \Phi^m X$, then $g = \Phi^m Y$ for some $Y \in \mathfrak{M}_{k+m}$.

Here induction is not needed for the proof, which is a straightforward calculation based on 4.2(6).

As an immediate consequence of (5), $\Phi^m$ is "fully homogeneous" in the following sense:

(6) If $X \in \mathfrak{M}$, $g \in C^+, g \leq \Phi^m X$, then $g = \Phi^m Y$ for some $Y \in \mathfrak{M}$.

Finally $\Phi^m$ has the following $\sigma$-finiteness property:

(7) Given $k (= 0, 1, 2, \ldots)$, there exist disjoint sets $G_n \subseteq \mathfrak{M}_{k+1}$ ($n = 1, 2, \ldots$) such that $R - \bigcup G_n \subseteq \mathfrak{M}_{k+1}$ and $\Phi^m G_n \subseteq 1$.

For, by 4.2(6), $\mathfrak{M}_{k+1}(\mathfrak{M}+1)^k$ is a fully homogeneous $F_1$-integral on $F_{k+1}$.

4.6 The measures $\mu_p$ and $\nu_p$. Using 4.5(7), we take a sequence of disjoint sets $G_n, n \in \mathfrak{N}$ such that $R - \bigcup G_n = Z \subseteq \mathfrak{M}$ and $\Phi X G_n \leq 1$; these sets will remain fixed throughout the rest of the proof. Let $p$ be any point of $R$, fixed for the moment. For each $X \in \mathfrak{M}$, put $\mu_p(X) = \text{value at } p$ of the function $\sum \Phi(X \cap G_n)$. From 4.5(3), $\mu_p$ is a finitely additive (non-negative) measure on $\mathfrak{M}$, and $\mu_p(G_n) \leq 1$. Further, if $X_1, X_2, \ldots$ is any sequence of disjoint sets in $\mathfrak{M}$, and if $X = \bigcup X_n \in \mathfrak{M}$, then $\mu_p(X) = \sum \mu_p(X_n)$ because $X$ is compact and each $X_n$ is open, so that all but a finite number of the sets $X_n$ must be empty. There is therefore [1, p. 2] an extension of $\mu_p$ to a complete countably additive measure $\mu_p^*$ on a Borel field containing $\mathfrak{M}$.

(1) Given $X \in \mathfrak{M}$, $\nu_p(X) = (\Phi X)(p)$ for each $p \in R - N_X$, where $N_X \subseteq \mathfrak{M}^r$.

For say $X \in \mathfrak{M}_k$; then $G_n \subseteq \mathfrak{M}_k \subseteq \mathfrak{M}$ (we may assume $k > 0$), so $X \cap G_n \subseteq \mathfrak{M}_k$ and $\nu_p(X) = \sum \mu_p(X \cap G_n) \in \mathfrak{M}_k$ and $\nu_p(X) = \sum \mu_p(X \cap G_n) = \sum \{\text{value at } p \text{ of } \Phi X(X \cap G_n)\}$. Now each $\chi(X \cap G_n) \in \mathfrak{M}_k$ and also $\chi(X \cap G_n) \in \mathfrak{M}_k$ and $\sum \chi(X \cap G_n) = \chi X$ on $R - Z$ where $Z \subseteq \mathfrak{M}_k$. Hence by 4.5(4) we have $\Phi(\sum \chi(X \cap G_n), 1 \leq n \leq m) = \Phi X$ on $R - N_X$ for some $N_X \subseteq \mathfrak{M}_k \subseteq \mathfrak{M}$, and the assertion follows.

4.7 The outer measure function $\nu$. For arbitrary $X \subseteq R$ and $p \in R$, let $\nu_p^* X$ denote the outer measure of $X$ with respect to the measure $\nu_p$. We write $\nu X$ for the function on $R$ whose value at $p$ is $\nu_p^* X$. If $X \in \mathfrak{M}$, $X$ is $\nu_p$-measurable for every $p \in R$. Generally, if $X$ is $\nu_p$-measurable except for a negligible set of $p$'s, we write $\nu X$ instead of $\nu^* X$.

(1) A "complete" measure is one for which all subsets of null sets are measurable.
(1) If \( X \in \mathcal{M}' \), \( \nu X \) exists and is negligible.

We have \( X = \bigcup X_k \) where \( X_k \in \mathcal{M}_k \), and we may suppose \( k \geq 1 \). As the sets \( G_n \) are disjoint and \( \nu_p \)-measurable for every \( p \), we have \( \nu^* X_k = \sum_n \nu^*(X_k \cap G_n) \), so it is enough to prove that \( \nu^*(X_k \cap G_n) \) is \( k \)-negligible. By 3.2, \( X_k \subseteq \bigcap G_n \cap R x_{mk} \) where \( x_{mk} \in E_k \) and, for fixed \( m \), \( x_{m1} \geq x_{m2} \geq \cdots \) and \( x_{mk} = 0 \). We have \( G_n = R y_n \) where \( y_n \in E \subseteq E_k \). Let \( Z_{mn} = \bigcap R x_{mn} \cap G_n = \bigcap R (x_{mn} \setminus y_n) \). The sequence \( \{ R(x_{mn} \setminus y_n) \} \) (\( h = 1, 2, \cdots \)) of functions of \( \mathcal{M}_k \) decreases monotonically, and its limit is 0 outside a \( k \)-negligible set; further, \( \Phi X R(x_{mn} \setminus y_n) \leq \Phi X G_n \leq 1 \). Hence, by 4.5(4), \( \nu R(x_{mn} \setminus y_n) = \Phi^* R(x_{mn} \setminus y_n) \to 0 \) except on a \( k \)-negligible set as \( h \to \infty \), proving \( \nu^* R x_{mn} \) is \( k \)-negligible. As \( X_k \cap G_n \subseteq \bigcup_{m} Z_{mn} \), this proves \( \nu^* (X_k \cap G_n) \) \( k \)-negligible, as required.

(2) If \( X \in \mathcal{M}' \), \( X \) is \( \nu_p \)-measurable for all \( p \in R - N_X \), where \( N_X \in \mathcal{M}' \); and \( \nu X \in \mathcal{F}(\mathcal{M}') \).

First suppose \( X \in \mathcal{E} \). Then \( X \) is \( \nu_p \)-measurable for all \( p \in R \). Again, write \( X_n = X \cap \bigcap G_n \); we have \( \nu X = \sum \nu X_n = \sum \Phi X_n \) where \( \Phi X_n \in \mathcal{E}' \subseteq \mathcal{F}(\mathcal{M}) \), from 3.3(3), showing that \( \nu X \in \mathcal{F}(\mathcal{M}') \).

Now suppose \( X \in \mathcal{E} \). Again \( X \) is \( \nu_p \)-measurable for all \( p \in R \), and as above it is enough to prove that each \( \nu (X \cap G_n) \in \mathcal{F}(\mathcal{M}') \). Thus we may assume \( X \subseteq G_n \). The \( \mathcal{M}' \)-measurability of \( \nu X \) now follows by transfinite induction over the rank \( \alpha \) of \( X \) considered as in the Borel field generated by sets in \( \mathcal{E} \) which are subsets of \( G_n \); we use the facts that \( X \) is a limit of a monotone sequence of sets of smaller rank and of finite measure, and that a (pointwise) limit of a sequence of functions in \( \mathcal{F}(\mathcal{M}') \) is in \( \mathcal{F}(\mathcal{M}') \).

Finally, if \( X \in \mathcal{M}' \), \( X = Y + Z \) where \( Y \in \mathcal{E} \), \( Z \in \mathcal{M}' \); if \( p \in R - N_X \) where \( N_X \) is negligible, \( \nu^* (Z) = 0 \) by (1), and \( X \) is \( \nu_p \)-measurable and \( \nu_p (X) = \nu_p (Y) \). Thus \( \nu (X) = \nu (Y) \in \mathcal{F}(\mathcal{M}') \).

4.8 The operator \( \Phi^* \). As a corollary to the last result, we have

(1) If \( f \in \mathcal{F}(\mathcal{M}^+) \), then \( f \) is \( \nu \)-measurable except for a negligible set of \( p \)’s.

We define \( \Phi^* f = \inf \{ f g d \nu_p \mid g \) is \( \nu_p \)-measurable and \( g \geq f \} \). The function on \( R \) whose value at \( p \) is \( \Phi^* f \) is denoted by \( \Phi^* f \). It is easy to verify that, for arbitrary \( X \subseteq R \),

\[
\Phi^* X = \nu^* X.
\]

We deduce

(3) If \( f \in \mathcal{F}(\mathcal{M}^+) \), then \( \Phi^* f \in \mathcal{F}(\mathcal{M}^+) \).

For, by a familiar argument, \( f = \sum \alpha_n X_n \) where \( \alpha_n > 0 \), \( X_n \in \mathcal{M}' \). By 4.7(2), \( X_n \) is \( \nu_p \)-measurable for all \( p \in R - N_n \) where \( N_n \in \mathcal{M}' \). Put \( N = U N_n \); then, if \( p \in R - N \), \( f \) is \( \nu_p \)-measurable and consequently \( \Phi^* f = \int_R f d \nu_p = \sum \alpha_n \nu_p (X_n) \) where, for each \( n \), the function \( \nu_p (X_n) \) of \( p \) is in \( \mathcal{F}(\mathcal{M})^+ \) by 4.7(2). Thus \( \Phi^* f \) differs from an \( \mathcal{M}' \)-measurable function at most on \( N \), and is therefore \( \mathcal{M}' \)-measurable.

(4) If \( f \geq 0 \) and \( [f] \in \mathcal{M}' \), then \( [\Phi^* f] \in \mathcal{M}' \).

Let \( [f] = X \). By 4.7(1), \( \nu^* X = 0 \) for all \( p \in R - N \) where \( N \in \mathcal{M}' \). If \( p \in R - N \),
we may take $g = \infty \chi X$ in the definition of $\Phi^*_f$, showing $\Phi^*_f = 0$ for $p \in R - N$.  
(5) If $f_n \in \mathcal{F}(\mathfrak{M}^+)$ ($n = 1, 2, \cdots$), then $\Phi^*_p(\sum f_n) = \sum \Phi^*_p f_n$ except for a negligible set of $p$'s.

The proof resembles that of (3).  We immediately deduce:

(6) If $f_n \in \mathcal{F}(\mathfrak{M}^+)$ ($n = 1, 2, \cdots$), $f_1 \geq f_2 \geq \cdots$, and $\Phi^*_p f_1 < \infty$ except on a negligible set, then $\Phi^*_p(\lim f_n) = \lim \Phi^*_p f_n$ except on a negligible set.

If $f \in \mathfrak{G}^+$, the value of $\Phi f$ at $p \in R$ is denoted by $\Phi_p f$; similarly we define $\Phi^*_p f$.

(7) If $f \in \mathfrak{G}^+$, then $\Phi^*_p f = \Phi_p f$ if $p \in R - N$ where $N \in \mathfrak{M}$.

Say $f \in \mathfrak{G}_k$.  By 3.4(5) we may write $f = \sum \sigma_n X_n + g$ where $\sigma_n > 0$, $X_n \in \mathcal{G}_k$, $g \geq 0$ and $[g] \in \mathfrak{M}_k \subset \mathfrak{M}$.  From 4.5(4), if $p \in R - N_1$ where $N_1 \in \mathfrak{M}_k$, $\Phi_p f = \sum \sigma_n X_n = \sum \sigma_n \nu_p(X_n)$ by 4.6, if $p \in N_2$, where $N_2 \in \mathfrak{M}_k$.  On the other hand, from (4), $\Phi^*_p g = 0$ except on $N_3 \in \mathfrak{M}$, and outside $N_3$ we have that $f$ is $\nu_p$-measurable and consequently $\Phi^*_p f = \sum \sigma_n X_n d \nu_p = \sum \sigma_n \nu_p(X_n)$.  Thus $\Phi^*_p f = \Phi_p f$ if $p \in R - (N_1 \cup N_2 \cup N_3)$.

Since $\Phi^*$ maps $\mathfrak{F}(\mathfrak{M})^+$ in itself, the iterates $\Phi^{*m}$ ($m = 1, 2, \cdots$) are all defined; it is easy to see that properties (3)–(6) apply to $\Phi^{*m}$, and similarly (7) gives (with a little more trouble):

(7') If $f \in \mathfrak{G}^+$, $\Phi^{*m} f$ and $\Phi f$ differ only on a negligible set.

4.9 Support properties of $\Phi^*$.  We list the following properties of $\Phi^*$ for later use; they all follow easily from the foregoing.  Throughout, it is assumed that $f, f_1, f_2, \cdots \in \mathcal{F}(\mathfrak{M}^+)$.  

(1) $[\Phi^{*m} f] \in \mathfrak{M}$.  

(2) If $[f] \in \mathfrak{M}$, $[\Phi^{*m} f] \in \mathfrak{M}$.  

(3) $\cup_n [\Phi^{*m} f_n] \subset [\Phi^{*m} \sup f_n] \subset [\Phi^{*m} \sum f_n]$, and  

$[\Phi^{*m} \sum f_n] - \cup_n [\Phi^{*m} f_n] \in \mathfrak{M}$.  

(4) If $[f_1] + [f_2] \in \mathfrak{M}$, then $[\Phi^{*m} f_1] + [\Phi^{*m} f_2] \in \mathfrak{M}$.

In particular:

(5) $[\Phi^{*m} f] + [\Phi^{*m} [f]] \in \mathfrak{M}$.  

4.10 The set-operator $I$.  As a preliminary to defining the final ideal $\mathfrak{M}$ of "null sets," we define $I(X)$, for $X \in \mathfrak{M}$, by: $I(X) = [\sum \Phi^{*m} \chi X]$, where $m = 0, 1, 2, \cdots$.  Taking $m = 0$ shows

(1) $I(X) \supset X$.

The following results follow easily with the aid of 4.9. It is assumed throughout that $X, X_1, X_2, \cdots \in \mathfrak{M}$.

(2) $I(X) \in \mathfrak{M}$.

(3) If $X \in \mathfrak{M}$, $I(X) \in \mathfrak{M}$.

(4) $I(X) = \cup [\Phi^{*m} \chi X] (m \geq 0)$.

(5) If $X \subset Y$, $I(X) \subset I(Y)$.

(6) $I(\cup X_n) = \cup I(X_n) \cup N$, where $N \in \mathfrak{M}$.
(7) \( I(X + Y) \supset I(X) + I(Y) \mod \mathfrak{A}' \); hence if \( X + Y \in \mathfrak{A}' \),
\[ I(X) + I(Y) \in \mathfrak{A}' . \]

(8) \( I(I(X)) = I(X) \mod \mathfrak{A}' . \)

(9) If \( f \in \mathfrak{F}(\mathfrak{A})^+ \), then \( \left[ \sum \Phi^* m f \right] = I[f] \mod \mathfrak{A}' \), and hence
\[ I[\Phi^* f] \subset I[f] \mod \mathfrak{A}' . \]

As the last four of these statements are less trivial than the others, we sketch their proofs.

**Proof of (6).** By (4) and 4.9(3),
\[ I(\bigcup X_n) = \bigcup \left\{ \left[ \Phi^* m \cdot \chi X_n \right] \cap Z_m \right\} \]
where \( Z_m \subseteq \mathfrak{A}' \), \( = \bigcup_{m,n} \left[ \Phi^* m \cdot \chi X_n \right] \cap Z' \) where \( Z' \subseteq \mathfrak{A}' \). But \( I(\bigcup X_n) \supset I(X_n) \), by (5).

**Proof of (7).** By (6), \( I(X) = I(X \cap Y) \cup I(X - Y) \cup N_1, I(Y) = I(X \cap Y) \cup I(Y - X) \cup N_2 \), where \( N_1, N_2 \subseteq \mathfrak{A}' \); so \( I(X) + I(Y) \subseteq I(X + Y) \cup (N_1 \cup N_2) \) by (5).

**Proof of (8).** Using (4), (6) and 4.9(5), we find
\[ I(I(f)) = U \left\{ \left[ \Phi^* m \cdot \chi \right] \right\} \mod \mathfrak{A}' = I(f) \mod \mathfrak{A}' . \]

**Proof of (9).** From (4) and 4.9(5), \( I[f] = \left[ \Phi^* m \cdot \chi [f] \right] = U \left[ \Phi^* m f \right] \mod \mathfrak{A}' = \left[ \sum \Phi^* m f \right] \). Hence \( \left[ \Phi^* f \right] = I[f] \mod \mathfrak{A}' \), giving (from (3) and (8)) \( I[\Phi^* f] \subset I[f] \mod \mathfrak{A}' \).

5. **Proof of Theorem 1 concluded.**

5.1 *The \( \sigma \)-ideal \( \mathfrak{A} \).* Define \( \mathfrak{A} = \{ X \mid \text{there exists } Y \in \mathfrak{A}' \text{ such that } X \subseteq Y \quad \text{and} \quad I(Y) \in \mathfrak{A} \mathfrak{S} \}. \) We have at once:

1. \( \mathfrak{A} \) is a \( \sigma \)-ideal. (From 4.10(6).)
2. \( \mathfrak{A}' \subseteq \mathfrak{A} \subseteq \mathfrak{A} \mathfrak{S} \). (From 4.10(3) and 4.10(1).)
3. \( X \subseteq \mathfrak{A}' \cap \mathfrak{A} \), \( X \) is empty. (From (2).)
4. \( \text{If } f \in \mathfrak{F}(\mathfrak{A}'^+) \text{ and } [f] \in \mathfrak{A}, \text{ then } [\Phi^* f] \in \mathfrak{A}. \) (From 4.10.)
5. \( \text{If } f, g \in \mathfrak{F}(\mathfrak{A}'^+) \text{ and } [f] + [g] \in \mathfrak{A}, \text{ then } [\Phi^* f] + [\Phi^* g] \in \mathfrak{A}. \)

For let \( R - X = [f] \cap [g] \), and put \( f = f_X X + f', \ g = g_X X + g' \). Then \( [f'] = [g'] \), so \( [\Phi^* f'] = [\Phi^* g'] \mod \mathfrak{A}' \), by 4.9(4). Also \( \Phi^* f = \Phi^* f' + \Phi^* f_X X \mod \mathfrak{A}' = \Phi^* f' \mod \mathfrak{A} \) by (4) and (2). Thus, modulo \( \mathfrak{A}, [\Phi^* f] = [\Phi^* f'] = [\Phi^* g'] = [\Phi^* g] \).

5.2 *The algebra \( E \).* Now put \( \mathfrak{F} = \mathfrak{F}' + \mathfrak{A} \); this is a Borel field containing \( \mathfrak{A} \). Define \( E = \mathfrak{F} / \mathfrak{A} \), a Boolean \( \sigma \)-algebra. Since \( \mathfrak{A}' = \mathfrak{A}' + \mathfrak{A}' \) and \( \mathfrak{A} \supset \mathfrak{A}' \), we have \( \mathfrak{F} = \mathfrak{F}' + \mathfrak{A} \), and a typical element of \( E \) is thus the class of sets \( (X) + \mathfrak{A} \) \( = \{ X + N \mid N \in \mathfrak{A} \} \) where \( X \in \mathfrak{F}' = \mathfrak{A} \mathfrak{S} \). We now prove

1. \( E \) satisfies the countable chain condition.

Suppose \( \mathfrak{A} \) is an uncountable family of sets \( A \in \mathfrak{A}' \), none of which is in \( \mathfrak{A} \),
but such that whenever \( A_1, A_2 \) are distinct members of \( \mathfrak{a} \) then \( A_1 \cap A_2 \in \mathfrak{R} \); we must derive a contradiction. We may suppose that \( \mathfrak{a} \) consists of just \( \mathbb{N}_1 \) sets; well-order \( \mathfrak{a} \) as \( \{ A_\alpha \mid \alpha < \omega_1 \} \), and let \( A'_\alpha = A_\alpha - \bigcup \{ A_\beta \mid \beta < \alpha \} \); then \( A'_\alpha \in \mathfrak{B}' \), \( A_\alpha - A'_\alpha \in \mathfrak{R} \), and distinct sets \( A'_\alpha \) are disjoint. Thus, replacing \( \mathfrak{a} \) by \( \{ A'_\alpha \mid \alpha < \omega_1 \} \), we may further assume that \( \mathfrak{a} \) consists of disjoint sets.

For each \( A \in \mathfrak{a} \), there is a least \( n \geq 0 \) such that \( \Phi^{**} \chi A \) is of second category in \( \mathfrak{R} \) (else \( I(A) \in \mathfrak{R} \) and \( A \in \mathfrak{R} \)). Let \( \mathfrak{a}_n \) be the subfamily of \( \mathfrak{a} \) for which this \( n \) has the value \( k \); then \( \mathfrak{a}_n \) must be uncountable for some \( k \). If \( k = 0 \), we have that each \( A \in \mathfrak{a}_0 \) is itself of second category; but (2.1(2)) each \( A \in \mathfrak{B}' \) has the form \( Ra + RH \) where \( a \in \mathfrak{B}' \) and \( H \in \mathfrak{R} \), and if \( A \) is of second category then \( a \neq 0 \). So if \( \mathfrak{a}_n \) is uncountable, \( \mathfrak{B}' \) would not satisfy the countable chain condition. We may therefore assume that \( \mathfrak{a}_n \) is uncountable for some \( k \geq 1 \).

By 4.5(7) and 4.8(7) there exist sets \( Y_1, Y_2, \ldots \in \mathfrak{a}_n \) such that \( R - \bigcup Y_n \in \mathfrak{R} \) and \( \Phi^{**} \chi Y_n \leq 1 \mod \mathfrak{B}' \). For every \( A \in \mathfrak{B}' \) we have \( \Phi^{**} \chi A = \bigcup_n \{ \Phi^{**} \chi (A \cap Y_n) \} \mod \mathfrak{B}' \), by 4.9(2) and 4.9(3); hence if \( A \in \mathfrak{a}_n \) we have that \( \Phi^{**} \chi (A \cap Y_n) \) is of second category for some \( n \). There is therefore some \( n \), which we may assume to be 1, to which uncountably many sets \( A \in \mathfrak{a}_n \) correspond; we replace each such \( A \) by \( A \cap Y_1 \), and thus obtain an uncountable family \( \mathfrak{a}' \subset \mathfrak{a} \) of disjoint sets satisfying:

(2) If \( A \in \mathfrak{a}' \), then \( A \subset Y_1 \) and \( \Phi^{**} \chi A \) is of second category.

From 4.9(1), there exists for each \( A \in \mathfrak{a}' \) a positive integer \( n(A) \) and a set \( W(A) \in \mathfrak{w}' \) of second category such that \( \Phi^{**} \chi A \geq (1/n(A)) \chi W(A) \). There is a positive integer \( h \) such that \( n(A) = h \) for uncountably many sets in \( \mathfrak{a}' \); we may thus assume further that

(3) If \( A \in \mathfrak{a}' \), \( \Phi^{**} \chi A \geq (1/h) \chi W(A) \).

For each subset \( \mathfrak{X} \subset \mathfrak{a}' \), put \( W(\mathfrak{X}) = \bigcap \{ W(A) \mid A \in \mathfrak{X} \} \). Then:

(4) If \( \mathfrak{X} \) has more than \( h \) elements, \( W(\mathfrak{X}) \) is of first category.

It is enough to prove this when \( \mathfrak{X} \) has \( h + 1 \) elements \( A_0, A_1, \ldots, A_h \). As these sets are disjoint, we have, modulo \( \mathfrak{X} \)-negligible functions, from (3),

\[
(h + 1) \chi W(\mathfrak{X}) \leq \sum \{ h \Phi^{**} \chi A_i \mid 0 \leq i \leq h \} = h \Phi^{**} (\sum \chi A_i) \quad (\text{see end of 4.8})
\]

\[
\leq h \Phi^{**} \chi (A_0) \leq h \Phi^{**} \chi Y_1 \leq h,
\]

proving \( W(\mathfrak{X}) \in \mathfrak{X} \subset \mathfrak{R} \).

Consider now the family \( \{ \mathfrak{Y} \} \) of maximal subsets \( \mathfrak{Y} \) of \( \mathfrak{a}' \) for which \( W(\mathfrak{Y}) \) is of second category; each \( A \in \mathfrak{a}' \) is in at least one \( \mathfrak{Y} \) (from (4), since \( A \) is itself of second category), and each \( \mathfrak{Y} \) contains at most \( h \) sets \( A \in \mathfrak{a}' \). Further, if \( \mathfrak{Y}_1 \neq \mathfrak{Y}_2 \), \( W(\mathfrak{Y}_1) \) and \( W(\mathfrak{Y}_2) \) are in \( \mathfrak{w}' \), are both of second category, and have intersection of first category. By an argument similar to that used above for \( k = 0 \), there are only countably many sets \( W(\mathfrak{Y}) \), and therefore only countably many sets \( \mathfrak{Y} \), each with at most \( h \) elements. Thus \( \mathfrak{a}' = \bigcup \mathfrak{Y} \) is countable, giving the desired contradiction.

5.3 The operator \( \Phi \). Let \( F = F(E) \); we define a mapping \( \Phi \) of \( F^+ \) in \( F^+ \) which will be proved to satisfy the requirements of Theorem 1 (4.1). The elements of \( F \) are of the form \( f = f + \mathcal{Z}(\mathfrak{X}) \), where \( f \in \mathcal{F}(\mathfrak{W}') \); here \( f + \mathcal{Z}(\mathfrak{X}) \) denotes the
family of all functions $f + h$ where $h$ is $\mathcal{H}$-negligible. Since $\mathcal{B} = \mathcal{B}' + \mathcal{H}$, we may require $f \in \mathcal{F}(\mathcal{B}')$ here.

Define $\tilde{f} = \Phi \ast f + Z(\mathcal{H})$, where $\tilde{f} = f + Z(\mathcal{H})$. From 4.8(3)-(5), this definition is single-valued and $\phi$ maps $F_+^+$ in itself. Further, $\phi$ is countably additive from 4.8(5), and $\sigma$-finite from 4.5(7) and 4.8(7); $\phi$ is thus an $F$-integral on $F$ (2.6). To verify that $\phi$ is a cylinder extension of $\phi_0$, we first observe that the correspondence $x \mapsto (Rx) + \mathcal{H}$ is (from 5.1(3)) a finitely additive isomorphism between $E'$ and a finitely additive subalgebra of $E$ which, restricted to $E_k$, is a complete isomorphism (because $\mathcal{A} \supseteq \mathcal{A}' \supseteq \mathcal{A}_k$). We may identify $E_k$ with the complete subalgebra $\{Rx + \mathcal{H} \mid x \in E_k\}$ of $E$ (equivalently, we take $E_k = (\mathcal{B}_k + \mathcal{H})/\mathcal{H}$); and similarly we may identify $F_k$ with the set of function classes $f + Z(\mathcal{H})$, $f \in \mathcal{F}(\mathcal{B}_k)$—that is, we realise $F_k$ as $\mathcal{F}(R, \mathcal{B}_k + \mathcal{H}, \mathcal{H})$. The cylinder mapping of $F_k$ in $F$ is now the identity mapping of $F_k$. If $f \in \mathcal{F}(\mathcal{B}_k)^+$, so that $\tilde{f} = f + Z(\mathcal{H})$ is a typical element of $F_k^+$, we again have that $R_k \tilde{f}$ is the unique $k$-continuous function in $\tilde{f}$ (compare 4.4)(9). Now if $f \in \mathcal{F}(\mathcal{B}_k)^+$, $\Phi \ast f = \Phi \ast f_k \mod \mathcal{H}$ by 4.8(4), (5), $= \Phi R_k \tilde{f} \mod \mathcal{H}$ by 4.8(7) $= \Phi \phi_k \tilde{f} \mod \mathcal{H}$ by definition of $\Phi$, so that $\phi_k \tilde{f} = (\Phi \ast f) = \Phi f$. That is, $\phi$ is a cylinder extension of $\phi_k$ ($k = 0, 1, 2, \ldots$), and in particular of $\phi_0$.

5.4 Full homogeneity. All that remains is to show that $\phi$ is fully homogeneous. Write $x = \phi x$ for $x \in E$; thus $\lambda$ is countably additive and $\sigma$-finite.

Let $s_0 = \bigvee \{z \mid z \in E, \lambda_s = 0\}$, $s_1 = e - s_0$, $E^1 = \{z \mid z \in E, z \leq s_1\}$. We first show:

(1) Given $x \in E$, there exists $\sigma x \in E^1$ such that, for all $y \in E$,

$$\lambda(y \land \sigma x) = (\lambda y) \chi x.$$  

(The element $\sigma x$ is in fact unique, but we do not need this.)

For let $H$ be the set of elements $x \in E$ for which such a $\sigma x$ exists. Then

(2) if $x \in H$ and $y \in E$, $\lambda\{y \land (z_1 - \sigma x)\} = (\lambda y) \chi (-x)$.

For suppose first that $\lambda y \ll \infty$. Then

$$\lambda(y) = \lambda(y \land z_1) = \lambda(y \land \sigma x) + \lambda(y \land (z_1 - \sigma x))$$

$$= (\lambda y) \chi x + \lambda(y \land (z_1 - \sigma x)),$$

so $\lambda(y \land (z_1 - \sigma x)) = (\lambda y)(1 - \chi x) = (\lambda y) \chi (-x)$.

In the general case, we know $y = \bigvee y_n \ (n = 1, 2, \cdots)$ where $\lambda y_n \ll \infty$, and the elements $y_n$ can be assumed disjoint. Then $\lambda(y_n \land (z_1 - \sigma x)) = (\lambda y_n) \chi (-x)$, and summation gives (2).

This shows that if $x \in H$ then $-x \in H$, with $\sigma(-x) = z_1 - \sigma x$.

Next let $x_n \in H \ (n = 1, 2, \cdots)$, $y \in E$, and suppose $\lambda(y) \ll \infty$. Then $\lambda(y \land \sigma x_n) \leq \lambda(y \land \sigma x_n) = (\lambda y) \chi x_n$ for all $n$, and therefore

$$\lambda(y \land \sigma x_n) \leq (\lambda y) \chi(x_n).$$

(9) This depends on the observation that if $g \in \mathcal{F}(\mathcal{B}_k)$ is $\mathcal{H}$-negligible, then $g$ is $\mathcal{H}_k$-negligible. For $g = \phi \mod \mathcal{H}$ where $\phi$ is $k$-continuous; being continuous and $\mathcal{K}R$-negligible, $\phi$ must be $0$. 

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But \( \lambda \{ y \land (z_1 - \Lambda \sigma x_1) \} = \lambda \{ y \land V(z_1 - \sigma x_1) \} \leq \sum \lambda \{ y \land (z_1 - \sigma x_1) \} = \sum (\lambda y)(-x_1) \) by (2) = \( \lambda y \sum (-x_1) \), and thus \( \lambda \{ y \land (z_1 - \Lambda \sigma x_1) \} \leq \inf \{ \lambda y, \lambda y \sum (-x_1) \} \); that is,

(4) \( \lambda(y \land (z_1 - \Lambda \sigma x_1)) \leq (\lambda y)\chi (-x_1). \)

But \( \lambda \{ y \land \Lambda \sigma x_1 \} + \lambda \{ y \land (z_1 - \Lambda \sigma x_1) \} = \lambda(y \land z_1) = \lambda y \ll \infty \). Adding (3) and (4), we see that both inequalities must be equalities; in particular (3) becomes

(5) \( \lambda(y \land \Lambda \sigma x_1) = (\lambda y)\chi(z_1), \)

if \( \lambda y \ll \infty \).

The restriction \( \lambda y \ll \infty \) is easily removed, as before, so (5) shows that \( \Lambda x_n \in H \), with \( \sigma(\Lambda x_n) = \Lambda \sigma x_n \).

Thus \( H \) is a \( \sigma \)-subalgebra of \( E \). Further,

(6) \( H \) contains each \( E_k \) \( (k = 0, 1, 2, \ldots) \).

For \( (4.2(3)) \) \( c_{k+1,\Lambda} \phi_{k+1} \) is a fully homogeneous \( F_k \)-integral on \( F_{k+1} \); we apply [4, p. 175, Lemma 4] to its strict form, taking \( \sigma x = \pi(z_1, x) \) for \( x \in E_{k+1} \), and obtain \( c_{k+1,\Lambda} \phi_{k+1} \chi_{k+1}(y \land \sigma x) = (c_{k+1,\Lambda} \phi_{k+1} \chi_{k+1} y) \chi_{k+1} \chi \), \( \chi_{k+1} \) denoting the characteristic function in \( F_{k+1} \). We use the same realisations of \( F_k \), \( F_{k+1} \) as at the end of 5.3; the cylinder mappings become identities, \( \chi_{k+1} y = \chi y \) for \( y \in E_{k+1} \), and because \( \phi \) is a cylinder extension of \( \phi_{k+1} \) it follows that \( \phi \chi(y \land \sigma x) = (\phi \chi)(y \land \sigma x) \) for \( x \in E_{k+1} \); thus \( E_k \subset E_{k+1} \subset H \).

Now let \( \mathfrak{C} \) = family of all sets \( X \in \mathfrak{X}' \) such that \( (X) + \mathfrak{X} \in H \). Then \( \mathfrak{C} \) is a Borel field containing \( UB_0 = \mathfrak{X}' \), so \( \mathfrak{C} \supset \mathfrak{X}' \). That is, \( (X) + \mathfrak{X} \in H \) for all \( X \in \mathfrak{X}' \), proving \( E \subset H \). This establishes (1).

Now, given \( x \in E \) and \( g \in F^+ \) such that \( g \leq \lambda x \), we must find \( y \leq x \in E \) such that \( \lambda y = g \). We may of course assume \( g > 0 \); and it is enough to show that there then exists a nonzero \( z \leq x \) in \( E \) such that \( \lambda z \leq g \), as then an “exhaustion” argument, based on the countable chain condition (cf. [2, p. 283]), produces the required element \( y \). We may further suppose that \( z \leq a_1 \) where \( a_1 \in E_1 \) and \( \lambda a_1 \leq 1 \). For \( c_1 \phi_1 = \phi_1 \) is fully homogeneous on \( F_1 \) and \( \phi \) extends \( \phi_1 \); hence (as in the proof of 4.6(7)) disjoint elements \( a_1, a_2, \ldots \in E_1 \) exist such that \( V a_n = e \) and \( \lambda a_n \leq 1 \). Since \( \sum (x \land a_n) \geq g > 0 \), there exists \( n \) such that \( [\lambda(x \land a_n)] \land [g] \neq \varnothing \); we may suppose \( n = 1 \), and then have \( g_1 = \lambda(x \land a_1) \land g > 0 \); we replace \( x \) by \( x \land a_1 \) and \( g \) by \( g_1 \). For some positive integer \( m \) we have \( g \geq (1/m)xw \) for some nonzero \( w \in E \). Because of the full homogeneity of \( \phi_1 \), there exist disjoint elements \( b_1, b_2, \ldots, b_m \in E_1 \) such that \( V b_i = a_1 \) and \( \lambda b_i = (1/m)\lambda a_1 \). Since \( x \leq V b_i \), \( \sum \lambda(b_i \land x \land \sigma w) = \sum \lambda(b_i \land x)w = (\lambda x)w \geq g \), \( (1/m) \lambda xw \geq g \). Hence, for some \( i \), \( 0 < \lambda(b_i \land x \land \sigma w) \leq \lambda(b_i \land \sigma w) = (\lambda b_i) \lambda w \leq (1/m) \lambda xw \leq g \), and we take \( z = b_i \land x \land \sigma w \). This completes the proof of Theorem 1.

6. Extensions of measures on measure algebras.

**Theorem 2.** Let \( A \) be a \( (\sigma-) \)-subalgebra of a measure algebra \( (E, \mu) \). (*) Let \( \lambda \)

(*) By saying that \( (E, \mu) \) is a measure algebra, we imply that \( \mu \) is \( \sigma \)-finite and positive on \( E \).
be a \(\sigma\)-finite positive measure on \(A\). Then there exists a \(\sigma\)-finite positive measure \(\lambda^*\) on \(E\) which extends \(\lambda\).

In what follows, it is understood that all measures are to be complete and \(\sigma\)-finite, and that the sets and functions used are measurable.

By [3, Theorem 2b, p. 149], \((E, \mu)\) has a realisation of the following form. We can realise \(A\) algebraically as the measure algebra of a measure space \((S, \nu)\) (the measure \(\nu\) has no simple relation to \(\mu\), and can find a measure space \((T, m)\) and a subset \(K\) of the product space \(S \times T\) (to which we give the usual product measure), in such a way that there is a measure-preserving isomorphism \(\theta\) of \((E, \mu)\) onto a certain Borel field of subsets of \(K\) modulo null sets, and further if \(a \in A\) then \(\theta a\) is the class of the "cylinder set" \((U \times T) \cap K\), where \(U\) is any subset of \(S\) in the class \(a\).

By the Radon-Nikodym theorem, there exists a non-negative function \(f\) on \(S\) such that, for each \(U \subset S\), \(\lambda(U) = \int_U f(s) d\nu(s)\). (Here \(\lambda(U)\) means \(\lambda a\) where \(a\) is the class of \(U\) modulo null sets.) Write \(T = U T_n\) \((n = 1, 2, \ldots)\), where the sets \(T_n\) are disjoint and \(m(T_n)\) is positive and finite. Define

\[
P(s) = \int_T \chi_X(s, t) \sum \left\{ \frac{(\chi_T \alpha)}{2^m(T_n)} \right\} dm(t);
\]

this is defined and \(\leq 1\) for almost all \(s \in S\). Further, \(P(s) > 0\) almost everywhere, since if \(P(s) = 0\) for all \(s \in U\), the set \((U \times T) \cap K\) is null, showing that \(U\) is in the class of \(\sigma \in A\)—that is, \(\nu(U) = 0\). Now define, for \(X \subset S \times T\),

\[
\lambda^*(X) = \int_{S \times T} \frac{(f(s)/P(s))(\chi X)}{dm(t)} \sum \left\{ \frac{(\chi_T \alpha)}{2^m(T_n)} \right\} dm(t).
\]

Then, applying \(\theta\), we see that \(\lambda^*\) gives a \(\sigma\)-finite positive measure on \(E\). To show that \(\lambda^*\) extends \(\lambda\) on \(A\), we verify (by a straightforward calculation) that if \(U \subset S\), \(\lambda^*((U \times T) \cap K) = \lambda U\).

7. Extensions of operators for measure algebras. In this section we prove that if we start with a measure algebra in Theorem 1, then we can arrange to end up with a measure algebra. More precisely:

**Theorem 3.** If, in Theorem 1, \(E_0\) is a measure algebra\((\sigma)\) with measure \(m_0\), the algebra \(E\) can be chosen so that it is also a measure algebra, with measure \(m\) extending \(m_0\).

**Proof.** Since \(m_0\) is \(\sigma\)-finite on \(E_0\), we can find an equivalent finite measure \(m_0^*\) on \(E_0\); say \(m_0^*(e) = 1\). We use the entire argument of \(\S 4\) (but not of \(\S 5\)), with the following additions. We first observe that \(E_1\) can be taken to be a measure algebra, say with measure \(m_1\). For let \(z_0 = \bigvee \{ x | x \in E_0, \phi_0 x = 0 \}\), \(z_1 = e - z_0\). The "strict form" \(\phi_0\) of \(\phi_0\) is defined on the function space on the principal ideal \(E_0(z_1) = \{ x | x \in E_0, x \leq z_1 \}\), and its range is the function space on the principal ideal \(E_0(\phi 1)\). The construction of \(E_1\) depended in the first
instance on applying the result of [4] to $\varphi_0$; this gives an algebra $E'_1$ containing $E_0(x_0)$ as a subalgebra, and a fully homogeneous strict extension $\varphi^*_0$ of $\varphi_0$ to an operator from $F(E'_1)$ to $F(E_0[\varphi 1])$. We then take $E_1$ = direct sum of $E'_1$ and $E_0(x_0)$, imbedding $E_0$ in $E_1$ in the obvious way; $\varphi_1$ is defined by $\varphi_1 f = c_0 \varphi_0(f)\varphi_0(x_0)$. By [4, Theorem 5], $E'_1$ is isomorphic to a principal ideal in a product $J \times E_0[\varphi 1]$, where $J$ is a numerical measure algebra. By [3; 2; 4], if we give $E_0[\varphi 1]$ the measure $m'_0$, then $J \times E_0[\varphi 1]$ with the usual product measure induces a (positive, $\sigma$-finite) measure (say) $m_1$ on $E'_1$. We extend $m_1$ to $E_1$ by using $m'_0$ on $E_0(x_0)$. By Theorem 2, there is a (positive) measure $m'_1$ on $E_1$ which extends $m'_0$. Note that $m'_1 \leq 1$ on $E_1$, because $m'_1(e) = m'_0(e) = 1$.

In this way, we may suppose that all the algebras $E_k$ of 4.2 are measure algebras, the measure $m_{k+1}$ on $E_{k+1}$ extending $m_k$ on $E_k$. Their common value gives a finitely additive measure $m'$ on $E'$, and hence on the family $\mathcal{E}'$ of sets $Rx, x \in E'$, in the representation space $R$ of $E'$ (cf. 4.3). By the same reasoning as in 4.6, we extend $m'$ to a complete, countably additive measure (still denoted by $m'$) on a Borel field containing $\mathcal{E}'$; note that $m'(R) = 1$. Let $\mathcal{R}'$ denote the family of subsets of $R$ with zero $m'$-measure. We show:

(1) $\mathcal{R} \supset \mathcal{R}_k$,

(2) $\mathcal{R} \supset \mathcal{R}'$.

For, by 2.3(4), each $X \in \mathcal{R}_k$ is contained in a set of the form $\xi^{-1}Y$, where $Y \in \mathcal{R} \cap B. S$. By 3.2, $Y \subseteq \bigcap_n Sx_{mn}$ where $x_{mn} \in E_k$, $x_{m1} \geq x_{m2} \geq \cdots$ and $\Lambda_n x_{mn} = o$. Thus $X \subseteq \xi^{-1}Y \subseteq \bigcap_n Rx_{mn}$ by 2.2(1); now, as $m'_k$ is finite, $m'_k x_{mn} \rightarrow 0$ as $n \rightarrow \infty$, so $m'_k \bigcap_n Rx_{mn} = 0$ for each $m$, proving $m'X = 0$ if $X \in \mathcal{R}_k$.

It follows at once that

(2') $\mathcal{R} \supset \mathcal{R}_0$,

Define $\mathcal{R} = \{ X \mid X \subset Y$ for some $Y \in \mathcal{R}'$ such that $I(Y) \in \mathcal{R}_0 \}$. It is easily verified that all the properties in 5.1 continue to apply for this modified definition of $\mathcal{R}$, except that in 5.1(2) we no longer have $\mathcal{R} \subset \mathcal{R} R$. But instead we have

(2') $\mathcal{R} \subset \mathcal{R}_0$,

because if $X \in \mathcal{R}$ then $X \subset Y \subset I(Y)$ where $I(Y) \in \mathcal{R}_0$. Hence 5.1(3) continues to hold.

For each $f \in \mathcal{F}(\mathcal{R}')^+$, put

$$\psi_f = \sum_{1}^{N} 2^{-n}\Phi^*(fG_x),$$

the sets $G_1, G_2, \cdots$, being those of 4.6. Then $\psi \leq 1, \psi_f \in \mathcal{F}(\mathcal{R}')^+$, $\psi$ is linear and countably additive mod $\mathcal{R}'$, and $[\psi_f] = [\Phi^* f] \mod \mathcal{R}'$. Hence $\psi f \in \mathcal{Z}(\mathcal{R}')$ if $f \in \mathcal{Z}(\mathcal{R}')$, and from this an easy induction shows that $[\psi^k f] = [\Phi^* f] \mod \mathcal{R}'$ (where $f \in \mathcal{F}(\mathcal{R}')^+$ and $k = 0, 1, 2, \cdots$), and that $\psi^k$ is countably additive mod $\mathcal{R}'$. Further,
(3) \[ \left( \sum 2^{-k} \psi_X X \right) \equiv I(X) \mod X', \quad (X \in X'), \]

because \( I(X) = \left[ \sum \Phi^k X \right] = U \left[ \Phi^k X \right] = U \left[ \psi_X X \right] \)
= \( \left[ \sum 2^{-k} \psi_X X \right] \).

Now define, for \( X \in X' \),

\[ m^* X = \int_R \sum 2^{-k} \psi_X X dm'. \]

The integrand is \( X' \)-measurable, non-negative, and \( \leq 1 \), so \( m^* \) is well defined
and is a finite, countably additive measure on \( X' \). We complete this measure
as usual, still calling it \( m^* \), and show

(4) \[ m^* Y = 0 \text{ if and only if } Y \in X. \]

For if \( m^* Y = 0 \), we have \( Y \subseteq X \) where \( X \in X' \) and \( m^* X = 0 \). In the definition
of \( m^* X \), the integrand must be zero almost everywhere \((m')\), which from
(3) and (2) gives \( I(X) \in X' \) and therefore \( Y \in X \). Conversely, if \( Y \in X \),
\( Y \subseteq X \in X' \) where \( I(X) \in X' \), and so \( m^* X = 0 \).

The algebra \( E \) is now defined exactly as in 5.2, except that we use the
new meaning of \( X \); and the measure \( m^* \) on \( X' + X \) induces a positive \( \sigma \)-finite measure \( m^* \) on \( E \). The proof of 5.2(1) no longer applies (as there we
used \( X \subseteq X R \)), but the result itself (the countable chain condition) is a trivial
consequence of the existence of \( m^* \). The operator \( \phi \) is defined just as in 5.3;
the arguments in 5.3 and 5.4 apply unchanged, so that \( \phi \) is a fully homo-
geneous cylinder extension of \( \phi_0 \). Finally, by Theorem 2, we replace the
measure \( m^* \) on \( E \) by a (positive, \( \sigma \)-finite) measure \( m \) which extends \( m_0 \) on \( E_0 \),
and the proof is complete.

References


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