SOME SEMIGROUPS ON AN \( n \)-CELL

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The purpose of this paper is to prove a theorem which is a generalization of a theorem proved by the author in [5]. The latter theorem is a special case of the one presented here. The theorem to be proved is:

**Theorem.** Let \( S \) be a semigroup which is topologically a closed \( n \)-cell, \( n \geq 2 \). Suppose for \( x \) and \( y \) in \( B \), the bounding \((n-1)\)-sphere of \( S \), \( xy = x \).

Then: (1) If \( S = K \), the minimal ideal of \( S \), then \( S \) consists entirely of left zeros, that is, \( xS = x \) for each \( x \) in \( S \).

(2) If \( S \neq K \), then \( K \) is a deformation retract of \( S \) and \( K \) consists entirely of left zeros for \( S \). Also there exists in \( S \) an \( I \)-semigroup \( T \) with the following properties:

(i) \( S \setminus K^0 = BT \), where \( K^0 \) denotes the interior of \( K \).

(ii) If \( b_1 \) and \( b_2 \) are in \( B \) and \( t_1 \) and \( t_2 \) belong to \( T \) and if \( b_1t_1 = b_2t_2 \) then \( t_1 = t_2 \).

(iii) For \( b_1 \) and \( b_2 \) in \( B \), \( t_1 \) and \( t_2 \) in \( T \), \( (b_1t_1)(b_2t_2) = b_1(b_2t_2) \).

For definitions and background material the reader is referred to [6; 11.]

The proof of the theorem is divided into a sequence of lemmas throughout which the hypotheses of the theorem are assumed to hold. The case \( S = K \) is easily disposed of in Lemmas 1, 2 and 3. The remainder of the lemmas is devoted to the case \( S \neq K \). In this case, the general idea is to prove that the relation, \( \leq \), on \( Q \) the Rees quotient of \( S \) by the ideal \( K \), defined by \( a \leq b \) if and only if \( a = bc \) for some \( c \) in \( Q \) is a partial order on \( Q \). Knowing this relation is a partial order, it is possible to construct an \( I \)-semigroup \( J \) in \( Q \) so that \( Q = \pi(B)J \) where \( \pi \) is the natural map from \( S \) onto \( Q \). This \( I \)-semigroup \( J \) is then “lifted” into \( S \) and it is shown that the \( I \)-semigroup \( T \) where \( \pi(T) = J \) satisfies the conclusion of the theorem.

**Lemma 1.** Each element of \( B \) is a right identity for \( S \). If \( s \in S \) and \( n \) a positive integer then there exists an element \( a \in S \) such that \( a^n = s \).

**Proof.** The proof of this lemma depends on the following theorem [4]:

If \( \alpha \) is a continuous function from \( S \) to \( S \) such that \( \alpha \) is the identity on \( B \), then \( \alpha \) maps \( S \) onto \( S \).

To prove the first part of the lemma, let \( b \in B \) be a fixed element of \( B \)
and define \( \alpha: S \rightarrow S \) by \( \alpha(x) = xb_0 \). Then for \( b \) in \( B \), by hypothesis, \( \alpha(b) = bb_0 = b \), hence by the above theorem, \( \alpha \) maps \( S \) onto \( S \). Thus \( Sb_0 = S \) and since \( b_0 \) is an idempotent it follows immediately that \( b_0 \) is a right identity for all of \( S \). Since \( b_0 \) was arbitrary in \( B \), the first part of the lemma follows.

For the remainder of the lemma let \( n \) be a fixed positive integer and define \( \alpha: S \rightarrow S \) by \( \alpha(x) = x^n \) for \( x \in S \). Since \( B \) consists of idempotents \( \alpha \) is the identity on \( B \) and hence maps \( S \) onto \( S \). This, however, implies that each element of \( S \) has an \( n \)th root in \( S \) which is the statement of the lemma.

**Lemma 2.** For \( x \) in \( S \) there exists an idempotent \( e \) in \( S \) such that \( ex = x = xe \).

**Proof.** Let \( p \) belong to \( S \) and let \( \{p_n\} \) be a sequence of elements in \( S \) defined in the following way: \( p_0 = p \), and \( (p_n)^2 = p_{n-1} \). Such a sequence exists by Lemma 1. Let \( Z(\{p_n\}) \) be defined as in \([5]\) and let \( e \) be the idempotent in \( Z(\{p_n\}) \). The author proves in \([5]\) that \( e \) acts as a two-sided identity for all of \( \{p_n\} \) and, in particular \( ep = p = pe \) which is as required by the lemma.

**Lemma 3.** If \( S = K \), then \( xS = x \) for each \( x \) in \( S \).

**Proof.** Since \( S \) is topologically a closed \( n \)-cell, each proper retract of \( S \) has fixed-point property. By Wallace \([9]\) therefore \( S \) is a group or \( K \subseteq E \). Clearly \( S \) is not a group, so \( S = K \) consists entirely of idempotents. Also by Wallace \([9]\), \( eSe = e \) for each \( e \in E \), thus for \( b \in B \), it follows that \( b = bSB = bS \). Now for arbitrary \( x \) in \( S \) by Lemma 1, \( xb = x \) for \( b \in B \), hence \( xS = (xb)S = x(bS) = xb = x \) and the lemma is established.

In the remainder it will be assumed that \( S \not= K \).

**Lemma 4.** \( S \backslash K \) is connected.

**Proof.** Wallace proved in \([8]\) that \( H^p(S) \approx H^p(K) \) and since \( S \) is a closed \( n \)-cell we have \( H^p(K) = 0 \) for all \( p > 0 \). In particular \( H^{n-1}(K) = 0 \), hence \( K \) does not cut \( R^n \) \([4]\) and since \( K \) is contained in the interior of \( S \), \( K \) does not cut \( S \).

**Definition.** For \( x \) and \( y \) in \( S \) with \( x \notin Py \) define \( n(By, x) \), the index of \( By \) relative to \( x \), as defined by Mostert and Shields in \([6]\). That is:

When \( x \in \overline{By} \), the mapping \( f: B \rightarrow S \backslash x \) defined by \( f(b) = by \) induces a homomorphism \( f^*: H^{n-1}(S \backslash x) \rightarrow H^{n-1}(B) \) where \( H^{n-1}(A) \) denotes the \( (n-1) \)-Čech cohomology group of \( A \) with integer coefficients. Since \( H^{n-1}(B) \) is isomorphic to the integers there exists a least positive integer \( k \) such that \( k \) generates \( f^*(H^{n-1}(S \backslash x)) \). For such a pair \( x \) and \( y \) in \( S \) define \( n(By, x) \) to be \( k \).

**Lemma 5.** If \( A \) is a connected space and \( \sigma: A \rightarrow S \) and \( \tau: A \rightarrow S \) are continuous functions such that \( \tau(t') \in Bx(t) \) for each \( t \) and \( t' \) in \( A \), if \( \sigma(A) \) is compact or if \( \tau \) is a constant, then \( n(Bx(t), \sigma(t)) = n(Bx(t'), \sigma(t')) \) for \( t \) and \( t' \) in \( A \).

**Proof.** Assume \( \sigma(A) \) is compact. Since \( A \) is connected it suffices to show that for each \( t \) in \( A \) there exists an open set \( U \) containing \( t \) such that for \( x \)
and \( y \) in \( U \), \( n(B\sigma(x), \tau(x)) = n(B\sigma(y), \tau(y)) \). To show the existence of such \( U \), let \( t_0 \) belong to \( A \). By hypothesis \( \tau(t_0) \) is not an element of \( B\sigma(A) \) so there exists an open \( n \)-cell \( O_1 \) in \( S \) such that \( \tau(t_0) \in O_1 \) and \( O_1^* \cap B\sigma(A) = \emptyset \). Hence \( B\sigma(A) \subseteq S \setminus O_1^* \). By hypothesis \( \tau \) is a continuous function so there exists an open set \( U \) in \( A \) containing \( t_0 \) with \( \tau(U) \subseteq O_1 \). The claim is now made that \( n(B\sigma(t_0), \tau(t_0)) = n(B\sigma(s), \tau(s)) \) for each \( s \) in \( U \). To establish the claim let \( s \) belong to \( U \) and define maps \( \lambda, \lambda_{t_0}, m_0, I \) and \( J \) in the following way:

\[
\lambda_s : B \to B \times A \quad \text{by} \quad \lambda_s(b) = (b, s),
\]

\[
\lambda_{t_0} : B \to B \times A \quad \text{by} \quad \lambda_{t_0}(b) = (b, t_0),
\]

\[
m_0 : B \times A \to S \quad \text{by} \quad m_0(b, t) = b \sigma(t),
\]

and \( I \) and \( J \) are the injection maps from \( S \setminus O_1^* \) to \( S \setminus \tau(s) \) and \( S \setminus \tau(t_0) \) respectively. Then it is easily seen that the mappings

\[
\theta_s : B \to S \setminus \tau(s) \quad \text{defined by} \quad \theta_s(b) = b \sigma(s)
\]

and

\[
\theta_{t_0} : B \to S \setminus \tau(t_0) \quad \text{defined by} \quad \theta_{t_0}(b) = b \sigma(t_0)
\]

are given by

\[
\theta_s = I m_1 \lambda_s \quad \text{and} \quad \theta_{t_0} = J m_0 \lambda_{t_0}
\]

where \( m_1 \) is \( m_0 \) with the range restricted to \( S \setminus O_1^* \).

The following sequences now arise from these functions:

\[
\begin{align*}
H^{n-1}(S \setminus \tau(s)) & \xrightarrow{I^*} H^{n-1}(S \setminus O_1^*) \xrightarrow{\lambda_s^* m_1^*} H^{n-1}(B) \\
\text{and the same sequence obtained by replacing \( s \) by \( t_0 \) and \( I^* \) by \( J^* \).}
\end{align*}
\]

Since \( O_1 \) is an open \( n \)-cell for any \( y \) in \( O_1 \) the injection map from \( S \setminus O_1^* \) into \( S \setminus y \) induces an isomorphism from \( H^{n-1}(S \setminus y) \) onto \( H^{n-1}(S \setminus O_1^*) \) [1]. Hence \( I^* \) and \( J^* \) are isomorphisms onto. By S. T. Hu [3], \( \lambda_s^* = \lambda_{t_0} \) so it follows that

\[
\lambda_{t_0}^* m_1^* = \lambda_{t_0}^* m_1^*.
\]

Looking at the above sequences it is easily seen that

\[
\theta_{t_0}^*(H^{n-1}(S \setminus \tau(s))) = \theta_{t_0}^*(H^{n-1}(S \setminus \tau(t_0))).
\]

Since \( I^* \) and \( J^* \) are isomorphisms onto and

\[
\theta_s^* = \lambda_s^* m_1^* I^*, \quad \theta_{t_0}^* = \lambda_{t_0}^* m_1^* J^*.
\]

From this we obtain that

\[
n(B\sigma(t_0), \tau(t_0)) = n(B\sigma(s), \tau(s))
\]

and the first part of the proof of the lemma is complete. The remainder of the proof follows similarly.
**Lemma 6.** If $x$ belongs to $S \setminus B$, then $n(Bb, x) = 1$ for each $b \in B$.

**Proof.** Let $b_0$ belong to $B$ and let $x \in S \setminus B$. Define $\theta$ from $B$ to $S \setminus x$ by $\theta(b) = bb_0$. By hypothesis on the multiplication in $B$, $\theta(b) = b$ for each $b$ in $B$. Let $\delta: S \setminus x \to B$ be a continuous function from $S \setminus x$ onto $B$ such that $\delta(b) = b$ for each $b$ in $B$. If $\phi$ denotes the function from $B$ onto $B$ defined by $\phi(b) = \delta(b)$ then $\phi$ is the identity function so that

$$\phi^*: H^{n-1}(B) \to H^{n-1}(B)$$

is an isomorphism. From this it follows that

$$\theta^*: H^{n-1}(S \setminus x) \to H^{n-1}(B)$$

is onto since

$$\phi^* = \theta^* \delta^*.$$ 

Thus by the definition of $n(Bb_0, x)$ we have $n(Bb_0, x) = 1$ and the lemma is established.

**Lemma 7.** For $b$ in $B$ and $x$ in $S$ with $b \in Bx$, $n(Bx, b) = 0$.

**Proof.** Let $\theta: B \to S \setminus b$ be defined by $\theta(s) = sx$. Since $b \in Bx$ it follows that $Bx \subset S \setminus B$. For if $Bx \cap B$ were nonvoid, then for $y \in Bx \cap B$ there would exist $b_0 \in B$ such that $y = b_0x$ and in virtue of the multiplication in $B$, that $b = by = b(b_0x) = (bb_0)x = bx$ contrary to the assumption that $b \in Bx$. Hence $Bx$ is a closed subset of $S$ contained in $S \setminus B$. Since $B$ is the boundary of $S$ relative to $R^n$ there exists a subset $S_0$ of $S$ with the following properties: $S_0$ is closed, $S_0$ is topologically equivalent to $S$ and $Bx \subset S_0 \subset S \setminus B$. Now define functions $i_1$ and $i_2$ by

$$i_1: Bx \to S_0 \quad \text{and} \quad i_1(y) = y \quad \text{for} \ y \in Bx,$$

$$i_2: S_0 \to S \setminus b \quad \text{and} \quad i_2(y) = y \quad \text{for} \ y \in S_0.$$ 

Also define

$$\theta_1: B \to Bx \quad \text{by} \quad \theta_1(y) = yx \quad \text{for} \ y \in B.$$

Clearly $\theta = i_2 \theta_1 i_1$ so that $\theta^* = \theta_1^* i_1^* i_2^*$. Looking at the sequence defined by these functions it follows that $\theta^*$ is the zero homomorphism, for we have:

$$H^{n-1}(S \setminus b) \xrightarrow{i_2^*} H^{n-1}(S_0) \xrightarrow{i_1^*} H^{n-1}(Bx) \xrightarrow{\theta_1^*} H^{n-1}(B)$$

and $H^{n-1}(S_0) = 0$. From this it follows that $n(Bx, b) = 0$.

**Lemma 8.** For $a \in S \setminus K$, $a$ belongs to $BS$. Thus each element of $S \setminus K$ has a two-sided identity belonging to $B$.

**Proof.** Suppose there exists an element $a_0$ in $S \setminus K$ such that $a_0 \in BS$. Let
If $k \in K$ and $f \in B$ be fixed. Clearly $Bk \cap S \setminus K = \emptyset$ and since $S \setminus K$ is connected it follows from Lemma 5, taking $A = S \setminus K$, $\tau =$ identity and $\sigma =$ constant map $k$, that $n(Bk, x) = n(Bk, f)$ for each $x \in S \setminus K$. But $a_0$ belongs to $S \setminus K$ so that $n(Bk, f) = n(Bk, a_0) = 0$ by Lemma 7.

Now using the assumption that $a_0 \in BS$, it follows in a similar way from Lemma 5, taking $A = S \setminus K$, $\sigma =$ identity, and $\tau =$ constant map $a_0$, that $n(Bf, a_0) = n(Bk, a_0)$. Hence by Lemma 6, $n(Bk, a_0) = 1$. This contradiction establishes the fact that $a_0 \in BS$. The remainder of the lemma follows quite easily since each element of $B$ is an idempotent and a right identity for all of $S$.

**Lemma 9.** If $a \in S \setminus K$, then $Ba \neq a$.

**Proof.** To prove this lemma let us assume that $Ba = a$ for some element $a$ in $S \setminus K$. The claim is now made that with this assumption $B(S \setminus K) = S$. If this were not the case then there would exist an element $p \in S$ with $B(S \setminus K) \subset S \setminus p$. Since $B \subset B(S \setminus K)$ it follows that $p \in B$ hence it is possible to define a function $\delta : S \setminus p \to B$ such that $\delta(p) = b$ for each $b$ in $B$.

Now for each $x$ in $S \setminus K$ define a function $\theta_x : B \to B$ by $\theta_x(b) = \delta(\delta_x(b))$. For each $b$ in $B$, $\theta_b$ is the identity and for $a$, $\theta_a$ is a constant. From this it can be concluded that the identity function on $B$ is null-homotopic, since $S \setminus K$ is connected. This contradiction establishes the fact that $B(S \setminus K) = S$.

Since $B(S \setminus K) = S$ and $K$ is nonempty, there exists an element $g \in B$ and $x \in S \setminus K$ such that $g \cdot x = x$. Hence $x = b \cdot x = (bg) \cdot x = b(gx) \in BK \subset K$ contrary to the fact that $x \in S \setminus K$. From this we obtain that $Ba \neq a$ for each $a$ in $S \setminus K$.

**Lemma 10.** For $a$ in $S \setminus K$, $Ja = Ba$ where $Ja$ denotes the set of elements in $S$ generating the same two-sided ideal as $a$.

**Proof.** Before proving this lemma let us note that the ideal generated by an element $x$ in $S \setminus K$ is $SxS$. If $J(x)$ denotes the ideal generated by $x$ then $J(x) = x \cup xS \cup Sx \cup SxS = SxS$ since $x$ has a two-sided identity in $S$.

It follows from Lemma 1 that $Ba \subset J_a$ for if $b \in B$ then $J(ba) = S(ba)S = (Sb)aS = SaS = J(a)$ so that $ba \in J_a$.

It remains only to show that $J_a \subset Ba$. First let us note that $Ba \cap K = \emptyset$ since $a \notin K$, as in the proof of Lemma 9. Hence $K \subset S \setminus Ba$, and if $P$ denotes the component of $S \setminus Ba$ containing $K$ it follows from Wallace [9] that $P^* \cap P = Ba$. For an element $p$ in $P \setminus K$, $BP \cap Ba = \emptyset$ for if not then $b_1p = b_2a$ for elements $b_1$ and $b_2$ in $B$. By Lemma 8 there exists $b$ in $B$ such that $bp = p$, hence $p = bp = (bb_1)p = b(b_2a) = b_2a = ba$ contrary to the fact that $p \in P$. Hence $BP$ does not meet $Ba$ and since $BP \cap P$ contains $p$, $BP$ is connected and $P$ is a component of $S \setminus Ba$ we have $BP \subset P$. By assumption $p \in K$, hence $K \subset S \setminus BP$, as in the proof of Lemma 9. Let $Q$ be the component of $S \setminus BP$ containing $K$. Clearly $K \subset Q \subset P$ and as before $Q^* \cap Q = BP \subset P$. Let $I(p) = J(p) \setminus J_p$. Then $I(p)$ must contain $K$, $I(p)$ does not meet $BP$ and by
Wallace [9], $I(p)$ is connected and $I(p)^* = J(p)$. The last statement follows from the fact that $Bp \subseteq J_p$ and by Lemma 9, $Bp \neq p$ so that $J_p \neq p$. Since $I(p)$ is connected and contains $K$, $I(p) \subseteq Q$, hence $J(p) = I(p)^* \subseteq Q^* = Q \cup Bp \subseteq P$. From this discussion we obtain that $J(p) \subseteq P$ for each $p \in P \setminus K$, hence $J_p \subseteq P$. But $I(a) \subseteq P$ so that $J(a) = I(a)^* \subseteq P^* = P \cup Ba$, therefore $J_a \subseteq Ba$ and Lemma 10 is established.

**Definition.** For $a$ and $b$ in $S \setminus K$ define $a \leq b$ if and only if there exists an element $c$ in $S \setminus K$ such that $a = bc$.

**Lemma 11.** $\leq$ as defined above is a partial order on $S \setminus K$.

**Proof.** (i) Since $a \in S$, $a = af$ for $f \in B$, so that $a \leq a$ and $\leq$ is reflexive.

(ii) If $a$ and $b$ belong to $S \setminus K$ and $a \leq b$, and $b \leq a$, then there exist elements $c$ and $d$ in $S \setminus K$ such that $a = bc$ and $b = ad$. Thus $aS = (bc)S = b(cS) \subseteq bS = (ad)S = a(dS) \subseteq aS$, or $aS = bS$. Hence $SaS = SbS$ so that $J_a = J_b$ and by Lemma 10, $Ba = Bb$. Since $a$ and $b$ both belong to $S \setminus K$ there exist elements $e$ and $f$ in $B$ such that $ea = a$ and $fb = b$. Now $a \in Ba = Bb$ so that $a = gb$ for some $g \in B$. From these equalities it follows that $a = ea = e(gb) = (eg)b = eb = e(ad) = (ea)d = ad = b$ so that $\leq$ is antisymmetric.

(iii) Clearly $\leq$ is transitive.

(i), (ii) and (iii) show that $\leq$ is a partial order on $S \setminus K$.

**Notation.** For the minimal ideal $K$ in $S$, let $Q$ denote the Rees Quotient of $S$ by $K$ and let $\pi$ denote the natural map from $S$ to $Q$. By Rees [7], $Q$ is a compact connected semigroup with zero, $\pi(K)$, and $\pi$ is continuous and a homomorphism.

It should be noted at this point that $\pi$ restricted to $S \setminus K$ is an isomorphism. For this reason, in the discussion that follows $S \setminus K$ and $\pi(S \setminus K)$, the former a subset of $S$ and the latter a subset of $Q$ will be considered the same. This identification will make the discussion simpler and somewhat shorter.

**Lemma 12.** There exists an $I$-semigroup $J \subseteq Q$ such that $Q = BJ$.

**Proof.** Let $f$ be a fixed element in $B$. Then $fQ$ is a compact connected semigroup with identity $f$ and zero $\pi(K)$. Define a partial order on $fQ$ by $a \leq b$ if and only if $a = bc$ for some $c \in fQ$. By Lemma 11, the fact that $f$ is a right identity for all of $S$ and the fact that $\pi(K)$ is a zero for $fQ$, it is easily seen that $\leq$ is a closed partial order on $fQ$. Hence by Koch [2] there exists an $I$-semigroup $J \subseteq fQ$ with endpoints $f$ and $\pi(K)$.

The next step in the proof is to show that $BJ = Q$. If it were the case that $S = BJ \cup K$, where $J_0 = J \setminus \pi(K)$, it would follow immediately that $Q = \pi(S) = \pi(BJ_0 \cup \pi(K)) = BJ$. Hence it suffices to show that $S = BJ_0 \cup K$.

Let us assume, to the contrary, that there exists an element $p$ in $S$ with $p$ not in $BJ_0 \cup K$. Since $J_0$ is a half-open interval and $J = J_0 \cup \pi(K)$ is closed there exists an element $k_0$ in $K$ with $J_0 \subseteq J_0 \cup k_0 \subseteq J_0^*$, where $J_0^*$ denotes the closure of $J_0$ in $S$. Since $J_0$ is connected, $J_0 \cup k_0$ is connected and by assump-
tion \( p \in B(J_0 \cup k_0) \). Thus by Lemmas 5 and 7, \( n(Bp, k_0) = n(Bp, f) = 0 \). Now \( p \in S \setminus K \) and since \( S \setminus K \) is connected and \((B(S \setminus K)) \cap K = \emptyset\), it follows that \( n(Bp, k_0) = n(Bf, k_0) = 1 \), again by Lemmas 5 and 6. This is a contradiction so \( p \) must belong to \( B(J_0 \cup k_0) \). With the preceding remarks the lemma is established.

**Lemma 13.** There exists an element \( k_0 \) in \( K \setminus K^0 \) such that if \( T \) denotes \( J_0^* \), then \( T = J_0 \cup k_0 \) and \( K \setminus K^0 = Bk_0 \).

**Proof.** From the definition of \( J_0 \) we see that \( \pi(J_0^* \setminus J_0) = \pi(K) \), hence \( J_0^* \setminus J_0 \subseteq K \). Now let \( k_0 \in J_0^* \setminus J_0 \). The claim is made that \( K \setminus K^0 = Bk_0 \). To prove this claim let \( k = gk_0 \) for some \( g \in B \) and assume \( k \in U \), an open set. Since \( k = gk_0 \) and \( k \in U \), there must exist open sets \( V_0 \) and \( V_1 \) containing \( g \) and \( k_0 \), respectively, such that \( V_0 \cap V_1 \subseteq U \). Now \( k_0 \in J_0^* \setminus J_0 \) and \( V_1 \) is open containing \( k_0 \), hence there exists an element \( t \) in \( J_0 \) with \( t \in V_1 \). Since \( t \in S \setminus K \), it follows that \( gt \) also belongs to \( S \setminus K \) so that \( U \cap S \setminus K \neq \emptyset \). Since \( k \) was an arbitrary element in \( Bk_0 \), it follows that \( Bk_0 \subseteq K \setminus K^0 \).

Conversely, let \( k \in K \setminus Bk_0 \). If it can be shown that \( k \in K^0 \) then it will be established that \( Bk_0 = K \setminus K^0 \). To prove \( k \in K^0 \), let \( P \) be the component of \( S \setminus Bk_0 \) containing \( k \). As before, since \( J_0 \cup k_0 \) is connected \( n(Bk_0, k) = n(Bf, k) = 1 \). If it were the case that \( B \subseteq P \), then it would be true that \( n(Bk_0, k) = n(Bk_0, f) = 0 \) since \( P \) is connected and does not meet \( Bk_0 \). This is a contradiction to the above statement that \( n(Bk_0, k) = 1 \), hence \( B \) does not meet the component \( P \). Thus the boundary of \( P \) relative to \( R^* \) is contained in \( Bk_0 \) which is a subset of \( K \). Now if \( P \) is not contained in \( K \), then \( K \) is a closed proper subset of \( P \cup K \) containing the boundary of \( P \cup K \). Hence

\[
i^*: H^{n-1}(P \cup K) \to H^{n-1}(K)
\]

is not onto where \( i^* \) is induced by the injection map [4]

\[
i: K \to P \cup K.
\]

By Wallace [8], however, \( H^{n-1}(K) \approx H^{n-1}(S) = 0 \), so that \( i^* \) is onto. Thus \( P \cup K = K \), that is \( P \subseteq K \). Since \( P \) is a component of an open set in \( S \), \( P \) is also open and therefore \( k \in P \). This completes the proof of the statement that \( Bk_0 = K \setminus K^0 \).

In order to complete the proof of this lemma it remains only to show that \( T = J_0 \cup k_0 \). By definition of \( T \) we have \( T \subseteq fS \) since \( J_0 \subseteq fS \) and therefore \( T = J_0^* \cap (fS)^* = fS \). This shows that \( f \) is a two-sided identity for \( T \). In the above argument it was shown that \( J_0^* \setminus J_0 \subseteq K \setminus K^0 = Bk_0 \). Now let \( k \in T \setminus J_0 \), then \( k = gk_0 \) for some \( g \in B \) and \( fk = k, f(k_0) = k_0 \). Hence \( k = f(kgk_0) = (fg)k_0 = f(k_0) = k_0 \) so that \( T \setminus J_0 = k_0 \). Thus \( T = J_0 \cup k_0 \) and the proof of the lemma is complete.

**Lemma 14.** \( T \) is an \( I \)-semigroup with zero \( k_0 \) and identity \( f \). Also \( BT = S \setminus K^0 \).
Proof. Clearly $T$ is a semigroup and an arc with zero $k_0$ and identity $f$. Also $S\setminus K^0 = S \setminus K \cup K \setminus K^0 = BJ_0 \cup Bk_0 = B(J_0 \cup k_0) = BT$. This concludes Lemma 14.

Lemma 15. For $k$ in $K$, $kS = k$.

Proof. First let us note that by Wallace [9], $K \subseteq E$ and $kSk = k$ for each $k \in K$. If $K^0 = \emptyset$, then $Bk_0 = K$ so that $k_0K = k_0(Bk_0) \subseteq k_0Sk_0 = k_0$. Thus $k_0S = k_0$ since $k_0S \subseteq k_0K$. If $K^0 \neq \emptyset$ then $Bk_0$, since it is the boundary of $K$ relative to $R^n$ is an $((n-1), G)$-rim for $K$, (see [10]). Hence by the dual of Wallace’s theorem [10], if $k \in K$ and $(Bk_0)k = Bk_0$ it follows that $Kk = k$. Since $k_0^2 = k_0$, we have $(Bk_0)k_0 = Bk_0 = Bk_0$ so that $Kk_0 = K$. Hence $k_0S \subseteq k_0K = k_0(Kk_0) = k_0$.

In either case, $K^0 = \emptyset$ or $K^0$ nonempty it has been shown that $k_0S = k_0$. Now let $k$ be an arbitrary element of $K$. Then $k_0k = k_0$ so that $kk_0 = k(k_0k) = k$ since $kSk = k$. Hence $kK = (kk_0)K = k(k_0K) = kk_0 = k$ and it follows that $kS = k$ which concludes the proof of the lemma.

Lemma 16. Let $t_0$ and $t_1$ belong to $T$ and let $b_0$ and $b_1$ be elements of $B$. Then $(b_0t_0)(b_1t_1) = b_0(t_0t_1)$ and if $b_0t_0 = b_1t_1$ then $t_0 = t_1$.

Proof. This lemma follows immediately from the fact that $f$ is an identity for $T$ and $fb = f$ for each $b$ in $B$.

Lemma 17. $K$ is a deformation retract of $S$.

Proof. Define $\theta: S \times T \rightarrow S$ by $\theta(s, t) = st$. $T$ is a closed interval with endpoints $f$ and $k_0$, $\theta(s, f) = sf = s$ and $\theta(s, k_0) = sk_0 \in K$. Also for $k \in K$, $\theta(k, k_0) = kk_0 = k$. Since $\theta$ is continuous it follows that $K$ is a deformation retract of $S$.

With Lemma 17 the proof of the theorem is now complete.

Example. An example of a semigroup described by the theorem and having a nontrivial kernel for $n = 2$ can be constructed as follows.

Let $K_0$ be a closed two-cell and $B_0$ the bounding 1-sphere of $K_0$. Define multiplication in $K_0$ by $xy = x$ for all $x$ and $y$ in $K_0$. Let $T_0$ be the closed unit interval with real multiplication. Then if $S = (K_0 \times \{0\}) \cup (B_0 \times T_0)$ and products are defined in $S$ by coordinate-wise multiplication, $S$ is a semigroup as described by the theorem, where $B_0$, of course, is $B_0 \times \{1\}$.

Clearly $S$ is topologically a closed two-cell and is a semigroup with a nontrivial kernel $K = K_0 \times \{0\}$. If $k_0$ is a fixed element of $B_0$, then $T = \{k_0\} \times T_0$ is an $I$-semigroup which has the property that $S \setminus K^0 = BT$.

In this example, for $a \in S \setminus K$, the representation of $a = bt$ for $b \in B$ and $t \in T$ is unique. In [5], the author gives an example of such a semigroup described above but in it there exists an element in $S \setminus K$ for which this representation is not unique.

For $n = 2$, different examples may be constructed by varying the multiplication of the $I$-semigroup $T_0$. (See [6].)
For any integer $n > 2$, examples can be constructed in a similar way. That is, let $K_0$ be a closed $n$-cell with $B_0$ the bounding $(n-1)$-sphere and follow the same construction as above.

**Bibliography**


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