SIMPLIFYING THE STRUCTURE OF SECOND ORDER
PARTIAL DIFFERENTIAL EQUATIONS

BY

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1. The problem. Consider the differential equation

\[(1) \Delta_2 A + b^i A_i + cA = 0\]

where the summation convention is being used and

\[(2) \quad \Delta_2 A = \frac{1}{g^{1/2}} \frac{\partial}{\partial x^i} \left( g^{1/2} g^{ij} \frac{\partial A}{\partial x^j} \right) = g^{ij} A_{,ij} + g^{ij} (A_{,ij} - \Gamma^k_{ij} A_k)\]

is the Laplace-Beltrami operator. Here a comma indicates covariant differentiation, a dot indicates (ordinary) differentiation, \(A_i = A_{,i}, g = \det(g_{ij})\) and is always assumed to be positive, and \(\Gamma^k_{ij}\) are the Christoffel symbols of the second kind with respect to \((g_{ij})\). All the functions in this paper are assumed to be sufficiently smooth. If (1) has a positive solution \(A_0\) in a domain \(D\), then the structure of (1) can be simplified. Indeed, setting \(\tilde{A} = A_0 A\), we reduce (1) to

\[(3) \quad \Delta_2 \tilde{A} + b^i \tilde{A}_i = 0\]

where the \(b^i\) depend on \(b^i, A_0\).

In this paper we consider the equation

\[(4) \quad \Delta_2 A + cA = 0 \quad (c \neq 0)\]

and wish to obtain an equivalent equation

\[(5) \quad \Delta_2 \tilde{A} + c_0 \tilde{A} = 0\]

where \(c_0\) is a constant, preferably zero. Here, \(\tilde{A}\) is the Laplace-Beltrami operator with respect to a different metric. We shall prove that it is possible to reduce (4) to (5), and that we can take \(c_0 = 0\) if \(c \leq 0\). We also get similar results for parabolic equations.

Ishii [6] has considered conformal mappings which transfer solutions of \(\Delta_2 A = 0\) into solutions of \(A_i A = 0\). Ingraham [4] considered the question of eliminating the \(b^i\) from equation (1) when \(c = 0\), on account of replacing the
covariant derivatives in $\Delta_2$ by covariant derivatives with respect to some non-Riemannian affine connections. In [5] he considered the parabolic case.

2. The transformation. We try to perform conformal mapping

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}$$

and set

$$\bar{A} = e^{h(\sigma)} A$$

where $\sigma$ and $h(\sigma)$ are to be determined. Using the formulas [3, p. 89]

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_{ij} - g_{ij} g^{km} g_{mk}$$

we find, after some elementary calculation,

$$\Delta_2 \bar{A} + c_0 \bar{A} = e^{-2\sigma} e^{h(\sigma)} \left\{ (\Delta_2 A + c A) + (2h'(\sigma) + n - 2) g^{ik} g_{ik} A_k 
+ [h''(\sigma) \Delta_1 \sigma + h'(\sigma) \Delta_2 \sigma + h'(\sigma)(h'(\sigma) + n - 2) \Delta_1 \sigma - c + e^{2\sigma} c_0] A \right\}$$

where $\Delta_\sigma = g^{ij} \sigma_{ij}$. Hence, (4) and (5) are equivalent in any subdomain of $D$ if and only if the coefficients of $A$ and $A_k$ vanish. The vanishing of the coefficients of the $A_k$ is equivalent to

$$h(\sigma) = \frac{2 - n}{2} \sigma.$$  

Assuming $n \geq 3$ we find that the coefficient of $A$ vanishes if and only if

$$\Delta_2 \sigma + \frac{n - 2}{2} \Delta_1 \sigma + \frac{2c}{n - 2} - \frac{2c_0}{n - 2} e^{2\sigma} = 0.$$  

Setting

$$u = \exp \left\{ \frac{n - 2}{2} \sigma \right\},$$

(10) becomes

$$\Delta_2 u + cu = c_0 u^{(n+3)/(n-3)}, \quad u > 0.$$  

We have thus proved:

**Theorem 1.** Let $n \geq 3$. Equation (4) is reducible in a domain $D$ to equation (5) by means of the conformal transformation (6), (7) if and only if equation (12) has a positive solution in $D$.

Due to the invariance of the operators of (4), (5) with respect to local change of the $x$-coordinates, **Theorem 1 holds also when $D$ is a domain on a manifold.**
Quite incidentally, if $c$ and $c_0$ are the scalar curvatures corresponding to the metric tensors $g_{ij}$ and $\tilde{g}_{ij}$ related by (6), then (10) is known to hold (see [3, p. 90]). Recently Yamabe [9] proved that on a compact manifold, for any positive definite metric there exists a conformal transformation which yields a new metric with constant scalar curvature. His proof does not make use of the specific nature of the function $c$, that is, he proves that for any smooth function $c$ on a compact Riemannian manifold with a positive definite metric there exists a constant $c_0$ and a positive smooth function $u$, defined on the whole manifold, such that (12) is satisfied. Hence we have:

**Theorem 2.** Given a compact Riemannian manifold $\mathbb{R}^n$, $n \geq 3$, with a positive definite metric, equation (4) can globally be reduced to equation (5) by means of (6), (7), (9), where $c_0$ is a constant depending on $c$.

If $c_0 \neq 0$ then the transformation is uniquely determined, up to a constant multiple $\neq 0$. Indeed, if $u_1$, $c_1$ is another solution, then $w = u_1/u$ satisfies $\Delta w + c_0 w = c_1 w^{(n+2)/(n-2)}$. Setting $w(x^0) = \max w(x)$, we then have $\Delta w \leq 0$, $c_0 w \leq 0$ at $x^0$; hence $c_1 \leq 0$. It follows that $\Delta w + c_0 w \leq 0$. The minimum principle now yields $w = \text{const}$.

**Corollary.** Let $D$ be an $n$-dimensional bounded domain, $n \geq 3$, and let $(g_{ij})$ be a positive matrix. Then (4) is reducible to (5), in the whole domain $D$, by means of (6), (7), (9), where the constant $c_0$ depends on $c$.

Under some conditions on $c(x)$ we can even take $c_0 = 0$. Thus we have:

**Theorem 3.** Let $D$ be an $n$-dimensional domain, $n \geq 3$, either bounded or unbounded but with finite boundary, and let $(g_{ij})$ be a positive matrix. If $c(x) \leq 0$ then (4) is reducible, in the whole domain $D$, to (5) with $c_0 = 0$, by means of a transformation (6), (7), (9).

**Proof.** We only have to establish the existence, in $D$, of a positive solution of the equation

$$\Delta u + cu = 0.$$ 

If $D$ is bounded, we solve (13) for any positive boundary values and thus obtain a positive solution, using the maximum principle. If $D$ is unbounded then the existence of a positive solution follows by recent results of Meyers and Serrin [7].

We note that if the diameter of $D$ is sufficiently small it is not necessary to make any assumption on $c$. We also remark that for a given bounded domain $D$ we can get $c_0 = 0$ if $c(x) \leq \epsilon$ where $\epsilon$ is sufficiently small, depending on $D$. However, if $\epsilon$ is not small there are in general no positive solutions of (13) in $D$.

For simplicity, we produce a counter-example for $n = 3$, $\Delta$ being the Laplace operator $\Delta$, and $c(x) = k^2 > 0$. Let $x^0$ be a point in $D$ whose distance from
the boundary of $D$ is $H$. We claim that there are no positive solutions of
$\Delta u + k^2 u = 0$ in $D$ if $k > \pi / H$.

Indeed, we apply the Pizetti formula [1, p. 259]
\[
\frac{1}{4\pi R^3} \int_{S_R} u(x) dS = \sum_{\lambda=0}^{n} \frac{R^{2\lambda} \Delta^{\lambda} u(x^0)}{(2\lambda + 1)!} + \frac{1}{4\pi(2m + 1)!} \int_{K_R} \frac{(R - r)^{2m+1} \Delta^{m+1} u(x)}{R^r} dV
\]
where $K_R$ is a ball of radius $R$, center $x^0$ and surface $S_R$, and $r = |x - x^0|$. Taking $m \to \infty$ we obtain
\[
\frac{1}{4\pi R^3} \int_{S_R} u(x) dS = \frac{\sin(kR)u(x^0)}{kR}.
\]

Now, if $u$ is positive in $D$ then it follows that $\sin kR > 0$; hence $kH \leq \pi$ which is a contradiction.

The case $n = 2$. From (8) we conclude that a reduction of (4) to (5) is possible if $h(\sigma) = 0$ and if
\[
c = e^{2\sigma} c_0.
\]

Hence, the reduction by (6), (7) is possible if and only if $sgn c(x) =$ const. and $c_0$ can be taken to be either any positive or any negative number, depending on $sgn c(x)$. Note that $(g_{ij})$ can be taken to be any nonsingular matrix.

3. Parabolic equations. Consider the parabolic equation
\[
a \frac{\partial A}{\partial t} = \Delta_2 A + cA \quad (a > 0)
\]
where $x$ varies in an $n$-dimensional domain $D$ ($n \geq 3$) and $0 \leq t < \infty$. The coefficients are functions of $(x, t)$ and $\Delta_2$ is elliptic. Performing the transformation (6), (7), (9) with $\sigma$ depending also on $t$, we find that (15) is reduced to
\[
\dot{\tilde{A}} = \Delta_2 \tilde{A} + c_0 \tilde{A} \quad (\tilde{a} > 0)
\]
if and only if there exists a positive solution of
\[
a \frac{\partial u}{\partial t} = \Delta_2 u + cu - c_0 u^{(n+2)/(n-2)}
\]
and then $\sigma$ is defined by (11). Also,
\[
\dot{\tilde{a}} = e^{-2\sigma} a.
\]

From now on we assume that if $D$ is unbounded then the functions $a$, $1/a$, $c$, $g_{ij}$, $g^{ij}$, $\partial a / \partial x^i$, $\partial^2 a / \partial x^i \partial x^j$, $\partial / \partial t$, $\partial g^{ij} / \partial x^i$, $\partial^2 g^{ij} / \partial x^i \partial x^j$, $\partial g^{ij} / \partial t$ are Hölder continuous and bounded for $x$ in $D$ and $t$ in finite intervals. We then claim
that for any $D$, there exists a positive solution of (17) with $c_0 = 0$. It is enough to establish it when $D$ is the whole $n$-dimensional space $E_n$.

Let $K(x, t; \xi, \tau)$ be the fundamental solution of (17) constructed by Dressel [2]. Then for any positive function $\phi(\xi)$ which tends to a positive limit $B$ as $|\xi| \to \infty$ we have a solution of (17) with $c_0 = 0$ for $x \in E_n$, $0 < t < \infty$, in the form

$$u(x, t) = \int_{E_n} K(x, t; \xi, 0) \phi(\xi) d\xi.$$ 

This solution is positive on $t=0$, and it tends to $B$ as $|x| \to \infty$, uniformly in $t$ in finite intervals $0 \leq t \leq T$, $T > 0$. Hence by appropriately applying the maximum principle [8] we conclude that $u(x, t) > 0$. We have thus proved:

**Theorem 4.** Let $D$ be any domain in $E_n$, $n \geq 3$ and let the coefficients of the parabolic equation (15) satisfy the boundedness assumptions mentioned above. Then equation (15) is reducible to equation (16) with $c_0 = 0$, by means of (6), (7), (9).

Note that no assumption is being made on the signature of $c$.

If $n = 2$, we first make a transformation of the form $A = e^{nu}A$ and obtain a new coefficient $c$ which is positive. We then can reduce (15) to (16) since equation (14) can then be solved.

**References**


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