p-ADIC GROUPS OF TRANSFORMATIONS

BY

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Introduction. There is the important conjecture that every compact group G of transformations on a manifold M is a Lie group. As is well known, if G is not a Lie group then G must contain a p-adic group which in turn acts effectively on M.

Recently, Yang has shown [7] that if a p-adic group does act on M, then the orbit map raises the integral cohomology dimension by 2. To obtain this result Yang extends the Smith special homology theory to maps of prime power period, using the reals mod 1 as coefficients.

In this paper the authors compute the cohomology of the universal classifying space B_G, for $G = A_p$, the p-adic group, and $G = \Sigma_p$, the p-adic solenoid. These results are then used with methods fitting the general scheme of [1] to prove the dimension theorems of Yang.

We conclude the paper with fixed point theorems for p-adic solenoids.

To calculate the cohomology of B_{A_p} and B_{Σ_p} we use Milnor's method of joins. The n-fold join of the p-adic solenoid is obtained as the limit of an inverse sequence of n-fold joins of the circle. The n-fold join of the circle is fibered (a principal fibration) by the circle with base space complex projective space. Equivariant maps from one fibration in the sequence to another are constructed. The inverse limit of these fibrations gives a principal fibration by the p-adic solenoid of the n-fold join of the p-adic solenoid. The universal classifying space is the inverse limit of the universal classifying space of the circle group. Then we obtain the cohomology of B_{Σ_p} by taking the direct limit of the cohomology of the complex projective spaces in the inverse sequence of fibrations.

In §2 we show how to construct a space M' on which the p-adic solenoid acts from a space M on which the p-adic group acts. This is important because the p-adic solenoid acts trivially on the cohomology of M'. The spectral sequences associated to the actions of A_p on M are complicated by the lack of simplicity of the coefficients. However, by considering the action of $A_p \subset \Sigma_p$ on M' the spectral sequences become more manageable. The induced action of $A_p \subset M'$ corresponds to a certain global situation. Results from this global situation when localized give the theorems on dimension for the action of A_p on M.

Throughout this paper Alexander-Spanier cohomology with compact supports will be used. Coefficients will be the ring of integers Z unless other-

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wise specified. The field of integers modulo a prime q will be denoted by Z_q. The group of p-adic rationals (i.e., fractions of the form a/p^i, $a \in Z$) will be denoted by R_p and the group R_p/Z of p-adic rationals mod one, by $R_p(1)$. Cohomology dimension will be taken in the sense of H. Cohen [3] (also see [1, Chapter I]). Thus $\dim_L X \leqq n$ if and only if $H_c^{n+1}(U, L)=0$ for all open subsets U of X, and, as is shown in [3], this implies that $H_c^i(U, L)=0$ for all $i>n$.

In §§4 and 5 we have stated the results in terms of generalized (cohomology) manifolds (see [1] or [6]) instead of the more restrictive locally euclidean manifolds. We require, once and for all, that all generalized manifolds be locally orientable.

Independently, C. N. Lee has treated these same questions for the case of a free action. He will publish his results in a paper entitled *Compact 0-dimensional transformation groups*. We also understand that J. P. Serre has (unpublished) proofs for some of these results for free actions. It is to be noted that most of the complications in the present treatment are due to the fact that we consider general actions instead of free ones.

We wish to express our thanks to C. T. Yang for having given us a copy of his manuscript, *p-adic transformation groups*. We also wish to acknowledge Yang's priority to Theorems 2 and 3, which he stated and proved in terms of homology.

1. **Calculation of $H^*(B_{A_p})$ and $H^*(B_{\Sigma_p})$.** Consider an infinite sequence S_j of $(2n+1)$-dimensional spheres, $j=0, 1, 2, \cdots$. Each sphere S_j may be regarded as the n-fold join of the 1-sphere. Each point $x \in S_j$ can be described by its *join coordinates* $(z_0, z_1, \cdots, z_n, \theta_0, \theta_1, \cdots, \theta_n)$ where the z_i are complex numbers of norm 1 and the θ_i are the barycentric coordinates, (i.e., the θ_i are real non-negative numbers whose sum is 1). The circle group C_j acts freely on S_j by $c(z_0, z_1, \cdots, z_n, \theta_0, \theta_1, \cdots, \theta_n)=(cz_0, cz_1, \cdots, cz_n, \theta_0, \theta_1, \cdots, \theta_n)$. This action defines a principal fibration (the Hopf fibration) of S_j onto the $2n$-dimensional complex projective space P_j. Let π_j denote this map. Milnor has shown [5], that the bundle is trivial over

$$\{\pi_j(z_0, \cdots, z_n, \theta_0, \theta_1, \cdots, \theta_n) \mid \theta_i \neq 0\}, \text{ for each fixed } i \in \{0, 1, \cdots, n\}.$$

Let $h_j\colon C_j \to C_{j-1}$ be the homomorphism defined by $h_j(c)=c^p$. Define an *equivariant* map $f_j\colon S_j \to S_{j-1}$ by $f_j(c(z_0, z_1, \cdots, z_n, \theta_0, \theta_1, \cdots, \theta_n)) = h_j(c)(z_0^p, z_1^p, \cdots, z_n^p, \theta_0, \theta_1, \cdots, \theta_n)$. Let f_j' denote the induced map $f_j'\colon P_j \to P_{j-1}$. Clearly, $\{(S_j, P_j, C_j\pi_j), f_j\}$ forms an inverse sequence of fibrations. Let $S=\operatorname{Inv}\lim\{S_j, f_j\}$ and $P=\operatorname{Inv}\lim\{P_j, f_j\}$. By looking at the inverse limits in terms of their coordinates, one easily sees that $\operatorname{Inv}\lim\{C_j, h_j\} =\Sigma_p$, S is homeomorphic to the n-fold join of Σ_p, and the induced map $\pi\colon S \to P$ is a principal fibration with fiber Σ_p. Moreover, the bundle (S, P, Σ_p, π) is trivial over $\{\pi(\{z_0, z_1, \cdots, z_n, \theta_0, \theta_1, \cdots, \theta_n\}_j) \mid \theta_i \neq 0\}$, for each fixed $i \in \{0, 1, 2, \cdots, n\}$.

In order to compute $H^*(P; Z)$ it suffices to compute $f'^*_{j+1}(u_j)$ in terms of u_{j+1}, where u_j, (respectively u_{j+1}), is the generator of $H^2(P_j; Z)$, (respectively $H^2(P_{j+1}; Z)$), because $H^*(P; Z) = \text{Dir lim}\{H^*(P_j; Z), f'^*_{j+1}\}$. (Recall that $H^{2r}(P_j; Z) \approx Z$, $0 \leqq r \leqq n$, and 0 otherwise. Moreover, $H^{2r}(P_j; Z)$ is generated by u_j^r, where exponentiation means cup product multiplication.) Let ${}_jE_2^{s,t}$ denote the E_2 term of the spectral sequence of the fibration (S_j, P_j, C_j, π_j); then d_2 maps the generator of $H^1(C_j; Z)$ onto u_j. The map f_{j+1}: ${}_jE_2^{s,t} \to {}_{j+1}E_2^{s,t}$, induced by f'^*_{j+1} sends the generator of $H^1(C_j; Z)$ onto p-times the generator of $H^1(C_{j+1}; Z)$. Thus $f'^*_{j+1}(u_j) = pu_{j+1}$. Since S has no homotopy up to dimension $2n+1$, S may be regarded as $2n$-universal and the cohomology of P to be the cohomology of B_{Σ_p} up to dimension $2n$. But n may be chosen arbitrarily large, and therefore $H^{2r}(B_{\Sigma_p}; Z) \approx R_p$, $r = 1, 2, \cdots$, $H^0(B_{\Sigma_p}; Z) \approx Z$ and 0 otherwise.

In each group C_j there is naturally imbedded the cyclic group Z_{p^j} of order p^j. The group Z_{p^j} acts freely on S_j. Let ϕ_j: $S_j \to L_j$ denote the decomposition map induced by this action; this is a principal fibration and L_j is a generalized lens space. The homomorphism h_j: $C_j \to C_{j-1}$ induces a homomorphism from Z_{p^j} to $Z_{p^{j-1}}$ and the equivariant map f_j induces a map g_j: $L_j \to L_{j-1}$, $j > 0$. Furthermore, on L_j the quotient group C_j/Z_{p^j} acts in such a way that $(L_j, P_j, C_j/Z_{p^j}, \pi_j')$ is a principal circle fibration. Note that $\pi_j = \pi_j' \phi_j$, and that the induced homomorphism $C_j/Z_{p^j} \to C_{j-1}/Z_{p^{j-1}}$ is of degree 1. As above one obtains inverse sequences of principal fibrations, $\{(S_j, L_j, Z_{p^j}, \phi_j), f_j\}$ and $\{(L_j, P_j, C_j/Z_{p^j}, \pi_j'), g_j\}$. Let $L = \text{Inv lim}\{L_j, g_j\}$. Clearly, $\text{Inv lim}\{Z_{p^j}\} = A_p$ which acts freely on S. Therefore L is the base space of this (locally trivial) principal fibration and its cohomology is the cohomology of B_{A_p} up to dimension $2n$.

Consider the commutative diagram of Gysin sequences

$$\begin{array}{ccccccc}
0 \to H^{2r}(P_j) & \xrightarrow{\cup\Omega_j} & H^{2r+2}(P_j) & \xrightarrow{\pi_j'^*} & H^{2r+2}(L_j) & \longrightarrow & 0 \\
\downarrow f'^*_{j+1} & & \downarrow f'^*_{j+1} & & \downarrow g^*_{j+1} & & \\
0 \to H^{2r}(P_{j+1}) & \xrightarrow{\cup\Omega_{j+1}} & H^{2r+2}(P_{j+1}) & \xrightarrow{\pi'^*_{j+1}} & H^{2r+2}(L_{j+1}) & \to & 0.
\end{array}$$

If $j=0$ then $\cup\Omega_j$ is an isomorphism onto, $(L_0 = S_0)$. Using the fact that $f'^*_{j+1}(u_j^r) = p^r u_{j+1}^{r+1}$ it follows by induction that $H^{2r+2}(L_j; Z) \approx Z_{p^j}$, $0 \leqq r < n-1$, and is isomorphic to 0 in the odd dimensions less than $2n+1$. Therefore, the diagram reduces to

$$\begin{array}{ccccccc}
0 \to Z & \xrightarrow{p^j} & Z & \xrightarrow{\pi_j'^*} & Z_{p^j} & \longrightarrow & 0 \\
\Big\downarrow p^r & & \Big\downarrow p^{r+1} & & \Big\downarrow g^*_{j+1} & & \\
0 \to Z & \to & Z & \xrightarrow{\pi_j'^*} & Z_{p^{j+1}} & \to & 0.
\end{array}$$

If $v_{j,r+1}$ denotes the generator of $H^{2(r+1)}(L_j; Z)$ then $g_{j+1}^*(v_{j,r+1}) = p^{r+1}(v_{j+1,r+1})$. Thus, passing to the direct limit it follows that $H^0(L; Z) \approx Z$, $H^2(L; Z) \approx R_p(1)$, and $H^r(L; Z) = 0$, $r \leqq 2n$, $r \neq 0, 2$.

The results of this section are summarized in the following theorem.

THEOREM 1. *Let B_{Σ_p} and B_{A_p} denote the universal classifying spaces for Σ_p and A_p, respectively. Then,*

$$H^r(B_{\Sigma_p}; Z) \approx \begin{cases} Z, & r = 0, \\ R_p, & r = 2, 4, 6, \cdots, \\ 0, & r = 1, 3, 5, \cdots, \end{cases}$$

$$H^r(B_{A_p}; Z) \approx \begin{cases} Z, & r = 0, \\ R_p(1), & r = 2, \\ 0, & r \neq 0, 2. \end{cases}$$

REMARKS. The ring structures are preserved under the direct limits and in case the coefficient domain is a field, this might be of some importance. It follows just as above that $H^*(B_{\Sigma_p}; L)$ is a polynomial algebra on one generator in dimension 2, provided L, (the coefficient ring) is a field of characteristic $\neq p$. If the characteristic of L is p then $H^r(B_{\Sigma_p}; L) = 0$ for all $r > 0$. On the other hand, $H^r(B_{A_p}; L) = 0$, for $r > 0$ if the characteristic of $L \neq p$, and

$$H^r(B_{A_p}; L) \approx \begin{cases} L, & r = 0, 1, \\ 0, & r \neq 0, 1, \end{cases}$$

if the characteristic of L is p.

If G is the limit of an inverse sequence of finite cyclic groups and has no element of finite order, then G is an *"adic group."* The cohomology of the universal classifying space B_G may be computed in exactly the way we have described above. In fact,

$$H^r(B_G; Z) \approx \begin{cases} Z, & r = 0, \\ \text{Character group of } G, & r = 2, \\ 0, & r \neq 0 \text{ or } 2. \end{cases}$$

2. **Preliminaries.** Let A_p act on a space M and consider A_p as a subgroup of Σ_p with $\Sigma_p/A_p \approx S^1$. It was noticed by D. Montgomery that one can define a space M' and an action of Σ_p on M' in such a way that $M'/\Sigma_p \approx M/A_p$, and also such that M' is an $(n+1)$-manifold if M is an n-manifold. The construction is roughly as follows. If g is a "generator" of A_p then let M' be the space obtained from $M \times [0, 1]$ by identifying $(x, 1)$ with $(g(x), 0)$. One can then obtain an action of Σ_p on M' in an obvious manner. In Proposition 2.1 below we present another construction of this space in a more general case. It is equivalent to the preceding in the special case.

PROPOSITION 2.1. *Let G be a closed subgroup of a compact topological group G' with G central in G'. Assume that G' has local cross-sections for G so that G' is a locally trivial G-bundle over G'/G. Put $M' = M \times_G G'$ (see [1, IV, 1.3]) and let G' act on M' by $g(m, h) = (m, hg^{-1})$ (where (m, h) denotes the element of M' represented by the element $m \times h \in M \times G'$, so that if $g \in G$ then $(m, h) = (g(m), gh)$). Then we have*

(1) *M' is a locally trivial bundle over G'/G with fibre M and structural group G.*

(2) *The sub-transformation group (G, M') of (G', M') leaves each fibre of M' invariant and on this fibre is equivalent to the transformation group (G, M).*

(3) *The inclusion $M \subset M'$ as a fibre induces the natural homeomorphisms $M'/G \approx M/G \times G'/G$ and $M'/G' \approx M/G$.*

(4) *M' is an $(n+d)$-gm if and only if M is an n-gm and G'/G is a d-manifold.*

Proof. (1) follows from general facts in [1, IV, 1.3]. (2) follows from the equality $g(m, h) = (m, hg^{-1}) = (m, g^{-1}h) = (g(m), h)$ for $g \in G$. The first part of (3) follows from the fact that for G central the orbit space M'/G is naturally homeomorphic to $M \times G'/G \times G \approx M/G \times G'/G$, and the second part of (3) follows from the first part in an obvious way or also can be seen from the fact that the map $M' \to M/G$ induced from the projection $M \times G' \to M$ is equivalent to the orbit map $M' \to M'/G'$. (4) follows from (1) and the fact that any fibration of a space is an n-gm if and only if both the fibre and the base are generalized manifolds (see [1, I, 4.10], or [6, Theorem 6]).

We will be mainly concerned with the case in which $G = A_p$ and $G' = \Sigma_p$ which satisfies the hypotheses of Proposition 2.1. In this case M' is an $(n+1)$-gm if and only if M is an n-gm, since $G'/G \approx S^1$.

LEMMA 2.2. *If G' is a compact connected topological group and G' acts on a space M', then G' acts trivially on $H_c^*(M')$.*

Proof. This lemma is well known if G' is a Lie group since every $g \in G'$ lies on a one-parameter group and hence its action is that of a homotopy. In general, G' is the inverse limit $G' = \text{Inv lim } G'/N_\alpha$ of a system of compact connected Lie groups G'/N_α, where the N_α are normal subgroups of G' with $\text{Inv lim } N_\alpha = \bigcap N_\alpha = (e)$. Furthermore, $M' = \text{Inv lim } M'/N_\alpha$ and the projection $M' \to M'/N_\alpha$ induces a transformation group G'/N_α on M'/N_α. The Lie group G'/N_α acts trivially on $H_c^*(M'/N_\alpha)$ and it follows that G' acts trivially on $H_c^*(M') = \text{Dir lim } H_c^*(M'/N_\alpha)$. More precisely, the diagram

$$\begin{array}{ccc} M' & \xrightarrow{\pi_\alpha} & M'/N_\alpha \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle gN_\alpha} \\ M' & \xrightarrow{\pi_\alpha} & M'/N_\alpha \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} H_c^*(M'/N_\alpha) & \xrightarrow{\pi_\alpha^*} & H_c^*(M') \\ \Big\downarrow (gN_\alpha)^* = \text{identity} & & \Big\downarrow g^* \\ H_c^*(M'/N_\alpha) & \xrightarrow{\pi_\alpha^*} & H_c^*(M'). \end{array}$$

For any $a \in H_c^*(M')$ and $b \in H_c^*(M'/N_\alpha)$ such that $a = \pi_\alpha^*(b)$, $g^*(a) = g^*\pi_\alpha^*(b) = \pi_\alpha^*(gN_\alpha)^*(b) = \pi_\alpha^*(b) = a$.

In Proposition 2.1, if $G = A_p$ and $G' = \Sigma_p$, then G acts trivially on $H_c^*(M')$ since its action is extendable to the connected group G'. This fact will be of considerable importance in the following sections.

LEMMA 2.3. *If G is a compact totally disconnected abelian group and H is a closed subgroup of G, then $H = \bigcap K_\alpha$, where the K_α are the open subgroups of G containing H.*

Proof. It is well known that the lemma holds for $H = (e)$. Therefore, in general, one has that $H = \bigcap (K_\alpha H)$ where the K_α are all the open subgroups of G. Since $K_\alpha H$ is also open the lemma follows.

LEMMA 2.4. *Let $G \approx A_p$ and let $G_i = G^{p^i}$. Then any open subgroup K is one of the G_i and any closed subgroup H is either (e) or one of the G_i.*

Proof. Let g be a "generator" of G. Then G_i is the (closure of the) group generated by g^{p^i}. Since the homomorphism $g \rightarrow g^a$ is an isomorphism onto if $(a, p) = 1$, g^b generates the same subgroup as does g^{p^i} where $b = p^i a$, $(a, p) = 1$. Thus let K be open and let b be minimal such that $g^b \in K$. It follows easily that $b = p^i$ for some i so that $G_i \subset K$. Since G_i is closed (as well as open) and g^i is dense in G, K cannot be larger than G_i. The second part follows easily from the first part and Lemma 2.3 if one notes that $G = G_0 \supset G_1 \supset G_2, \cdots$, $\bigcap G_i = (e)$.

We will restrict ourselves in the remainder of this paper to the case in which $G = A_p$, $G' = \Sigma_p$. Let E_G be the N-fold join of Σ_p. That is, we shall take bundles universal up to a very large N, in fact, so large that it does affect the arguments in this paper. We have shown by the construction used in obtaining Theorem 1 that there exist equivariant maps

$$E_{(e)} = E_{G/G_0} \leftarrow \cdots \leftarrow E_{G/G_i} \leftarrow E_{G/G_{i+1}} \leftarrow \cdots \leftarrow E_G$$

where, for convenience, we take $E_{(e)}$ to be a point. (Each E_{G/G_i} is the N-fold join of circles. Moreover, $E_G = \operatorname{Inv} \lim E_{G/G_i}$, $B_G = \operatorname{Inv} \lim B_{G/G_i}$.)

Let us suppose that G acts as a topological transformation group on the locally compact Hausdorff space M. Then, $M = \operatorname{Inv} \lim M/G_i$, $M \times E_G = \operatorname{Inv} \lim (M/G_i \times E_{G/G_i})$. Let G operate on $M \times E_G$ by the diagonal action and denote $M \times_G E_G = (M \times E_G)/G$ by M_G as in [1, IV, 3.1]. Then, $M_G = \operatorname{Inv} \lim (M/G_i)_{G/G_i}$. Let $\pi_{\infty,i}$ and $\pi_{i,j}$ denote the natural maps M_G

$\to(M/G_i)_{G/G_i}$ and $(M/G_i)_{G/G_i}\to(M/G_j)_{G/G_j}$; $i>j$. Note that $(M/G_0)_{G/G_0} = M/G$, since we have taken $E_{(e)}$ to be a point.

Let $\mathcal{S}_i$ be the Leray sheaf in degree two of $\pi_{i,0}$, and $\mathcal{S}$ the Leray sheaf in degree two of $\pi_{\infty,0}$. That is, $\mathcal{S}_i$ is the sheaf on M/G associated with the presheaf $U\to H_c^2(\pi_{i,0}^{-1}(\overline{U}); Z)$ for U open and $\overline{U}$ compact (see [1, XII, 2.1]). ($\mathcal{S}$ has stalks $H^2(B_{G_x}; Z)$.)

The maps $\pi_{i+1,i}$ and $\pi_{\infty,i}$ induce maps $f_{i+1,i}\colon \mathcal{S}_i\to\mathcal{S}_{i+1}$ and $f_{\infty,i}\colon \mathcal{S}_i\to\mathcal{S}$ and therefore $\mathcal{S}=\text{Dir lim } \mathcal{S}_i$. (This is clearly true for the presheaves and because the direct limit of sheaves is defined in terms of the presheaves it follows for the sheaves.)

Let $F_j=F(G_j, M)/G\subset M/G$, where $F(G_j, M)$ denotes the fixed point set of G_j on M. Then clearly $\mathcal{S}_i$ has stalks isomorphic to $Z_{p^{i-j}}$ over F_j-F_{j-1} for $j<i$ and is trivial over $M/G-F_{i-1}$.

The following lemma will be used in §§3 and 4.

Lemma 2.5. *The sheaf $\mathcal{S}_i$ is constant over F_j-F_{j-1}, for all j.*

Proof. The sheaf $\mathcal{S}_i$ is trivial over $M/G-F_{i-1}$. Consequently it is constant over $F_j - F_{j-1}$, for $j > i - 1$. For $j \leqq i - 1$, $F(G_j, M) - F(G_{j-1}, M) = (F(G_j, M)-F(G_{j-1}, M))/G_i$ and is a locally trivial fibre bundle over F_j-F_{j-1} with fibre $G/G_j\approx Z_{p^j}$. Therefore, $(F(G_j, M)-F(G_{j-1}, M))_{G/G_i}$ is a locally trivial fibre bundle (via $\pi_{i,0}$) over $F_j - F_{j-1}$ with fibre B_{G_j/G_i} $(=E_{G/G_i/G_j/G_i})$ and structure group G/G_i (see [1, IV, 1.3]). However, since the action of G/G_i on E_{G/G_i} can be extended to an action of $G'/G_i\approx S^1$ by our choice of $E_{G/G_i}=E_{G'/G_i}$, we also have that the action of the structural group G/G_i on B_{G_j/G_i} can be extended to an action of $G'/G_i\approx S^1$. Since the action of a circle group is trivial on cohomology the lemma follows.

3. Effective actions of A_p on locally compact spaces. In this section we consider the general case in which $G=A_p$, M is a locally compact Hausdorff space and $\dim_Z M=n<\infty$.

By Proposition 2.1, G also acts on a space M' of dimension $m=n+1$ over Z and $M'/G\approx M/G\times S^1$. The action of G on M' is extendable to an action of $G'=\Sigma_p$ so that G acts trivially on $H_c^*(M')$ by Lemma 2.2. We first obtain global results about this action and then interpret them to obtain results for the action of G on M.

We will use the notation $G_i=G^{p^i}$ and $F_i=F(G_i, M')/G\subset M'/G$ of §2.

Lemma 3.1. $H_c^{m+4}(M'/G)=0$ *(resp., if* $\dim_Z F(G_i, M')\leqq m-1$ *for all* i, *then* $H_c^{m+3}(M'/G)=0$).

Proof. Consider the diagram $M'/G\xleftarrow{\pi_1}M'_G\xrightarrow{\pi_2}B_G$ (see [1, IV, 3.1], for general facts concerning this diagram) and denote by ${}_1E_r^{s,t}$ and ${}_2E_r^{s,t}$ the Leray spectral sequences of π_1 and π_2 respectively. Recall that π_2 is a locally trivial fibre map with fibre M' and structural group G. Since G acts trivially on $H_c^*(M')$ the coefficients of ${}_2E_2^{s,t}=H_c^s(B_G; H_c^t(M'))$ are constant. Thus it follows from §1 that ${}_2E_2^{s,t}=0$ for $s>2$ or $t>m$. Hence $H_c^k(M'_G; Z)=0$ for $k>m+2$.

REMARK 3.2. The dimension theorem of Yang (see Theorem 2) for free actions can now be obtained easily without use of any of our information about $\mathcal{S}_i$ or $\mathcal{S}$ as follows. Since the fibres of π_1 are E_G which are acyclic (in small dimensions), π_1 is a Vietoris map in small dimensions. Thus $H_c^{m+3}(M'/G;\ Z) \approx H_c^{m+3}(M'_G;\ Z) = 0$. Since $M'/G \approx M/G \times S^1$ it follows that $H_c^{n+3}(M/G;\ Z) \approx H_c^{m+3}(M'/G;\ Z) = 0$ and the same must be true for any open subset of M/G, whence $\dim_Z M/G \leqq n+2$.

Returning to the general case we also have that ${}_1E_2^{s,t} = H_c^s(M'/G; H^t(B_{G_x}))$ $= 0$ for $t \neq 0$, 2 and ${}_1E_2^{s,2} = H_c^s(M'/G; \mathcal{S}) = \text{Dir lim } H_c^s(M'/G; \mathcal{S}_i)$ by the remarks following Lemma 2.4 and [4, Théorème 4.12.1].

The sheaf $\mathcal{S}_i$ is trivial over the open set $M'/G - F_{i-1}$. Therefore $H_c^s(M'/G; \mathcal{S}_i)$ $\approx H_c^s(F_{i-1}; \mathcal{S}_i|_{F_{i-1}}) = 0$ for $s > m$ (resp. $s \geqq m$), since $\dim_Z F_{i-1} = \dim_Z F(G_{i-1}, M')$ $\leqq m$ (resp. $m-1$), and G is effectively finite on $F(G_{i-1}, M')$.

Thus ${}_1E_2^{s,2} = 0$, for $s > m$, (resp. $s \geqq m$). It follows that the map, (with $s = m+4$ (resp. $s = m+3$)),

$$H_c^s(M'/G) = {}_1E_2^{s,0} \to {}_1E_\infty^{s,0} \subset H_c^s(M'_G) = 0$$

is injective and this finishes the proof.

THEOREM 2 (YANG). $\dim_Z M/G \leqq n+3$ (*resp., if* $\dim_Z F(G_i, M) \leqq n-1$ *for all* i *then* $\dim_Z M/G \leqq n+2$).

Proof. Since $M'/G \approx M/G \times S^1$ we have that $H_c^{n+4}(M/G) \approx H_c^{m+4}(M'/G) = 0$. This must also be true for every open subset of M/G and hence $\dim_Z M/G$ $\leqq n+3$, by definition.

The second part follows in the same manner, since

$$F(G_i, M') \approx F(G_i, M) \times_G G'$$

and therefore $\dim_Z F(G_i, M') = \dim_Z F(G_i, M) + 1$.

REMARK. If instead of using cohomology with coefficients in the integers we were to use coefficients in a field L then $\dim_L M/G = n$ if the characteristic of $L \neq p$ and $\dim_L M/G \leqq n+2$, (respectively $\leqq n+1$) if the characteristic of $L = p$. One needs only to repeat the argument above using the facts about $H^*(B_{A_p}; L)$ as stated in the remarks following Theorem 1. For fields whose characteristic $\neq p$ the argument is especially easy because the map π_1 is a Vietoris map and therefore induces an isomorphism of $H^*(M'/G)$ onto $H^*(M'_G)$.

Thus it follows that $\dim_L M/G \leqq n$. That $\dim_L M/G \geqq n$ follows from the fact that $\dim_L M$ is assumed to be n and that a proper light map cannot lower dimension. For fields whose characteristic is p the argument cannot be simplified. One only needs to observe that the Leray sheaf $\mathcal{S}$ in degree 1 is the sheaf that must be studied in translating the argument from the integers to a field L whose characteristic is p.

4. **Actions of A_p and Σ_p on an n-gm.** We assume now that M is an n-gm

over Z. Then M' is an m-gm with $m=n+1$. From the spectral sequence of the fibering $M'\to^M S^1$ it follows that M' is orientable if M is orientable and G preserves the orientation of M. Note that G^2 always preserves orientation and that $G^2=G$ or G_1 according as p is odd or $p=2$. Moreover, given $x\in M$ and an orientable neighborhood U of x in M, we can always find a neighborhood $V\subset U$ of x and an index i such that $G_i(V)\subset U$. Since G/G_i is finite, $\dim_Z M/G_i = \dim_Z M/G$, it clearly suffices to consider the case in which M is orientable and G preserves its orientation. We shall assume this in the remainder of this section. (Alternatively the canonical orientable double covering M^d of M (see [2, §§4.1, 5.2 and 6.1]) enables one to lift the action of G so that G preserves the orientation of M^d. One easily obtains $M/G\approx(M^d/G)/D$ ($D=$deck transformation $\approx Z_2$) and hence $\dim_Z M/G=\dim_Z M^d/G$.)

Consider the proof of Lemma 3.1 and note that ${}_2E_2^{2,m}=H^2(B_G;\ H_c^m(M')) = H_c^2(B_G;\ Z)\approx R_p(1)$. Therefore $R_p(1)\approx{}_2E_2^{2,m}={}_2E_\infty^{2,m}=H_c^{m+2}(M'_G)$ since ${}_2E_2^{s,t}=0$ for $s>2$ or $t>m$.

REMARK. For free actions it follows as in Remark 3.2 that $H_c^{m+2}(M'/G;\ Z) \approx H_c^{m+2}(M'_G;\ Z)\approx R_p(1)$ and consequently that $\dim_Z M/G=n+2$.

LEMMA 4.1. *The group* $H_c^s(M'/G;\ \mathcal{S}_i)=0$ *for* $s\geqq m$.

Proof. As in Lemma 3.1, $H_c^s(M'/G;\ \mathcal{S}_i)=H_c^s(F_{i-1};\ \mathcal{S}_i|_{F_{i-1}})$. Since $H_c^s(F_{i-1};\ \mathcal{S}_i|_{F_{i-1}})=0$ for $s>m$ one needs only to show the group to be zero for $s=m$. In Lemma 2.5 it was shown that the sheaf $\mathcal{S}_i$ was constant over F_j-F_{j-1}, for all j.

Consider the exact cohomology sequence

$$\to H_c^m(F_j - F_{j-1};\mathcal{S}_i|_{F_j-F_{j-1}}) \to H_c^m(F_j;\mathcal{S}_i|_{F_j}) \to H_c^m(F_{j-1};\mathcal{S}_i|_{F_{j-1}}) \to 0.$$

By induction, it suffices to show that the first term in the sequence is 0.

If $f\colon M'\to M'/G$ is the natural map then $f^{-1}(F_j-F_{j-1})$ is a closed subset of the m-gm $M'-f^{-1}(F_{j-1})$. Suppose $V_1\subset f^{-1}(F_j-F_{j-1})$ were a component of $M'-f^{-1}(F_{j-1})$. Then $f(V_1)$ would be a component of $M'/G-F_{j-1}$ since f is open and closed. Let $V=f^{-1}(f(V_1))$. Then $V\subset f^{-1}(F_j-F_{j-1})$ and G_{j-1} would act effectively on V as Z_p. Moreover, $\overline{V}-V\subset f^{-1}(F_{j-1})$. Hence one could define an effective action of Z_p on M' leaving $M'-V$ pointwise stationary. Then by the Smith theorems $M'=\overline{V}$ and therefore G would not act effectively on M' contrary to assumption. Thus without any loss of generality we may assume that $F=f^{-1}(F_j-F_{j-1})$ is a closed invariant subset of the orientable m-gm $X=M'-f^{-1}(F_{j-1})$ and that F does not contain any component of X so that $H_c^m(F;\ L)=0$ for any coefficient domain L. If $j=0$, then $F/G=F$ so that $H_c^m(F/G;\ Z)=0$. If $j>0$ then $G/G_j\approx Z_{p^j}$ acts freely on F. The Smith sequence, applied to the subgroup Z_p of $Z_{p^j}\approx G/G_j$ yields the exact sequence

$$H_c^m(F;\ Z_p)\to H_\sigma^m(F;\ Z_p)\to 0$$

which implies that $H_c^m(F/Z_p;\ Z_p) \approx H_\sigma^m(F;\ Z_p) = 0$. The transfer homomorphism σ^*: $H_c^m(F;\ Z) \to H_c^m(F/Z_p;\ Z)$ and the homomorphism f^*: $H_c^m(F/Z_p;\ Z) \to H_c^m(F;\ Z)$ satisfy $\sigma^* f^*(a) = pa$ and thus multiplication by p is trivial in $H_c^m(F/Z_p;\ Z)$. Consider the exact Bockstein sequence

$$\to H_c^m(F/Z_p;\ Z) \xrightarrow{i^*} H_c^m(F/Z_p;\ Z) \xrightarrow{j^*} H_c^m(F/Z_p;\ Z_p) \to 0.$$

The homomorphism i^* is multiplication by p and is trivial. Since $H_c^m(F/Z_p;\ Z_p) = 0$ it follows that $H_c^m(F/Z_p;\ Z) = 0$. Continuing inductively one now obtains easily that $H_c^m(F/G;\ Z) = H_c^m(F/Z_{p^j};\ Z) = 0$ and by the universal coefficient sequence we get that $H_c^m(F/G;\ L) = 0$ for any constant coefficient sheaf L. Thus, since $\mathcal{S}_i|_{F_j - F_{j-1}}$ is constant and since $F_j - F_{j-1} = F/G$, we have that $H_c^m(F_j - F_{j-1};\ \mathcal{S}_i|_{F_j - F_{j-1}}) = 0$ as was to be shown.

Lemma 4.2. *The group* $H_c^{m+3}(M'/G) = 0$ *and there is a subgroup* K *of* $H_c^{m+2}(M'/G)$ *such that* $H_c^{m+2}(M'/G)/K \approx R_p(1)$.

Proof. The first part follows as in the proof of the second part of Lemma 3.1. For the second part observe that ${}_1E_2^{s,2} = 0$ for $s \geqq m$ by Lemma 4.1. Thus, $R_p(1) \approx H_c^{m+2}(M'_G) \approx {}_1E_\infty^{m+2,0} = {}_1E_4^{m+2,0} = {}_1E_2^{m+2,0}/d_3({}_1E_2^{m-1,2})$. Since ${}_1E_2^{m+2,0} = H_c^{m+2}(M'/G)$ the lemma follows.

Theorem 3 (Yang). *Let* G *be a* p*-adic group which acts effectively on an* m*-gm* M, *over* Z. *Then* $\dim_Z M/G = n+2$. *Moreover, if* M *is orientable and* G *does not reverse orientation then* $H_c^{n+2}(M/G)/K \approx R_p(1)$ *for some* p*-torsion subgroup* $K \subset H_c^{n+2}(M/G)$.

Proof. This follows from Lemma 4.2 just as Theorem 2 followed from Lemma 3.1. That K is a p-torsion subgroup follows from the facts that $K \approx d_3(E_2^{m-1,2})$ and the sheaf $\mathcal{S}_i$ is constant over $F_j - F_{j-1}$ with stalks $Z_{p^{i-j}}$.

Remark. As in the remark following Theorem 2 we can say something about $\dim_L M/G$ when L is a field. If the characteristic of $L \neq p$ and M is an n-gm over L then the remark following Theorem 2 states that $\dim_L M/G = n$. If M is an n-gm over L and L is a field of characteristic p then the argument above can easily be modified to show that $\dim_L M/G = n+1$.

Corollary 4.3. *If a* p*-adic solenoid* Σ_p *acts effectively on an* n*-gm* M, *over* Z, *then* $\dim_Z M/\Sigma_p = n+1$.

5. **Fixed points for actions of Σ_p.** In this section M will denote a locally compact Hausdorff space and L_q a field of characteristic $q \neq p$.

Lemma 5.1. *If* $G \approx A_p$ *acts on* M *then* π^*: $H_c^*(M/G;\ L_q) \to H_c^*(M;\ L_q)$ *is an isomorphism onto the subgroup* $[H_c^*(M;\ L_q)]^G$ *of invariant elements.*

Proof. The finite group G/G_i acts on the space M/G_i. It follows from [1, III, 2.3] that $H_c^*(M/G;\ L_q) \to H_c^*(M/G_i;\ L_q)$ is an isomorphism onto

$[H_c^*(M/G_i;\ L_q)]^G$. The lemma follows upon passage to the limit $M=\text{Inv lim } M/G_i$.

COROLLARY 5.2. *If $H_c^*(M;\ L_q)\approx H_c^*(E^n;\ L_q)$ or $H_c^*(S^n;\ L_q)$ and if $p\neq 2$, $q=0$ or if G acts trivially on $H_c^n(M;\ L_q)$ then π^* is an isomorphism onto.*

COROLLARY 5.3. *If $G'=\Sigma_p$ acts on M, $\dim_{L_0} M<\infty$ and $H_c^*(M;\ L_0)\approx H_c^*(E^n;L_0)$, (respectively $\approx H_c^*(S^n;L_0)$), then $H_c^*(F(G;\ M);L_0)\approx H_c^*(E^r;\ L_0)$, (respectively $\approx H_c^*(S^r;\ L_0)$), and $n-r$ is even.*

Proof. Let $A_p=G\subset G'$. By Corollary 5.2, $H_c^*(M/G;\ L_0)\approx H_c^*(E^n;\ L_0)$ (respectively $H_c^*(S^n;\ L_0)$). Thus since $M/G'\approx(M/G)/(G'/G)$, $G'/G\approx S^1$, and $F(G',\ M)\approx F(G'/G,\ M/G)$, the lemma follows from [1, IV, 5.7].

COROLLARY 5.4. *If $G'=\Sigma_p$ acts on M, $\dim_{L_0} M<\infty$, and $H_c^*(M;\ L_0)\approx H_c^*(E^n;L_0)$ then G' has a fixed point. (Similarly for $H_c^*(M;L_0)\approx H_c^*(S^n;L_0)$, n even.)*

In a forthcoming paper by one of the authors it is shown that if a p-adic group G acts on a connected orientable n-gm M over L_q such that G acts trivially on $H_c^n(M;L_q)$, then the orbit space is an n-gm over the field L_q, $q\neq p$.

COROLLARY 5.5. *If $G'=\Sigma_p$ acts on a connected orientable n-gm M over L_q, $q\neq p$ then each component of $F(G',\ M)$ is an orientable r-gm over L_q and $n-r$ is even, $r\geqq 0$.*

Proof. Let $A_p=G\subset G'$. The space M/G is an orientable n-gm over L_q. The circle group G'/G acts on M/G and its orbit space is M/G'. Since G' is connected and G is totally disconnected, $F(G',\ M)\approx F(G'/G,\ M/G)$. Then by a theorem of Conner-Floyd (see for example [1, V, 3.2]), each component of $F(G';\ M)$ is an orientable r-gm over L_q and $n-r$ is even.

REFERENCES

1. A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies, vol. 46, Princeton University Press, 1960.

2. G. E. Bredon, *Orientation in generalized manifolds and applications to the theory of transformation groups*, Michigan Math. J. vol. 7 (1960) pp. 35–64.

3. H. Cohen, *A cohomological definition of dimension for locally compact Hausdorff spaces*, Duke Math. J. vol. 21 (1954) pp. 209–224.

4. R. Godement, *Théorie des faisceaux*, Paris, Hermann, 1958.

5. John Milnor, *Construction of universal bundles*. II, Ann. of Math. vol. 63 (1956) pp. 430–436.

6. Frank Raymond, *Separation and union theorems for generalized manifolds with boundary*, Michigan Math. J. vol. 7 (1960) pp. 7–21.

7. C. T. Yang, *p-adic transformation groups*, Michigan Math. J. vol. 7 (1960) pp. 201–218.

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