ON DIFFERENTIABLY SIMPLE ALGEBRAS

BY

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1. Introduction. It is known (see Albert [1]) that every simple commutative power-associative algebra of degree \( t > 2 \) over an algebraically closed field \( \mathbb{F} \) of characteristic \( p > 5 \) is a Jordan algebra. Moreover, in the partially stable case, a characterization of the simple algebras of degree two is given by Albert in [3]. In his theory Albert expresses the structure of simple partially stable algebras in terms of certain commutative associative algebras \( \mathcal{B} \) over \( \mathbb{F} \). These commutative associative algebras have unity elements, and each algebra \( \mathcal{B} \) is differentiably simple relative to some set of derivations of \( \mathcal{B} \) over \( \mathbb{F} \). In this paper we shall determine the structure of the algebras \( \mathcal{B} \) and derive a property of simple partially stable algebras which follows from Albert’s characterization.

Let \( \mathcal{B} \) be a commutative associative algebra with unity element \( e \) over \( \mathbb{F} \). We shall now define a commutative power-associative algebra \( \mathcal{T} \) over \( \mathbb{F} \) which is the essential subalgebra of a partially stable commutative power-associative algebra \( \mathcal{S} \) as defined by Albert in [3]. Let \( m \geq 2 \) and let \( \gamma_i \mathcal{B} \) denote a homomorphic image of the vector space \( \mathcal{B} \) for \( i = 0, \ldots, m \). Then \( \mathcal{T} \) will be the vector space direct sum

\[
\mathcal{T} = \mathcal{B} + \mathcal{F}
\]

where \( \mathcal{F} \) is the sum, not necessarily direct, of the component spaces \( \gamma_0 \mathcal{B}, \ldots, \gamma_m \mathcal{B} \). Select elements \( b_{ij} \) in \( \mathcal{B} \) and derivations \( D_{ij} \) of \( \mathcal{B} \) over \( \mathbb{F} \) such that

\[
\begin{align*}
  b_{ij} &= b_{ji}, & b_{00} &= e, & b_{0j} &= 0 \quad (j \neq 0), \\
  D_{ij} &= -D_{ji},
\end{align*}
\]

for \( i, j = 0, \ldots, m \) where then \( D_{ii} = 0 \) for \( i = 0, \ldots, m \). We now define products in \( \mathcal{T} \) by assuming that \( \mathcal{B} \) is a subalgebra of \( \mathcal{T} \), that

\[
(y_i a)b = y_i(ab) = b(y_ia) \quad (i = 0, \ldots, m)
\]

for all elements \( a \) and \( b \) of \( \mathcal{B} \), and finally that

\[
(y_i a)(y_j b) = b_{ij}ab + (aD_{ij})b - a(bD_{ij})
\]

for all \( a \) and \( b \) of \( \mathcal{B} \) and \( i, j = 0, \ldots, m \). The result will be a commutative power-associative algebra of degree two over \( \mathbb{F} \).

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We shall also require that the $b_{ij}$ and the $D_{ij}$ be chosen so that:

(A) The algebra $B$ is \{ $D_{ij}$\}-simple.

(B) If $g$ is in $\mathcal{L}$ and $gu = 0$ for all $u$ in $\mathcal{L}$, then $g = 0$.

It is one of the principal results of Albert in [3] that these conditions are equivalent to the simplicity of the partially stable algebra $\mathcal{S}$ mentioned above. It is known [2] that condition (A) implies that

$$B = e\mathcal{F} + \mathcal{R}$$

where $\mathcal{R}$ is the radical of $B$ and $x^p = 0$ for every element $x$ in $\mathcal{R}$. We shall completely determine the structure of $B$, and we state our main result as

**Theorem 1.** Let $B$ be a commutative associative algebra with unity $e$ over an algebraically closed field $\mathcal{F}$, and let $B$ be differentiably simple relative to a set of derivations of $B$ over $\mathcal{F}$. Then $B = \mathcal{F}[e, x_1, \cdots, x_n]$ is an algebra with generators $x_1, \cdots, x_n$ over $\mathcal{F}$ which are independent except for the relations $x_1^p = \cdots = x_n^p = 0$ where $p > 0$ is the characteristic of $\mathcal{F}$.

In all examples of the algebras $\mathcal{X}$ given to date the space $\mathcal{L}$ has been a direct sum of the components $y_0 B, \cdots, y_m B$. As our final result we shall construct a class of examples of the algebras $\mathcal{X}$ in which $\mathcal{L} = (y_0 B, \cdots, y_m B)$ with $m = 2$ and $\mathcal{L}$ is not a direct sum and cannot be represented as a direct sum in this manner.

2. **The algebra $B$.** Let $B$ be a commutative associative algebra with unity $e$ over $\mathcal{F}$, and let $B$ be $\mathcal{F}$-simple for some set $\mathcal{D}$ of derivations of $B$ over $\mathcal{F}$. Then by (6) we may write

$$B = e\mathcal{F} + \mathcal{R}$$

where $x^p = 0$ for each $x$ in $\mathcal{R}$. The algebra $B$, being finite dimensional, is finitely generated. Let \{ $e, x_1, \cdots, x_n$ \} be a set of generators of $B$ which is minimal in the sense that no set containing $e$ and having fewer elements generates $B$. Also let

$$\mathcal{A} = \mathcal{F}[e, x_1, \cdots, x_n]$$

be the commutative associative algebra generated over $\mathcal{F}$ by generators $e, x_1, \cdots, x_n$ which are independent except for the relations $e^2 = e$, $ex_i = x_i$, and $x_i^p = 0$ which hold for $i = 1, \cdots, n$. It is clear that the mappings $e \rightarrow e$, $x_i \rightarrow x_i$ ($i = 1, \cdots, n$) define a homomorphism $\phi$ of $\mathcal{A}$ onto $B$. We let $\mathcal{M}$ be the kernel of $\phi$. We see that Theorem 1 will be proved if we can show that $\mathcal{M} = 0$.

We now note some properties of $\mathcal{A}$. We may write

$$\mathcal{A} = e\mathcal{F} + \mathcal{R}$$
where $R = \mathbb{F}[x_1, \ldots, x_n]$ is the radical of $A$ and consists of all polynomials in the $x_i$ with constant term zero. We observe that every element of $A$ which is not in $R$ has an inverse. For if $a = \alpha + u$ with $\alpha$ in $R$, $u$ in $R$, $\alpha \neq 0$, then $a^{-1} = (\alpha^p)^{-1}(\alpha + u)^{-1}$. Also it is known [4] that the derivation algebra of $A$ consists of all linear transformations $D = D(a_1, \ldots, a_n)$ of $A$ defined by

$$aD = (\partial a/\partial x_1)a_1 + \cdots + (\partial a/\partial x_n)a_n$$

where $a_1, \ldots, a_n$ are in $A$ and $\partial a/\partial x_i$ denotes the ordinary partial derivative of the polynomial $a$ with respect to $x_i$ ($i = 1, \ldots, n$). Thus $x_iD = a_i$ and the derivations of $A$ are completely determined by the images of the $x_i$ and these images may be arbitrarily chosen.

**Theorem 2.** Let $D$ be a derivation of $A$. Then the transformation $\overline{D}$ defined by

$$\varphi(u)\overline{D} = \varphi(uD)$$

is a derivation of $B$ if and only if $\mathcal{M}D \subseteq \mathcal{M}$. Moreover, every derivation of $B$ is induced in this manner by a derivation of $A$.

**Proof.** Every $\tilde{u}$ in $B$ is the image under $\varphi$ of some $u$ in $A$, whence $\overline{D}$ is defined on all of $B$. Now assume $\mathcal{M}D \subseteq \mathcal{M}$. Suppose $\tilde{u} = \varphi(u) = \varphi(v)$ for elements $u$ and $v$ in $A$. Then $u = v + a$ where $a$ is in $\mathcal{M}$, $uD = vD + aD$, and $\varphi(uD) = \varphi(vD) + \varphi(aD)$. But $aD$ is in $\mathcal{M}$, so $\varphi(aD) = 0$, $\varphi(uD) = \varphi(vD)$. Thus $\overline{D}$ is well-defined. Conversely, if $\overline{D}$ is well-defined, then $\varphi(u) = \varphi(v)$ implies $\varphi(uD) = \varphi(vD)$. Thus, if $a$ is any element of $\mathcal{M}$ we have

$$\varphi(uD) = \varphi((u + a)D) = \varphi(uD) + \varphi(aD)$$

from which it follows that $\varphi(aD) = 0$ and $aD$ is in $\mathcal{M}$. We conclude that $\overline{D}$ is well-defined if and only if $\mathcal{M}D \subseteq \mathcal{M}$. We will now show that $\overline{D}$ is a derivation of $B$.

Let $\tilde{u}, \tilde{v}$ be elements of $B$ and let $\alpha, \beta$ be in $\mathbb{F}$. Then $\tilde{u} = \varphi(u), \tilde{v} = \varphi(v)$ for some $u$ and $v$ in $A$, and

$$(\alpha \tilde{u} + \beta \tilde{v})\overline{D} = [\varphi(\alpha u + \beta v)]\overline{D} = \varphi((\alpha u + \beta v)D)$$

$$= \alpha \varphi(uD) + \beta \varphi(vD) = \alpha (u\overline{D}) + \beta (v\overline{D}).$$

Hence $\overline{D}$ is linear. We also have

$$(\tilde{u}\tilde{v})\overline{D} = [\varphi(uv)]\overline{D} = \varphi((uv)D)$$

$$= \varphi((uD)v + u(vD)) = (u\overline{D})\tilde{v} + \tilde{u}(v\overline{D}),$$

so $\overline{D}$ is a derivation.

Now let $\overline{D}$ be any derivation of $B$. We shall show that $\overline{D}$ is the induced derivation $\overline{D}$ of some derivation $D$ of $A$. Any element $\tilde{u}$ of $B$ may be written as a polynomial in the generators $\tilde{x}_1, \ldots, \tilde{x}_n$. And, as in $A$, $\overline{D}$ is completely
determined by its action on the $\bar{x}$, according to the formula

$$u\bar{D} = (\partial u/\partial \bar{x}_1)(\bar{x}_1 \bar{D}) + \cdots + (\partial u/\partial \bar{x}_n)(\bar{x}_n \bar{D}).$$

Choose elements $y_i$ in $\mathfrak{A}$ so that $\phi(y_i) = \bar{x}_i \bar{D}$ for $i = 1, \cdots, n$. We can define a derivation $D$ of $\mathfrak{A}$ by specifying that $x_i D = y_i$ ($i = 1, \cdots, n$). Now let $\overline{D}$ be induced by $D$ according to formula (10). Then $\bar{x}_i \overline{D} = \bar{x}_i \bar{D}$ for $i = 1, \cdots, n$. Thus if $\overline{D}$ is a derivation we shall have $\overline{D} = \bar{D}$. Therefore it remains only to show that $\mathfrak{M} D \subseteq \mathfrak{M}$.

It is readily seen that if $f = f(x_1, \cdots, x_n)$ is any polynomial over $\mathfrak{A}$ in $x_1, \cdots, x_n$, then $\bar{f} = \phi(f) = f(x_1, \cdots, x_n)$ is the same polynomial with $x_i$ replaced by $\bar{x}_i$ for $i = 1, \cdots, n$. Thus we may write $\partial f/\partial x_i = g_i(x_1, \cdots, x_n)$ and

$$\phi(\partial f/\partial x_i) = \phi(g_i) = g_i(x_1, \cdots, x_n) = \partial \bar{f}/\partial \bar{x}_i$$

for $i = 1, \cdots, n$. Now let $u$ be any element of $\mathfrak{M}$. Then $u = \phi(u) = 0$, and by (9), (11), and (12) we have

$$\phi(uD) = \phi(\partial u/\partial x_1)y_1 + \cdots + \phi(\partial u/\partial x_n)y_n$$

$$= (\partial u/\partial \bar{x}_1)y_1 + \cdots + (\partial u/\partial \bar{x}_n)y_n = u\bar{D} = 0.$$

Therefore $uD$ is in $\mathfrak{M}$ and the theorem is proved.

We noted earlier that every element of $\mathfrak{A}$ which is not in $\mathfrak{N}$ has an inverse. From this it follows that every proper ideal of $\mathfrak{A}$ is contained in $\mathfrak{M}$. Thus $\mathfrak{M} \subseteq \mathfrak{R}$. Recalling that $\mathfrak{B}$ is $\mathfrak{D}$-simple for a set $\mathfrak{D}$ of derivations of $\mathfrak{B}$ we now state

**Theorem 3.** Let $\mathfrak{D}$ be the set of all derivations $D$ of $\mathfrak{A}$ over $\mathfrak{B}$ such that the induced derivations $\overline{D}$ are in $\mathfrak{D}$. Then $\mathfrak{M}$ is a maximal $\mathfrak{D}$-ideal of $\mathfrak{A}$, and an element $u$ of $\mathfrak{A}$ is in $\mathfrak{M}$ if and only if $u$ is in $\mathfrak{M}$ and the elements $u D_1 \cdots D_k$ are in $\mathfrak{M}$ for all values of $k$ and all derivations $D_i$ in $\mathfrak{D}$.

**Proof.** By Theorem 2, if $D$ induces $\overline{D}$ then $\mathfrak{M}$ is a $D$-ideal. Thus $M$ is a $D$-ideal for every $D$ in $\mathfrak{D}$ and hence is a $\mathfrak{D}$-ideal. Let $\mathfrak{R} \neq \mathfrak{A}$ be a $\mathfrak{D}$-ideal properly containing $\mathfrak{M}$ and let $\mathfrak{R} = \phi(\mathfrak{R})$. It is easily verified that $\mathfrak{R}$ is a nontrivial $\mathfrak{D}$-ideal in $\mathfrak{B}$ contradicting $\mathfrak{D}$-simplicity. Hence $\mathfrak{M}$ is a maximal $\mathfrak{D}$-ideal. It is also easily seen that the sum of two $\mathfrak{D}$-ideals is a $\mathfrak{D}$-ideal from which it follows that $\mathfrak{M}$ is maximal in the strong sense that it contains all other $\mathfrak{D}$-ideals.

Now let $u$ be any element of $\mathfrak{M}$. Then $u$ is in $\mathfrak{M}$ and $u D_1 \cdots D_k$ is in $\mathfrak{M}$ for all $k$ and all $D_i$ in $\mathfrak{D}$. Conversely, suppose $u$ is in $\mathfrak{M}$ and $u D_1 \cdots D_k$ is in $\mathfrak{M}$ for every product $D_1 \cdots D_k$ of derivations in $\mathfrak{D}$. The ideal generated by $u$ and all $u D_1 \cdots D_k$ is a $\mathfrak{D}$-ideal and hence is contained in $\mathfrak{M}$. Thus $u$ is in $\mathfrak{M}$, and the proof is complete.

To Theorem 3 we have the following immediate
Corollary. If \( u \) is an element of \( \mathfrak{M} \), then \( \tilde{u} = \phi(u) \) is a nonzero element of \( \mathfrak{B} \) if and only if there is some product \( D_1 \cdots D_k \) of derivations \( D_i \) in \( \mathfrak{D} \) such that \( uD_1 \cdots D_k \) is nonsingular.

Theorem 4. Let \( u \) be an element of \( \mathfrak{M} \) whose terms of degree one are not all zero. Then \( u \) is not in \( \mathscr{F} \).

Proof. We assume without loss of generality that \( u \) is in \( \mathfrak{M} \) and its term of degree one in \( x_1 \) is not zero. Then we may write \( u = ax_1 + v \) where \( a \) is nonsingular and \( v \) is in \( \mathfrak{F}[x_3, \cdots, x_n] \). Thus

\begin{equation}
\begin{aligned}
x_1 &= a^{-1}u - a^{-1}v = u_0 + v_0
\end{aligned}
\end{equation}

where \( u_0 = a^{-1}u \) is in \( \mathfrak{M} \) and \( v_0 = -a^{-1}v \) is in the ideal \( \mathfrak{B} \) generated by \( x_3, \cdots, x_n \). We observe that every element \( f \) of \( \mathfrak{B} \) is a polynomial with terms of the form \( x_1^r y \) where \( y \) is a monomial in \( \mathfrak{F}[x_3, \cdots, x_n] \) of degree \( t \geq 1 \). If \( f \) is not in \( \mathfrak{F}[x_3, \cdots, x_n] \) we associate with \( f \) the number \( N(f) \) which is the minimum of the degrees of \( y \) for all terms \( x_1^r y \) of \( f \) with \( r \neq 0 \). Note that \( 1 \leq N(f) \leq (n-1)(p-1) \).

Now assume it impossible to write \( x_1 = u_1 + v_1 \) where \( u_1 \) is in \( \mathfrak{M} \) and \( v_1 \) is in \( \mathfrak{F}[x_3, \cdots, x_n] \). Then we may write \( x_1 = u_2 + v_2 \) where \( u_2 \) is in \( \mathfrak{M} \), \( v_2 \) is in \( \mathfrak{B} \), and \( N(v_2) \) is maximal. Let \( x_1^r y \) be any term of \( v_2 \) with \( r \neq 0 \) and \( y \) of degree \( N(v_2) \). By (13) we have for each such \( x_1^r y \)

\begin{equation}
\begin{aligned}
x_1^r y &= x_1^{r-1}(u_0 + v_0)y = x_1^{r-1}u_0y + x_1^{r-1}v_0y
\end{aligned}
\end{equation}

where \( x_1^{r-1}u_0y \) is in \( \mathfrak{M} \) and \( x_1^{r-1}v_0y \) is a polynomial each term of which has the form \( x_1^s z \) with \( z \) in \( \mathfrak{F}[x_3, \cdots, x_n] \) and the degree of \( z \) greater than \( N(v_2) \). Hence by means of substitutions as in (14) we may obtain \( x_1 = u_3 + v_3 \) with \( u_3 \) in \( \mathfrak{M} \), \( v_3 \) in \( \mathfrak{B} \), and \( N(v_3) > N(v_2) \). Thus \( x_1 = u_4 + v_1 \) with \( u_4 \) in \( \mathfrak{M} \) and \( v_1 \) in \( \mathfrak{F}[x_3, \cdots, x_n] \). But from this it follows that \( x_1 = \phi(x_1) = \phi(v_1) \) is in \( \mathfrak{F}[x_3, \cdots, x_n] \) contradicting the hypothesis that \( \{x_1, \cdots, x_n\} \) is a minimal set of generators of \( \mathfrak{B} \). This proves our theorem.

Before we can prove our next theorem we must develop some notation and prove two lemmas on the combinatorial properties of derivations. Let \( S = \{n_1, \cdots, n_s\} \) be an ordered set of positive integers, the ordering being the natural one. Let \( \pi_1, \cdots, \pi_r \) be ordered subsets of \( S \) such that \( \pi_1 \cup \cdots \cup \pi_r = S \) and \( \pi_i \cap \pi_j = 0 \) (the empty set) if \( i \neq j \). We shall call the ordered \( r \)-tuple \( \pi = (\pi_1, \cdots, \pi_r) \) an \( r \)-partition of \( S \).

Lemma 1. Let \( a_1, \cdots, a_r \) be elements of \( \mathfrak{A} \) and let \( D_1, \cdots, D_r \) be derivations in \( \mathfrak{D} \). Let

\[ T(\pi_i) = D_{i_1} \cdots D_{i_r} \quad (i = 1, \cdots, r) \]

if \( \pi_i = \{i_1, \cdots, i_r\} \) is a nonempty ordered subset of the ordered set \( S = \{1, \cdots, s\} \); and if \( \pi_i = 0 \), then \( T(\pi_i) \) is to be the identity transformation \( I \) of \( \mathfrak{A} \). We now assert that
where $\pi$ ranges over all $r$-partitions of $S$.

**Proof.** We induct on $s$. If $s = 1$, each partition has the form $\pi = (0, \cdots, 0, 1, 0, \cdots, 0)$ and formula (15) becomes

$$(a_1 \cdots a_r) D_1 \cdots D_s = \sum_{r} [a_1 T(\pi_1)] \cdots [a_r T(\pi_r)]$$

which is correct. Now assume (15) correct for $s$ derivations. Then

$$(a_1 \cdots a_r) D_1 \cdots D_{s+1} = \sum_{r} \{ [a_1 T(\pi_1)] \cdots [a_r T(\pi_r)] \} D_{s+1}$$

$$= \sum_{r} \sum_{i} [a_1 T(\pi_i)] \cdots [a_{i-1} T(\pi_{i-1})]$$

$$\cdot [a_i T(\pi_i \cup \{s + 1\})][a_{i+1} T(\pi_{i+1})] \cdots [a_r T(\pi_r)].$$

But if $\pi = (\pi_1, \cdots, \pi_r)$ is a general $r$-partition of $\{1, \cdots, s\}$, then $\theta = (\pi_1, \cdots, \pi_{i-1}, \pi_i \cup \{s + 1\}, \pi_{i+1}, \cdots, \pi_r)$ is a general $r$-partition of $\{1, \cdots, s+1\}$. Hence

$$(a_1 \cdots a_r) D_1 \cdots D_{s+1} = \sum_{\theta} [a_1 T(\theta_1)] \cdots [a_r T(\theta_r)]$$

where $\theta = (\theta_1, \cdots, \theta_r)$ ranges over all $r$-partitions of $\{1, \cdots, s+1\}$. This is formula (15) for $s+1$ derivations, and the lemma is proved.

We also have

**Lemma 2.** Let $S_1 = \{i_1, \cdots, i_q\}$ and $S_2$ be ordered subsets of the set $S = \{1, \cdots, s\}$ such that $S_1 \cap S_2 = 0$ and $S_1 \cup S_2 = S$, and let $R$ be the set of all $r$-partitions $\pi$ of $S$ with $\pi_i = S_2$ for some fixed $t$. If $a_1, \cdots, a_r$ are in $\mathfrak{A}$ and $D_1, \cdots, D_s$ in $\mathfrak{D}$, then

$$\sum_{\pi \in R} [a_1 T(\pi_1)] \cdots [a_r T(\pi_r)]$$

$$= [(a_1 \cdots a_{i-1} a_{i+1} \cdots a_r) D_{i_1} \cdots D_{i_q}][a_i T(\pi_i)].$$

**Proof.** If $\pi = (\pi_1, \cdots, \pi_r)$ with $\pi_i = S_2$, then $\theta = (\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_r)$ is an $(r-1)$-partition of $S_1$. Moreover, the correspondence $\pi \leftrightarrow \theta$ is a 1-1 correspondence of $R$ with the set of all $(r-1)$-partitions of $S_1$. Our result now follows from Lemma 1.

We are now able to prove

**Theorem 5.** The ideal $\mathfrak{M}$ contains no monomial.

**Proof.** Theorem 4 asserts that $\mathfrak{M}$ contains no monomial of total degree one in $x_1, \cdots, x_n$. Assume that $\mathfrak{M}$ contains no monomial of degree $r-1$ but
that \( u = x_1^{r_1} \cdots x_n^{r_n} \) has degree \( r = r_1 + \cdots + r_n \) and is in \( \mathbb{M} \). Then for each \( i \) for which \( r_i \neq 0 \) we may write \( u = a_i x_i \) where \( a_i \) is not in \( \mathbb{M} \). Thus, by the corollary to Theorem 3, there is a product \( G_i \) of \( t_i \) derivations in \( \mathcal{D} \) such that \( a_i G_i \) is nonsingular. We let \( i_0 \) be a value of \( i \) for which \( t_i = t_i \) is minimal. There is clearly no loss of generality if we assume \( i_0 = 1 \) and \( G = G_1 = D_1 \cdots D_t \) so that \( a_1 G \) is nonsingular. We now apply \( G \) to the element \( u \), and by Lemma 1 we obtain

\[
\begin{align*}
\sum x_i T(\pi_{ij}) & \cdots x_n T(\pi_{in})
\end{align*}
\]

We observe that the constant term of \( uG \) is zero since \( u \) is in \( \mathbb{M} \) and \( uG \) is in \( \mathbb{M} \). Let us now compute the linear term in \( x_1 \) of \( uG \). Consider first all summands in (17) with \( \pi_{ij} = 0 \) for some fixed index \( j \). By Lemma 2 the sum of these summands is \((a_1 G)x_1\) which has a term \( a_1 x_1 \) where \( a_1 \neq 0 \) is the constant term of the nonsingular element \( a_1 G \). Letting \( j = 1, \cdots, r_1 \) we find that the total coefficient of \( x_1 \) from this source is \( r_1 a_1 \neq 0 \). Note that any summand in (17) in which \( \pi_{ij} = 0 \) with \( i \neq 1 \) has \( x_1 \) as a factor and therefore does not have a linear term in \( x_1 \). Thus there remains only the consideration of those summands of (17) in which all \( \pi_{ij} \) are nonempty. For such a summand to have a linear term in \( x_1 \) it must be that some \( x_i T(\pi_{ij}) \) has a linear term in \( x_1 \) and all other \( x_k T(\pi_{ik}) \) are nonsingular. But again it follows from Lemma 2 that the sum of all summands in (17) having \( x_i T(\pi_{ij}) \) as a factor is \( w = (a_i H) [x_i T(\pi_{ij})] \) where \( H \) is a product of fewer than \( t \) derivations. Hence \( a_i H \) is singular and \( w \) has no linear term. We conclude that \( uG \) has a linear term \( r_1 a_1 x_1 \) contrary to Theorem 4.

We are now essentially through. Albert has shown [2] that for any non-zero element \( u \) of \( \mathbb{N} \) there exists an element \( v \) of \( \mathbb{F} \) such that \( uv = x_1^{-1} \cdots x_n^{-1} \). Thus if \( \mathbb{M} \neq 0 \) then \( \mathbb{M} \) contains a monomial, contrary to Theorem 5. Therefore \( \mathbb{M} = 0 \) from which Theorem 1 follows.

3. Some consequences of condition (B). Let \( \mathcal{X} \) be the commutative power-associative algebra described in §1. By (1) we see that

\[
\mathcal{X} = \mathcal{B} + \mathcal{B} = \mathcal{B} + (y_0 \mathcal{B}, \cdots, y_m \mathcal{B}),
\]

and, having determined the structure of \( \mathcal{B} \), we are now in a position to investigate that of \( \mathcal{X} \).

Let \( u \) be any element of \( \mathcal{B} \). Then \( u = \sum_{j=0}^{n} y_j b_j \) where \( b_j \) is in \( \mathcal{B} \). From condition (B) we see that \( u = 0 \) if and only if \( (y_0 a)u = \sum_{j=0}^{n} (y_j a)(y_j b_j) = 0 \) for all \( a \) in \( \mathcal{B} \) and \( i = 0, 1, \cdots, m \). From this it follows that \( u = 0 \) if and only if the relations

\[
\sum_{j=0}^{m} (b_{ij} b_j - b_j D_{ij}) = 0 \quad (i = 0, \cdots, m),
\]

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\[ \sum_{j=0}^{m} (aD_{ij})b_j = 0 \quad (i = 0, \ldots, m) \]

hold for every \( a \) in \( \mathfrak{B} \).

It should be noted that the requirement that the algebra \( \mathfrak{T} \) satisfy condition (B) is never inconsistent with the definition of multiplication in \( \mathfrak{T} \). The effect of condition (B) is to completely determine the algebra \( \mathfrak{T} \) by determining the kernels of the vector space homomorphisms \( \mathfrak{B} \to \mathfrak{B}_i \) for \( i = 0, \ldots, m \) and the nature of the sum \( \mathfrak{L}_i \). To demonstrate this we let

\[ \mathfrak{T}^* = \mathfrak{B} + \mathfrak{L}^* = \mathfrak{B} + z_0\mathfrak{B} + \cdots + z_m\mathfrak{B} \]

where each vector space \( z_i\mathfrak{B} \) is an isomorphic copy of \( \mathfrak{B} \). Let products in \( \mathfrak{T}^* \) be defined in terms of the same elements \( b_{ij} \) and derivations \( D_{ij} \) of \( \mathfrak{B} \) which determined products in \( \mathfrak{T} \). Since \( \mathfrak{T}^* \) is a direct sum we see that multiplication is well-defined. Now let \( \mathfrak{U} \) be the set of all elements \( u \) in \( \mathfrak{L}^* \) such that \( uw = 0 \) for all \( w \) in \( \mathfrak{L}_i \). The set \( \mathfrak{U} \) is an ideal of \( \mathfrak{T}^* \). The algebra \( \mathfrak{T} \) is equivalent to \( \mathfrak{T}^*/\mathfrak{U} \) and hence exists and is uniquely determined by condition (B) and the choice of the elements \( b_{ij} \) and derivations \( D_{ij} \) of \( \mathfrak{B} \).

4. A special case with \( m = 2 \). In this section we shall construct a class of examples of the algebras \( \mathfrak{T} \) in which \( \mathfrak{L} = (y_0\mathfrak{B}, \ldots, y_m\mathfrak{B}) \) with \( m = 2 \) and \( \mathfrak{L} \) is not a direct sum. We let \( \mathfrak{B} = \mathfrak{F}[e, x, y], \mathfrak{T} = \mathfrak{B} + (y_0\mathfrak{B}, y_1\mathfrak{B}, y_2\mathfrak{B}) \) and let

\begin{align*}
(20) & \quad xD_{01} = e, \quad yD_{01} = x^{p-1}, \\
(21) & \quad xD_{02} = x^2y, \quad yD_{02} = xy, \\
(22) & \quad xD_{12} = -x, \quad yD_{12} = xy^2, \\
(23) & \quad b_{11} = 0, \quad b_{12} = e, \quad b_{22} = -x^2.
\end{align*}

The algebra \( \mathfrak{B} \) is \( D_{01} \)-simple \([2]\) and hence is \( \{D_{ij}\} \)-simple. Thus \( \mathfrak{T} \) satisfies condition (A). We complete the definition of \( \mathfrak{T} \) by imposing condition (B). As a routine consequence of formulas (18) and (19) we now have Lemma 3 which we state without proof.

**Lemma 3.** In this special case an element \( y_0b_0 + y_1b_1 + y_2b_2 \) of \( \mathfrak{L} \) is zero if and only if \( b_0 = -b_2x \), \( b_1 = 0 \), and \( b_2 = x^{p-1}f(y) + y^{p-1}g(x) \) where \( f(y) \) and \( g(x) \) are polynomials over \( \mathfrak{F} \) in \( y \) and \( x \) respectively.

It follows from Lemma 3 that \( \mathfrak{T} \) is not a direct sum. In fact we may write

\[ \mathfrak{T} = \mathfrak{B} + (y_0\mathfrak{B} + y_1\mathfrak{B}, y_2\mathfrak{B}) \]

and we see that \( (y_0\mathfrak{B} + y_1\mathfrak{B}) \cap y_2\mathfrak{B} \) is spanned by the independent vectors \( y_2x^{p-1}y^i \) and \( y_2x^iy^{p-1} \) where \( i = 0, \ldots, p-1 \) and \( j = 0, \ldots, p-2 \). Hence \( \mathfrak{T} \) has dimension \( 4p^2 - 2p + 1 \). We will show next that \( \mathfrak{T} \) not only fails to be a direct sum as presently represented, but, furthermore, *Albert's construction cannot yield a representation of \( \mathfrak{T} \) as a direct sum.*

Let \( \mathfrak{S} = \mathfrak{F}[\bar{e}, \bar{z}_1, \ldots, \bar{z}_i] = \bar{e}\bar{S} + \bar{R} \) be a polynomial algebra over \( \mathfrak{F} \) with
unity \( \mathcal{E} \) and generators \( \tilde{e}_1, \ldots, \tilde{e}_r \), such that \( \tilde{e}_i^p = 0 \) for \( i = 1, \ldots, r \) but which are otherwise independent. Suppose there exist \( x_0, \ldots, x_m \) such that \( \mathcal{I} = \mathcal{B} + \mathcal{I} \) where \( \mathcal{I} = x_0\mathcal{B} + \cdots + x_m\mathcal{B} \) is a direct sum. We will denote by \( a, b, \) etc. elements of \( \mathcal{B} \) and by \( b_{ij} \) and \( D_{ij} \) the elements and derivations of \( \mathcal{B} \) which define multiplication in this new representation of \( \mathcal{I} \). We observe that \( \mathcal{E} = e \) since \( \mathcal{B} \) and \( \mathcal{B} \) have the same unity element as \( \mathcal{I} \). We may write expressions for the \( x_k e \) in terms of the original representation of \( \mathcal{I} \). Thus

\[
(25) \quad x_k e = a_k + y_0 b_k + y_1 c_k + y_2 d_k
\]

where \( a_k, b_k, c_k, d_k \) are in \( \mathcal{B} \) and \( k = 0, \ldots, m \). Since \( \mathcal{I} \) is power-associative and \( p \) is an odd prime we have

\[
(26) \quad (x_k b)^p = \left( x_k b \right)^{2^{(p-1)/2}} = x_k (b_k^{(p-1)/2} b^p)
\]

for \( k = 0, \ldots, m \), and similarly

\[
(27) \quad (y_k b)^p = y_k (b_k^{(p-1)/2} b^p)
\]

for \( k = 0, 1, 2 \). From (23), (25), (26), (27) and Lemma 3 we see that

\[
(28) \quad x_k e = (x_k e)^p = a_k^p + y_0 b_k^p = \alpha + y_0 \beta
\]

where \( \alpha \) and \( \beta \) are in \( \mathcal{B} \). Since \( (x_k e)^2 = e \) it follows that \( \alpha = 0 \) and \( \beta = \pm 1 \). Thus \( x_k e = \pm y_0 e \) and we can now prove

**Lemma 4.** The algebras \( \mathcal{B} \) and \( \mathcal{B} \) coincide as do the spaces \( \mathcal{I} \) and \( \mathcal{I} \).

**Proof.** Since \( x_k e = \pm y_0 e \) and \( (x_k e)(x_k e) = 0 \) for \( k = 1, \ldots, m \), we see by (25) that \( a_k = 0 \) for \( k = 0, \ldots, m \). Hence \( \mathcal{I} \subseteq \mathcal{B} \). Since \( \mathcal{B} \) is \( \{ D_{ij} \} \)-simple there is a derivation \( D_{ik} \) of \( \mathcal{B} \) such that \( \mathcal{B} \) is not a \( D_{ik} \)-ideal. From this it follows that each of \( \mathcal{B}, x_0 \mathcal{B}, \) and \( x_1 \mathcal{B} \) has dimension \( p^r \). Thus \( 3p^r \leq 4p^2 - 2p + 1 \) which implies \( r \leq 2 \). If \( r < 2 \) the dimension of \( \mathcal{B} \) is seen to be greater than that of \( \mathcal{B} \). Thus \( r = 2 \) and \( \mathcal{B} = \mathcal{B} \).

We have shown that \( \mathcal{B} + \mathcal{V} = \mathcal{B} + \mathcal{B} \). Now let \( b \) be any element of \( \mathcal{B} \). Then \( b = b + u \) for elements \( b \) in \( \mathcal{B} \) and \( u \) in \( \mathcal{V} \). Let \( w \) be an arbitrary element of \( \mathcal{V} \). We see that \( uw = bw - bw \) is in \( \mathcal{B} \) and in \( \mathcal{V} \). Thus \( uw = 0 \) for all \( w \) in \( \mathcal{V} \). Therefore \( u = 0 \), and the lemma is proved.

We now have \( \mathcal{B} = \mathcal{B} = \mathcal{B} [e, x, y] \) and

\[
\mathcal{I} = \mathcal{B} + (y_0 \mathcal{B} + y_1 \mathcal{B}, y_2 \mathcal{B}) = \mathcal{B} + x_0 \mathcal{B} + \cdots + x_m \mathcal{B}.
\]

We shall show that this leads to a contradiction. Setting \( a_k = 0 \) \( (k = 0, \ldots, m) \) in (25) we obtain

\[
(28) \quad x_k e = y_0 b_k + y_1 c_k + y_2 d_k \quad (k = 0, \ldots, m).
\]

**Lemma 5.** If \( \mathcal{I} = \mathcal{B} + x_0 \mathcal{B} + \cdots + x_m \mathcal{B} \) is a direct sum, then \( m = 2 \) and there exist elements \( v \) and \( w \) in \( \mathcal{B} \) such that \( b_2 = -d_2 x + xyw \) and \( c_2 = xyw \).
Proof. The dimension of \( x_0 \mathfrak{B} \) is always the same as that of \( \mathfrak{B} \), and we noted earlier that this must also be true for at least one other \( x_k \mathfrak{B} \). Hence we may assume that \( x_1 \mathfrak{B} \) has dimension \( p^2 \). It follows that \( x_k \mathfrak{B} \) has dimension less than \( p^2 \) for \( k = 2, \ldots, m \). Hence if \( U = x^{p-1}y^{p-1} \), then \( x_k U = 0 \) for \( k \geq 2 \), and we see from (28) and Lemma 3 that \( b_k \) and \( c_k \) are in \( \mathfrak{R} \) for \( k \geq 2 \). We see also that \( x_1 U + x_0 b_1 U = y_1 c_1 U \neq 0 \) since \( x_0e = \pm y_0e \) and \( x_0 \mathfrak{B} + x_1 \mathfrak{B} \) is a direct sum. Hence \( c_1 \) is not in \( \mathfrak{R} \).

Now let \( u_0, \ldots, u_m \) be elements of \( \mathfrak{B} \) such that \( y_0e = x_0 u_0 + \cdots + x_m u_m \). From (28) and Lemma 3 we see that \( c_1 u_1 + \cdots + c_m u_m = 0 \), and, since \( c_1 \) is not in \( \mathfrak{R} \) but \( c_2, \ldots, c_m \) are in \( \mathfrak{R} \), this implies that \( u_1 \) is in \( \mathfrak{R} \). Now, also by Lemma 3, \( d_1 u_1 + \cdots + d_m u_m - e \) is in \( \mathfrak{R} \) and hence \( d_1 \) is not in \( \mathfrak{R} \) for some \( k \geq 2 \). Without loss of generality we assume \( d_2 \) is not in \( \mathfrak{R} \).

Let \( \mathfrak{C} \) be the subspace of \( \mathfrak{B} \) consisting of all elements \( u \) such that \( x_2 u = 0 \). Since \( d_2 \) is nonsingular, it follows from (28) and Lemma 3 that \( u = x^{p-1} F(y) + y^{p-1} G(x) \) for some polynomials \( F(y) \) and \( G(x) \). Let \( s \) be the dimension of \( \mathfrak{C} \). Then \( x_2 \mathfrak{B} \) has dimension \( p^2 - s \), and we see that \( s \geq 2p - 1 \). Thus \( x_2 u = 0 \) for all possible choices of the polynomials \( F(y) \) and \( G(x) \). Thus \( x_2 \mathfrak{B} \) has dimension \( p^2 - 2p + 1 \) and \( m = 2 \).

We have shown that the space \( \mathfrak{C} \) consists of all \( u \) in \( \mathfrak{B} \) of the form \( u = x^{p-1} F(y) + y^{p-1} G(x) \). From Lemma 3 we see that \( c_1 u_0 = 0 \) and \( (b_2 + d_2 x) u = 0 \) for all \( u \) in \( \mathfrak{C} \). It therefore follows that \( c_2 = x y v \) and \( b_2 = -d_2 x + x y v \) for some \( v \) and \( w \) in \( \mathfrak{B} \). We can now obtain our main result which we state as

**Theorem 6.** The algebra \( \mathfrak{X} \) cannot be represented as a direct sum.

Proof. We compute \( b_{02} \) and single out those terms which possibly give rise to a linear term in \( x \) alone. We recall that \( b_{02} = 0 \) by (2). Using (20), \( \cdots, (23) \) and Lemma 5 we see that

\[
b_{02} = (x_0 e)(x_2 e) = \pm (y_0 e)(y_0 b_2 + y_1 c_2 + y_2 d_2)
= \pm (b_2 - c_2 D_{01} - d_2 D_{02}) = \mp d_2 x + \Omega
\]

where \( \Omega \) is a sum of terms each having \( x^2 \) or \( y \) as a factor. Since \( d_2 \) is nonsingular \( b_{02} \neq 0 \). This contradiction proves the theorem.

**References**


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