SEMI-SIMILARITY INVARIANTS FOR SPECTRAL OPERATORS ON HILBERT SPACE(1)

BY

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1. **Introduction.** Classical spectral multiplicity theory generalizes to normal operators on Hilbert space the unitary determination of a normal matrix by the multiplicities of its eigenvalues. Somewhat more refined notions are required in the finite dimensional case for the equivalence theory of the non-normal matrix or operator. Here the problem is most fruitfully considered in terms of similarity, and the characterization can be given by the elementary divisors, invariant factors, various canonical matrix forms, or either of the numerical invariants, the Weyr or Segre characteristics. Occasionally discussed in matrix theory textbooks (see, for example, [22; 30]) these numerical invariants are generally defined in combinatorial terms: *the Segre characteristic* (after Corrado Segre [28]) *of an eigenvalue* λ_0 *of the operator* T is the sequence, in descending order, of the exponents of the elementary divisors of T that contain λ_0; this sequence of integers sums to the multiplicity $\mathbf{u}(\lambda_0)$ of λ_0 as a root of the characteristic polynomial, and the conjugate partition in decreasing order of $\mathbf{u}(\lambda_0)$ is the *Weyr characteristic* (after Eduard Weyr [33; 34]) *of* λ_0. Both characteristics are easily read off the Jordan canonical form(2), and either one, for all eigenvalues, is a complete set of similarity invariants for T.

The aim of the present work (whose results have in part been summarized in [12]) is to present in detail the beginnings of an equivalence theory for operators on Hilbert and Banach spaces patterned after this elegant and complete finite dimensional theory. Before turning to such operators however, and to the known results of spectral multiplicity theory, it is convenient to reclothe the characteristics in more modern garb. For this we let $\mathcal{W}(\lambda_0, k)$ be the kth integer in the Weyr characteristic of λ_0 for T, and $\mathcal{S}(\lambda_0, k)$ be the number of occurrences of k as an exponent of the elementary divisors of T that contain λ_0—that is, the number of occurrences of k in the Segre characteristic. Then these are related in a simple manner:

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(2) For example, if the λ_0-block of the Jordan canonical representation for T has superdiagonal entries $\{1110111011101101000\}$, then $\mathbf{u}(\lambda_0)$ is 20, the Segre characteristic of λ_0 is 44432111, and the Weyr characteristic is 8543.

$$\mathcal{S}(\lambda_0, k) = \mathcal{W}(\lambda_0, k) - \mathcal{W}(\lambda_0, k+1),$$

and,

$$\sum_k \mathcal{W}(\lambda_0, k) = \sum_k k\mathcal{S}(\lambda_0, k) = \mathfrak{u}(\lambda_0).$$

These functions also have simple spatial interpretations: $\mathcal{W}(\lambda_0, k)$ is the maximum number of linearly independent vectors annihilated by $(T-\lambda_0 I)^k$ but not by $(T-\lambda_0 I)^{k-1}$, and $\mathcal{S}(\lambda_0, k)$ is the maximum number of independent k-dimensional subspaces completely reducing T on which $T-\lambda_0 I$ has index k.

We intend to present, as a basic complete set of invariants for a certain class of operators on Hilbert space, a function $\mathcal{W}$ with properties like those of the finite dimensional Weyr characteristic above, defined for two arguments, measures and cardinal numbers. Operators with the same generalized Weyr characteristic will be "piecewise" similar; the equivalence relation obtained this way is called *semi-similarity.*

Operator equivalence on Hilbert space, usually treated as a problem of unitary equivalence, has a history of more than fifty years. One can mention the original solution for a self-adjoint operator on a separable Hilbert space by Hellinger [18] and Hahn [16], the extension by Stone [29] to unbounded operators, the formulation in terms of a cardinal valued function with Borel set argument by Friedrichs [15], and the extension to the nonseparable case by Wecken [31], whose work, together with that of Nakano [23; 24], laid the foundation for the modern approach. Here the completion of the Boolean algebra of projections in the resolution of the identity of the given operator is basic; following this line there have been treatments by Plessner and Rohlin [25] and Halmos [17] and attention given by a variety of authors to unitary invariants for systems of operators: Kelley [20] treating commutative W^* algebras, Segal [27] commutative and noncommutative W^* algebras, Mackey [21] representations of commutative C^* algebras on a separable Hilbert space, and Kadison [19] representations of noncommutative C^* algebras on an arbitrary Hilbert space.

We shall return to some aspects of this subject after considering another part of operator theory: the attempt, developed most extensively by Dunford and co-workers (Dunford's review article [10] covers the subject and literature exhaustively), to extend the reduction theory arising from the spectral theorem for normal operators to operators on Banach space.

The basic notion here (formal definitions will be given in §2) is that of a completely reducible or *spectral* operator—roughly, one which has a resolution of the identity like that of a normal operator. A spectral operator is uniquely decomposable as the sum of two commuting operators, a quasi-nilpotent and a *scalar* operator (or the *scalar part*), this latter an operator expressible as an integral $\int \lambda E(d\lambda)$, with E the resolution of the identity. (We note the obvious parallel between this decomposition and that of a Jordan canonical matrix as the sum of a diagonal matrix and a nilpotent matrix commuting with it.) This has led to studies by Bade of unbounded spectral operators and Boolean algebras of projections on Banach space in a series of papers [1; 2; 3; 4] on which our work will primarily be based.

In the equivalence theory presented here, though the invariants are constructed in the Banach space context, for application to operators both abstract structural knowledge of the operators and a multiplicity theory of the classical kind are required. To meet the first of these requirements our approach is oriented toward spectral operators; the second limits us to Hilbert space.

To clarify the second restriction, we call attention to the current state of multiplicity theory on Banach space. Dieudonné [6] has constructed a multiplicity theory for representations of a function algebra as an algebra of operators on a Banach space whose conjugate space is separable, and Bade a multiplicity theory for a complete Boolean algebra of projections on an arbitrary Banach space in [4], which will be used extensively here. One would like to derive from these a multiplicity theory of the classical kind for operators—i.e., producing equivalence conclusions for, say, scalar operators with respect to some form of similarity—but there seem to be substantial difficulties involved. One of these difficulties, for example, illustrated by Dieudonné's example [7], is that there is in general no bounded projection onto the pieces of simple multiplicity. Another is that, though Bade's work is the natural generalization to Banach space of the Hilbert space commutative weakly closed algebra case, even on Hilbert space the single operator case is considerably more complicated than that of the weakly closed algebra, as Kadison has recently pointed out in [19]. Kadison's example of two unitarily equivalent W^* algebras on a separable Hilbert space, each generated by unitarily inequivalent C^* algebras, illustrates the difficulty.

Thus though the Weyr characteristic is here actually defined for a Banach space of uniform finite multiplicity with respect to a complete Boolean algebra of projections and a commuting quasi-nilpotent, for application to single operators we turn to Hilbert space and use the multiplicity theory of a normal operator. This approach necessitates some awkwardness. Most of the statements and definitions for spectral operators will be made via normal operators associated in a certain manner with them, this roundabout method being necessary to tie in to the multiplicity theory.

An example is the next (and final) restriction: the equivalence theory applies only to spectral operators on Hilbert space whose scalar parts are similar to normal operators with no part of infinite uniform multiplicity. (A scalar operator on Hilbert space is always similar to a normal operator—by the unmodified word "similar" we always mean the conjugacy induced by a nonsingular operator.) Such spectral operators are called *essentially finite*, though it is not assumed the multiplicities are bounded. This restriction permits the handling of the essentially finite single operator case by piecing together the results obtained for spaces of uniform finite multiplicity, but is a serious incompleteness in the theory and its major weakness at this stage.

The unitary invariants of a normal matrix, or the similarity invariants of a diagonal one, can be viewed as given by a cardinal valued function vanishing off eigenvalues (the function $\mathbf{u}$ above). For a self-adjoint operator on a separable Hilbert space, Borel set arguments suffice in the multiplicity function, but for the nonseparable case equivalence classes of functions, as in Wecken's original work [31], or of measures, as in Halmos' treatment [17], are required. (Actually, in all these cases the projections in some Boolean algebra are fundamental, and the multiplicity function defined directly or indirectly for these.)

The Weyr characteristic will first be defined, when we deal with a complete spectral measure and commuting quasi-nilpotent on Banach space, for Borel set and cardinal number arguments. For each fixed cardinal, considered as a set function, it is then shown to be a multiplicity function in the sense of Halmos, and consequently to decompose the support of the spectral measure into sets of *uniform characteristic*. Our chief result here is that a nonnegligible Borel set δ has uniform characteristic if and only if

$$\sum_k \mathcal{W}(\delta, k) = n,$$

where n is the multiplicity of the space. The definition is then altered to permit measure arguments, the characteristic defined for pairs $\langle N, Q\rangle$ of commuting operators on Hilbert space, where N is normal and essentially finite and Q is quasi-nilpotent, and this extended to essentially finite spectral operators. The Weyr characteristic is similarity invariant, but not a complete set of such invariants, nor does the addition of a piecewise boundedness condition produce similarity for operators with the same characteristic. Two such operators are, however, related by an unbounded similarity, but this notion is not sufficiently precise for an equivalence relation. Thus we call two spectral operators *semi-similar* if they can be decomposed, by projections in the completions of their resolutions of the identity, into the same number of similar parts. Semi-similarity is an equivalence relation for spectral operators, and for essentially essentially finite spectral operators on Hilbert space the Weyr characteristic is a complete set of semi-similarity invariants. Finally, the results extend easily and naturally to the adjoint operators.

In work on multiplicity theory it is customary to apologize for the abstract nature of the invariants produced, and the difficulty (or impossibility) of their computation. After making this apology, we also feel impelled to remark on the naturalness of semi-similarity. Unitary equivalence is the natural and successful equivalence relation for normal operators. But for normal operators, the notions of unitary equivalence and similarity coincide (cf. Putnam [26]). Since the similarity equivalence class determined by a scalar operator on Hilbert space always includes a normal operator, a similarity equivalence theory for such scalar operators can be obtained merely by transferring the unitary invariants of the associated normal operators. The addition of the quasi-nilpotent has necessitated relinquishing, in this work, the boundedness of the similarity.

Semi-similarity does have several desirable properties: the spectrum and essential finiteness are semi-similarity invariant, semi-similar scalar operators are actually similar, and semi-similar normal operators are unitarily equivalent. Reflection on the spatial meaning of the Jordan canonical reduction for matrices also enhances its naturalness. The 1's that were counted, for example, in the superdiagonal appear as 1's as a result of a finite number of norm changes; in the infinite-dimensional case, even under the most restrictive assumptions (separable Hilbert space, pure point spectrum of uniform multiplicity 2), this process cannot be duplicated, and spectral operators with the same spatial action need not be similar.

Thus some form of unbounded similarity arises naturally, and it may even be suggested that an investigation of systems of operators (noncommutative C^* and W^* algebras, for example) in terms of some form of similarity equivalence might prove fruitful. In this connection it should be mentioned that Bade has shown in [4] that an essentially finite (defined somewhat differently) scalar operator on a separable Banach space is always related to a normal operator on Hilbert space by a similarity induced by an operator that, together with its inverse, is closed and densely defined.

Three further remarks are necessary before we outline the arrangement of material to follow. First, no assumption of separability is required or made. Second, no results from the dimension theory of rings of operators or comparison theory for projections will be needed; our main tools are the multiplicity theory of Bade [4] for a complete Boolean algebra of projections, and certain of his results on completeness from [3], and the classical spectral multiplicity theory for a single normal operator—here we follow Halmos' exposition [17]. Third, the reader, noticing the primacy of the Weyr characteristic in the statements and proofs, may wonder as to the inclusion of the Segre characteristic. It is included because it is readily defined by means of the Weyr characteristic, and because it seems clear that the Weyr characteristic is unsuited to the nonessentially finite case, where the Segre characteristic, defined directly, may succeed.

The next section is devoted to background material—spectral operator theory, the multiplicity theories required, the results on Boolean algebras of projections—and to notational conventions. In §3 we investigate, from a spatial standpoint, a Banach space of uniform finite multiplicity with respect to a complete spectral measure, and examine the action of a commuting quasi-nilpotent. (Actually the full force of quasi-nilpotency is never required;

quasi-nilpotency with respect to the weak topology is sufficient to prove all our results.) Here a notion of vector independence is introduced, patterned after linear independence, that characterizes the multiplicity. The orbit of any vector under the quasi-nilpotent is shown to be independent; this implies the quasi-nilpotent is actually nilpotent (a result also proved by Foguel in [13; 14], using direct matrix methods to study the operators commuting with such a Boolean algebra of projections), and permits the definition, in §4, of the Weyr and Segre characteristics with Borel set and cardinal number arguments. The Weyr characteristic, for fixed cardinals, is shown to be a multiplicity function and sets of uniform characteristic defined; then the crucial summability condition characterizing such sets is proved, and measure arguments introduced, for which similar results hold. In §5 we turn to the single operator situation on Hilbert space, define the Weyr characteristic for essentially finite spectral operators and prove it is similarity invariant. In §6 it is shown not to be a complete set of similarity invariants; here the desired properties of semi-similarity are proved, and the summability condition relating sets or measures of uniform characteristic to the multiplicity function enables us to conclude that the Weyr characteristic is a complete set of semi-similarity invariants for these operators. We discuss the imposition of additional conditions to produce similarity, a problem left open in this work. Finally, in §7, the adjoint situation is treated, first in the Banach space context and then the single operator Hilbert space case.

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2. **Preliminaries.** In this section background material is collected, and notational conventions established. We begin with a brief outline of some of the principal results from N. Dunford's theory of spectral operators, taken largely from [9; 10], and certain related material about Boolean algebras of projections due to Bade [2; 3]. A complete discussion of most of this material, except the multiplicity theory, will appear in [11].

A homomorphism $E(\cdot)$ from the Boolean algebra $\mathcal{B}$ of Borel subsets of the complex plane $\mathcal{C}$ onto a bounded Boolean algebra of idempotent operators on a Banach space $\mathfrak{X}$ is called a *spectral measure*; that is,

$$\left.\begin{aligned} E(\delta \cap \pi) = E(\delta) \wedge E(\pi), \quad E(\delta \cup \pi) = E(\delta) \vee E(\pi) \\ E(\mathcal{C}) = I, \quad E(\mathcal{C} - \delta) = I - E(\delta), \quad |E(\delta)| \leqq M \end{aligned}\right\} \qquad \delta, \pi \in \mathcal{B},$$

where $E(\delta)\wedge E(\pi)=E(\delta)E(\pi)$ and $E(\delta)\vee E(\pi)=E(\delta)+E(\pi)-E(\delta)E(\pi)$ are the infimum and supremum respectively in the natural order of commuting projections (*viz.*: $E_1 \geqq E_2$ if $E_1E_2=E_2E_1=E_2$), and M is a constant. A (bounded) linear operator T on $\mathfrak{X}$ is a *spectral operator* if there is a spectral measure E that commutes with T, is countably additive in the strong topology, and satisfies $spec(T|E(\delta)\mathfrak{X})\subseteq\bar{\delta}$, $\delta\in\mathfrak{B}$, where $spec(T|E(\delta)\mathfrak{X})$ denotes the spectrum of the restriction of T to the subspace $E(\delta)\mathfrak{X}$, and $\bar{\delta}$ denotes the closure of δ. The spectral measure E, called the *resolution of the identity for* T, is unique, vanishes outside the spectrum of T, and commutes with every operator commuting with T. If the spectral operator $T=\int\lambda E(d\lambda)$, the integral in the uniform topology, then T is called a *scalar operator*. Every spectral operator T has a unique canonical decomposition $T=S+Q$ where S is a scalar operator and Q is a quasi-nilpotent (i.e., $|Q^k|^{1/k}\to 0$ as $k\to\infty$, or, equivalently, $spec(Q)=\{0\}$) commuting with S. The operators T and S have the same spectrum and resolution of the identity. It is customary to call S the *scalar part*, and Q the *quasi-nilpotent* part, of T.

If the underlying vector space is a Hilbert space, then the (bounded) normal operators are the scalar operators whose resolutions of the identity are self-adjoint. Wermer [32], applying a result of Mackey [21, Theorem 55], has shown that every scalar operator on Hilbert space is similar to a normal operator; this result will be used frequently.

Repeated use will be made of the simple properties of the integral of a complex-valued bounded Borel function f with respect to a (countably additive) spectral measure E. This integral can be defined in the uniform topology using simple functions in the usual manner, and is a bounded operator on $\mathfrak{X}$. The mapping

$$f\to S(f)=\int f(\lambda)E(d\lambda)$$

is an algebraic homomorphism; that is, $S(fg)=S(f)S(g)$, and, in particular, $S(\chi_\delta f)=E(\delta)S(f)=\int_\delta f(\lambda)E(d\lambda)$, where χ_δ is the characteristic function of the Borel set δ. Also, $S(f)x=\int f(\lambda)E(d\lambda)x$, $x\in\mathfrak{X}$, and $x^*S(f)x=\int f(\lambda)x^*E(d\lambda)x$, $x\in\mathfrak{X}$, $x^*\in\mathfrak{X}^*$ (the conjugate space of $\mathfrak{X}$). For each f and δ, the integral satisfies

$$\operatorname*{ess\,inf}_{\lambda\in\delta}|f(\lambda)| \leqq \left|\int_\delta f(\lambda)E(d\lambda)\right| \leqq 4M \operatorname*{ess\,sup}_{\lambda\in\delta}|f(\lambda)|,$$

M being the uniform bound for $\{|E(\cdot)|\}$. The definition can be extended to unbounded Borel functions (cf. [1] for details); in this case the mapping $f\to S(f)$ is an operational calculus exactly analogous to that of an unbounded normal operator on Hilbert space. The operator $S(f)$ has an inverse $S(1/f)$ if and only if $E(f^{-1}\{0\})=0$, and of course the inverse of a bounded operator may be unbounded.

If $\{\mathfrak{X}_\alpha | \alpha \in A\}$ is a family of subspaces of the Banach space $\mathfrak{X}$, the closure of the intersection of these subspaces will be denoted by $\bigwedge_{\alpha\in A} \mathfrak{X}_\alpha$, and the smallest closed subspace containing every $\mathfrak{X}_\alpha$ by $\bigvee_{\alpha\in A} \mathfrak{X}_\alpha$. Then a Boolean algebra E of commuting projections is called *(σ-complete) complete* if every (countable) subfamily $\{E_\alpha | \alpha\in A\} \subseteq E$ has an infimum $\bigwedge_{\alpha\in A} E_\alpha$ and supremum $\bigvee_{\alpha\in A} E_\alpha$ in E, with ranges $\bigwedge_\alpha E_\alpha \mathfrak{X}$ and $\bigvee_\alpha E_\alpha \mathfrak{X}$ respectively. A Boolean algebra is *countably decomposable* if every disjoint subfamily is at most countable. Thus a countably additive spectral measure is σ-complete, and a countably decomposable σ-complete spectral measure is complete.

The following results lie somewhat deeper. A σ-complete Boolean algebra of projections is always bounded in norm [3, Theorem 2.2], and its strong closure is complete [3, Theorem 2.7] (see also [8]). A complete Boolean algebra of projections contains every projection in the weakly (or equivalently, strongly) closed algebra it generates [3, Theorem 2.8]. In the Hilbert space case it follows readily from these results and the continuity of adjunction in the weak topology, that the projections in the second commutator of a self-adjoint spectral measure are self-adjoint, and precisely comprise the Boolean algebra completion of the spectral measure. (If the Hilbert space is separable, each of these possible extensions is vacuous: the completion, projections in the strong or weak topological closure, and projections in the second commutator are already the given spectral measure.)

To return to the Banach space situation, if $E(\cdot)$ is a spectral measure on $\mathfrak{X}$ fixed in the context of discussion, and $x\in\mathfrak{X}$, then $\mathfrak{M}(x)$ will denote the smallest closed subspace spanned by $\{E(\delta)x | \delta\in\mathcal{B}\}$. In [4] it is shown that if E is countably additive and complete, then the cyclic subspace $\mathfrak{M}(x)$ has L_1 structure:

$$\mathfrak{M}(x) = \{S(f)x \mid x \in \text{domain } S(f)\}.$$

In our attempt to avoid using the properties of Hilbert space, we will make repeated use of the following result [3, Theorem 3.1], which furnishes a replacement on a Banach space, under certain circumstances, for the inner product: if $E(\cdot)$ is a σ-complete spectral measure on $\mathfrak{X}$ and x is any fixed nonzero vector in $\mathfrak{X}$, then there is a bounded linear functional $x^*\in\mathfrak{X}^*$ such that

(a) $x^*E(\delta)x \geqq 0, \quad \delta \in \mathcal{B}$,

(b) $x^*E(\delta)x = 0$ implies $E(\delta)x = 0$.

(In Hilbert space, if E is self-adjoint, the functional x^* can be taken to be x, and the measure $|E(\cdot)x|^2$ is called the *measure determined by x.*) In general we call such a functional a *Bade functional for x with respect to E.*

Before turning to the multiplicity theory itself, we recall (cf. [17, p. 79]) that under the partial order established by absolute continuity ($\ll$) the family of equivalence classes of regular, totally-finite, non-negative, countably additive set functions (=*measures*) on the Borel subsets of the complex plane is a Boolean σ-ring with the property that a bounded orthogonal nonzero subfamily is at most countable. Here two measures μ and ν are called *orthogonal* if $\mu \wedge \nu = 0$ (and the word misused in the customary fashion when applied to families), and we call μ and ν *equivalent*, written $\mu \equiv \nu$, if both $\mu \ll \nu$ and $\nu \ll \mu$.

The relation between regular measures partially ordered this way and Borel sets is particularly simple: if $\nu \ll \mu$ then there is a subset δ of the support Λ of μ such that $\nu \equiv \mu(\delta \cap \cdot)$, and there is a natural and obvious one-one order preserving correspondence between equivalence classes of measures bounded by μ and equivalence classes of Borel subsets of Λ, where we call two such sets δ and π equivalent if $\mu(\delta \Delta \pi) = 0$. In such fashion any regular measure establishes a natural order and equivalence relation on the Borel subsets of its support.

A countably additive spectral measure also relates readily to the Borel subsets of Λ: if $E(\pi) \neq 0$, then $\pi \subseteq \delta$ exactly when $E(\pi) \leqq E(\delta)$, and π and δ are equivalent, $\pi \equiv \delta$, if and only if $E(\delta) = E(\pi)$. If there is a vector $x \in \mathfrak{X}$ with the property that $E(\delta)x = 0$ implies $E(\delta) = 0$ (called a *separating vector* by Segal [27], this notion is due to Nakano, who proved the existence of such a vector is equivalent to the countable decomposability of E), and if x^* is any Bade functional for x and E, then the scalar measure $x^*E(\cdot)x$ and the spectral measure $E(\cdot)$ have the same null sets and determine the same equivalence relation on the subsets of their common support.

Bade's multiplicity theory in [4] can now be described. Let E be a fixed complete countably additive spectral measure, operating on the Banach space $\mathfrak{X}$. A countably decomposable projection P in the range of E has *multiplicity n*, an arbitrary cardinal, if there is a family $\{x_\alpha \mid \alpha \in A\} \subset \mathfrak{X}$ of cardinality n, with $P\mathfrak{X} = \bigvee_{\alpha \in A} \mathfrak{M}(x_\alpha)$, and no family of smaller cardinality has this property. The multiplicity is said to be *uniform* if every nonzero subprojection of P in E also has multiplicity n. These definitions extend abstractly, in an order preserving fashion, to arbitrary (not necessarily countably decomposable) projections in E, and moreover, there is a unique decomposition of the identity as a disjoint supremum of projections P_n in E, with P_n of uniform multiplicity n if $P_n \neq 0$. *We will say the space $\mathfrak{X}$ itself has* (*uniform*) *multiplicity n if the identity has* (*uniform*) *multiplicity n.*

Satisfactory conclusions as to the structure of $\mathfrak{X}$ relative to E have been derived only when n is finite, and this case, rather than the preceding global formulation, is of primary interest to us. Under this assumption, if $\mathfrak{X}$ is a space of uniform multiplicity n and $\mathfrak{X} = \bigvee_{i=1}^{n} \mathfrak{M}(x_i)$, then each x_i is a separating vector for E, the spanning manifolds $\mathfrak{M}(x_i)$ are disjoint in the sense that

$$\mathfrak{M}(x_j) \wedge \left(\bigvee_{i \neq j} \mathfrak{M}(x_i) \right) = 0,$$

and there exist Bade functionals x_i^* for x_i such that

$$x_j^*\left(\bigvee_{i\neq j}\mathfrak{M}(x_i)\right)=0$$

and $x_i^*S(f)x_i=0$ implies $S(f)x_i=0$. The measures $x_i^*E(\cdot)x_i$ are all equivalent. Every vector $x\in\mathfrak{X}$ can be written

$$x=\lim_{k\to\infty}\sum_{i=1}^n\int_{\pi_k}f_i(\lambda)E(d\lambda)x_i,$$

where $\pi_k=\{\lambda|\ |f_i(\lambda)|\leqq k,\ i=1,\cdots,n\}$, $k=1,2,\cdots$. The L_1 structure of the cyclic manifolds $\mathfrak{M}(x)$ thus permits the imbedding of such a space of uniform finite multiplicity in a direct sum of L_1 spaces.

Only a minor topological change is required to formulate an analogous scheme in the conjugate space for the adjoint Boolean algebra. Here the relevant topology is the weak-* or $\mathfrak{X}$-topology on $\mathfrak{X}^*$; systematic replacement of the strong topology by this one in the foregoing description produces essentially the same results. Thus if E is complete on $\mathfrak{X}$ in the sense previously described, then E^*, the family of adjoints of elements of E, is complete on $\mathfrak{X}^*$ in the sense that an arbitrary subfamily $\{E_\alpha^*|\alpha\in A\}$ has a supremum $\bigvee_\alpha E_\alpha^*$ and infimum $\bigwedge_\alpha E_\alpha^*$ in E^* with ranges $\bigvee_\alpha E_\alpha^*\mathfrak{X}^*$ and $\bigwedge_\alpha E_\alpha^*\mathfrak{X}^*$ respectively, where the lattice operations in $\mathfrak{X}^*$ refer to the appropriate $\mathfrak{X}$-closed manifolds. In this situation, a Bade functional for $x^*\in\mathfrak{X}^*$ with respect to E^* can always be chosen in the space $\mathfrak{X}$. We write $S^*(f)$ for the adjoint of $S(f)$ and consider the cyclic manifolds $\mathfrak{N}(x^*)$, defined to be the least $\mathfrak{X}$-closed manifold spanned by $\{E^*(\delta)x^*|\delta\in\mathfrak{B}\}$. Then

$$\mathfrak{N}(x^*)=\{S^*(f)x^*\mid x^*\in\text{domain }S^*(f)\}$$

and a multiplicity function defined in the same way on E^* decomposes the identity similarly. The structure of a space of uniform finite multiplicity is also similar: if $\mathfrak{X}^*=\bigvee_{i=1}^n\mathfrak{N}(x_i^*)$, then each x_i^* is a separating vector,

$$\mathfrak{N}(x_j^*)\wedge\left(\bigvee_{i\neq j}\mathfrak{N}(x_i^*)\right)=0^*,$$

and for each i there is a Bade functional $x_i\in\mathfrak{X}$ with $x^*x_i=0$ if $x^*\in\bigvee_{i\neq j}\mathfrak{N}(x_j^*)$. Also every $x^*\in\mathfrak{X}^*$ has the weak-* representation

$$x^*x=\lim_{k\to\infty}\sum_{i=1}^n\int_{\pi_k}f_i(\lambda)E^*(d\lambda)x_i^*x,\qquad x\in\mathfrak{X},$$

with π_k defined as before.

The relation between the two multiplicities is known only for projections or spaces of finite multiplicity, and in this case is the expected one: the projection $P\in E$ has (uniform) finite multiplicity n in $\mathfrak{X}$ if and only if $P^*\in E^*$ has (uniform) finite multiplicity n in $\mathfrak{X}^*$. Thus the space $\mathfrak{X}$ has uniform finite multiplicity n with respect to E if and only if $\mathfrak{X}^*$ has the same property with respect to E^*.

Some remarks may place this multiplicity theory in perspective. First, the definition of multiplicity for a space differs somewhat from the usual conception: for example, in the nonseparable case it is possible for a spectral measure to be *simple* in the customary sense, and yet the space need not have multiplicity 1. (In the countably decomposable case, and only in this case, does this simplicity coincide with multiplicity 1.) Second, if the underlying space is Hilbert space and the spectral measure self-adjoint, then the cyclic manifolds can be given a canonical L_2 structure. That is, for any $x\in\mathfrak{X}$, the manifold $\mathfrak{M}(x)$ is easily seen to be unitarily equivalent to

$$L_2(\Lambda, \mathfrak{B}, |E(\cdot)x|^2),$$

where $\mathfrak{B}$ is the family of Borel subsets of Λ, the support of $|E(\cdot)x|^2$. The unitary equivalence carries the operation of $E(\delta)$ on $\mathfrak{M}(x)$ to multiplication by the characteristic function χ_δ on the L_2 space, and, if the given spectral measure is the resolution of the identity of the normal operator N, carries the action of N on $\mathfrak{M}(x)$ to "multiplication by λ" on the L_2 space. If the Hilbert space is separable as well, we can outline an alternate approach to a decomposition of the space relative to the spectral measure, not involving a uniformity concept. A vector $x_1\in\mathfrak{X}$ can be chosen so that $|E(\cdot)x_1|^2$ is maximal (in the order determined by absolute continuity) among all such vector measures, then x_2 chosen with the same property in the space $\mathfrak{M}(x_1)^\perp$, and the process continued. This produces a decomposition of the space as an orthogonal direct sum, $\sum\oplus\mathfrak{M}(x_i)$, and a descending sequence of measures, $\mu_1, \mu_2, \cdots$ (or equivalently, one measure μ, and a descending sequence of sets $\delta_2, \delta_3, \cdots$) which characterizes E or N to unitary equivalence. A multiplicity function can be defined as a complete set of unitary invariants, whose value at a measure μ (or set δ) is the maximum n for which $\mu\wedge\mu_n\neq 0$ (or $E(\delta\cap\delta_n)\neq 0$), or ∞ if there is no maximum. This is the approach in the separable case described, for example, in [21].

In the nonseparable Hilbert space case it is no longer possible to choose vectors with this maximality property and the multiplicity theory is more difficult. Neither of the preceding global formulations is suited to our purposes. Roughly speaking, the methods described above for imposing structure on a ziggurated configuration count all the manifolds above the measure or set, while that of Halmos [17] is to count only those that cover the measure or set completely. Thus the multiplicity function will be order reversing, rather than order preserving. We content ourselves next with a description of the results, omitting the L_2 structure, as it will not be used.

Let N be a (bounded) normal operator, with resolution of the identity E, on the arbitrary Hilbert space $\mathfrak{H}$. There is associated with N a unique mapping $\mathbf{u}$ from the equivalence classes of (regular, non-negative, totally-finite, countably additive) measures on the complex plane to the cardinal numbers satisfying:

(a) if μ is identically zero, then $\mathbf{u}(\mu)=0$,

(b) $0\neq\nu\ll\mu$ implies $\mathbf{u}(\nu)\geqq\mathbf{u}(\mu)$, and

(c) if μ is the supremum of an orthogonal (hence countable) family $\{\mu_i\}$ of nonzero measures, then

$$\mathbf{u}(\mu) = \min_i \{\mathbf{u}(\mu_i)\}.$$

Such a mapping, for which the name *multiplicity function* is reserved hereafter, is a complete set of unitary invariants for N or E. A measure μ has *multiplicity* $\mathbf{u}(\mu)$, and has *uniform multiplicity* if $0\neq\nu\ll\mu$ implies $\mathbf{u}(\nu)=\mathbf{u}(\mu)$.

To each measure μ there corresponds a self-adjoint projection $C(\mu)$, which we call the *carrier of* μ, in the second commutator of E, and hence in the completion $\overline{E}$ of E. This correspondence preserves order, takes orthogonal measures into disjoint projections, and has the explicit formulation

$$C(\mu) = \bigwedge_{P\in\overline{E}} \{P \mid Px = x \text{ whenever } |E(\cdot)x|^2 \ll \mu\}.$$

Each spectral measure $C(\mu)E(\cdot)$ is countably decomposable (cf. Kelley [20, Theorem 2.2]) and thus complete; each subspace $C(\mu)\mathfrak{H}$ has multiplicity $\mathbf{u}(\mu)$ with respect to the complete Boolean algebra $C(\mu)E(\cdot)$ —in the sense of Bade previously described—and has uniform multiplicity $\mathbf{u}(\mu)$ if μ has uniform multiplicity. In the case of uniform multiplicity the measures determined by the spanning vectors (x determines $|E(\cdot)x|^2$) are all equivalent.

Associated with $\mathbf{u}$, though not uniquely, is an orthogonal family $\{\mu_\alpha \mid \alpha\in A\}$ of nonzero measures of uniform multiplicity such that for any measure μ, we have $\mathbf{u}(\mu)=0$ unless μ is covered by the μ_α in the sense that $\mu\equiv\bigvee_\alpha (\mu\wedge\mu_\alpha)$, and in this case,

$$\mathbf{u}(\mu) = \min_{\alpha\in A} \{\mathbf{u}(\mu_\alpha) \mid \mu\wedge\mu_\alpha \neq 0\}.$$

The closure of the union of the supports of the μ_α is *spec*(N). The projections $C(\mu_\alpha)$, $\alpha\in A$, are a disjoint family whose supremum is the identity.

In the sequel we shall attempt to adhere to the notational conventions and definitions already established, directly or tacitly, in this section. In general, *operator* will mean bounded linear operator (except that we permit operators of the form $S(f)$ to be unbounded), *projection* will always mean idempotent operator (that is, we never assume, without explicit mention, that a projection is self-adjoint), and *set* or *subset* (except in the phrase "complete set of invariants") will be reserved for a Borel subset of the complex plane, and other nouns denoting aggregates (e.g., family, class) will be used for collections of other objects—vectors, measures, operators, etc. *Measure* (unmodified) will always mean a regular, totally-finite, non-negative, countably additive measure on such sets; *function* (unmodified) will always mean a complex valued Borel measurable function on the complex plane.

An operator will be called *nonsingular* if it has an everywhere defined, bounded inverse, and by *similar* is always meant a similarity induced by a nonsingular operator. We shall be forced to define both projections and sets for various objects (vectors, measures, operators) and shall distinguish between these by calling the former the *carrier* of the object, written $C(\cdot)$, and the latter the *support*, written $s(\cdot)$. The null set will be denoted by $\emptyset$, and the characteristic function of the set δ by χ_δ. As it will never be necessary to consider the same object under more than one norm, and that always the natural one, the symbol "$|\ \ |$" will be used for all norms. Occasionally it will be necessary to use superscripts as indices, and these will be used without parentheses when exponentiation would be meaningless (as for a vector) or redundant (as for a projection). We continue to permit ourselves the luxury of confusing a spectral measure with its Boolean algebra range, and a measure with its equivalence class. We number results consecutively, without regard for type or section. That is, the first theorem, Theorem 6, will be found in §3.

3. **Uniform finite multiplicity.** *Throughout this and the following section $\mathfrak{X}$ will be a Banach space of uniform multiplicity $n<\infty$ with respect to the countably additive and decomposable (hence complete) spectral measure E defined on the Borel subsets $\mathfrak{B}$ of the compact set Λ in the complex plane.* Let Q be a fixed quasi-nilpotent commuting with $E(\cdot)$, and $\{x_1, \cdots, x_n\} \subset \mathfrak{X}$ be a fixed family of vectors such that $\mathfrak{X} = \bigvee_{i=1}^n \mathfrak{M}(x_i)$. Then the disjointness condition

$$\mathfrak{M}(x_j) \wedge \left(\bigvee_{i \neq j} \mathfrak{M}(x_i) \right) = 0, \qquad\qquad = 1, \cdots, n,$$

holds and every $x \in \mathfrak{X}$ can be written

$$x = \lim_{k \to \infty} \sum_{i=1}^{n} \int_{\pi_k} f_i(\lambda) E(d\lambda) x_i,$$

where $\pi_k = \{\lambda \mid |f_i(\lambda)| \leqq k,\ i = 1, \cdots, n\}$, $k = 1, 2, \cdots$.

The *carrier* of a vector x, written $C(x)$, is defined by

$$C(x) = \bigwedge_{\delta \in \mathfrak{B}} \{E(\delta) \mid E(\delta)x = x\}.$$

The elementary consequences of this definition are that $C(x) = 0$ if and only if $x = 0$, that $C(E(\delta)x) = E(\delta)C(x)$ for each $\delta \in \mathfrak{B}$, and that $C(x_i) = I$ for $i = 1, \cdots, n$. It is also easy to see that if $C(x)C(y) = 0$ then $C(x+y) = C(x) + C(y)$. The completeness of E implies that this infimum is actually an element of E; thus there is a set $\delta \in \mathfrak{B}$, the *support* of x, written $s(x)$, such that $E(\delta) = C(x)$. (Actually the support is not uniquely defined; what we have in mind is an equivalence class of sets, but this looseness will cause no difficulty.) Similarly $s(f)$ will denote the support of the function f on Λ: $s(f) = \{\lambda \mid f(\lambda) \neq 0, \lambda \in \Lambda\}$. If $E(\delta) = 0$, we call δ *negligible* and write $\delta \equiv \emptyset$.

A vector $x \in \mathfrak{X}$ will be called *full over* δ if $C(x) \geqq E(\delta)$, and simply *full* if $C(x) = I$ (that is, a full vector is a separating vector). A set $\delta \in \mathfrak{B}$ is an *inverting set* for the nonzero operator

$$S(f) = \int f(\lambda) E(d\lambda)$$

if the operator

$$\int_\delta \frac{1}{f(\lambda)} E(d\lambda)$$

is nonzero and bounded. This latter operator will usually be written $S(1/f)E(\delta)$. It is clear that every nonzero operator of the form $S(f)$ has an inverting set.

LEMMA 1. *If $x \in \mathfrak{X}$ is full and $S(f)x = 0$, then $S(f) = 0$.*

Proof. Let δ be an inverting set for $S(f)$; then $S(1/f)E(\delta)S(f)x = E(\delta)x = 0$. Thus $E(\Lambda - \delta)x = x$, and, since x is full, $E(\Lambda - \delta) = I$. This implies δ is negligible, and therefore $S(f)$ must be zero. Q.E.D.

Our first objective is to generalize the dimension theory of finite dimensional vector spaces. However the space $\mathfrak{X}$ is being considered as a finitely generated free module whose scalar ring $\{S(f)\}$ contains zero divisors. The procedure will be to reduce the space by an inverting set whenever the invertibility of a scalar is required in the standard elementary arguments. The basic definition of this section is the following.

A finite family of vectors $\{y_1, \cdots, y_m\} \subset \mathfrak{X}$ will be called *independent* if there is a family $\{z_1, \cdots, z_m\} \subset \mathfrak{X}$ such that

$$\left.\begin{array}{l}\text{(a) } \mathfrak{M}(y_i) \subseteq \mathfrak{M}(z_i) \\ \text{(b) } C(z_i) = I\end{array}\right\} i = 1, \cdots, m,$$

(c) $\sum_{i=1}^m S(f_i)z_i = 0$ implies each $S(f_i)z_i = 0$, for every family $\{f_1, \cdots, f_m\}$ of Borel functions with $z_i \in$ domain $S(f_i)$.

There will be some computation to follow with expressions of the form in (c), and we will henceforth use these with the unwritten assumption that they are meaningful—that is, that the vectors are in the domains of the corresponding operators.

It is clear that any subfamily of an independent family is independent. It is not true in general (but only for full vectors) that the vanishing of a "linear" combination of independent vectors implies the coefficients vanish, but it does follow in general by the next lemma, whose proof is obvious, that the summands vanish in this case.

LEMMA 2. *If $\{y_1, \cdots, y_m\}$ is independent and $\sum_i S(f_i)y_i = 0$, then $S(f_i)y_i = 0$ for each i.*

Independence is not precisely the same as disjointness of the manifolds $\mathfrak{M}(y_i)$. The following lemma states that this disjointness is equivalent to condition (c) above, and thus implies that *for full vectors* the two notions, independence and disjointness of their cyclic manifolds, coincide.

LEMMA 3. *If* $\{y_1, \cdots, y_m\} \subset \mathfrak{X}$, *then* $\sum_i S(f_i)y_i = 0$ *implies each* $S(f_i)y_i = 0$ *if and only if*

$$\mathfrak{M}(y_j) \wedge \left(\bigvee_{i \neq j} \mathfrak{M}(y_i) \right) = 0, \qquad j = 1, \cdots, m.$$

Proof. The disjointness condition clearly implies the other. To prove the converse, let j be fixed and assume $z_0 \in \mathfrak{X}$ is contained in the indicated intersection. Then

$$z_0 = S(f_j)y_j = \lim_{k \to \infty} \sum_{i \neq j} S(f_i)E(\pi_k)y_i,$$

where $\pi_k = \{\lambda \mid |f_i(\lambda)| \leqq k,\ i \neq j\}$. As $\{\pi_k\}$ is an increasing sequence of sets, for each fixed k,

$$S(f_j)E(\pi_k)y_j - \sum_{i \neq j} S(f_i)E(\pi_k)y_i = 0.$$

Thus by assumption, $S(f_j)E(\pi_k)y_j = 0$ for all k, and hence $S(f_j)y_j = z_0 = 0$. Q.E.D.

Lemma 3 also implies that the fixed family $\{x_1, \cdots, x_n\}$, whose cyclic manifolds span $\mathfrak{X}$, is independent. There is now a simple characterization of the support of a vector.

LEMMA 4. *If* $x = \lim_{k \to \infty} \sum_{i=1}^{n} S(f_i)E(\pi_k)x_i$ *and* $\delta_i = s(f_i)$, $i = 1, \cdots, n$, *then* $s(x) \equiv \bigcup_i \delta_i$.

Proof. It is clear that $E(\bigcup_i \delta_i) \geqq C(x)$. On the other hand, if $E(\delta)x = x$, then

$$x = \lim_{k \to \infty} \sum_{i=1}^{n} S(f_i)E(\pi_k)E(\delta)x_i,$$

and for each k,

$$\sum_{i=1}^{n} [S(f_i)E(\pi_k) - S(f_i)E(\pi_k)E(\delta)]x_i = 0.$$

By Lemmas 1 and 3, $S(f_i)E(\pi_k) = S(f_i)E(\pi_k \cap \delta)$ for each i and k, and thus $E(\delta) \geqq E(\delta_i)$ for each i. Q.E.D.

Next, using a standard elementary idea, we prove the "dimension" theorem that the maximum cardinality of an independent family of vectors in $\mathfrak{X}$ is n. This requires a preliminary lemma.

LEMMA 5. *If* $\{y_1, \cdots, y_m\}$ *is an independent family of full vectors and* $y_1 \in \text{domain } S(f)$, *then* $\{y_1, \cdots, y_{m-1}, y_m + S(f)y_1\}$ *is an independent family of full vectors.*

Proof. Let $C(y_m+S(f)y_1)=P$. Then $0=(I-P)(y_m+S(f)y_1)=(I-P)y_m +(I-P)S(f)y_1$. Thus Lemmas 1 and 2 imply $I=P$, and $y_m+S(f)y_1$ is full. Similarly, if

$$\sum_{i=1}^{m-1} S(f_i)y_i + S(f_m)(y_m + S(f)y_1) = 0,$$

then

$$(S(f_1) + S(f_m)S(f))y_1 + \sum_{i=2}^{m} S(f_i)y_i = 0,$$

so $S(f_i)=0$ for $i=1, \cdots, m$.

THEOREM 6. *The maximum cardinality of an independent family of vectors in* $\mathfrak{X}$ *is* n.

Proof. What must be shown is that no family of $n+1$ full vectors in $\mathfrak{X}$ can be independent, and we do this by induction on n, the multiplicity of $\mathfrak{X}$. If $n=1$ and $\mathfrak{X}=\mathfrak{M}(x_1)$, let $y_1=S(f_1)x_1$ and $y_2=S(f_2)x_1$ be two full vectors in $\mathfrak{X}$. Then $x_1=S(1/f_2)y_2$, so $y_1=S(f_1/f_2)y_2$ and $\{y_1, y_2\}$ is not independent.

Next, suppose the assertion is true for spaces of uniform multiplicity $n-1$, and let $y_1, \cdots, y_{n+1}$ be full vectors in $\mathfrak{X}=\bigvee_{i=1}^{n} \mathfrak{M}(x_i)$, a space of uniform multiplicity n. We assume $\{y_1, \cdots, y_{n+1}\}$ is independent and derive a contradiction. Let

$$y_i = \lim_{k\to\infty} \sum_{j=1}^{n} S(f_{ij})E(\pi_k)x_j, \qquad i = 1, \cdots, n+1.$$

Then some coefficient for y_1, say $S(f_{1n})$, is not zero. Let δ_1 be an inverting set for $S(f_{1n})$ on which f_{1n} is bounded away from ∞ as well as away from zero. If $S(1/f_{1n})E(\delta_1)y_1$ is in the domain of $S(f_{2n})$, let $\delta_2=\delta_1$; otherwise $S(f_{2n})E(\delta_1) \neq 0$ and let δ_2 be any non-negligible subset of δ_1 on which f_{2n} is bounded and nonzero. Similarly we define $\delta_3=\delta_2$ if $S(1/f_{1n})E(\delta_2)y_1 \in \text{domain } S(f_{3n})$, and otherwise let δ_3 be any non-negligible subset of δ_2 on which f_{3n} is bounded and nonzero. Continued so, this process produces a non-negligible set $\delta=\delta_{n+1}$ such that the nonzero vector $S(1/f_{1n})E(\delta)y_1$ is in the domain of $S(f_{in})$, $i=2, \cdots, n+1$.

The remainder of the argument takes place in the space $E(\delta)\mathfrak{X}$, in which our assumptions permit us to assert $\{E(\delta)y_i \mid i=1, \cdots, n+1\}$ is an independent family of full vectors. Let $\tilde{y}_1=E(\delta)y_1$, $\tilde{y}_i=E(\delta)y_i-S(f_{in}/f_{1n})E(\delta)y_1$ for $i=2, \cdots, n+1$. By repeated application of Lemma 5,

$$\{\tilde{y}_i \mid i = 1, \cdots, n+1\}$$

is an independent family of full vectors with respect to $E(\delta)E(\cdot)$, but

$$\{\tilde{y}_i \mid i = 2, \cdots, n+1\} \subset \bigvee_{j=2}^{n} \mathfrak{M}(E(\delta)x_j),$$

a space of uniform multiplicity $n-1$. Since this latter family is independent and full in $\bigvee_{j=2}^{n} \mathfrak{M}(E(\delta)x_j)$, this contradicts the induction hypothesis and completes the proof.

We have proved slightly more than the theorem explicitly asserts.

COROLLARY 7. *The maximum cardinality of an independent family of vectors in a subspace of the form* $\bigvee_{i=1}^{m} \mathfrak{M}(z_i)$ *is* m.

The most useful criterion for independence is the following.

THEOREM 8. *Let* $\{y_1, \cdots, y_m\}$ *be a family of nonzero vectors in* $\mathfrak{X}$ *for which* $\bigwedge_{i=1}^{m} C(y_i) \neq 0$, *and with the property that* $\sum_i S(f_i)y_i = 0$ *implies* $S(f_i)y_i = 0$ *for each* i. *Then* $\{y_1, \cdots, y_m\}$ *is independent.*

Proof. If each y_i is full, there is nothing to prove. Otherwise, let $s(y_i) = \sigma_i$, and let $\{\delta_j \mid j \in J\}$ be the (finite) class of non-negligible minimal sets in the ring generated by Λ and $\{\sigma_i \mid i = 1, \cdots, m\}$. That is, $\{\delta_j\}$ is the collection of non-negligible minimal sets in the family of all sets obtainable from $\{\Lambda, \sigma_i \mid i = 1, \cdots, m\}$ by intersection and relative complementation. Then $\{\delta_j\}$ is disjoint, J is finite, $\Lambda \equiv \bigcup_{j \in J} \delta_j$, and $\sigma_i \cap \delta_j \not\equiv \emptyset$ implies $\sigma_i \supseteq \delta_j$. We can suppose these equivalences to be equalities, and the indices so chosen that $\delta_1 = \bigcap_i \sigma_i$ and $\delta_2 = \Lambda - \bigcup_i \sigma_i$, if this complement is non-negligible. By assumption, $E(\delta_1) \neq 0$, and Theorem 6, applied to the space $E(\delta_1)\mathfrak{X}$, shows $m \leqq n$.

We must construct $\{z_1, \cdots, z_m\}$ satisfying the definition of independence. This will be done by constructing $E(\delta_j)z_i$ for each i and j, and then defining z_i as $\sum_j E(\delta_j)z_i$ for each i. To begin, let $E(\delta_1)z_i = E(\delta_1)y_i$ and $E(\delta_2)z_i = E(\delta_2)x_i$, $i = 1, \cdots, m$, if $\delta_2 \in \{\delta_j\}_{j \in J}$. On any other set $\delta \in \{\delta_j\}_{j \in J}$, hereafter fixed, there are q vectors from the family $\{y_1, \cdots, y_m\}$, which we assume to be $y_1, \cdots, y_q$ for notational convenience, such that $C(y_i) \geqq E(\delta)$, $i = 1, \cdots, q$, and $C(y_i)E(\delta) = 0$, $i = q+1, \cdots, m$. Let $E(\delta)z_i = E(\delta)y_i$, $i = 1, \cdots, q$. It remains to define $E(\delta)z_i$, $i = q+1, \cdots, m$, and we do this inductively, demonstrating the construction of $E(\delta)z_{q+1}$.

Corollary 7 shows that, for any non-negligible $\pi \subseteq \delta$, we cannot have $\mathfrak{M}(E(\pi)x_k) \subseteq \bigvee_{i=1}^{q} \mathfrak{M}(E(\pi)z_i)$ for every $k = 1, \cdots, n$. Letting (cf. [4, Lemma 4.7])

$$\pi_k = \bigcup_{\pi \in \mathfrak{B}} \left\{\pi \mid E(\pi)x_k \notin \bigvee_{i=1}^{q} \mathfrak{M}(E(\delta)z_i), \pi \subseteq \delta\right\},$$

it is clear that $\delta - \bigcup_{k=1}^{n} \pi_k$ is negligible, and we can suppose $\delta = \bigcup_{k=1}^{n} \pi_k$. We can also suppose, without loss of generality, that $\{\pi_k | k=1, \cdots, n\}$ is a disjoint family (this is the worst case). Then let $E(\pi_k)z_{q+1}=E(\pi_k)x_k$, for $k=1, \cdots, n$; then $E(\delta)z_{q+1} = \sum_{k=1}^{n} E(\pi_k)z_{q+1}$.

Continuing by induction, let $E(\delta)z_{q+2}, \cdots, E(\delta)z_m$ be constructed in this manner, and, as this is possible for every $\delta \in \{\delta_j\}_{j\in J}$, let $z_i = \sum_j E(\delta_j)z_i$, $i=1, \cdots, m$. By construction $\{z_i | i=1, \cdots, m\}$ has the desired properties. Q.E.D.

Our aim now is to study the action of Q, the fixed quasi-nilpotent commuting elementwise with E. Here the main result, to be used in defining the invariants of the next section, is that for each nonzero $x \in \mathfrak{X}$, the nonzero elements of $\{x, Qx, Q^2x, \cdots\}$, the orbit of x under Q, are an independent family. Also, from this it will follow that Q is actually nilpotent, with index at most n.

The next theorem and corollary, though stated for Q, describe a basic property of any operator commuting with E.

THEOREM 9. *For every* $x \in \mathfrak{X}$, $C(x) \geqq C(Qx)$.

Proof. Otherwise there is a non-negligible set $\delta \subseteq s(Qx) - s(x)$. But then $0 \neq E(\delta)Qx = QE(\delta)x = 0$.

COROLLARY 10. *For every* $x \in \mathfrak{X}$, $C(Q^ix) \geqq C(Q^{i+1}x)$, $i=1, 2, \cdots$.

THEOREM 11. *If* $Qx = S(f)x$, *then* $Qx=0$.

Proof. We assume $Qx \neq 0$ and derive a contradiction. Thus $S(f)x \neq 0$ and there must exist $\epsilon > 0$ and a non-negligible set $\delta \subseteq s(x)$ such that $1/\epsilon > |f(\lambda)| > \epsilon$ for $\lambda \in \delta$. As Q commutes with E and f is bounded on δ, it follows that $Q^2E(\delta)x = S(f)QE(\delta)x = S(f^2)E(\delta)x$ and, more generally, that $Q^kE(\delta)x = S(f^k)E(\delta)x$. If $\mathfrak{X}$ is Hilbert space and E is self-adjoint, the proof is easy:

$$\begin{aligned} |Q^k|^2 |E(\delta)x|^2 &\geqq |Q^kE(\delta)x|^2 = |S(f^k)E(\delta)x|^2 \\ &= \int_\delta |f(\lambda)|^{2k} |E(d\lambda)x|^2 > \epsilon^{2k} |E(\delta)x|^2, \end{aligned}$$

and, as $E(\delta)x \neq 0$, we have $|Q^k| > \epsilon^k$, which contradicts the quasi-nilpotency of Q.

In the general case, we let x^* be a Bade functional for x with respect to E, and recall that there is a real constant M for which

$$4M \operatorname*{ess\,sup}_{\lambda \in \pi} |g(\lambda)| \geqq \left| \int_\pi g(\lambda)E(d\lambda) \right|$$

for any $\pi \in \mathfrak{B}$ and essentially bounded measurable function g.

Then, if $f(\lambda) = r(\lambda) \exp\{i\theta(\lambda)\}$ is a polar decomposition of f, we have

$$Q^k E(\delta)x = \int_\delta r^k(\lambda) e^{ik\theta(\lambda)} E(d\lambda)x,$$

and

$$x^* \int_\delta e^{-ik\theta(\lambda)} E(d\lambda) Q^k x = \int_\delta r^k(\lambda) x^* E(d\lambda)x.$$

Then, as $4M \geqq |E(\delta)S(e^{-ik\theta(\cdot)})|$, it follows that

$$4M|x^*| \, |E(\delta)x| \, |Q^k| \geqq \int_\delta r^k(\lambda) x^* E(d\lambda)x > \epsilon^k x^* E(\delta)x.$$

Taking kth roots and the limit as $k \to \infty$, this produces the desired contradiction. Q.E.D.

The preceding theorem states that no subspace of the form $\mathfrak{M}(x)$ can be invariant under Q. It is of interest to note that our only use of the quasi-nilpotency of Q per se in this work occurs in the proof of this theorem, and even here the full force of this property is not required, quasi-nilpotency in the weak topology being sufficient.

To prove the chief result, a function-theoretic lemma somewhat apart from the main line of argument is required.

LEMMA 12. *Let $P(t, \lambda) = t^m + h_{m-1}(\lambda)t^{m-1} + h_{m-2}(\lambda)t^{m-2} + \cdots + h_0(\lambda)$ be a polynomial in t whose coefficients are bounded measurable functions with compact support. Then there are bounded measurable functions g_i, $i = 1, \cdots, m$, with compact support such that for almost all λ,*

$$P(t, \lambda) = \prod_{i=1}^{m} [t - g_i(\lambda)].$$

It is classically known that the roots of a monic polynomial with complex coefficients vary continuously with the coefficients. If the roots are suitably ordered, then this, together with the fact that a continuous function of a Borel measurable function is Borel measurable, can be used to prove the lemma. A direct proof of the existence and measurability of the root functions can be found in Foguel [13, Lemma 3.1], and their boundedness is easy to see (cf. Bourbaki [5, p. 97]).

THEOREM 13. *If $\sum_{i=0}^{m} S(f_i)Q^i x = 0$, then $S(f_i)Q^i x = 0$ for each i.*

Proof. We assume there is a relation of the type above with $S(f_m)Q^m x \neq 0$ and derive a contradiction. First, we assert there is an inverting set δ for $S(f_m)$ such that $\delta \subseteq s(Q^m x)$ and $S(f_i)E(\delta)$ is a bounded operator for $i = 0, 1, \cdots, m$. For there is certainly an inverting set δ_m for $S(f_m)$ with $\delta_m \subseteq s(Q^m x)$. Then let δ_{m-1} be any non-negligible subset of δ_m on which f_{m-1}

is bounded, unless $S(f_{m-1})\ E(\delta_m)=0$, in which case let $\delta_{m-1}=\delta_m$. Continuing so, for $j=1, \cdots, m$ consecutively, let δ_{m-j} be any non-negligible subset of δ_{m-j+1} on which f_{m-j} is bounded, unless $S(f_{m-j})E(\delta_{m-j+1})$ is zero, in which case let $\delta_{m-j}=\delta_{m-j+1}$. Then $\delta=\delta_0$ has the asserted properties.

Next, we have $\sum_{i=0}^{m} S(f_i)E(\delta)Q^ix=0$ with $S(f_m)E(\delta)Q^mx\neq 0$, and applying $S(1/f_m)E(\delta)$, obtain, with $Q^my\neq 0$,

$$Q^my + S(h_{m-1})Q^{m-1}y + S(h_{m-2})Q^{m-2}y + \cdots + S(h_0)y = 0,$$

where $y=E(\delta)x$ and $h_i=(\chi_\delta f_i)/f_m$. On δ the functions h_i are bounded and measurable, and, applying the preceding lemma to $P(t, \lambda)=t^m+h_{m-1}(\lambda)t^{m-1} + \cdots +h_0(\lambda)$, let g_i, $i=1, \cdots, m$, be the bounded measurable root functions.

Then, writing

$$z_1 = \prod_{i=2}^{m} [Q - S(g_i)]y,$$

it follows that

$$(Q - S(g_1))z_1 = \prod_{i=1}^{m}[Q - S(g_i)]y = 0,$$

and thus $Qz_1=S(g_1)z_1$. By Theorem 10, $Qz_1=0$. That is, $\prod_{i=2}^{m} [Q-S(g_i)]Qy = 0$. Now let $z_2 = \prod_{i=3}^{m} [Q - S(g_i)]Qy$; in the same way, $Qz_2 = \prod_{i=3}^{m} [Q-S(g_i)]Q^2y=0$. Continuing so, we have $Q^my=0$, the desired contradiction. Q.E.D.

Thus, from Theorem 8 and Corollary 10 we can conclude:

COROLLARY 14. *For every $x\in\mathfrak{X}$, the orbit of x under Q is independent.*

COROLLARY 15. *The index of Q is at most n. That is, $Q^n\mathfrak{X}=0$.*

4. **The Weyr and Segre characteristics.** We continue under the assumptions of the previous section. The characteristics will be defined by the cardinalities of certain families of vectors. The first definition in this section will be fundamental for all our considerations.

A family of vectors $\{x_\alpha | \alpha\in A\}$ is a *kth index system over δ*, for a positive integer k and non-negligible set $\delta\in\mathfrak{B}$, if

$$\left.\begin{array}{l}\text{(a) } Q^kx_\alpha=0 \\ \text{(b) } C(Q^{k-1}x_\alpha)\geqq E(\delta)\end{array}\right\} \alpha\in A, \text{ and}$$

(c) $\{Q^ix_\alpha | i=0, 1, \cdots, k-1, \alpha\in A\}$ is independent.

We note that if π is a non-negligible subset of δ, then $\{E(\pi)x_\alpha | \alpha\in A\}$ is a kth index system over π.

THEOREM 16. *For each integer k and non-negligible $\delta\in\mathfrak{B}$, two maximal kth index systems over δ have the same cardinality.*

Proof. Let $\{x_\alpha | \alpha \in A\}$ and $\{y_\beta | \beta \in B\}$ be maximal kth index systems over the non-negligible set δ. Fix α and consider the family

$$\{Q^i E(\delta) x_\alpha \mid i = 0, 1, \cdots, k-1\}.$$

The union of this family with $\{Q^i y_\beta | i = 0, 1, \cdots, k-1, \beta \in B\}$ cannot be independent by the maximality of $\{y_\beta | \beta \in B\}$, and thus (Theorem 8 and Corollary 10) there is a nontrivial relation of the form

$$\text{(i)} \qquad 0 = \sum_{\beta \in B} \sum_{i=0}^{k-1} S(f_{i\beta\alpha}) Q^i y_\beta + \sum_{i=0}^{k-1} S(f_{i\alpha}) Q^i E(\delta) x_\alpha.$$

It is now asserted that there is a non-negligible set $\delta_\alpha \in \mathcal{B}$, for which there is a relation of the form

$$\text{(ii)} \qquad 0 \neq E(\delta_\alpha) Q^{k-1} x_\alpha = \sum_{\beta \in B} S(f_{\beta\alpha}) Q^{k-1} y_\beta.$$

For, let p be the least integer such that $S(f_{p\alpha}) Q^p E(\delta) x_\alpha$ is nonzero in (i). Let δ_α be the intersection of an inverting set for $S(f_{p\alpha})$ with δ. Applying $S(1/f_{p\alpha}) E(\delta_\alpha) Q^{k-p-1}$ to (i), one obtains a relation

$$\text{(iii)} \qquad 0 \neq E(\delta_\alpha) Q^{k-1} x_\alpha = \sum_{\beta \in B} \sum_{i=0}^{k-1} S(g_{i\beta\alpha}) Q^i E(\delta_\alpha) y_\beta.$$

Applying Q to (iii), we have

$$0 = \sum_{\beta \in B} \sum_{i=0}^{k-2} S(g_{i\beta\alpha}) Q^{i+1} E(\delta_\alpha) y_\beta,$$

which implies $S(g_{i\beta\alpha}) E(\delta_\alpha) = 0$ for $i = 0, 1, \cdots, k-2$, $\beta \in B$. (All the vectors are full over δ_α.) Therefore,

$$0 \neq E(\delta_\alpha) Q^{k-1} x_\alpha = \sum_{\beta \in B} S(g_{k-1,\beta,\alpha}) E(\delta_\alpha) Q^{k-1} y_\beta,$$

which is of the form (ii), and the assertion is demonstrated.

Then, for each $\alpha \in A$ we have a relation of the form (ii). It is clear that the family $\{E(\delta_\alpha) Q^{k-1} x_\alpha | \alpha \in A\}$ is independent. As $\{Q^{k-1} y_\beta | \beta \in B\}$ is also independent, and $E(\delta_\alpha) Q^{k-1} x_\alpha \in \bigvee_\beta \mathfrak{M}(Q^{k-1} y_\beta)$ for each α, it follows from Corollary 7 that the cardinality of A does not exceed the cardinality of B. By symmetry the reverse is true. Q.E.D.

The *Weyr characteristic* $\mathcal{W}$ can now be defined: let $\mathcal{W}(\delta, k)$ be the cardinality of Theorem 16, and be zero if $\delta \equiv \emptyset$. Then by Theorem 6 and Corollary 15, $\mathcal{W}(\delta, k) \leqq n$ for all k, and $\mathcal{W}(\delta, k) = 0$ when $k > n$.

THEOREM 17. (a) *For every* $\delta \in \mathcal{B}$, $\mathcal{W}(\delta, k) \geqq \mathcal{W}(\delta, k+1)$ *for each* $k = 1, 2, \cdots$.

(b) *If* $\delta, \pi \in \mathcal{B}$, $\emptyset \not\equiv \pi \subset \delta$, *then* $\mathcal{W}(\pi, k) \geqq \mathcal{W}(\delta, k)$ *for each* k.

(c) *If δ is the union of a countable disjoint family $\{\delta_i\} \subset \mathcal{B}$ of non-negligible sets, then for each k,*

$$\mathcal{W}(\delta, k) = \min_i \{\mathcal{W}(\delta_i, k)\}.$$

Proof. Assertion (a) follows from the fact that if $\{x_\alpha \mid \alpha \in A\}$ is a $(k+1)$st index system over δ, then $\{Qx_\alpha \mid \alpha \in A\}$ is a kth index system over δ. Assertion (b) is equally trivial, and to prove (c), let q be the minimum described for some fixed k. By (b), $\mathcal{W}(\delta_i, k) \geqq \mathcal{W}(\delta, k)$ for every i, and hence $q \geqq \mathcal{W}(\delta, k)$. On the other hand, as each $\mathcal{W}(\delta_i, k) \geqq q$, for each i there is a kth index system $\{x_{\alpha i} \mid \alpha \in A\}$ over δ_i of cardinality q. Let

$$y_\alpha = \sum_{i=1}^{\infty} \frac{E(\delta_i) x_{\alpha_i}}{2^i \left| x_{\alpha_i} \right|}, \qquad \alpha \in A.$$

Then $\{y_\alpha \mid \alpha \in A\}$ has cardinality q and is readily seen to be a kth index system over δ. Hence $\mathcal{W}(\delta, k) \geqq q$. Q.E.D.

Thus we can define the *Segre characteristic* $\mathcal{S}$ for each $\delta \in \mathcal{B}$ and k by $\mathcal{S}(\delta, k) = \mathcal{W}(\delta, k) - \mathcal{W}(\delta, k+1)$. The values of $\mathcal{W}$ are readily recoverable from those of $\mathcal{S}$, and the relation

$$\sum_k \mathcal{W}(\delta, k) = \sum_k k\mathcal{S}(\delta, k), \qquad \delta \in \mathcal{B},$$

is immediate.

Theorem 17 also states that for each k, $\mathcal{W}(\cdot, k)$ has the formal properties of a multiplicity function in the sense of Halmos, described in §2, except for the presence of set, rather than measure, arguments. Call a set $\delta \in \mathcal{B}$ *k-uniform* if $\emptyset \neq \pi \subseteq \delta$, $\pi \in \mathcal{B}$ implies $\mathcal{W}(\pi, k) = \mathcal{W}(\delta, k)$. Then it follows from the arguments of [17, §49], which we shall not reproduce, that for fixed k, the set Λ can be partitioned into disjoint k-uniform sets. (The proofs in [17], given for the Boolean σ-ring of measures under the partial ordering $\ll$ apply without real change to the Boolean σ-ring of Borel subsets of Λ under the partial ordering defined by E.) As $\mathcal{W}(\cdot, k)$ can only take the values $0, 1, \cdots, n$, there can be at most $n+1$ k-uniform sets for each k. Thus, for each k there is a finite disjoint class $\{\delta_{ik} \mid i = 0, 1, \cdots, n\}$ with each (possibly negligible) set δ_{ik} being k-uniform, and with $\Lambda = \bigcup_{i=0}^n \delta_{ik}$. A set $\delta \in \mathcal{B}$ will be said to be of *uniform characteristic* if it is k-uniform for every k. Then, by considering the minimal sets in the finite ring generated (intersection and relative complementation) by Λ and the sets δ_{ik}, we can conclude the following.

THEOREM 18. *There is a partition of Λ as a finite union of disjoint sets of uniform characteristic.*

Next, suppose m is the index of Q (that is, $Q^m\mathfrak{X} = 0$, $Q^{m-1}\mathfrak{X} \neq 0$, or equivalently, m is the maximum integer for which $\mathcal{W}(\cdot, m) \neq 0$). Let $\{x_\alpha^m \mid \alpha \in A_m\}$

be a maximal mth index system over a non-negligible set $\delta \in \mathfrak{B}$, with $C(x_\alpha^m) = E(\delta)$ for each $\alpha \in A_m$ (cf. Corollary 10). Then there is a maximal $(m-1)$st index system over δ containing $\{Qx_\alpha^m | \alpha \in A_m\}$, say $\{x_\alpha^{m-1} | \alpha \in A_{m-1}\}$, such that $C(x_\alpha^{m-1}) = E(\delta)$, $\alpha \in A_{m-1}$. Then there is a maximal $(m-2)$nd index system over δ, say $\{x_\alpha^{m-2} | \alpha \in A_{m-2}\}$, containing $\{Qx_\alpha^{m-1} | \alpha \in A_{m-1}\}$, and with $C(x_\alpha^{m-2}) = E(\delta)$, $\alpha \in A_{m-2}$, etc. Continuing so, we obtain a family of vectors

$$\{x_\alpha^k \mid \alpha \in A_k, k = 1, \cdots, m\},$$

whose cardinality is $\sum_k \mathfrak{W}(\delta, k)$. A family constructed in this way, which will be frequently used in the sequel, will be called a *complete index system over* δ; its distinguishing properties are that $\{x_\alpha^k | \alpha \in A_k\}$ is a maximal kth index system over δ for $k = 1, \cdots, m$, that $C(x_\alpha^k) = E(\delta)$ for each α and k (and hence $C(Q^j x_\alpha^k) = E(\delta)$ or 0, depending on whether $j < k$ or $j \geqq k$), and that $\{Qx_\alpha^k | \alpha \in A_k\} \subseteq \{x_\alpha^{k-1} | \alpha \in A_{k-1}\}$ for $k = 2, \cdots, m$. We can always suppose the elements indexed in such a fashion that $A_k \subseteq A_{k-1}$, $k = 2, \cdots, m$.

THEOREM 19. *If $\delta \in \mathfrak{B}$ is non-negligible, a complete index system over δ is independent.*

Proof. If $\{x_\alpha^k | \alpha \in A_k, k = 1, \cdots, m\}$ is a complete index system over δ, each x_α^k is full over δ. Thus (Theorem 8) suppose that

$$\sum_{k=1}^{m} \sum_{\alpha \in A_k} S(f_\alpha^{(k)}) x_\alpha^k = 0.$$

Applying Q^{m-1}, we obtain

$$\sum_{k=1}^{m} \sum_{\alpha \in A_k} S(f_\alpha^{(k)}) Q^{m-1} x_\alpha^k = \sum_{\alpha \in A_m} S(f_\alpha^{(m)}) x_\alpha^1 = 0,$$

and the independence of $\{x_\alpha^1 | \alpha \in A_1\}$ implies that $S(f_\alpha^{(m)})E(\delta)$ is zero for each $\alpha \in A_m$. Then application of Q^{m-2} to the original equation shows that $S(f_\alpha^{(m-1)})E(\delta) = 0$ for each $\alpha \in A_{m-1}$, etc.

COROLLARY 20. *For any $\delta \in \mathfrak{B}$, $\sum_k \mathfrak{W}(\delta, k) \leqq n$.*

The next theorem, characterizing the sets of uniform characteristic, will be used later to relate the Weyr characteristic to the multiplicity function of a normal operator. This will be required, in §6, to prove that the characteristic is a complete set of invariants.

THEOREM 21. *A non-negligible set $\delta \in \mathfrak{B}$ has uniform characteristic if and only if*

$$\sum_k \mathfrak{W}(\delta, k) = n.$$

Proof. The sufficiency of the summability condition is not difficult to prove. For, suppose δ does not have uniform characteristic, and let $\pi \in \mathfrak{B}$, $\pi \subset \delta$ with $\mathcal{W}(\pi, k_0) > \mathcal{W}(\delta, k_0)$ for some k_0. Then, since $\mathcal{W}(\pi, k) \geqq \mathcal{W}(\delta, k)$ for every k, it follows that

$$n \geqq \sum_k \mathcal{W}(\pi, k) > \sum_k \mathcal{W}(\delta, k).$$

To verify the necessity, it will be shown(³), by downward induction on k, that the closure of $Q^k E(\delta)\mathfrak{X}$ satisfies

$$\text{CL}\{Q^k E(\delta)\mathfrak{X}\} = \bigvee_{i=k+1}^{m} \bigvee_{\alpha \in A_i} \mathfrak{M}(Q^k \overset{i}{x}_\alpha), \qquad k = 0, 1, \cdots, m-1,$$

where $\{x^i_\alpha | \alpha \in A_i, i = 1, \cdots, m\}$ is a complete index system over δ. For $k=0$ we will then have

$$E(\delta)\mathfrak{X} = \bigvee_{i=1}^{m} \bigvee_{\alpha \in A_i} \mathfrak{M}(\overset{i}{x}_\alpha),$$

and since $E(\delta)\mathfrak{X}$ has uniform multiplicity n, it will follow that $\sum_k \mathcal{W}(\delta, k) \geqq n$.

First let $k = m-1$. Then, if the inclusion

$$\text{(i)} \qquad \text{CL}\{Q^{m-1} E(\delta)\mathfrak{X}\} \supseteqq \bigvee_{\alpha \in A_m} \mathfrak{M}(Q^{m-1} \overset{m}{x}_\alpha)$$

is proper, by [4, Lemma 4.7] there must exist a vector $x_0 \in E(\delta)\mathfrak{X}$ such that $Q^{m-1}x_0 \neq 0$ and

$$\text{(ii)} \qquad \mathfrak{M}(Q^{m-1} x_0) \wedge \left(\bigvee_{\alpha \in A_m} \mathfrak{M}(Q^{m-1} \overset{m}{x}_\alpha) \right) = 0.$$

Let σ be the support of $Q^{m-1}x_0$. Suppose there exists a relation

$$\sum_{j=0}^{m-1} \sum_{\alpha \in A_m} S(f_{\alpha j}) E(\sigma) Q^j \overset{m}{x}_\alpha + \sum_{i=0}^{m-1} S(g_i) Q^i x_0 = 0.$$

Multiplying by Q^{m-1} we get

$$\sum_{\alpha \in A_m} S(f_{\alpha 0}) E(\sigma) Q^{m-1} \overset{m}{x}_\alpha + S(g_0) Q^{m-1} x_0 = 0,$$

so by (ii) it follows that g_0 and $f_{\alpha 0}$, $\alpha \in A_m$, vanish essentially on σ. Multiplying successively by Q^{m-2}, Q^{m-3}, etc., it follows that all the functions $f_{\alpha j}$, $\alpha \in A_m$, $j = 0, 1, \cdots, m-1$, and g_i, $i = 0, 1, \cdots, m-1$ vanish essentially on σ. Thus the vectors

(³) The author would like to express his, and the reader's, indebtedness to the referee for this proof, a substantial improvement in clarity and brevity over the original cumbersome version.

$$\{E(\sigma)Q^{j}x_{\alpha}^{m},\ Q^{i}x_0 \mid i, j = 0, 1, \cdots, m-1\}$$

are independent and $\mathcal{W}(\sigma, m) > \mathcal{W}(\delta, m)$, contradicting the assumption of uniform characteristic. Hence (i) must be an equality, which was to be proved.

To conclude, we suppose

$$\text{(iii)} \qquad \mathrm{CL}\{Q^{k}E(\delta)\mathfrak{X}\} = \bigvee_{i=k+1}^{m} \bigvee_{\alpha\in A_i} \mathfrak{M}(Q^{k}x_{\alpha}^{i}),$$

and prove

$$\mathrm{CL}\{Q^{k-1}E(\delta)\mathfrak{X}\} = \bigvee_{i=k}^{m} \bigvee_{\alpha\in A_i} \mathfrak{M}(Q^{k-1}x_{\alpha}^{i}).$$

Were this false there would be a nonzero vector $Q^{k-1}y_0 \in Q^{k-1}E(\delta)\mathfrak{X}$ such that

$$\text{(iv)} \qquad \mathfrak{M}(Q^{k-1}y_0) \wedge \left(\bigvee_{i=k}^{m} \bigvee_{\alpha\in A_i} \mathfrak{M}(Q^{k-1}x_{\alpha}^{i})\right) = 0.$$

If $Q^k y_0 = 0$, then the preceding argument can be used to show $\mathcal{W}(s(y_0), k) > \mathcal{W}(\delta, k)$, a contradiction. If $Q^k y_0 \neq 0$, there is a non-negligible subset π of $s(Q^k y_0)$ and a relation, by (iii), of the form

$$E(\pi)Q^{k}y_0 = \sum_{i=k+1}^{m} \sum_{\alpha\in A_i} S(f_{\alpha}^{(i)})Q^{k}E(\pi)x_{\alpha}^{i}.$$

Let

$$z_0 = E(\pi)y_0 - \sum_{i=k+1}^{m} \sum_{\alpha\in A_i} S(f_{\alpha}^{(i)})E(\pi)x_{\alpha}^{i}.$$

Then it follows easily from (iv) that

$$\text{(a)} \qquad Q^k z_0 = 0,$$

$$\text{(b)} \qquad Q^{k-1} z_0 \neq 0,$$

$$\text{(c)} \qquad \mathfrak{M}(Q^{k-1}z_0) \wedge \left(\bigvee_{i=k}^{m} \bigvee_{\alpha\in A_i} \mathfrak{M}(Q^{k-1}x_{\alpha}^{i})\right) = 0.$$

Again by exactly the argument for the case of $k = m-1$, we see the family

$$\{Q^{j}E(\pi)x_{\alpha}^{k},\ Q^{i}z_0 \mid \alpha \in A_k,\ i, j = 0, 1, \cdots, k-1\}$$

is independent, and therefore a kth index system over π. Thus we have the contradiction: $\mathcal{W}(\pi, k) > \mathcal{W}(\delta, k)$. Q.E.D.

We conclude this section by altering the definition of the Weyr characteristic to permit measure arguments. Let x be any full vector and x^* be a Bade functional for x and E. If the measure μ is bounded by the measure

$x^*E(\cdot)x$ in the sense that $\mu \ll x^*E(\cdot)x$, we define $\mathcal{W}(\mu, k)$ using the support of μ by

$$\mathcal{W}(\mu, k) = \mathcal{W}(s(\mu), k), \qquad k = 1, 2, \cdots.$$

Otherwise, if μ is not bounded by $x^*E(\cdot)x$, let $\mathcal{W}(\mu, k) = 0$. Then $\mathcal{W}$ is actually defined for equivalence classes of measures bounded by $x^*E(\cdot)x$, and the correspondence discussed in §2 between these equivalence classes and those of Borel subsets of Λ under the partial order defined by E makes it easy to rephrase our recent results for measure arguments. Thus the obvious measure analog of Theorem 17 is immediate:

(a) *If $\mu \neq 0$, then $\mathcal{W}(\mu, k) \geqq \mathcal{W}(\mu, k+1)$ for $k = 1, 2, \cdots$.*

(b) *If $0 \neq \nu \ll \mu$, then $\mathcal{W}(\nu, k) \geqq \mathcal{W}(\mu, k)$ for each k.*

(c) *If μ is equivalent to the supremum of the orthogonal (hence necessarily countable) family $\{\mu_i\}$ of nonzero measures and $\mathcal{W}(\mu, k)$ is defined, then $\mathcal{W}(\mu, k) = \min_i\{\mathcal{W}(\mu_i, k)\}$ for each k.*

The Segre characteristic $\mathcal{S}$ can now be defined:

$$\mathcal{S}(\mu, k) = \mathcal{S}(s(\mu), k) = \mathcal{W}(\mu, k) - \mathcal{W}(\mu, k + 1),$$

and we can say a measure μ has *uniform characteristic* if $0 \neq \nu \ll \mu$ implies $\mathcal{W}(\nu, k) = \mathcal{W}(\mu, k)$ for each k. It follows that the measure $x^*E(\cdot)x$ is equivalent to the supremum of a finite orthogonal family of measures of uniform characteristic, that $\sum_k \mathcal{W}(\mu, k) \leqq n$ for every μ in the domain of $\mathcal{W}$, and that *each such μ has uniform characteristic if and only if*

$$\sum_k \mathcal{W}(\mu, k) = n.$$

5. **Spectral operators on Hilbert space.** Hereafter we abandon the assumption of uniform multiplicity that governed the preceding sections, and limit ourselves to the case in which the underlying space is Hilbert space. Suppose N is a (bounded) normal operator on the arbitrary Hilbert space $\mathfrak{H}$, with resolution of the identity E and multiplicity function $\mathbf{u}$ characterizing N within unitary equivalence. If Q is a quasi-nilpotent commuting with N, then, for any measure μ, the operator Q is completely reduced by each subspace $C(\mu)\mathfrak{H}$. Thus if μ is a measure of uniform finite multiplicity, the preceding results permit us to define, on the space $C(\mu)\mathfrak{H}$ and with respect to the complete spectral measure $C(\mu)E(\cdot)$, the Weyr characteristic for all measures ν satisfying $\nu \ll \mu$.

The operator N will be called *essentially finite* if $\mathbf{u}(\mu) < \infty$ for every measure μ of uniform multiplicity. Our first objective is to define the Weyr characteristic for an essentially finite N and commuting Q. For this let $\{\mu_\alpha \mid \alpha \in A\}$ be an orthogonal family of nonzero measures of uniform multiplicity associated with $\mathbf{u}$ satisfying $\bigvee_\alpha C(\mu_\alpha) = I$, as described in §2. If μ is any measure, we let $\mathcal{W}(\mu, k) = 0$ if $\mu \not\equiv \bigvee_\alpha (\mu \wedge \mu_\alpha)$; otherwise let

$$\mathcal{W}(\mu, k) = \min_{\alpha \in A} \{\mathcal{W}(\mu \wedge \mu_\alpha, k) \mid \mu \wedge \mu_\alpha \neq 0\}$$

for each k.

It is not clear that this definition is independent of the particular family $\{\mu_\alpha\}$. To prove this, let $\{\nu_\beta \mid \beta \in B\}$ be an another orthogonal family associated with $\mathbf{u}$ of nonzero measures of uniform multiplicity with $\bigvee_\beta C(\nu_\beta) = I$, and suppose $\mathcal{W}'$ has been defined in a similar way by this family. If μ is any measure then $\bigvee_\alpha (\mu \wedge \mu_\alpha) \equiv \bigvee_\beta (\mu \wedge \nu_\beta)$. Next let k be fixed and suppose $\mu \wedge \mu_{\alpha'} \neq 0$, and $\mathcal{W}(\mu \wedge \mu_{\alpha'}, k)$ is a minimum defining $\mathcal{W}(\mu, k)$. The nonzero members of $\{\mu \wedge \mu_{\alpha'} \wedge \nu_\beta \mid \beta \in B\}$ are a countable orthogonal family with supremum equivalent to $\mu \wedge \mu_{\alpha'}$. Hence, by the already discussed measure analog of Theorem 17,

$$\mathcal{W}(\mu \wedge \mu_{\alpha'}, k) = \min_\beta \{\mathcal{W}(\mu \wedge \mu_{\alpha'} \wedge \mu_\beta, k) \mid \mu \wedge \mu_{\alpha'} \wedge \nu_\beta \neq 0\}.$$

If β' is an index for which this last family is a minimum, then

$$\begin{aligned} \mathcal{W}(\mu, k) &= \mathcal{W}(\mu \wedge \mu_{\alpha'}, k) = \mathcal{W}(\mu \wedge \mu_{\alpha'} \wedge \nu_{\beta'}, k) \\ &= \mathcal{W}'(\mu \wedge \mu_{\alpha'} \wedge \nu_{\beta'}, k) \\ &\geqq \mathcal{W}'(\mu \wedge \nu_{\beta'}, k) \\ &\geqq \mathcal{W}'(\mu, k). \end{aligned}$$

Thus by symmetry $\mathcal{W} = \mathcal{W}'$, and $\mathcal{W}$ is well-defined, depending only on the pair $\langle N, Q \rangle$.

We now wish to prove results analogous to those of §4 for $\mathcal{W}$. It is clear that, for each k, $\mathcal{W}(\mu, k) \geqq \mathcal{W}(\mu, k+1)$ and that $0 \neq \nu \ll \mu$ implies $\mathcal{W}(\nu, k) \geqq \mathcal{W}(\mu, k)$. If $\{\mu_i\}$ is a countable orthogonal family with supremum equivalent to $\mu \equiv \bigvee_\alpha (\mu \wedge \mu_\alpha)$, then for each k,

$$\begin{aligned} \mathcal{W}(\mu, k) &= \min_\alpha \{\mathcal{W}(\mu \wedge \mu_\alpha, k) \mid \mu \wedge \mu_\alpha \neq 0\} \\ &= \min_\alpha \left\{\min_i \{\mathcal{W}(\mu_i \wedge \mu \wedge \mu_\alpha, k) \mid \mu_i \wedge \mu \wedge \mu_\alpha \neq 0\} \mid \mu \wedge \mu_\alpha \neq 0\right\} \\ &= \min_\alpha \left\{\min_i \{\mathcal{W}(\mu_i \wedge \mu_\alpha, k) \mid \mu_i \wedge \mu_\alpha \neq 0\} \mid \mu \wedge \mu_\alpha \neq 0\right\} \\ &= \min_i \left\{\min_\alpha \{\mathcal{W}(\mu_i \wedge \mu_\alpha, k) \mid \mu_i \wedge \mu_\alpha \neq 0\}\right\} \\ &= \min_i \{\mathcal{W}(\mu_i, k)\}, \end{aligned}$$

because $\mu \wedge \mu_\alpha \neq 0$ if and only if $\mu_i \wedge \mu_\alpha \neq 0$ for some i.

Thus the measure analog of Theorem 17 holds in general, and we can define $\mathcal{S}(\mu, k)=\mathcal{W}(\mu, k)-\mathcal{W}(\mu, k+1)$ and speak of k-uniform measures and measures of *uniform characteristic*, and it follows that each measure is equivalent to a supremum of an orthogonal (at most) *countable* family of nonzero measures of uniform characteristic.

We state formally the next two efforts along this line.

THEOREM 22. *For any measure μ, $\sum_k \mathcal{W}(\mu, k) \leqq \mathbf{u}(\mu)$.*

Proof. As previous considerations imply, the theorem is true for measures of uniform multiplicity,

$$\begin{aligned}\sum_k \mathcal{W}(\mu, k) &= \sum_k \min_{\alpha\in A} \{\mathcal{W}(\mu \wedge \mu_\alpha, k) \mid \mu \wedge \mu_\alpha \neq 0\} \\ &\leqq \min_{\alpha\in A} \left\{ \sum_k \mathcal{W}(\mu \wedge \mu_\alpha, k) \mid \mu \wedge \mu_\alpha \neq 0 \right\} \\ &\leqq \min_{\alpha\in A} \{\mathbf{u}(\mu_\alpha) \mid \mu \wedge \mu_\alpha \neq 0\} \\ &= \mathbf{u}(\mu).\ \text{Q.E.D.}\end{aligned}$$

The function $\mathcal{W}$ is decreasing in k as well as μ, and so this theorem implies $\mathcal{W}(\mu, k)=0$ whenever $k>\mathbf{u}(\mu)$. Thus, for any μ,

$$\sum_k k\mathcal{S}(\mu, k) = \sum_k \mathcal{W}(\mu, k).$$

The proper general measure analog of Theorem 21 is the following.

THEOREM 23. *The measure μ has uniform characteristic if and only if μ has uniform multiplicity and $\sum_k \mathcal{W}(\mu, k)=\mathbf{u}(\mu)$.*

Proof. If μ has uniform multiplicity, then, as we observed in §4, the desired conclusion follows from Theorem 21. Conversely, if μ has uniform characteristic, then each $\mu\wedge\mu_\alpha$ with $\mu\wedge\mu_\alpha\neq 0$ has uniform characteristic. As each of these measures has uniform multiplicity as well, Theorem 21 and the uniform characteristic of μ imply

$$\sum_k \mathcal{W}(\mu, k) = \sum_k \mathcal{W}(\mu \wedge \mu_\alpha, k) = \mathbf{u}(\mu \wedge \mu_\alpha)$$

for each of these α. Then all the $\mathbf{u}(\mu\wedge\mu_\alpha)$ with $\mu\wedge\mu_\alpha\neq 0$ are equal, and equal to $\mathbf{u}(\mu)$, and μ has uniform multiplicity. Q.E.D.

The next objective is to define essential finiteness and the Weyr characteristic, already defined for pairs $\langle N, Q\rangle$ of commuting operators with N normal and essentially finite and Q quasi-nilpotent, for spectral operators. For these and other future purposes the following theorem is required. The second assertion of the theorem, that similar normal operators are unitarily equivalent, was first proved by Putnam [26] in a different manner.

THEOREM 24. *If S_1 is a scalar operator, with resolution of the identity E_1, on the Hilbert space $\mathfrak{H}_1$, and L is a nonsingular operator from $\mathfrak{H}_1$ onto the Hilbert space $\mathfrak{H}_2$ and $S_2=LS_1L^{-1}$, then S_2 is a scalar operator on $\mathfrak{H}_2$ with resolution of the identity $E_2(\cdot)=LE_1(\cdot)L^{-1}$. If S_1 and S_2 are normal as well, then they are unitarily equivalent.*

Proof. The first assertion will be proved by showing E_2 as defined above is a resolution of the identity for S_2 (in the sense of §2) and $S_2=\int\lambda E_2(d\lambda)$, and applying the uniqueness of such a resolution. Most of the desired properties of E_2 follow directly from those of E_1: it is clear that E_2 is countably additive in the strong topology, uniformly bounded, and a homomorphism from the Boolean algebra $\mathfrak{B}$ of Borel subsets of the complex plane $\mathfrak{C}$ to a Boolean algebra of idempotents on $\mathfrak{H}_2$ with $E(\emptyset)=0$ and $E(\mathfrak{C})=I$. Also, $E_2(\delta)S_2=LE_1(\delta)L^{-1}S_2=LE_1(\delta)S_1L^{-1}=LS_1E_1(\delta)L^{-1}=S_2LE_1(\delta)L^{-1}=S_2E_2(\delta)$ for each $\delta\in\mathfrak{B}$, so S_2 commutes with $E_2(\cdot)$. Moreover, $S_2=LS_1L^{-1}$ $=L[\int\lambda E_1(d\lambda)]L^{-1}=\int\lambda LE_1(d\lambda)L^{-1}=\int\lambda E_2(d\lambda)$. Thus if $\lambda_0\notin\bar{\delta}$, the closure of $\delta\in\mathfrak{B}$, then the bounded operator

$$\int_\delta(\lambda_0-\lambda)^{-1}E_2(d\lambda)$$

is an inverse on $E_2(\delta)\mathfrak{H}_2$ for the restriction of (λ_0I-S_2) to this space, since

$$(\lambda_0I-S_2)|\,E_2(\delta)\mathfrak{H}_2=\int_\delta(\lambda_0-\lambda)E_2(d\lambda).$$

Therefore $spec(S_2|\,E_2(\delta)\mathfrak{H}_2)\subseteq\bar{\delta}$, and it follows that E_2 is the unique resolution of the identity for S_2.

If S_1 and S_2 are normal, then E_1 and E_2 are self-adjoint, and we can apply a standard algebraic argument. Let $(L^*L)^{1/2}$ be the positive definite self-adjoint square root of the positive definite self-adjoint operator L^*L. Simple computation shows that $U=L(L^*L)^{-1/2}$ is unitary, and, since $E_2(\cdot)L=LE_1(\cdot)$ and $L^*E_2(\cdot)=E_1(\cdot)L^*$, that L^*L commutes with $E_1(\cdot)$. It follows that $(L^*L)^{1/2}$ commutes with $E_1(\cdot)$, and therefore $E_2(\delta)=UE_1(\delta)U^*$, $\delta\in\mathfrak{B}$. As U is independent of δ, it follows that $S_2=US_1U^*$. Q.E.D.

We recall that every scalar operator S on Hilbert space has a *normal conjugate*, LSL^{-1}, L nonsingular. Theorem 24 shows that two normal conjugates of S are unitarily equivalent, and will have the same multiplicity function. Thus S can unambiguously be called *essentially finite* if any normal conjugate is essentially finite, and the spectral operator $T=S+Q$ (in canonical decomposition) can be called essentially finite if S has this property. As the spectrum is a similarity invariant, for any nonsingular L the operator Q is quasi-nilpotent if and only if LQL^{-1} is quasi-nilpotent; this remark and Theorem 24 imply that, for any nonsingular L, the operator $LTL^{-1}=LSL^{-1}$

$+LQL^{-1}$ is spectral with scalar part LSL^{-1}, quasi-nilpotent part LQL^{-1}, and (if E is the resolution of the identity of T) resolution of the identity $LE(\cdot)L^{-1}$. Therefore T is essentially finite if and only if LTL^{-1} is, and essential finiteness is similarity invariant.

THEOREM 25. *Let $T_1=N_1+Q_1$ and $T_2=N_2+Q_2$ be spectral operators in canonical decomposition on $\mathfrak{H}$, with N_1 and N_2 normal. Suppose T_1 is essentially finite, and $T_2=LT_1L^{-1}$, with L nonsingular. Then T_2 is essentially finite, and the Weyr characteristics defined by $\langle N_1, Q_1\rangle$ and $\langle N_2, Q_2\rangle$ are the same.*

Proof. That T_2 and N_2 are essentially finite follows from the preceding discussion, and it is clear that $N_2=LN_1L^{-1}$ and $Q_2=LQ_1L^{-1}$. We also know that N_1 and N_2 are unitarily equivalent, and that, if E_i is the (self-adjoint) resolution of the identity for N_i $(i=1, 2)$, then $E_2(\cdot)=LE_1(\cdot)L^{-1}$ and E_1 and E_2 are unitarily equivalent. Therefore N_1 and N_2 have the same multiplicity function and measures of uniform multiplicity. In view of the manner of definition of the characteristic, it is sufficient to confine our attention to a pair of spaces of the form $C_1(\mu)\mathfrak{H}$ and $C_2(\mu)\mathfrak{H}$, where μ is a measure of uniform multiplicity and C_i the carrier operation with respect to E_i, $i=1, 2$.

Before proceeding further, some preliminary remarks about Boolean algebras of projections and similarity are necessary. The first, which follows from an obvious and simple computation, is that similarity preserves the order relation among commuting projections. That is, if P_1 and P_2 are commuting projections with $P_1\geqq P_2$, then $LP_1L^{-1}\geqq LP_2L^{-1}$. Then a computation only slightly less simple shows that if $\{P\}$ and $\{LPL^{-1}\}$ are complete Boolean algebras, and $\{P_\beta|\beta\in B\}$ is an arbitrary subset of $\{P\}$, we have

$$\bigwedge_{\beta\in B}\{LP_\beta L^{-1}\} = L\left\{\bigwedge_{\beta\in B} P_\beta\right\}L^{-1},$$

and

$$\bigvee_{\beta\in B}\{LP_\beta L^{-1}\} = L\left\{\bigvee_{\beta\in B} P_\beta\right\}L^{-1}.$$

In our case, though $\{C_i(\mu)E_i(\cdot)\}$, $i=1, 2$, are complete, the Boolean algebras E_1 and E_2 are only σ-complete; in general these will not be complete if the Hilbert space is non-separable.

The assertion now is that $C_2(\mu)=LC_1(\mu)L^{-1}$. To see this, suppose the projection $P_1\in\overline{E}_1$, the completion of E_1, has the property that $P_1x=x$ whenever $|E_1(\cdot)x|^2\ll\mu$, and let $P_2=LP_1L^{-1}$. Then P_1 is a strong limit of operators in the range of E_1, and the continuity of multiplication in the strong topology implies $P_2\in\overline{E}_2$. Moreover, P_2 has the property that $L^{-1}P_2Lx=x$ whenever $|L^{-1}E_2(\cdot)Lx|^2\ll\mu$. That is, $P_2Lx=Lx$ whenever $|L^{-1}E_2(\cdot)Lx|^2\equiv|E_2(\cdot)Lx|^2\ll\mu$. By definition $C_1(\mu)$ is the infimum of all such P_1, and the nonsingularity of L implies $C_2(\mu)$ is the infimum of all such P_2. The assertion then follows from the preceding paragraph. Thus by Theorem 24, $C_1(\mu)$ and $C_2(\mu)$ are unitarily equivalent and L maps $C_1(\mu)\mathfrak{H}$ onto $C_2(\mu)\mathfrak{H}$.

For the spaces $C_1(\mu)\mathfrak{H}$ and $C_2(\mu)\mathfrak{H}$, the characteristic is defined by the cardinalities of index systems. Therefore to prove the theorem it suffices to prove that if $\{x_\alpha | \alpha \in A\}$ is a kth index system over $\delta \in \mathfrak{B}$ (the Borel subsets of $s(\mu)$) for the pair $\langle N_1, Q_1\rangle$ on the space $C_1(\mu)\mathfrak{H}$, then the family $\{Lx_\alpha | \alpha \in A\}$ with the same cardinality is a kth index system over δ for the pair $\langle N_2, Q_2\rangle$ on the space $C_2(\mu)\mathfrak{H}$. We can suppose, without loss of generality, that $x_\alpha \in E_1(\delta)\mathfrak{H}$ for each $\alpha \in A$, and also, for notational simplicity, that $C_1(\mu) = C_2(\mu) = I$. Then the family $\{x_\alpha | \alpha \in A\}$ has the properties:

$$\left.\begin{aligned} &\text{(a)} \qquad Q_1^k x_\alpha = 0 \\ &\text{(b)} \quad C_1(Q_1^{k-1} x_\alpha) = E_1(\delta) \end{aligned}\right\} \alpha \in A,$$

$$\text{(c)} \quad \{Q_1^i x_\alpha | i = 0, 1, \cdots, k-1, \alpha \in A\} \text{ is } E_1\text{-independent},$$

E_1 and E_2 now being complete. Then (b) and (c) can be rewritten:

$$\text{(b}'\text{)} \qquad \bigwedge_{\pi \in \mathfrak{B}} \{E_1(\pi) | E_1(\pi) Q_1^{k-1} x_\alpha = Q_1^{k-1} x_\alpha\} = E_1(\delta), \ \alpha \in A,$$

$$\text{(c}'\text{)} \qquad \sum_{\alpha \in A} \sum_{i=0}^{k-1} \int f_{i\alpha}(\lambda) E_1(d\lambda) Q_1^i x_\alpha = 0 \text{ implies } f_{i\alpha}(\lambda) = 0$$

for $[E_1]$-almost all $\lambda \in \delta$, $\alpha \in A$, $i = 0, \cdots, k-1$ (applying Theorem 8, as all the vectors are E_1-full over δ).

That $\{Lx_\alpha | \alpha \in A\}$ is a kth index system over δ for the pair $\langle N_2, Q_2\rangle$ now follows from the arguments:

$$\text{(a}''\text{)} \qquad Q_2^k L x_\alpha = L Q_1^k x_\alpha = 0, \ \alpha \in A.$$

$$\begin{aligned} \text{(b}''\text{)} \qquad & \bigwedge_{\pi \in \mathfrak{B}} \{E_2(\pi) | E_2(\pi) Q_2^{k-1} L x_\alpha = Q_2^{k-1} L x_\alpha\} \\ &= \bigwedge_{\pi \in \mathfrak{B}} \{L E_1(\pi) L^{-1} | L E_1(\pi) Q_1^{k-1} x_\alpha = L Q_1^{k-1} x_\alpha\} \\ &= L\left(\bigwedge_{\pi \in \mathfrak{B}} \{E_1(\pi) | E_1(\pi) Q_1^{k-1} x_\alpha = Q_1^{k-1} x_\alpha\}\right) L^{-1} \\ &= L E_1(\delta) L^{-1} \\ &= E_2(\delta), \ \alpha \in A. \end{aligned}$$

$$\text{(c}''\text{)} \qquad \sum_{\pi \in A} \sum_{i=0}^{k-1} \int f_{i\alpha}(\lambda) E_2(d\lambda) Q_2^i L x_\alpha = 0$$

implies

$$L\left(\sum_{\alpha\in A}\sum_{i=0}^{k-1}\int f_{i\alpha}(\lambda)E_1(d\lambda)Q_1^i x_\alpha\right) = 0,$$

and hence $f_{i\alpha}(\lambda)=0$ for $[E_1]$ a.a. $\lambda\in\delta$ and each i and α. It follows by unitary equivalence that $f_{i\alpha}(\lambda)=0$ for $[E_2]$ a.a. $\lambda\in\delta$ and each i and α. Thus the fact, following from (b''), that each nonzero $Q_2^iLx_\alpha$ is E_2-full over δ implies that $\{Q_2^iLx_\alpha \mid i=0, 1, \cdots, k-1, \alpha\in A\}$ is E_2-independent. Q.E.D.

The Weyr characteristic of an essentially finite spectral operator T on $\mathfrak{H}$ can now be specified as that defined by any pair $\langle LSL^{-1}, LQL^{-1}\rangle$, where LSL^{-1} is normal. It is thus well-defined and similarity invariant.

6. **Semi-similarity.** If an essentially finite spectral operator is actually scalar (that is, has quasi-nilpotent part zero), then its Weyr characteristic vanishes for $k>1$, while for $k=1$, Theorem 23 shows it to be the multiplicity function of a normal conjugate, with the measures of uniform characteristic being those of uniform multiplicity. The unitary invariants of a normal operator can be interpreted, by Theorem 24, as the similarity invariants of a similar scalar operator. Thus our results reduce to ordinary multiplicity theory, phrased in terms of similarity, when the spectral operator is scalar.

To return to the general situation, the Weyr characteristic is similarity invariant. We next present a simple and suggestive example to show that, even under the most restrictive assumptions (separable Hilbert space, self-adjoint and equal scalar parts, pure point spectrum of uniform multiplicity 2), essentially finite spectral operators with the same characteristic need not be similar.

For this, let $\{z_n \mid n=-1, 0, 1, 2, \cdots\}$ be a complete orthonormal system in the separable Hilbert space $\mathfrak{H}$ and define N, Q_1, and Q_2 by $Nz_{-1}=Nz_0 = Q_iz_{-1} = Q_iz_0 = 0$ for $i = 1, 2$, and on each two dimensional subspace $[z_{2n-1}, z_{2n}]$, $n=1, 2, \cdots$, by the 2×2 matrix blocks:

$$N:\begin{pmatrix}\frac{1}{n} & 0\\ 0 & \frac{1}{n}\end{pmatrix},\qquad Q_1:\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad Q_2:\begin{pmatrix}0 & \frac{1}{n}\\ 0 & 0\end{pmatrix},$$

the matrices written in the standard algebraic manner (so the rows give the coordinates of the images of the basis vectors). Clearly, $Q_1^2=Q_2^2=0$, and the operators $T_1=N+Q_1$ and $T_2=N+Q_2$ are essentially finite spectral operators with the same resolution of the identity E. The only measures that need be considered are those with point mass on any nonvoid subset of the points $0, 1, 1/2, 1/3, \cdots$, and each of these has uniform multiplicity 2. Both T_1 and T_2 have the same Weyr characteristic $\mathcal{W}$, where for any measure μ of the above type, we have $\mathcal{W}(\mu, 1)=\mathcal{W}(\mu, 2)=1$, and $\mathcal{W}(\mu, k)=0$ for $k>2$. Every such measure has uniform characteristic.

But T_1 and T_2 are not similar. For if $T_2 = LT_1L^{-1}$ for some nonsingular L, then $E(\cdot) = LE(\cdot)L^{-1}$, and L must be completely reduced by each two dimensional subspace $[z_{2n-1}, z_{2n}]$, $n = 0, 1, \cdots$. As $Q_2E(\cdot) = LQ_1E(\cdot)L^{-1}$, an elementary calculation shows that on each such subspace with $n > 0$, L and L^{-1} must have the form:

$$L: \begin{pmatrix} a_n & b_n \\ 0 & na_n \end{pmatrix}, \qquad L^{-1}: \begin{pmatrix} \dfrac{1}{a_n} & \dfrac{-b_n}{na_n^2} \\ 0 & \dfrac{1}{na_n} \end{pmatrix}$$

with b_n arbitrary and each $a_n \neq 0$. If L and L^{-1} are to be bounded operators, then the sequence $\{|a_n|\}$ must be bounded above and away from zero; but in this case $|Lz_{2n}| \to \infty$ as $n \to \infty$. It is clear, however, that T_1 and T_2 are "piecewise similar" in a simple way, and this is what we shall mean by semi-similarity.

To be precise, two spectral operators T_1 and T_2 on the Hilbert space $\mathfrak{H}$ with respective resolutions of the identity E_1 and E_2 will be called *semi-similar* if, for $i = 1, 2$, there is a family $\{P^i_\alpha | \alpha \in A\}$ of disjoint nonzero projections in $\overline{E}_i$ with $\bigvee_\alpha P^i_\alpha = I$, and such that for each $\alpha \in A$, there is a nonsingular operator L_α from the subspace $\mathfrak{H}^1_\alpha = P^1_\alpha\mathfrak{H}$, invariant under T_1, onto the subspace $\mathfrak{H}^2_\alpha = P^2_\alpha\mathfrak{H}$, invariant under T_2, satisfying

$$T_2P^2_\alpha = L_\alpha T_1 L^{-1}_\alpha P^2_\alpha, \qquad \alpha \in A.$$

We note some elementary consequences of this definition. First, for $i = 1$ or 2, the projections P^i_α commute with each other and (where $T_i = S_i + Q_i$ in canonical decomposition) with S_i, Q_i, and E_i, and the spaces $\mathfrak{H}^i_\alpha$ completely reduce each of these. The argument of Theorem 24 shows that for each α, we have $T_iP^i_\alpha$ a spectral operator on $\mathfrak{H}^i_\alpha$ with scalar part $S_iP^i_\alpha$, quasi-nilpotent part $Q_iP^i_\alpha$, resolution of the identity $E_i(\cdot)P^i_\alpha$, and each of these pairs of restricted operators satisfies a corresponding similarity induced by L_α. In particular, $E_2(\cdot)P^i_\alpha = L_\alpha E_1(\cdot)P^1_\alpha L^{-1}_\alpha P^2_\alpha$, and thus $P^2_\alpha = L_\alpha P^1_\alpha L^{-1}_\alpha P^2_\alpha$. Moreover,

$$\begin{aligned} T_1P^1_\alpha &= L^{-1}_\alpha(L_\alpha T_1 P^1_\alpha L^{-1}_\alpha P^2_\alpha)L_\alpha P^1_\alpha \\ &= L^{-1}_\alpha T_2 L_\alpha P^1_\alpha, \qquad \alpha \in A, \end{aligned}$$

and similar equalities hold for $S_1P^1_\alpha$, $Q_1P^1_\alpha$, and $E_1(\cdot)P^1_\alpha$.

THEOREM 26. *Semi-similarity is an equivalence relation for spectral operators.*

Proof. Only the transitivity is in question. Suppose T_1 and T_2 are semi-similar, with projections P^i_α and partial similarities L_α, $\alpha \in A$, $i = 1, 2$, as above and also that T_2 and T_3 are semi-similar, with projections P^i_β and partial similarities L_β, $\beta \in B$, $i=2, 3$, defined in the same manner, with $T_3P^3_\beta = L_\beta T_2 L_\beta^{-1} P^3_\beta$ and $T_2P^2_\beta = L_\beta^{-1}T_3L_\beta P^2_\beta$, $\beta \in B$.

To show that T_1 and T_3 are semi-similar we consider the nonzero members of $\{P^2_\alpha P^2_\beta \mid \alpha \in A, \beta \in B\}$ and for these define

$$P^1_{\alpha\beta} = L_\alpha^{-1} P^2_\alpha P^2_\beta L_\alpha P^1_\alpha,$$
$$P^3_{\alpha\beta} = L_\beta P^2_\alpha P^2_\beta L_\beta^{-1} P^3_\beta.$$

For notational simplicity we can assume that each $P^2_\alpha P^2_\beta \neq 0$.

As each P^2_β is a strong limit of operators in the range of E_2, it is clear that $P^1_{\alpha\beta} \in \overline{E}_1$. It is also clear that $\{P^1_{\alpha\beta} \mid \alpha \in A, \beta \in B\}$ is a disjoint family, and, applying the remarks about similarity and complete Boolean algebras in the proof of Theorem 25, for each fixed α we have $\bigvee_\beta P^1_{\alpha\beta} = L_\alpha^{-1} P^2_\alpha (\bigvee_\beta P^2_\beta) L_\alpha P^1_\alpha = L_\alpha^{-1} P^2_\alpha L_\alpha P^1_\alpha = P^1_\alpha$, so $\bigvee_{\alpha,\beta} P^1_{\alpha\beta} = I$. Straight-forward computation readily verifies that T_1 commutes with $P^1_{\alpha\beta}$. The same arguments apply throughout for $P^3_{\alpha\beta}$ and T_3.

Next, let $L_{\alpha\beta} = L_\beta L_\alpha$; it is easily seen that $L_{\alpha\beta}$ is a nonsingular map of $P^1_{\alpha\beta}\mathfrak{H}$ onto $P^3_{\alpha\beta}\mathfrak{H}$ with inverse $L_\alpha^{-1}L_\beta^{-1}$. Finally,

$$\begin{aligned}
L_{\alpha\beta}T_1L_{\alpha\beta}^{-1}P^3_{\alpha\beta} &= (L_\beta L_\alpha)T_1(L_\alpha^{-1}L_\beta^{-1})(L_\beta P^2_\alpha P^2_\beta L_\beta^{-1}P^3_\beta) \\
&= L_\beta(L_\alpha T_1 L_\alpha^{-1} P^2_\alpha) P^2_\beta L_\beta^{-1} P^3_\beta \\
&= L_\beta(T_2 P^2_\alpha) P^2_\beta L_\beta^{-1} P^3_\beta \\
&= L_\beta T_2 (P^2_\beta) P^2_\alpha L_\beta^{-1} P^3_\beta \\
&= L_\beta T_2 (L_\beta^{-1} P^3_\beta L_\beta P^2_\beta) P^2_\alpha L_\beta^{-1} P^3_\beta \\
&= T_3 P^3_\beta (L_\beta P^2_\beta P^2_\alpha L_\beta^{-1} P^3_\beta) \\
&= T_3 P^3_{\alpha\beta},
\end{aligned}$$

and $L_{\alpha\beta}$ is the desired partial similarity. Q.E.D.

That the Weyr characteristic is a complete set of semi-similarity invariants for essentially finite spectral operators will be proved via normal operators. For this we require the following theorem, which also permits immediate deduction of other properties of semi-similarity.

THEOREM 27. *If $T_1 = N_1 + Q_1$ and $T_2 = N_2 + Q_2$ are two spectral operators in canonical decomposition with normal scalar parts, and T_1 is semi-similar to T_2, then N_1 and N_2 are unitarily equivalent.*

Proof. As the resolutions of the identity are self-adjoint, the projections $\{P^i_\alpha \mid \alpha \in A\} \subseteq \overline{E}_i$ establishing a semi-similarity are self-adjoint as well, and we can, for $i=1, 2$, write $\mathfrak{H}$ as the orthogonal direct sum $\sum_\alpha \oplus \mathfrak{H}^i_\alpha$. By Theorem 24, for each $\alpha \in A$ there is a unitary operator U_α from $\mathfrak{H}^1_\alpha$ onto $\mathfrak{H}^2_\alpha$ for which $N_2P^2_\alpha = U_\alpha N_1 U^*_\alpha P^2_\alpha$. Clearly we can define U to be $\sum_\alpha \oplus U_\alpha$, and then $N_2 = UN_1U^*$.

COROLLARY 28. *Semi-similar spectral operators have similar scalar parts and the same spectrum.*

Proof. This follows from Theorems 26 and 27, and the observation that the spectrum is a similarity invariant.

We can now prove the semi-similarity analog of Theorem 25.

THEOREM 29. *If T_1 and T_2 are semi-similar spectral operators and T_1 is essentially finite, then so is T_2. If $\mathfrak{W}_i$ is the Weyr characteristic of T_i, $i=1,2$, then $\mathfrak{W}_1$ is identically equal to $\mathfrak{W}_2$.*

Proof. The first conclusion follows from the preceding corollary. To prove the second, we note first that by Theorem 26 it is sufficient to consider the case in which T_1 and T_2 have normal scalar parts. Suppose then that $T_1 = N_1 + Q_1$ and $T_2 = N_2 + Q_2$ have respective resolutions of the identity E_1 and E_2, and the semi-similarity is established by the self-adjoint projections P^i_α, $i=1, 2$, and the partial similarities L_α: $P^1_\alpha\mathfrak{H} \to P^2_\alpha\mathfrak{H}$, $\alpha \in A$.

Then N_1 and N_2 are unitarily equivalent, and have the same multiplicity function and measures of uniform multiplicity. Let μ be any such measure; we can confine our attention to the Borel subsets $\mathfrak{B}$ of the support of μ. Suppose $\{x_\beta \mid \beta \in B\}$ is a kth index system over $\delta \in \mathfrak{B}$ for the pair $\langle N_1, Q_1\rangle$. The proof will be complete if we can show there is a kth index system over δ for the pair $\langle N_2, Q_2\rangle$ with the same cardinality. We can assume, without loss of generality, that $x_\beta \in E_1(\delta)\mathfrak{H}$ for each $\beta \in B$.

Let β be fixed. As α varies, the nonzero $P^1_\alpha x_\beta$ are orthogonal vectors summing to x_β, and consequently at most countable. It follows that there are non-negative constants $a_{\alpha\beta}$, nonzero if and only if $P^1_\alpha x_\beta$ is nonzero, so that $\{a_{\alpha\beta}L_\alpha P^1_\alpha x_\beta \mid \alpha \in A\}$ is summable.

Next, let $a_\alpha = \min \{a_{\alpha\beta} \mid a_{\alpha\beta} \neq 0,\ \beta \in B\}$ for each α for which this class is nonvoid, and let a_α be zero otherwise. As the index class B is finite, it is clear that a_α is well-defined and is zero if and only if $P^1_\alpha x_\beta = 0$ for each $\beta \in B$. Furthermore, $\{a_\alpha \mid \alpha \in A,\ a_\alpha \neq 0\}$ is nonvoid and at most countable, and for each $\beta \in B$ the family $\{a_\alpha L_\alpha P^1_\alpha x_\beta \mid \alpha \in A\}$ is summable. Let

$$y_\beta = \sum_{\alpha \in A} a_\alpha L_\alpha P^1_\alpha x_\beta, \qquad \beta \in B.$$

Before it can be shown that $\{y_\beta \mid \beta \in B\}$ is the desired kth index system over δ for $\langle N_2, Q_2\rangle$ some preliminary remarks are necessary. First, by orthogonality it is clear that no y_β is zero, and that, for each α, $P^2_\alpha y_\beta = a_\alpha L_\alpha P^1_\alpha x_\beta$. This implies $a_\alpha = 0$ if and only if $P^2_\alpha y_\beta = 0$ for every $\beta \in B$.

Next, let $A_1 = \{\alpha \mid a_\alpha \neq 0\}$ and $A_2 = A - A_1$. Then A_1 is countable, and

$\alpha\in A_2$ if and only if either $P^1_\alpha x_\beta=0$, $\beta\in B$, or $P^2_\alpha y_\beta=0$, $\beta\in B$. As $\delta\subseteq s(\mu)$, for notational convenience we can consider the Boolean algebras E_1 and E_2 to be complete by assuming $C_1(\mu)=C_2(\mu)=I$, as in the proof of Theorem 25. Then for every $\alpha\in A_2$, $0=C_1(P^1_\alpha x_\beta)=P^1_\alpha C_1(x_\beta)=P^1_\alpha E_1(\delta)$, and thus

$$\sum_{\alpha\in A_1} P^1_\alpha E_1(\delta) = E_1(\delta).$$

Hence

$$\sum_{\alpha\in A_1} P^1_\alpha E_1(\pi) = E_1(\pi), \qquad \pi\subseteq\delta,\ \pi\in\mathfrak{B}.$$

Since $P^2_\alpha E_2(\cdot)=L_\alpha E_1(\cdot)P^1_\alpha L_\alpha^{-1}P^2_\alpha$ for all $\alpha\in A$, it is clear that $\alpha\in A_2$ implies $P^2_\alpha E_2(\delta)=0$, and therefore

$$\sum_{\alpha\in A_1} P^2_\alpha E_2(\pi) = E_2(\pi), \qquad \pi\subseteq\delta,\ \pi\in\mathfrak{B}.$$

We can now show $\{y_\beta|\beta\in B\}$ is a kth index system over δ for $\langle N_2, Q_2\rangle$. The proof, like that of Theorem 25, requires three steps.

(a)
$$\begin{aligned} Q_2^k y_\beta &= Q_2^k\Big(\sum_{\alpha\in A_1} P^2_\alpha y_\beta\Big)\\ &= \sum_{\alpha\in A_1} Q_2^k a_\alpha L_\alpha P^1_\alpha x_\beta\\ &= \sum_{\alpha\in A_1} a_\alpha L_\alpha P^1_\alpha Q_1^k x_\beta\\ &= 0, \qquad \beta\in B.\end{aligned}$$

(b) To show that $C_2(Q_2^{k-1}y_\beta)=E_2(\delta)$, $\beta\in B$, it suffices to show that any subset π of δ satisfying $E_2(\pi)Q_2^{k-1}y_\beta=0$ is E_2-negligible. For such a π, any fixed β, and each $\alpha\in A_1$, we have

$$\begin{aligned} 0 &= P^2_\alpha E_2(\pi)Q_2^{k-1}y_\beta\\ &= E_2(\pi)Q_2^{k-1}P^2_\alpha y_\beta\\ &= E_2(\pi)Q_\alpha^{k-1}a_\alpha L_\alpha P^1_\alpha x_\beta\\ &= a_\alpha L_\alpha E_1(\pi)Q_1^{k-1}P^1_\alpha x_\beta.\end{aligned}$$

As $a_\alpha\neq 0$ and L_α is nonsingular, it follows that $E_1(\pi)Q_1^{k-1}P^1_\alpha x_\beta=0$, and thus $E_1(\pi)Q_1^{k-1}x_\beta=0$. This implies $E_1(\pi)=0$, and the desired conclusion follows from the unitary equivalence of E_1 and E_2.

(c) We must show that

(i) $$\sum_{\beta\in B}\sum_{i=0}^{k-1}\int f_{i\beta}(\lambda)E_2(d\lambda)\overset{i}{Q_2}y_\beta = 0$$

implies $f_{i\beta}(\lambda)=0$ for $[E_2]$ a.a. $\lambda\in\delta$, $i=0, 1, \cdots, k-1$, $\beta\in B$. Equation (i) implies, for each $\alpha\in A_1$,

$$\sum_{\beta\in B}\sum_{i=0}^{k-1}\int f_{i\beta}(\lambda)E_2(d\lambda)\overset{i}{Q_2}\overset{2}{P_\alpha}y_\beta = 0;$$

hence

$$\sum_{\beta\in B}\sum_{i=0}^{k-1}\int f_{i\beta}(\lambda)E_2(d\lambda)\overset{i}{Q_2}a_\alpha L_\alpha\overset{1}{P_\alpha}x_\beta = 0;$$

hence

$$\sum_{\beta\in B}\sum_{i=0}^{k-1}\int f_{i\beta}(\lambda)a_\alpha L_\alpha\overset{1}{P_\alpha}E_1(d\lambda)\overset{i}{Q_1}x_\beta = 0;$$

hence

(ii) $$\overset{1}{P_\alpha}\left[\sum_{\beta\in B}\sum_{i=0}^{k-1}\int f_{i\beta}(\lambda)E_1(d\lambda)\overset{i}{Q_1}x_\beta\right] = 0.$$

Thus the bracketed expression in (ii) is zero, and, for $i=0, 1, \cdots, k-1$ and $\beta\in B$, it follows that $f_{i\beta}(\lambda)=0$ for $[E_1]$ a.a. $\lambda\in\delta$. The unitary equivalence of E_1 and E_2 now implies the conclusion for $[E_2]$ a.a. $\lambda\in\delta$, and the proof of (c) is complete. Q.E.D.

The last theorem of this section, asserting the Weyr characteristic is a complete set of semi-similarity invariants, will depend strongly on specific properties of Hilbert space geometry. To clarify the ideas, suppose $\mathfrak{H}$ is a Hilbert space of uniform multiplicity 2 with respect to the complete self-adjoint spectral measure E, with support Λ, and suppose $\mathfrak{H}=\mathfrak{M}(x_1)\vee\mathfrak{M}(x_2)$. (We envisage the situation in which $\{x_1, x_2\}$ is a complete index system for a nilpotent Q, with $Qx_1=x_2$, and seek to establish a canonical form within semi-similarity.) Then there are also orthogonal vectors z_1 and z_2 so that $\mathfrak{H}$ is the orthogonal direct sum $\mathfrak{M}(z_1)\oplus\mathfrak{M}(z_2)$, with $|E(\cdot)z_1|^2=|E(\cdot)z_2|^2=\mu$. It can be supposed without loss of generality in this case that $x_2=z_2$. Then there are two ways in which these two representations of $\mathfrak{H}$ differ.

First, the mass distribution of x_1 may differ from that of z_1. Though the subspace $\mathfrak{M}(x_1)$ is unitarily equivalent to $\mathfrak{M}(z_1)$ by the mapping $S(f)x_1\to S(f)S(g^{1/2})z_1$, where g is the Radon-Nikodym derivative $d|E(\cdot)x_1|^2/d\mu$ (since the measures are equivalent), our objective is really the "identity mapping" $S(f)x_1\to S(f)z_1$. But in general both vectors will not be in the same domains, and this mapping will be unbounded. However, Λ can easily be written as a disjoint countable union of non-negligible sets, say $\Lambda\equiv\bigcup\delta_i$, on each of which $|g(\cdot)|$ is bounded above and away from zero. Then the mappings $S(f)E(\delta_i)x_1 \to S(f)E(\delta_i)z_1$ are nonsingular from $\mathfrak{M}(E(\delta_i)x_1)$ onto $\mathfrak{M}(E(\delta_i)z_1)$. If $\mathfrak{M}(x_1)$ and $\mathfrak{M}(x_2)$ were orthogonal, these could be extended to nonsingular mappings of $E(\delta_i)\mathfrak{H}$ onto itself of the desired type.

The second difference of course is the lack of this orthogonality in general. The subspaces $\mathfrak{M}(x_1)$ and $\mathfrak{M}(x_2)$ may make an angle of zero degrees. But here another decomposition of Λ can be constructed, on each set of which the angle is bounded away from zero relative to E. That is, there is a countable family $\{\pi_i\}$ of disjoint non-negligible sets, and associated positive constants a_i, such that $\Lambda \equiv \bigcup \pi_i$ and if π is any non-negligible subset of π_i, then

$$\frac{|(E(\pi)x_i, E(\pi)x_2)|}{|E(\pi)x_1| \; |E(\pi)x_2|} \leqq \frac{1}{(1 + a_i)^{1/2}}.$$

For, recalling that $x_2 = z_2$, by orthogonality we can write

$$x_1 = \int f_1(\lambda)E(d\lambda)z_1 + \int f_2(\lambda)E(d\lambda)x_2.$$

The independence of $\{x_1, x_2\}$ implies $f_1 \neq 0$ a.e., and since $E(\sigma)x_1 \perp E(\sigma)x_2$ if $\sigma \subseteq \{\lambda | f_2(\lambda) = 0\}$, we can assume $f_2 \neq 0$ a.e. Then let $\{\delta_j\}$ be a countable disjoint partition of Λ such that $1/j \leqq |f_1(\lambda)| \leqq j$ for $\lambda \in \delta_j$, $j = 1, 2, \cdots$, let $\{\sigma_k\}$ be another such partition such that $|f_2(\lambda)| \leqq k$ for $\lambda \in \sigma_k$, $k = 1, 2, \cdots$, and π_i be the non-negligible subsets of a common refinement. If $\pi_i = \delta_j \cap \sigma_k$, let $a_i = 1/j^2k^2$.

Now suppose $\emptyset \neq \pi \subseteq \pi_i$. Then

$$|E(\pi)x_1|^2 = \int_\pi (|f_1(\lambda)|^2 + |f_2(\lambda)|^2)\mu(d\lambda)$$

and $|E(\pi)x_2|^2 = \mu(\pi)$, so

$$\frac{|(E(\pi)x_1, E(\pi)x_2)|}{|E(\pi)x_1| \; |E(\pi)x_2|} = \frac{\left|\int_\pi f_2(\lambda)\mu(d\lambda)\right|}{\left(\mu(\pi)\int_\pi (|f_1(\lambda)|^2 + |f_2(\lambda)|^2)\mu(d\lambda)\right)^{1/2}}$$

$$\leqq \frac{\int_\pi |f_2(\lambda)|\mu(d\lambda)}{\left(\mu(\pi)\int_\pi |f_1(\lambda)|^2\mu(d\lambda) + \mu(\pi)\int_\pi |f_2(\lambda)|^2\mu(d\lambda)\right)^{1/2}}.$$

By the Schwarz inequality,

$$\left(\int_\pi |f_2(\lambda)|\,\mu(d\lambda)\right)^2 \leqq \mu(\pi)\int_\pi |f_2(\lambda)|^2\mu(d\lambda);$$

hence

$$\frac{|(E(\pi)x_1, E(\pi)x_2)|}{|E(\pi)x_1|\,|E(\pi)x_2|} \leqq \frac{1}{\left\{1+\dfrac{\mu(\pi)\displaystyle\int_\pi |f_1(\lambda)|^2\mu(d\lambda)}{\left(\displaystyle\int_\pi |f_2(\lambda)|\,\mu(d\lambda)\right)^2}\right\}^{1/2}}$$

$$\leqq \frac{1}{(1+1/j^2k^2)^{1/2}},$$

the desired inequality.

It is interesting to note that in this situation $E(\pi)x_1$ and $E(\pi)z_1$ can nonetheless be orthogonal, but if z_1 is chosen "on the same side of x_2 as x_1"—that is, so that f_1 is real and positive—this cannot occur. For this one observes that $\tilde{z}_1$, defined by

$$\tilde{z}_1 = \int \frac{f_1(\lambda)}{|f_1(\lambda)|}\,E(d\lambda)z_1,$$

has all of the relevant properties of z_1. Then it is easy to compute that

$$\frac{|(E(\pi)x_1, E(\pi)\tilde{z}_1)|}{|E(\pi)x_1|\,|E(\pi)\tilde{z}_1|} \geqq \frac{1}{j(j^2+k^2)^{1/2}}.$$

The proof of the theorem will follow more formal lines.

THEOREM 30. *If T_1 and T_2 are essentially finite spectral operators on $\mathfrak{H}$ with the same Weyr characteristic, then T_1 and T_2 are semi-similar.*

Proof. Again it is sufficient to consider the case in which $T_1 = N_1 + Q_1$ and $T_2 = N_2 + Q_2$ have normal scalar parts. By Theorem 23, N_1 and N_2 have the same measures of uniform multiplicity, with corresponding equal multiplicities. Thus N_1 and N_2 are unitarily equivalent, and it is no loss of generality to assume them equal, say $N = N_1 = N_2$, with self-adjoint resolution of the identity E. Let $\{\mu_\alpha\}$ be an orthogonal family of nonzero measures of uniform *characteristic* with $\bigvee_\alpha C(\mu_\alpha) = I$. We intend to construct the desired semi-similarity in each subspace $C(\mu_\alpha)\mathfrak{H}$, and it is clearly sufficient to restrict attention to one such, say $C(\mu)\mathfrak{H}$, and prove the theorem under the assumption that $C(\mu) = I$.

Supposing this, then, let Λ be the support of μ (or E). As the characteristics are the same, we can choose two complete index systems, $\{x_i \mid i = 1, \cdots, n\}$ for the complete spectral measure E and nilpotent Q_1, and $\{y_i | i=1, \cdots, n\}$ for E and Q_2. (Single indices will be more convenient than the double indices used heretofore.) The indexing can be taken so that the two families are moved isomorphically by Q_1 and Q_2—that is, so that $Q_1x_i=x_j$ $(Q_1x_i=0)$ if and only if $Q_2y_i=y_j$ $(Q_2y_i=0)$. Then $\mathfrak{H}=\bigvee_i \mathfrak{M}(x_i) = \bigvee_i \mathfrak{M}(y_i)$ and all the measures determined by the vectors introduced are equivalent.

We next wish to assume the manifolds $\mathfrak{M}(y_i)$ are mutually orthogonal. This is merely a notational convenience, to avoid introducing a canonical system by writing $\mathfrak{H}=\sum_1^n \oplus \mathfrak{M}(z_i)$ with $|E(\cdot)z_i|^2=\mu$ for each i, defining Q_3 in the obvious way (i.e., so that $Q_3z_i=z_j$ $(Q_3z_i=0)$ exactly when $Q_2y_i=y_j$ $(Q_2y_i=0)$), and proving both $N+Q_1$ and $N+Q_2$ semi-similar to $N+Q_3$. Thus it can be assumed that $\mathfrak{H}=\sum \oplus \mathfrak{M}(y_i)$, and then we can write

$$x_i = \sum_{j=1}^{n} S(f_{ij})y_j, \qquad i = 1, \cdots, n.$$

Then let

$$\begin{aligned} \overset{0}{\delta}_{ij} &= \{\lambda \mid f_{ij}(\lambda) = 0\}, \\ \overset{2m}{\delta}_{ij} &= \{\lambda \mid m < |f_{ij}(\lambda)| \leqq m+1\}, \qquad m = 1, 2, \cdots, \\ \overset{2m-1}{\delta}_{ij} &= \left\{\lambda \,\middle|\, \frac{1}{2^m} < |f_{ij}(\lambda)| \leqq \frac{1}{2^{m-1}}\right\}, \qquad m = 1, 2, \cdots, \end{aligned}$$

and note that for each i and j we have $\Lambda=\bigcup_{m=0}^{\infty} \overset{m}{\delta}_{ij}$. Consider all non-negligible intersections of the form

$$\overset{m_1}{\delta}_{i_1j_1} \cap \overset{m_2}{\delta}_{i_2j_2} \cap \cdots \cap \overset{m_n}{\delta}_{i_nj_n},$$

where $i_1, \cdots, i_n$ and $j_1, \cdots, j_n$ are permutations of $1, \cdots, n$, and $m_1, \cdots, m_n$ are arbitrary non-negative integers. This is a countable disjoint family, say $\{\sigma_k | k=1, 2, \cdots\}$, with union equivalent to Λ. If σ_k is the intersection above, and f_{ij} does not vanish identically on σ_k, then $\lambda \in \sigma_k$ implies

$$\frac{1}{2^{(m+1)/2}} < |f_{ij}(\lambda)| \leqq \frac{m}{2} + 1,$$

where $m=\max\{m_1, \cdots, m_n\}$.

The remainder of the proof follows closely an elementary finite dimensional argument. As $\mathfrak{H}=\bigvee_i \mathfrak{M}(x_i)$, we can write

$$y_k = \lim_{p\to\infty} \sum_{i=1}^{n} \int_{\pi_p} g_{ki}(\lambda)E(d\lambda)x_i, \qquad k = 1, \cdots, n,$$

where $\pi_p=\{\lambda \mid |g_{ki}(\lambda)| \leqq p;\ i, k=1, \cdots, n\}$, $p=1, 2, \cdots$. Then

$$\begin{aligned} y_k &= \lim_{p\to\infty} \sum_{i=1}^{n} \int_{\pi_p} g_{ki}(\lambda) \left(\sum_{j=1}^{n} \int_{d\lambda} f_{ij}(\zeta) E(d\zeta) y_j \right) \\ &= \lim_{p\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\pi_p} g_{ki}(\lambda) f_{ij}(\lambda) E(d\lambda) y_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \int g_{ki}(\lambda) f_{ij}(\lambda) E(d\lambda) y_j, \end{aligned}$$

passing to the limit by orthogonality.

It follows by independence, that the operator defined by the matrix $\|f_{ij}\|$ with respect to the "basis" $y_1, \cdots, y_n$ is formally invertible, and the preceding equation and this statement remain valid in each subspace $E(\sigma_k)\mathfrak{H}$. Then the formal matrix inverse of $\|f_{ij}\chi_{\sigma_k}\|$, computed pointwise for each λ, exists for every k, and in fact differs only on a set of measure zero from $\|g_{ij}\chi_{\sigma_k}\|$. By construction, both of these can be taken to be matrices of bounded functions, and thus represent bounded, everywhere-defined operators. Consequently for any vector $z \in E(\sigma_k)\mathfrak{H}$ with $z = \sum_i S(h_i)E(\sigma_k)y_i$, we define

$$\begin{aligned} L_k z &= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\sigma_k} h_i(\lambda) f_{ij}(\lambda) E(d\lambda) y_j \\ &= \sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) \left(\sum_{j=1}^{n} \int_{d\lambda} f_{ij}(\zeta) E(d\zeta) y_j \right) \\ &= \sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) x_i, \end{aligned}$$

and L_k is a nonsingular operator of $E(\sigma_k)\mathfrak{H}$ onto itself.

Finally

$$\begin{aligned} T_1 L_k z &= T_1 L_k \left(\sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) y_i \right) \\ &= (N + Q_1) \sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) x_i \\ &= \sum_{i=1}^{n} \int_{\sigma_k} \lambda h_i(\lambda) E(d\lambda) x_i + \sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) Q_1 x_i \\ &= L_k \left(\sum_{i=1}^{n} \int_{\sigma_k} \lambda h_i(\lambda) E(d\lambda) y_i + \sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) Q_2 y_i \right) \\ &= L_k (N + Q_2) \left(\sum_{i=1}^{n} \int_{\sigma_k} h_i(\lambda) E(d\lambda) y_i \right) \\ &= L_k T_2 z. \text{ Q.E.D.} \end{aligned}$$

The question remains as to what additional conditions need be imposed on semi-similar spectral operators to produce actual similarity. It is clear from Corollary 28 that the failure of similarity can be traced, roughly speaking, to the quasi-nilpotent parts, and it is natural to conjecture that some form of relative piecewise boundedness condition of these parts is sufficient. That is, a condition of the form

$$0 < \frac{1}{K} \leqq \frac{|Q_1 P_\alpha^1|}{|Q_2 P_\alpha^2|} \leqq K < \infty, \qquad \alpha \in A,$$

for a positive constant K and families of projections $\{P_\alpha^1\}$ and $\{P_\alpha^2\}$, each with supremum the identity, related to the respective operators and to each other in a suitable way. But, as we see below, even with a slight strengthening of this condition, the projections of a semi-similarity are not a sufficiently fine decomposition of $\mathfrak{H}$ to permit the desired conclusion, and the question remains open.

The strengthened condition we have in mind, violated flagrantly in the example of nonsimilar but semi-similar operators with which this section began, is

$$0 < \frac{1}{K} \leqq \frac{|Q_1^k P_\alpha^1|}{|Q_2^k P_\alpha^2|} \leqq K < \infty, \qquad \alpha \in A,$$

where the $\{P_\alpha^i | \alpha \in A\}$ are the projections of a semi-similarity, and the inequalities are to hold for all positive integral k for which the denominator (or equivalently, the numerator) does not vanish. It is easy to see that this is a necessary condition for similarity; the following simple example, under the same restrictive conditions as the previous one, shows it to be insufficient.

Let $\mathfrak{H}$ be the direct sum of a countable family of 4-dimensional Hilbert spaces, and define, on the nth subspace and with respect to a complete orthonormal system, the operators Q_1 and Q_2 by the matrix blocks

$$Q_1: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_2: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and N by diag $\{1/n, 1/n, 1/n, 1/n\}$. As before, by adding a zero subspace, we can insure that N has pure point spectrum.

As $\mathfrak{H}$ is separable, the projections of a semi-similarity must be in the range of the resolution of the identity of N, and it is clear the conjectured boundedness condition is satisfied. But also as before, if $N + Q_2 = L(N + Q_1)L^{-1}$, then L must be completely reduced by each distinguished 4-dimensional subspace, and on each such must have the form

$$L: \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ e & f & g & h \\ 0 & en & 0 & gn \end{pmatrix},$$

where $a, b, \cdots, h$ are functions of n. If L is to be bounded and invertible, then $e, g \to 0$ as $n \to \infty$, and $ag \neq ce$ for all n. Then on the nth subspace L^{-1} has the form

$$L^{-1}: \begin{pmatrix} * & * & \dfrac{-c}{ag - ce} & * \\ 0 & * & 0 & * \\ * & * & \dfrac{a}{ag - ce} & * \\ 0 & * & 0 & * \end{pmatrix}.$$

If either a or c is zero for an infinity of n, this is unbounded since both e and g approach zero; in the contrary case, if L^{-1} is to be bounded we must have both $|a/c| \to \infty$ and $|c/a| \to \infty$ as $n \to \infty$.

7. **The adjoint Weyr characteristic**(4). In this section we consider the application of the preceding theory to the conjugate space and adjoint operators. We recall from §2 that the multiplicity theory for a complete Boolean algebra of projections on a Banach space extends, with the substitution of the weak-* or $\mathfrak{X}$-topology for the strong topology, to the $\mathfrak{X}$-complete Boolean algebra of adjoints, E^*, on the conjugate space $\mathfrak{X}^*$, and the multiplicity and uniformity of projections or spaces of finite multiplicity is preserved. Moreover, the adjoint of a quasi-nilpotent operator is also quasi-nilpotent, and adjunction preserves commutativity. Thus we are led to consider how the apparatus of §§3 and 4 can be transferred to the adjoint situation. But this is easily accomplished: the definitions and theorems of these sections are both meaningful and true after the adjunction of all vectors, operators, and spaces involved. Only very minor modifications are required in a minority of the proofs, and these are left to the reader.

Thus we have the notions of *independence*, *kth index systems*, and *complete index systems* in $\mathfrak{X}^*$ with respect to E^* and Q^*, and a Weyr characteristic defined here. Sets or measures of uniform characteristic are characterized by the summability condition of Theorem 21. The natural question is how this characteristic relates to the Weyr characteristic in $\mathfrak{X}$ and the answer is the simplest possible.

(4) The author is grateful to Professor W. G. Bade for suggesting this subject for investigation.

THEOREM 31. *Let $\mathfrak{X}$ be a Banach space of uniform multiplicity $n<\infty$ with respect to the complete spectral measure E, let Q be a quasi-nilpotent commuting with E, and $\mathcal{W}$ be the Weyr characteristic these define. Then in the conjugate space $\mathfrak{X}^*$, a space of uniform multiplicity $n<\infty$ with respect to the ($\mathfrak{X}$-complete) spectral measure E^*, the Weyr characteristic defined by E^* and Q^* is also $\mathcal{W}$.*

Proof. Let δ be a set of uniform characteristic in the $\mathfrak{X}$ context; it is sufficient to show the Weyr characteristic in $\mathfrak{X}^*$ has δ as a set of uniform characteristic and agrees with $\mathcal{W}$ on δ. Thus we can assume $E(\delta)=I$. Let $\{x_\alpha^k \mid \alpha\in A_k,\ k=1,\cdots,m\}$ be a complete index system over δ in $\mathfrak{X}$, with $A_k=1,\cdots,m_k$; then $m_j\leqq m_i$ if $j\geqq i$. For each x_α^k choose a Bade functional x_α^{*k} such that

$$x_\alpha^{*k}\left(\bigvee_{\beta\neq\alpha;\,j\neq k}\mathfrak{M}(x_\beta^j)\right)=0.$$

If π is a Borel subset of δ, then $x_\alpha^{*k}E(\pi)x_\alpha^k$ is non-negative, and is zero if and only if $E(\pi)=0$, or equivalently, $E^*(\pi)=0$. Our interest is in the family $\{x_\alpha^{*k}\mid \alpha\in A_k,\ k=1,\cdots,m\}$, but the order must be inverted in accordance with the following scheme.

Let

$$z_\alpha^{*m}=\left\{x_\alpha^{*1},\quad \alpha=1,\cdots,m_m,\right.$$

$$z_\alpha^{*m-1}=\begin{cases} x_\alpha^{*2}, & \alpha=1,\cdots,m_m,\\ x_\alpha^{*1}, & \alpha=m_m+1,\cdots,m_{m-1},\end{cases}$$

$$z_\alpha^{*m-2}=\begin{cases} x_\alpha^{*3}, & \alpha=1,\cdots,m_m,\\ x_\alpha^{*2}, & \alpha=m_m+1,\cdots,m_{m-1},\\ x_\alpha^{*1}, & \alpha=m_{m-1}+1,\cdots,m_{m-2},\end{cases}$$

and in general, for each $j=1,\cdots,m$, let

$$z_\alpha^{*m-j+1}=\begin{cases} x_\alpha^{*j}, & \alpha=1,\cdots,m_m,\\ x_\alpha^{*j-1}, & \alpha=m_m+1,\cdots,m_{m-1},\\ x_\alpha^{*j-2}, & \alpha=m_{m-1}+1,\cdots,m_{m-2},\\ \vdots & \vdots\\ x_\alpha^{*j-k}, & \alpha=m_{m-k+1}+1,\cdots,m_{m-k},\quad (k=0,1,\cdots,j-1),\\ \vdots & \vdots\\ x_\alpha^{*1}, & \alpha=m_{m-j+2}+1,\cdots,m_{m-j+1},\end{cases}$$

where $0=m_{m+1}$. Thus the cardinality of each $\{z_\alpha^{*k} \mid \alpha\in A_k\}$ is m_k.

Our assertion now is not that $\{z_\alpha^{*k} \mid \alpha\in A_k,\ k=1,\cdots,m\}$ is a *complete* index system over δ in $\mathfrak{X}^*$, but merely that each $\{z_\alpha^{*k} \mid \alpha\in A_k\}$ is a kth index system over δ in $\mathfrak{X}^*$. But this more modest assertion is sufficient to prove the theorem, for considering the total cardinality of all the families $\{z_\alpha^{*k} \mid \alpha\in A_k\}$ and Corollary 20, it is clear that it will imply each such family is in fact a *maximal* kth index system. Thus by Theorem 21 it will follow that δ is a set of uniform characteristic in $\mathfrak{X}^*$, and that the two characteristics are equal.

Fix j then, $j=1,\cdots,m$; we show $\{z_\alpha^{*m-j+1} \mid \alpha\in A_{m-j+1}\}$ is an $(m-j+1)$st index system over δ with respect to E^* and Q^*.

(a) First it must be verified that $Q^{*m-j+1}z_\alpha^{*m-j+1}=0^*$ for each $\alpha\in A_{m-j+1}$. Select and fix any such α, say $m_{m-k+1}<\alpha\leqq m_{m-k}$, for some fixed k, $k=0,1,\cdots,j-1$. Then $z_\alpha^{*m-j+1}=x_\alpha^{*j-k}$. As $Qx_\beta^i=x_\beta^{i-1}$ for each β and i (interpreting non-positive superscripts as producing zero), it follows that

$$Q^{*m-j+1}z_\alpha^{*m-j+1}x_\beta^{i}=x_\alpha^{*j-k}Q^{m-j+1}x_\beta^{i}=x_\alpha^{*j-k}x_\beta^{i-(m-j+1)}.$$

This is zero, save for the possible exceptional case when $\beta=\alpha$ and $i-(m-j+1)=j-k$—except, that is, when $\beta=\alpha$ and $i=m-k+1$. But as $\alpha>m_{m-k+1}$, no vector x_α^{m-k+1} exists for this α. Thus $Q^{*m-j+1}z_\alpha^{*m-j+1}$ annihilates each x_β^i, and it follows that

$$Q^{*m-j+1}z_\alpha^{*m-j+1}\Big(\bigvee_{\beta,i}\mathfrak{M}(x_\beta^i)\Big)=0.$$

(b) Next it must be verified that $C^*(Q^{*m-j}z_\alpha^{*m-j+1})=E^*(\delta)=I^*$ for $\alpha\in A_{m-j+1}$. Let α and k be fixed as above, with $m_{m-k+1}<\alpha\leqq m_{m-k}$. Suppose the assertion false; then there is a non-negligible Borel set $\pi\subset\delta$ such that $E^*(\pi)Q^{*m-j}z_\alpha^{*m-j+1}=0^*$. Since $\alpha\leqq m_{m-k}$, there exists a vector x_α^{m-k}; but then

$$\begin{aligned}0&=E^*(\pi)Q^{*m-j}z_\alpha^{*m-j+1}x_\alpha^{m-k}\\&=x^{*j-k}E(\pi)Q^{m-j}x_\alpha^{m-k}\\&=x_\alpha^{*j-k}E(\pi)x_\alpha^{j-k}.\end{aligned}$$

Thus $E(\pi)=0$, and then $E^*(\pi)=0$.

(c) Finally, we must show $\{Q^{*i}z_\alpha^{*m-j+1} \mid \alpha\in A_{m-j+1},\ i=0,1,\cdots,m-j\}$ is E^*-independent. As all the vectors are full, we suppose

$$0^*=\sum_{\alpha\in A_{m-j+1}}\sum_{i=0}^{m-j}S^*(f_{\alpha i})Q^{*i}z_\alpha^{*m-j+1}$$

and prove the coefficients $S^*(f_{\alpha i})$ are zero. It can be assumed that the functions themselves are bounded (by successively restricting attention, if necessary, to subspaces of the form $E^*(\pi_p)\mathfrak{X}^*$, $(\pi_p=\{\lambda \mid |f_{\alpha i}(\lambda)|\leqq p,\ \alpha\in A_{m-j+1},\ i=0,1,\cdots,m-j\},\ p=1,2,\cdots)$. Then

$$0^* = \sum_{\alpha=1}^{m_m} \sum_{i=0}^{m-j} S^*(f_{\alpha i}) Q^{*i} x_\alpha^{*j} + \sum_{\alpha=m_m+1}^{m_m-1} \sum_{i=0}^{m-j} S^*(f_{\alpha i}) Q^{*i} x_\alpha^{*j-1}$$
$$+ \sum_{\alpha=m_{m-k+1}+1}^{m_m-k} \sum_{i=0}^{m-j} S^*(f_{\alpha i}) Q^{*i} x_\alpha^{*j-k} + \cdots + \sum_{\alpha=m_{m-j+2}+1}^{m_{m-j+1}} \sum_{i=0}^{m-j} S^*(f_{\alpha i}) Q^{*i} x_\alpha^{*1}.$$

Now let β be arbitrary but fixed, $m_{m-k+1} < \beta \leqq m_{m-k}$, for fixed k, $k=0, 1, \cdots, j-1$, and apply the representation of the zero functional displayed above to x_β^q, where q is to be determined. As $Q^i x_\beta^q = x_\beta^{q-i}$, it follows that

$$0 = x_\beta^{*j-k} S(f_{\beta 0}) x_\beta^q + x_\beta^{*j-k} S(f_{\beta 1}) x_\beta^{q-1} + \cdots + x_\beta^{*j-k} S(f_{\beta, m-j}) x_\beta^{q-(m-j)}.$$

We recall that j is a fixed index, $j=1, \cdots, m$; then $m \geqq j > k$, or $m-k \geqq j-k \geqq 1$. As $\beta \leqq m_{m-k}$, it follows that $\beta \in A_p$ for each $p \leqq m-k$. Thus with this value of β, there are vectors $x_\beta^{m-k}, x_\beta^{m-k-1}, \cdots, x_\beta^{j-k}, \cdots, x_\beta^1$, and q may be chosen successively as $j-k, j-k+1, \cdots, m-k$. It follows that $S(f_{\beta i})=0$ for $i=0, 1, \cdots, m-j$, and the family $\{Q^{*i} z_\alpha^{*m-j+1} \mid \alpha \in A_{m-j+1}, i=0, 1, \cdots, m-j\}$ is indeed independent. Q.E.D.

Only a few remarks are now necessary to treat the Hilbert space case. If δ is a Borel set, let δ^* denote the conjugate set, $\delta^* = \{\lambda \mid \bar\lambda \in \delta\}$, and for a Borel measure μ, write μ^* for the measure on the conjugate domain, $\mu^*(\delta) = \mu(\delta^*)$, $\delta \in \mathfrak{B}$. If N is a normal operator with self-adjoint resolution of the identity E and multiplicity function $\mathbf{u}$, then the adjoint operator N^* has a self-adjoint resolution of the identity that (as there is no possibility of confusion with the adjoint) can be written E^*, with $E^*(\delta) = E(\delta^*)$, $\delta \in \mathfrak{B}$, and multiplicity function $\mathbf{u}^*$ given by $\mathbf{u}^*(\mu) = \mathbf{u}(\mu^*)$. Thus the ranges of E and E^* are the same; hence they have the same strong closure or completion $\overline{E}$. Similarly, the ranges of $\mathbf{u}$ and $\mathbf{u}^*$ are the same; hence N is essentially finite if and only if N^* is.

If $T=S+Q$ is a spectral operator in canonical decomposition, then the adjoint $T^* = S^* + Q^*$ is also. If $N = LSL^{-1}$ is a normal conjugate of S, then L^* is nonsingular and $N^* = (L^*)^{-1} S^* L^*$ is a normal conjugate of S^*. Thus T is essentially finite if and only if T^* is. The relation, in this case, between the Weyr characteristic defined by T and that defined in the intrinsic manner as an operator on $\mathfrak{H}$ by T^* is the one suggested by our notation.

THEOREM 32. *Let T be an essentially finite spectral operator on the Hilbert space $\mathfrak{H}$ with Weyr characteristic $\mathfrak{W}$. Then T^* is essentially finite, and if $\mathfrak{W}^*$ is the Weyr characteristic of T^*, then $\mathfrak{W}^*(\mu, k) = \mathfrak{W}(\mu^*, k)$ for every μ and k.*

Proof. The first conclusion follows from the preceding remarks, which also, with Theorem 25, imply that to consider the relation between $\mathfrak{W}$ and $\mathfrak{W}^*$ it is sufficient to suppose $T = N+Q$ to have a normal scalar part. For each measure μ, let $C(\mu)$ and $C^*(\mu)$ denote the carrier projections in $\overline{E}$ associated with the multiplicity functions $\mathbf{u}$ and $\mathbf{u}^*$ respectively. It is easy to see that for any $x \in \mathfrak{H}$, we have $|E(\cdot)x|^2 \ll \mu$ if and only if $|E^*(\cdot)x|^2 \ll \mu^*$. Thus

$$\begin{aligned} C(\mu) &= \bigwedge_{P \in \overline{E}} \{P \mid Px = x \text{ whenever } |E(\cdot)x|^2 \ll \mu\} \\ &= \bigwedge_{P \in \overline{E}} \{P \mid Px = x \text{ whenever } |E^*(\cdot)x|^2 \ll \mu^*\} \\ &= C^*(\mu^*). \end{aligned}$$

Hence it is sufficient to restrict attention to a space $C(\mu)\mathfrak{H} = C^*(\mu^*)\mathfrak{H}$ of uniform, finite, and necessarily equal multiplicities (with respect to the two multiplicity functions). In fact, μ can be chosen of $\mathcal{W}$-uniform characteristic. Then, with the obvious modification of Banach space adjoint to Hilbert space adjoint, the preceding theorem gives the desired conclusion.

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