A GENERAL CLASS OF LINEAR TRANSFORMATIONS OF WIENER INTEGRALS\(^{(1,2)}\)

BY

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Introduction. Let \( C \) be the space of continuous functions \( x(t), 0 \leq t \leq 1 \), where \( x(0) = 0 \). We consider one to one linear transformations of \( C \) onto \( C \) of the form

\[
y(t) = x(t) + \int_0^1 L(t, s)dx(s).
\]

N. Wiener \([4]\) has given an integral on \( C \). We obtain the transformation formula for this integral under transformations of type (0.1). The transformations of type (0.1) which we consider contain as a special case the transformations considered by Cameron and Martin \([1]\).

Let \( I \) be the interval \([0, 1]\) and \( I^2 \) the square \( I \otimes I \). If \( x \in C \), we define

\[
|||x||| = \max_{t \in I} |x(t)|.
\]

If \( K(t, s) \) is bounded and measurable on \( I^2 \) and \( K(t, t) \) is measurable on \( I \), we denote the Fredholm determinant of \( K(t, s) \) evaluated at \( \lambda = -1 \) by

\[
D(K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \cdots \int_0^1 \left| \begin{array}{ccccc} K(s_1, s_1) & \cdots & K(s_1, s_n) \\ & \ddots & \cdots & \ddots \\ & & K(s_n, s_1) & \cdots & K(s_n, s_n) \end{array} \right| ds_1 \cdots ds_n.
\]

We say that \( M(t, s) \) is of bounded variation (B.V.) on \( I^2 \) if there exists \((t_0, s_0) \in I^2\) such that \( M(t_0, s) \) and \( M(t, s_0) \) are of B.V. on \( I \) and

\[
\var_{(t, s) \in I^2} M(t, s) < \infty.
\]

Here

\[
\var_{(t, s) \in I^2} M(t, s) = \sup \sum_{j=1}^{\infty} \sum_{i=1}^{n} \left| M(t_i, s_j) - M(t_i, s_{j-1}) + M(t_{i-1}, s_{j-1}) - M(t_{i-1}, s_j) \right|,
\]

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where the supremum is taken over all partitions $0 \leq t_0 \leq \cdots \leq t_n \leq 1$ and $0 \leq s_0 \leq \cdots \leq s_m \leq 1$ of $I$. The following theorem is the main result of this paper.

**Theorem A.** Let $L(t, s)$ satisfy the following conditions

(0.4) $L(0, s) = 0$, $s \in I$.

(0.5) $L(t, s)$ is absolutely continuous in $t$ on $I$ for each $s \in I$ and there exist $M(t, s)$ and $J(t)$ such that

$$\frac{\partial}{\partial t} L(t, s) = M(t, s),$$

for almost all $t$ on $I$ for all $s \in I$, where

$$M(t, s) = \begin{cases} M(t, s), & 0 \leq t < s \leq 1, \\ M(t, s) + \frac{1}{2} J(t), & 0 \leq t = s \leq 1, \\ M(t, s) + J(t), & 0 \leq s < t \leq 1, \end{cases}$$

and where

(0.8) $M(t, s)$ is of B.V. on $I^2$ and $J(t)$ is of B.V. on $I$. Also

$$D(M) \neq 0.$$

Then the transformation (0.1) carries $C$ onto $C$ in a one to one manner and if $F(y)$ is a measurable functional such that either side of the following equation exists, both sides exist and are equal.

$$\int_0^\infty F(y)d\mu_y = \left| D(M) \right| \int_0^\infty F(x(\cdot) + \int_0^1 L(-,s)dx(s)) \exp \{-\Psi(x)\} d\mu_x$$

where

$$\Psi(x) = \int_0^1 \left[ \int_0^1 M(t, s) dx(s) \right]^2 dt + 2 \int_0^1 \int_0^1 M(t, s) dx(s) dx(t)$$

$$+ \int_0^1 J(t)d(x(t))^2.$$
independent of $s$. Assume also that

(0.13) $J(t)$ is continuous and of B.V. on $I$,
(0.14) $H(t, s)$ is continuous in $s$ on $I$ for every $t \in I$, and
(0.15) $D(K) \neq 0$ where

\[
K(t, s) = \begin{cases} 
\int_0^t H(u, s)du, & 0 \leq t < s \leq 1, \\
\int_0^t H(u, t)du + \frac{1}{2} J(t), & 0 \leq t = s \leq 1, \\
\int_0^t H(u, s)du + J(s), & 0 \leq s < t \leq 1.
\end{cases}
\]

Then if $F(y)$ is a measurable functional such that either side of the following equation exists, both sides exist and are equal.

\[
\int_0^\infty F(y)dy = |D(K)| \int_0^\infty F(x(\cdot)) + \int_0^1 \int_0^1 H(u, s)dx(s)ds \\
+ \int_0^1 J(s)x(s)ds \exp\{-\Phi(x)\}dx
\]

where

\[
\Phi(x) = \int_0^1 \left\{ \int_0^1 H(t, s)x(s)ds + J(t)x(t) \right\}^2 dt \\
+ 2 \int_0^1 \int_0^1 H(t, s)x(s)dx(t) + \int_0^1 J(t)d(x(t))^2.
\]

**Theorem C.** Let $M(t, s)$ and $J(t)$ satisfy (0.7)-(0.9), let $M(t, s)$ be continuous in $s$ on $I$ for each $t \in I$, and let

(0.19) $J(1) = M(t, 1) = M(1, t) = 0$,

$t \in I$.

Let $\epsilon > 0$ and define

(0.20) $M^*(t, s) = \frac{1}{\epsilon} \int_s^{t+\epsilon} M(t, v)dv$, $(t, s) \in I^3$;

where $M(t, v) = 0$ for $v > 1$;

(0.21) $J^*(t) = \frac{1}{\epsilon} \int_t^{t+\epsilon} J(u)du$, $t \in I$,

where $J(u) = 0$ for $u > 1$; and
(0.22) \[ (M^t)^*(t, s) = \begin{cases} M^t(t, s), & 0 \leq t < s \leq 1, \\ M^t(t, s) + \frac{1}{2} J^t(t), & 0 \leq s = t \leq 1, \\ M^t(t, s) + J^t(t), & 0 \leq s < t \leq 1. \end{cases} \]

Then there exists \( \delta > 0 \) such that if \( 0 < \epsilon < \delta \) and \( F(y) \) is a measurable functional such that either side of the following equation exists, both sides exist and are equal.

\[
\int_\epsilon^\infty F(y)dy = \left| D(M^t)^* \right| \int_\epsilon^\infty \left( x(\cdot) + \int_0^1 \int_0^1 (M\epsilon^t)^*(u, s)du dx(s) \right) \exp \left\{ -\int_0^1 \left[ \int_0^1 (M^t)^*(t, s)dx(s) \right] dt \right. \\
\left. - 2 \int_0^1 \int_0^1 M^t(t, s)dx(s)dx(t) - \int_0^1 J^t(t)d(x(t)) \right\} d\omega x.
\]

(0.23)

1. In this section we give some lemmas used in the proof of Theorem C.

**Lemma 1.** Let \( N_\epsilon(t, s) \) be measurable on \( I^2 \) and \( N_\epsilon(t, t) \) be measurable on \( I \) for all \( \epsilon > 0 \). Let \( N_\epsilon(t, s) \) be bounded on \( I^2 \) independent of \( \epsilon > 0 \). Let

\[
\lim_{\epsilon \to 0^+} N_\epsilon(t, s) = N(t, s)
\]

almost everywhere on \( I^2 \) and

\[
\lim_{\epsilon \to 0^+} N_\epsilon(t, t) = N(t, t)
\]

almost everywhere on \( I \). Then

(1.1) \[ \lim_{\epsilon \to 0^+} D(N_\epsilon) = D(N). \]

**Proof.** Let \( \eta > 0 \) be given. By application of Hadamard's lemma [3], it follows that there exists an integer \( K \) such that

\[
\left| D(N_\epsilon) - \sum_{n=0}^{K} \frac{1}{n!} \int_0^1 \cdots \int_0^1 N_\epsilon(s_1, s_1) \cdots N_\epsilon(s_1, s_n) ds_1 \cdots ds_n \right| < \frac{\eta}{3}
\]

independent of \( \epsilon > 0 \) and

\[
\left| D(N) - \sum_{n=0}^{K} \frac{1}{n!} \int_0^1 \cdots \int_0^1 N(s_1, s_1) \cdots N(s_1, s_n) ds_1 \cdots ds_n \right| < \frac{\eta}{3}
\]

It follows immediately from the hypothesis and bounded convergence that
\[
\lim_{\epsilon \to 0^+} \frac{1}{n!} \sum_{n=1}^{\infty} \int_0^1 \cdots \int_0^1 \begin{vmatrix}
N_1(s_1, s_1) & \cdots & N_1(s_1, s_n) \\
\vdots & \ddots & \vdots \\
N_1(s_n, s_1) & \cdots & N_1(s_n, s_n)
\end{vmatrix}
\, ds_1 \cdots ds_n
= \frac{1}{n!} \sum_{n=1}^{\infty} \int_0^1 \cdots \int_0^1 \begin{vmatrix}
N(s_1, s_1) & \cdots & N(s_1, s_n) \\
\vdots & \ddots & \vdots \\
N(s_n, s_1) & \cdots & N(s_n, s_n)
\end{vmatrix}
\, ds_1 \cdots ds_n.
\]

Therefore there exists \( \delta > 0 \) such that \( 0 < \epsilon < \delta \) implies
\[
| D(N) - D(N') | < \eta.
\]

This implies (1.1).

The following result was communicated to the author by R. H. Cameron.

**Lemma 2.** Let \( N(t, s) \) be bounded and measurable on \( I^2 \). Then

(1.2) \[
D\left(\int_0^t N(u, s)\, ds\right) = D\left(\int_0^1 N(t, v)\, dv\right).
\]

**Proof.** A typical term from the expansion of a determinant in the expansion (0.2) of \( D(\int_0^1 N(u, s)\, du) \) can be written

\[
\int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \prod_{i=1}^m \int_0^{s_i} N(u_i, s_i)\, du_i \int_0^{s_{m+1}} N(u_{m+1}, s_{m+1})\, du_{m+1}
\]

\[
\cdots \int_0^{s_{n-1}} N(u_{n-1}, s_n)\, du_{n-1} \int_0^{s_n} N(u_n, s_{m+1})\, du_n
\]

\[
= \int_0^1 \cdots \int_0^1 du_1 \cdots du_n \prod_{i=1}^m \int_0^{u_i} N(u_i, s_i)\, du_i \int_0^{u_{m+1}} N(u_{m+1}, s_{m+1})\, du_{m+1}
\]

\[
\cdots \int_0^{u_{n-1}} N(u_{n-1}, s_n)\, du_n
\]

(1.3)

\[
= \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \prod_{i=1}^m \int_0^{s_i} N(s_i, v_i)\, dv_i \int_0^{s_{m+1}} N(s_{m+1}, v_{m+1})\, dv_{m+1}
\]

\[
\cdots \int_0^{s_{n-1}} N(s_{n-1}, v_n)\, dv_n
\]

\[
= \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \prod_{i=1}^m \int_0^{s_i} N(s_i, v_i)\, dv_i \int_0^{s_{m+1}} N(s_{m+1}, v_{m+1})\, dv_{m+1}
\]

\[
\cdots \int_0^{s_{n-1}} N(s_{n-1}, v_n)\, dv_n \int_0^{s_{m+1}} N(s_{m+1}, v_{m+1})\, dv_{m+1}
\]

by use of the Fubini theorem, a change of dummy variables, and a change in
the order of multiplication. The extreme right hand member of (1.3) is the same corresponding term in the expansion of

\[ D\left(\int_{0}^{t} N(t, s) ds\right) \]

as the extreme left hand member of (1.3) is in the expansion of

\[ D\left(\int_{0}^{t} N(u, s) du\right). \]

This establishes (1.2).

**Lemma 3.** Let \( M(t, s) \) be bounded and measurable on \( I^2 \) and let \( J(t) \) be bounded and measurable on \( I \). Let \( J^*(t), M^*(t, s), \) and \((M^*)^*(t, s)\) be defined as in (0.20), (0.21) and (0.22) and

\[
(1.4) \quad K_e(t, s) = \begin{cases} 
- \int_{0}^{t} \frac{1}{e} \left[ M(u, s + e) - M(u, s) \right] du, & 0 \leq t < s \leq 1, \\
- \int_{0}^{1} \frac{1}{e} \left[ M(u, s + e) - M(u, s) \right] du + \frac{1}{2} J^*(s), & 0 \leq t = s \leq 1, \\
- \int_{0}^{1} \frac{1}{e} \left[ M(u, s + e) - M(u, s) \right] du + J^*(s), & 0 \leq s < t \leq 1.
\end{cases}
\]

Then for all \( e > 0, \)

\[
(1.5) \quad D(K_e) = D(M^*)^*.
\]

**Proof.** Let

\[
N(t, s) = \frac{1}{e} \left[ M(t, s + e) - M(t, s) \right],
\]

\[
\phi_{\eta}(u) = \begin{cases} 
0, & u \leq -\eta, \\
\frac{1}{\eta^2} (u + \eta), & -\eta \leq u \leq 0, \\
\frac{1}{\eta^2} (u - \eta), & 0 \leq u \leq \eta, \\
0, & u \geq \eta,
\end{cases}
\]

and

\[
N_\eta(t, s) = -N(t, s) + \phi_{\eta}(t - s)J^*(s).
\]

Then the following two equations hold insofar as their right hand sides are defined.
\[
\int_0^t N_\eta(u, s)du = -\int_0^t N(u, s)du + J^\eta(s) \begin{cases} 0, & \eta < t + \eta < s < 1, \\ \frac{1}{2}, & \eta < t = s < 1, \\ 1, & 2\eta < s + \eta < t < 1, \end{cases}
\]

and
\[
\int_0^1 N_\eta(t, u)du = -\int_0^1 N(t, u)du
\]
\[
\begin{cases} 0, & 0 < t < s - \eta < 1 - \eta, \\ \int_t^t + \eta \frac{1}{\eta^2} (t - u + \eta)J^\eta(u)du, & 0 < t = s < 1 - \eta, \\ \int_t^t + \eta \frac{1}{\eta^2} (t - u + \eta)J^\eta(u)du - \int_{t - \eta}^t \frac{1}{\eta^2} (t - u - \eta)J^\eta(u)du, & 0 < s < t - \eta < 1 - 2\eta. \end{cases}
\]

Since \( J^\eta(t) \) is continuous on \( I \), we have
\[
\begin{aligned}
\lim_{\eta \to 0^+} \int_0^t N_\eta(u, s)du &= -\int_0^t N(u, s)du + J^\eta(s) \begin{cases} 0, & 0 < t < s < 1, \\ \frac{1}{2}, & 0 < s = t < 1, \\ 1, & 0 < s < t < 1, \end{cases} \\
&= K_\eta(t, s)
\end{aligned}
\]

and
\[
\begin{aligned}
\lim_{\eta \to 0^+} \int_0^1 N_\eta(t, u)du &= -\int_0^1 N(t, u)du + J^\eta(t) \begin{cases} 0, & 0 < t < 2 < 1, \\ \frac{1}{2}, & 0 < t = s < 1, \\ 1, & 0 < s < t < 1. \end{cases} \\
\end{aligned}
\]

Since \( N_\eta(t, s) \) is bounded and measurable on \( I^2 \) for every \( \eta > 0 \), we use Lemma 2 to obtain
\[
D \left( \int_0^t N_\eta(u, s)du \right) = D \left( \int_0^1 N_\eta(t, u)du \right).
\]

Since
\[
\lim_{\eta \to 0^+} \int_0^1 N_\eta(t, v)dv
\]
exists almost everywhere on \( I^2 \),
exists almost everywhere on $I$, 
\[
\int_0^1 N_u(t, v)dv
\]
is measurable on $I^2$ by the Fubini theorem in three dimensions, 
\[
\int_0^1 N_u(t, v)dv
\]
is measurable on $I$ by the Fubini theorem, and 
\[
\int_0^1 N_u(t, v)dv
\]
is bounded independent of $\eta>0$, we can use Lemma 1 to obtain
\[
\lim_{\eta \to 0^+} D\left( \int_0^1 N_u(t, v)dv \right) = D\left( \lim_{\eta \to 0^+} \int_0^1 N_u(t, v)dv \right).
\]
Similarly,
\[
\lim_{\eta \to 0^+} D\left( \int_0^1 N_v(u, s)du \right) = D\left( \lim_{\eta \to 0^+} \int_0^1 N_v(u, s)du \right).
\]
Since
\[
- \int_0^1 N(t, v)dv = - \int_0^1 \frac{1}{\epsilon} [M(t, v + \epsilon) - M(t, v)]dv
\]
\[
= - \int_{s+\epsilon}^{s+\epsilon} \frac{1}{\epsilon} M(t, v)dv + \int_0^1 \frac{1}{\epsilon} M(t, v)dv
\]
\[
= \frac{1}{\epsilon} \int_{s+\epsilon}^{s+\epsilon} M(t, v)dv = M^*(t, s),
\]
we have from (1.7),
\[
\lim_{\eta \to 0^+} \int_0^1 N_v(t, v)dv = (M^*)^*(t, s).
\]
Therefore (1.6), (1.8), (1.9), (1.10), and (1.11) imply (1.5).

**Lemma 4.** Let $M(t, s)$ be of $B.V.$ on $I^2$. Then if $(t_1, s_1) \in I^2$, $M(t, s_1)$ and $M(t_1, s)$ are of $B.V.$ on $I$. Also $M(t, s)$ is measurable on $I^2$, $M(t, t)$ is measurable on $I$, and for almost every $t \in I$,
(1.12) \( \lim_{s \to t^*} M(t, s) = M(t, t). \)

Furthermore, if \( \varepsilon > 0 \) and \( M'(t, s) \) is defined by (0.20), then there exists \( B < \infty \) independent of \( \varepsilon > 0 \) such that

(1.13) \( \operatorname{var}_{(t, s) \in \mathbb{R}^2} M'(t, s) < B. \)

Proof. Let \( (t_1, s_1) \in I^2 \). Let \( (t_0, s_0) \in I^2 \) be such that \( M(t, s_0) \) and \( M(t_0, s) \) are of B.V. on \( I \) and \( 0 \leq u_0 \leq \cdots \leq u_n \leq 1 \) be a partition of \( I \). Then

\[
\sum_{i=1}^{n} |M(u_i, s_i) - M(u_{i-1}, s_i)| \leq \sum_{i=1}^{n} |M(u_i, s_i) - M(u_{i-1}, s_i) - M(u_i, s_0) + M(u_{i-1}, s_0)|
\]

\[
+ \sum_{i=1}^{n} |M(u_i, s_0) - M(u_{i-1}, s_0)|.
\]

Since the right hand side is bounded independent of the partition \( 0 \leq u_0 \leq \cdots \leq u_n \leq 1 \), \( M(t, s) \) is of B.V. on \( I \). Similarly, \( M(t_1, s) \) is of B.V. on \( I \).

Let \( (t, s) \in I^2 \) and \( \{0 \leq t_0 \leq \cdots \leq t_n \leq t; 0 \leq s_0 \leq \cdots \leq s_m < s\} \) be a partition of \( [0, t] \otimes [0, s] \). Let

\[
P(t, s) = \sup \sum \{M(t_i, s_j) - M(t_{i-1}, s_j) - M(t_{i-1}, s_{j-1}) + M(t_{i-1}, s_{j-1})\},
\]

\[
N(t, s) = \sup \sum \{-[M(t_i, s_j) - M(t_{i-1}, s_j) - M(t_{i-1}, s_{j-1}) + M(t_{i-1}, s_{j-1})]\},
\]

where the sums are taken over the positive terms only and the suprema are taken over all partitions of \( [0, t] \otimes [0, s] \). It is easy to see that

(1.14) \( M(t, s) = P(t, s) - N(t, s) + M(0, s) + M(t, 0) - M(0, 0) \)

and that if \( 0 \leq u \leq v \leq 1 \) and \( t \in I \), then

(1.15) \( P(u, u) \leq P(v, v) \),

(1.16) \( P(t, u) \leq P(t, v) \), and

(1.17) \( P(u, t) \leq P(v, t) \).

We will show that \( P(t, s) \) is measurable on \( I^2 \). Let \( \alpha > 0 \). Then

\[
\phi(t) = \sup_{s \in I} \{s: P(t, s) < \alpha\},
\]

is a monotone decreasing function because if \( t' < t'' \) and \( \phi(t') < \phi(t'') \), then there exists \( s'' \) such that \( \phi(t') < s'' < \phi(t'') \) and according to (1.17), \( P(t', s'') \leq P(t'', s'') \leq \alpha \) so that \( \phi(t') \geq s'' \), a contradiction. Therefore \( \phi(t) \) is measurable and its ordinate set

\[
\{(t, s) \in I^2: P(t, s) < \alpha\}
\]
is measurable for all real \( \alpha \). Therefore \( P(t, s) \) is measurable on \( I^2 \).

From (1.15) it follows that \( P(t, t) \) is measurable on \( I \). Similarly, \( N(t, s) \) is measurable on \( I^2 \) and \( N(t, t) \) is measurable on \( I \). Since \( M(t, 0) \) and \( M(0, s) \) are measurable on \( I \), it follows from (1.14) that \( M(t, s) \) is measurable on \( I^2 \) and \( M(t, t) \) is measurable on \( I \).

Because of (1.15), \( P(t, t) \) is continuous almost everywhere on \( I \). Therefore, from (1.16) and (1.17) it follows that for almost all \( t \in I \),

\[
\lim_{s \to t^+} P(t, s) = P(t, t).
\]

Similarly, for almost all \( t \in I \),

\[
\lim_{s \to t^+} N(t, s) = N(t, t).
\]

Since \( M(0, s) \) is of B.V. on \( I \),

\[
\lim_{s \to t^+} M(0, s) = M(0, t)
\]

for almost all \( t \in I \). Therefore (1.12) follows from (1.14).

Let \( \epsilon > 0 \) and define

\[
P^{\epsilon}(t, s) = \frac{1}{\epsilon} \int_s^{s+\epsilon} P(t, v) dv,
\]

\[N^{\epsilon}(t, s) = \frac{1}{\epsilon} \int_s^{s+\epsilon} P(t, v) dv,
\]

where \( P(t, s) = P(t, 1) \) and \( N(t, s) = N(t, 1) \) if \( s > 1 \). Then from (1.14),

\[
M^{\epsilon}(t, s) = P^{\epsilon}(t, s) - N^{\epsilon}(t, s) + M^{\epsilon}(0, s) + M(t, 0) - M(0, 0)
\]

and

\[
(1.18) \quad \var_{(t,s) \in I^2} M^{\epsilon}(t, s) < \var_{(t,s) \in I^2} P^{\epsilon}(t, s) + \var_{(t,s) \in I^2} N^{\epsilon}(t, s).
\]

Let \( 0 \leq t' \leq t'' \leq 1 \) and \( 0 \leq s' \leq s'' \leq 1 \). Then

\[
P^{\epsilon}(t'', s'') - P^{\epsilon}(t', s') - P^{\epsilon}(t'', s'') + P^{\epsilon}(t', s')
\]

\[= \frac{1}{\epsilon} \int_{s'}^{s''} [P(t'', v) - P(t', v) - P(t'', v - (s'' - s')) + P(t', v - (s'' - s'))] dv
\]

\[\geq 0
\]

since

\[
P(t'', v) - P(t', v) - P(t'', v - (s'' - s')) + P(t', v - (s'' - s')) \geq 0
\]

for all \( v, s'' \leq v \leq s'' + \epsilon \) as can be seen from the definition. It follows that
\[
\text{var}_{(t,s) \in I^2} P^*(t, s) = P^*(1, 1) - P^*(1, 0) + P^*(0, 1) - P^*(0, 0)
\]
\[
\leq P^*(1, 1) - P^*(1, 0)
\]
\[
\leq \frac{1}{\varepsilon} \int_1^{1+\varepsilon} P(1, 1) dv \leq P(1, 1)
\]

independent of \(\varepsilon > 0\). Similarly,

\[
\text{var}_{(t,s) \in I^2} N(t, s) = N(1, 1)
\]

independent of \(\varepsilon > 0\). Therefore (1.13) follows from (1.18).

**Lemma 5.** Let \(M(t, s)\) be of B.V. on \(I^2\) and \(J(t)\) be of B.V. on \(I\). Let \(M(t, s)\), \(M^*(t, s)\), and \((M^*)^*(t, s)\) be defined by (0.7), (0.20), (0.21), and (0.22). Then

\[
(1.19) \quad \lim_{\varepsilon \to 0^+} D(M^*)^* = D(M).
\]

**Proof.** We will show that \(M^*(t, s)\) is of B.V. on \(I^2\). Since \(M^*(t, 1) = 0, t \in I\), and \(M^*(0, s)\) is absolutely continuous on \(I\), it follows from Lemma 4 that \(M^*(t, s)\) is of B.V. on \(I^2\). Since \(J^*(t)\) is measurable, we conclude from Lemma 4 that \((M^*)^*(t, s)\) is measurable on \(I^2\) and \((M^*)^*(t, s)\) is measurable on \(I\). Since \(M(t, s)\) is bounded on \(I^2\), it follows that \(M^*(t, s)\) is bounded independent of \(\varepsilon > 0\). Since \(J(t)\) is bounded, \((M^*)^*(t, s)\) is bounded independent of \(\varepsilon > 0\).

From Lemma 4 we know that \(M(t, s)\) is of B.V. in \(s\) on \(I\) for all \(t \in I\). Therefore

\[
\lim_{\varepsilon \to 0^+} M^*(t, s) = M(t, s)
\]

almost everywhere on \(I^2\). From conclusion (1.12) of Lemma 4 it follows that

\[
\lim_{\varepsilon \to 0^+} M^*(t, t) = M(t, t)
\]

almost everywhere on \(I\). Since \(J^*(t)\) is of B.V. we have

\[
\lim_{\varepsilon \to 0^+} J^*(t) = J(t)
\]

almost everywhere on \(I\). Therefore

\[
\lim_{\varepsilon \to 0^+} (M^*)^*(t, s) = M^*(t, s)
\]

almost everywhere on \(I^2\) and

\[
\lim_{\varepsilon \to 0^+} (M^*)^*(t, t) = M^*(t, t)
\]

almost everywhere on \(I\). Therefore (1.19) follows from conclusion (1.1) of Lemma 1.

2. **Proof of Theorem C.** We make the following manipulations so that we can use Theorem B.
\[
\int_0^1 \int_0^t (M')^*(u, s)dudx(s) = \int_0^1 \int_0^t M'(u, s)dudx(s) + \int_0^1 \int_0^t J'(u)dudx(s)
\]

\[
= \int_0^1 \int_0^t \frac{1}{\epsilon} \int_s^{s+\epsilon} M(u, v)dvdudx(s) - \int_0^t x(s)ds \int_0^1 J'(u)du
\]

(2.1)

\[
= -\int_0^1 x(s)ds \frac{1}{\epsilon} \int_s^{s+\epsilon} M(u, v)dudv + \int_0^1 \int_s^{s+\epsilon} J'(s)ds - \int_0^1 \int_0^t \frac{1}{\epsilon} [M(u, s + \epsilon) - M(u, s)]du \] x(s)ds + \int_0^t J'(s)x(s)ds.
\]

Now for fixed \( \epsilon > 0 \),

\[
-\frac{1}{\epsilon} [M(t, s + \epsilon) - M(t, s)]
\]

satisfies (0.12) and satisfies (0.14) because \( M(t, s) \) is continuous on \( 0 < s < \infty \)
for all \( t \in I \). \( J'(t) \) satisfies (0.13). From Lemma 5 and condition (0.9) we
know there exists \( \delta_i > 0 \) such that for \( 0 < \epsilon < \delta_i \),

\[
D((M')^*(t, s)) \neq 0.
\]

Therefore, from (1.4) and conclusion (1.5) of Lemma 3, condition (0.15) of
Theorem B is satisfied when \( 0 < \epsilon < \delta_i \). Therefore we can write (0.17) with
\( H(t, s) \) replaced by \( -\frac{1}{\epsilon} [M(t, s + \epsilon) - M(t, s)]/\epsilon \) and \( J(t) \) replaced by \( J'(t) \) and
we have by (1.5), (0.17), and (2.1), if \( f(y) \) is measurable on \( C \) and \( 0 < \epsilon < \delta_i \),

\[
\int_0^\infty F(y)dy = |D((M')^*(t, s))| \int_0^\infty F(x(\cdot) + \int_0^1 \int_0^{x(\cdot)} (M')^*(u, s)dudx(s))
\cdot \exp \left\{ -\int_0^1 \left\{ \int_0^1 \left[ -\frac{1}{\epsilon} [M(t, s + \epsilon) - M(t, s)] \right] x(s)ds + J'(t)x(t) \right\}^2 dt
\right.

\left. -2 \int_0^1 \int_0^1 \left[ -\frac{1}{\epsilon} [M(t, s + \epsilon) - M(t, s)] x(s)dsdx(t) \right.
\right.

\left. -\int_0^1 J'(t)x(t) ds \right\} dx
\]

\[
= |D((M')^*)| \int_0^\infty F(x(\cdot) + \int_0^1 \int_0^{x(\cdot)} (M')^*(u, s)dudx(s))
\cdot \exp \left\{ -\int_0^1 \left\{ \int_0^1 (M')^*(t, s)ds(s) \right\}^2 dt
\right.

\left. -2 \int_0^1 \int_0^1 M'(t, s)ds(s)dx(t)
\right.

\left. -\int_0^1 J'(t)x(t) ds \right\} dx.
\]
provided either side exists. This proves Theorem C.

3. In this section we prove some lemmas which we use in letting \( \epsilon \to 0^+ \).

**Lemma 6.** Let \( N(t, s) \) be measurable in \( t \) on \( I \) for every \( s \in I \) and let

\[
| N(t, s) | < B < \infty
\]

and

\[
\operatorname{var} N(t, s) < B < \infty,
\]

where \( B \) is independent of \( (t, s) \in I^2 \). Then if \( x \in C \),

\[
\int_0^1 \int_0^t N(u, s) \, du \, dx(s)
\]

exists and equals

\[
\int_0^t \int_0^1 N(u, s) \, dx(s) \, du
\]

for all \( t \in I \).

**Proof.** For every \( u \in I \), we have

\[
\lim_{\operatorname{norm} P \to 0^+} \sum_{i=1}^n N(u, s_i) \left[ x(s_i) - x(s_{i-1}) \right] = \int_0^1 N(u, s) \, dx(s)
\]

where \( P \) is the partition

\[
0 = s_0 \leq \cdots \leq s_n = 1.
\]

From the hypothesis it follows that the limitand on the left is measurable and bounded independent of the partition. Then from the bounded convergence theorem, we have

\[
\int_0^1 \int_0^t N(u, s) \, du \, dx(s)
\]

exists and equals

\[
\int_0^t \int_0^1 N(u, s) \, dx(s) \, du.
\]

**Lemma 7.** Let \( M(t, s) \) be of B. V. on \( I^2 \) and \( J(t) \) be of B. V. on \( I \). Let \( (M^*)^*(t, s) \) and \( \overline{M}(t, s) \) be defined by (0.22) and (0.7) respectively. Then

\[
\int_0^1 (M^*)^*(t, s) \, dx(s)
\]

converges in Hilbert norm on \( C \otimes I \) to
\[ \int_0^1 M(t, s)dx(s) \]

as \( \epsilon \to 0^+ \).

**Proof.** By a theorem of [2] and Fubini's theorem,

\[ \int_0^\infty \left( \int_0^1 [(M^*)^*(t, s) - M(t, s)]dx(s) \right)^2 dx(t) \]

\[ = \int_0^1 \frac{1}{2} \int_0^1 [(M^*)^*(t, s) - \overline{M}(t, s)]^2 dsdt \]

\[ = \frac{1}{2} \int_0^1 [(M^*)^*(t, s) - \overline{M}(t, s)]^2 dsdt. \]

The integrand on the right is bounded and tends to zero almost everywhere on \( I^2 \) as \( \epsilon \) tends to \( 0^+ \) because for every \( t \in I \), \( M(t, s) \) is a continuous function of \( s \) almost everywhere on \( I \). The conclusion follows by bounded convergence.

**Lemma 8.** Let \( M(t, s) \) be of B. V. on \( I^2 \), \( M(t, 1) = M(1, t) = 0 \), \( t \in I \), and \( x \in C \). Then

\[ \int_0^1 \int_0^1 M(t, s)dx(s)dx(t) \]

exists and equals

\[ \int x(t,x(s))dM(t, s). \]

**Proof.** We need to show that

\[ \int_0^1 M(t, s)dx(s) \]

is of B.V. Let \( 0 \leq t_0 \leq \cdots \leq t_n \leq 1 \) be a partition of \( I \). Then \( M(t, 1) = 0 \), \( t \in I \), and we have

\[ \sum_{i=0}^n \left| \int_0^{t_i} M(t_i, s)dx(s) - \int_0^{t_{i-1}} M(t_{i-1}, s)dx(s) \right| \]

\[ = \sum_{i=0}^n \left| \int_0^{t_i} x(s)d[M(t_i, s) - M(t_{i-1}, s)] \right| \]

\[ \leq \|x\| \sum_{i=0}^n \left[ \int_0^{t_i} \text{var} [M(t_i, s) - M(t_{i-1}, s)] \right]. \]

\( M(t_i, s) \) is of B.V. in \( s \) on \( I \) for each \( i = 0, 1, \ldots, n \), by Lemma 4. Therefore,
we can find \( n \) partitions \( \{0 \leq s^{(i)} \leq \cdots \leq s_{m_i}^{(i)} \leq 1\} \) for \( i = 1, 2, \ldots, n \) such that

\[
\begin{align*}
\text{var} \left[ M(t, s) - M(t_{i-1}, s) \right] &< \frac{1}{n} + \sum_{i=1}^{n} \left| M(t_{i}, s_{j}^{(i)}) - M(t_{i-1}, s_{j}^{(i)}) - M(t_{i}, s_{j-1}^{(i)}) - M(t_{i-1}, s_{j-1}^{(i)}) \right|, \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left| M(t_{i}, s_{j}) - M(t_{i-1}, s_{j}) - M(t_{i}, s_{j-1}) + M(t_{i-1}, s_{j-1}) \right|
\end{align*}
\]

Then if \( 0 \leq s_0 \leq \cdots \leq s_m \leq 1 \) is the union of the partitions \( \{0 \leq s^{(i)} \leq \cdots \leq s_{m_i}^{(i)} \leq 1\} \)

\[
\sum_{i=1}^{n} \text{var} \left[ M(t_{i}, s) - M(t_{i-1}, s) \right] 
\]

(3.2) \[
\leq 1 + \sum_{i=1}^{n} \sum_{j=1}^{m} \left| M(t_{i}, s_{j}) - M(t_{i-1}, s_{j}) - M(t_{i}, s_{j-1}) + M(t_{i-1}, s_{j-1}) \right| 
\]

(3.2) \[
\leq 1 + \text{var} \left[ M(t, s) \right].
\]

Now (3.1) and (3.2) show that \( \int_{0}^{1} M(t, s) dx(s) \) is of B.V. Therefore

\[
\int_{0}^{1} \int_{0}^{1} M(t, s) dx(s) dx(t)
\]

exists.

Let \( \varepsilon > 0 \) be given. Since \( x \in C \) and

\[
\text{var} \left[ M(t, s) \right] < \infty,
\]

\[
\int_{I} x(t)x(s) dM(t, s)
\]

exists. Since \( M(1, s) = M(t, 1) = 0 \),

(3.3) \[
\int_{I} x(t)x(s) dM(t, s) = \int_{I} M(t, s) d(x(t)x(s)).
\]

Choose \( \delta \) such that if the partitions \( T, 0 = t_0 \leq \cdots \leq t_n = 1 \), and \( S, 0 = s_0 \leq \cdots \leq s_m = 1 \), have norm less than \( \delta \), then

\[
\left| \int_{I} M(t, s) d(x(t)x(s)) - \sum_{i=1}^{n} \sum_{j=1}^{m} M(t_i, s_j)(x(s_j) - x(s_{j-1}))(x(t_i) - x(t_{i-1})) \right| < \frac{\varepsilon}{3}
\]

and

(3.5) \[
\left| \int_{0}^{1} \int_{0}^{1} M(t, s) dx(s) dx(t) - \sum_{i=1}^{n} \int_{0}^{1} M(t_i, s) dx(s)(x(t_i) - x(t_{i-1})) \right| < \frac{\varepsilon}{3}.
\]
Let $T$ be fixed with norm $T < \delta$. Choose $\delta'$ such that if norm $S < \delta'$, then for $i = 1, 2, \ldots, n$,

\begin{equation}
\left| \int_{0}^{1} M(t, s) dx(s) - \sum_{j=1}^{m} M(t, s_{j})(x(s_{j}) - x(s_{j-1})) \right| < \frac{\epsilon}{6n \|x\| + 1}.
\end{equation}

Let norm $S < \min(\delta, \delta')$. Using (3.4)–(3.6) we obtain

\[
\left| \int_{0}^{1} M(t, s) dx(t) dx(s) - \int_{0}^{1} \int_{0}^{1} M(t, s) dx(s) dx(t) \right| < \epsilon.
\]

Since $\epsilon$ is arbitrary, we have

\[
\int_{0}^{1} M(t, s) dx(t) dx(s) = \int_{0}^{1} \int_{0}^{1} M(t, s) dx(s) dx(t).
\]

Therefore the conclusion follows from this and (3.3).

**Lemma 9.** Let $M(t, s)$ be of B.V. on $P$ and $M(t, 1) = M(1, s) = 0$ and let $M'(t, s)$ be defined as in (0.20). Then

\[
\lim_{\epsilon \to 0^+} \int_{0}^{1} \int_{0}^{1} M'(t, s) dx(s) dx(t) = \int_{0}^{1} \int_{0}^{1} M(t, s) dx(s) dx(t)
\]

over $C$.

**Proof.** Let $N_{\epsilon}(t, s) = M'(t, s) - M(t, s)$. By Lemma 4, $N_{\epsilon}(t, s)$ is of B.V. on $P$. Also $N_{\epsilon}(t, 1) = N_{\epsilon}(1, s) = 0$. By Lemma 8,

\[
\left[ \int_{0}^{1} \int_{0}^{1} N_{\epsilon}(t, s) dx(s) dx(t) \right]^{2} = \left[ \int_{P} x(t) x(s) dN_{\epsilon}(t, s) \right]^{2}
\]

(3.7)

\[
= \int_{P} \int_{P} x(t) x(s) x(u) x(v) dN_{\epsilon}(t, s) dN_{\epsilon}(u, v)
\]

\[
= \int_{P} \int_{P} x(t) x(s) x(u) x(v) (N_{\epsilon}(u, v) N_{\epsilon}(t, s))
\]

from the Fubini theorem. Also from the Fubini theorem,

\[
\int_{P} \left( \int_{0}^{1} \int_{0}^{1} N_{\epsilon}(t, s) dx(s) dx(t) \right)^{2} d\omega x
\]

(3.7)

\[
= \int_{P} \int_{P} x(t) x(s) x(u) x(v) d(N_{\epsilon}(u, v) N_{\epsilon}(t, s)) d\omega x
\]

\[
= \int_{P} U(t, s, u, v) d(N_{\epsilon}(u, v) N_{\epsilon}(t, s)),
\]

where
1961] LINEAR TRANSFORMATIONS OF WIENER INTEGRALS 475

\[ U(t, s, u, v) = \int_{c}^{w} x(t)x(s)x(u)x(v)dx, \quad (t, s, u, v) \in I^2 \otimes I^2. \]

The use of the Fubini theorem is justified since \(||x||^4\) is integrable on \(C\). Since if \(0 \leq t_1 \leq t_2 \leq t_3 \leq t_4\),

\[ \int_{c}^{w} x(t_1)x(t_2)x(t_3)x(t_4)d\omega x = t_1 \left( \frac{t_2}{2} + \frac{t_3}{4} \right), \]

\(U(t, s, u, v)\) is absolutely continuous on \(I^2 \otimes I^2\). Since \(U(t, s, u, v)\) is zero if any argument is zero and \(N_\epsilon(t, 1)\) and \(N_\epsilon(1, s)\) are zero, we have on integrating by parts,

\[ (3.8) \int_{I^2} U(t, s, u, v)d(N_\epsilon(u, v)N_\epsilon(t, s)) = \int_{I^2} N_\epsilon(t, s)N_\epsilon(u, v)dU(t, s, u, v). \]

For every fixed \(t \in I\), \(M(t, s)\) is continuous in \(s\) almost everywhere on \(I\). Therefore \(N_\epsilon(t, s)\) converges almost everywhere to zero as \(\epsilon \to 0^+\). Since \(N_\epsilon(t, s)\) is bounded on \(I^2\) and \(U(t, s, u, v)\) is absolutely continuous, the right hand side of (3.8) tends to zero as \(\epsilon \to 0^+\). Using (3.7) the lemma is established.

**Lemma 10.** Let \(J(t)\) be of B.V. on \(I\) and \(J(1) = 0\). Let \(J^*(t)\) be defined by (0.21). Then

\[ \lim_{\epsilon \to 0^+} \int_{0}^{1} J^*(t)d(x(t))^2 = \int_{0}^{1} J(t)d(x(t))^2 \text{ over } C. \]

**Proof.** The proof is similar to the proof of Lemma 9. Let \(N_\epsilon(t) = J^*(t) - J(t)\). By using Lemma 8 and the Fubini theorem,

\[ \int_{C} \left( \int_{0}^{1} N_\epsilon(t)d(x(t))^2 \right)^2 d\omega x = \int_{C} \left( \int_{0}^{1} N_\epsilon(t)d(x(t))^2 d(x(t))^2 \right)^2 d\omega x \]

\[ = \int_{C} \int_{I^2} (x(t)x(s))^2 d(N_\epsilon(t)N_\epsilon(s))d\omega x \]

\[ = \int_{I^2} \chi(t, s)d(N_\epsilon(t)N_\epsilon(s)) \]

\[ = \int_{I^2} N_\epsilon(t)N_\epsilon(s)d\chi(t, s), \]

where \(\chi(t, s) = \int_{C}(x(t)x(s))^2 d\omega x\) is absolutely continuous. The conclusion follows by bounded convergence since \(J(t)\) is bounded.

**4. Proof of Theorem A.** We assume condition (0.19) temporarily. Let \(F(y)\) be bounded and continuous in the uniform topology and vanish if \(||y|| \geq R\).
From Lemmas 7, 9, and 10, there exists a sequence \( \{ \epsilon_n \} \) such that

\[
\lim_{n \to \infty} \epsilon_n = 0
\]

and

(4.1) \[
\lim_{n \to \infty} \int_0^1 (M^{*n})^*(t, s) dx(s) = \int_0^1 M(t, s) dx(s)
\]

almost everywhere on \( C \otimes I \),

(4.2) \[
\lim_{n \to \infty} \int_0^1 \int_0^1 M^{*n}(t, s) dx(s) dx(t) = \int_0^1 \int_0^1 M(t, s) dx(s) dx(t)
\]

almost everywhere on \( C \), and

(4.3) \[
\lim_{n \to \infty} \int_0^1 J^{*n}(t) d(x(t))^2 = \int_0^1 J(t) d(x(t))^2
\]

almost everywhere on \( C \).

Consider

\[
\left| \int_0^1 \int_0^1 (M^{*n})^*(u, s) dx(s) du - \int_0^1 \int_0^1 M(u, s) dx(s) du \right| \leq \int_0^1 \left| \int_0^1 (M^{*n})^*(u, s) dx(s) - \int_0^1 M(u, s) dx(s) \right| du.
\]

Since by Lemma 4

\[
\text{var}_{(t, s) \in \mathbb{R}} M^{*}(t, s)
\]

is bounded independent of \( \epsilon \) and since \( M^{*}(1, s) = 0 \) and \( J^{*}(t) \) is bounded independent of \( \epsilon \) and \( t \),

\[
\text{var}_{x \in I} (M^{*})^*(t, s)
\]

is bounded independent of \( t \) and \( \epsilon \). Therefore the integrand on the right of (4.4) is bounded on \( I \) independent of \( \epsilon > 0 \) for every \( x \in C \). By (4.1) this integrand tends to zero as \( n \) tends to \( \infty \) for almost all \( u \) on \( I \) for almost all \( x \in C \). It follows from (4.4), Lemma 6, and bounded convergence that

(4.5) \[
\lim_{n \to \infty} \int_0^1 \int_0^1 (M^{*n})^*(u, s) dx(s) du = \int_0^1 \int_0^1 M(u, s) dx(s)
\]

uniformly in \( t \) on \( I \) for almost all \( x \in C \).

From Lemma 4 it follows that
\[ \int_0^1 M^*(t, s) \, dx(s) \]

is bounded on \( I \) independent of \( \epsilon \) for every \( x \in C \). Since \( J'(t) \) is bounded on \( I \) independent of \( \epsilon \), we have from (4.1) and bounded convergence,

\[ \lim_{n \to \infty} \int_0^1 \left[ \int_0^1 (M^*)^*(t, s) \, dx(s) \right]^2 \, dt = \int_0^1 \left( \int_0^1 M(t, s) \, dx(s) \right)^2 \, dt \]

for almost every \( x \in C \).

We will now show that there exists \( R_1 \) independent of \( n \) such that for sufficiently large \( n, n > N \),

\[ F \left( x(\cdot) + \int_0^1 \int_0^1 (M^*)^*(u, s) \, du \, dx(s) \right) \]

vanishes if

\[ |||x||| \geq R_1. \]

From (1.19) of Lemma 5,

\[ \lim_{t \to 0^*} D(M_*^*) = D(M). \]

From (0.9) there exists \( N \) such that if \( n > N \), \( D(M^*_n)^* \) is bounded from zero. We shall assume from now on that \( n > N \). From [3] we know there exists the Volterra reciprocal kernel \(((M^*)^*)^{-1}(t, s)\) such that

\[ ((M^n)^*)^{-1}(t, s) + (M^*)^*(t, s) = - \int_0^1 ((M^n)^*)^{-1}(t, u)(M^*)^*(u, s) \, du \]

\[ = - \int_0^1 (M^*)^*(t, u)((M^*)^*)^{-1}(u, s) \, du, \]

and \(((M^*)^*)^{-1}(t, s)\) is bounded on \( I^2 \) independent of \( n \). From (4.8) and Lemma 4, (1.13),

\[ \text{var}_{s \in I} (M^*)^*^{-1}(t, s) \]

\[ \leq \text{var}_{s \in I} (M^*)^*(t, s) + \text{var}_{s \in I} \int_0^1 ((M^*)^*)^{-1}(t, u)(M^*)^*(u, s) \, du \]

\[ \leq \text{var}_{s \in I} (M^*)^*(t, s) \left[ 1 + \sup_{(t, s) \in B} \left| ((M^*)^*)^{-1}(t, s) \right| \right] \]

\[ < B, \]

where \( B \) is independent of \( t \) and \( n \). Therefore from (4.9) and Lemma 6 which we use by virtue of (4.8) and (4.9), if \( |||y||| < R \), then
\begin{align}
\| y(t) + \int_0^1 \int_0^t (M^\ast)^{-1}(u, s) du \, dy(s) \| \\
\leq R \left[ 1 + B + \sup_{\tau \in I} | (M^\ast)^{-1}(t, 1) | \right] < R_1
\end{align}

which is bounded independent of \( n \).

Consider

\begin{equation}
y(t) = x(t) + \int_0^1 \int_0^t (M^\ast)^*(u, s) du \, dx(s).
\end{equation}

Then by Lemma 6 and (4.8),

\begin{align}
y(t) + \int_0^1 \int_0^t (M^\ast)^{-1}(u, s) du \, dy(s) \\
= x(t) + \int_0^1 \int_0^t (M^\ast)^*(u, s) du \, dx(s) + \int_0^1 \int_0^t (M^\ast)^{-1}(u, s) du \, dx(s) \\
+ \int_0^1 \int_0^t (M^\ast)^{-1}(u, s) du \, dx(s) \int_0^1 \int_0^t (M^\ast)^*(w, v) dv \, dx(v) \\
= x(t)
\end{align}

From (4.10)–(4.12) it follows that there exists \( R_1 \) independent of \( n \) such that if

\[ ||x|| > R_1, \]

then

\[ \| x(t) + \int_0^1 \int_0^t (M^\ast)^*(u, s) du \, dx(s) \| \geq R. \]

Therefore

\[ F \left( x(\cdot) + \int_0^1 \int_0^t (M^\ast)^*(u, s) du \, dx(s) \right) \]
vanishes if \( |||x||| > R_t \).  
From Lemmas 4 and 8
\[
\int_0^1 \int_0^1 M^{*n}(t, s) dx(s) dx(t)
\]
is bounded independent of \( n \) and \( x \in C \) as long as \( |||x||| < R_t \). Also
\[
\int_0^1 J^{*n}(t) d(x(t))^2
\]
is bounded independent of \( n \) and \( x \in C \) as long as \( |||x||| < R_t \) because
\[
\text{var } J^*(t) \in \tau
\]
is bounded independent of \( \epsilon > 0 \). Because
\[
F \left( x(\cdot) + \int_0^1 \int_0^1 (M^{*n})(u, s) du dx(s) \right)
\]
vanishes if \( |||x||| \geq R_t \) and is bounded and \( F(y) \) is continuous in the uniform topology and because of (4.5), (4.6), (4.2) and (4.3) we have, if (0.23) holds in some range \( 0 < \epsilon < \delta \), by bounded convergence,
\[
\int_\epsilon^\omega F(y) dw = |D(M)| \int_\epsilon^\omega F \left( x(\cdot) + \int_0^1 \int_0^1 (M^{*n})(u, s) du dx(s) \right)
\]
\[
\cdot \exp \left\{ - \int_0^1 \left( \int_0^1 M(t, s) dx(s) \right)^2 dt \right\}
\]
\[
\cdot 2 \int_0^1 \int_0^1 M(t, s) dx(s) dx(t) - \int_0^1 J(t) d(x(t))^2 \right\} dw x.
\]
(4.13)

Therefore, under the hypothesis of Theorem A, the restrictions on \( F(y) \), and condition (0.19), if (0.23) holds in a range \( 0 < \epsilon < \delta \), then (4.13) holds. By Lemmas 4 and 5, there exists \( \delta_0 > 0 \) such that if \( 0 < \eta < \delta_0 \) the hypothesis of Theorem C is satisfied when \( M(t, s) \) and \( J(t) \) are replaced by \( M^{*}(t, s) \) and \( J^{*}(t) \) respectively. Consequently there exists \( \delta_0 > 0 \) such that (0.23) holds for \( 0 < \epsilon < \delta_0 \) and \( 0 < \eta < \delta_0 \) with \( M(t, s) \) and \( J(t) \) replaced by \( M^{*}(t, s) \) and \( J^{*}(t) \) respectively so that (4.13) holds with \( M(t, s) \) and \( J(t) \) replaced by \( M^{*}(t, s) \) and \( J^{*}(t) \) respectively for \( 0 < \eta < \delta_0 \). Therefore (4.13) holds. We may remove condition (0.19) since the values of \( J(1), M(1, t), \) and \( M(1, s) \) do not affect the formula (4.13) provided the hypothesis of Theorem A still holds.

From (0.4), (0.5) and (0.6) it follows that
\[
L(t, s) = \int_0^t M(u, s) du
\]
(4.14)
for all \((t, s) \in I^2\). Therefore (0.10) follows from (4.13) when \(F(y)\) is continuous in the uniform topology, is bounded, and vanishes outside a uniform sphere. We can progressively enlarge the class of functionals for which (0.10) holds to the class of all measurable functionals as done in [1, p. 215].

Since \(D(M) \neq 0\) there is a Volterra reciprocal kernel, \(M^{-1}(t, s)\), such that

\[
M^{-1}(t, s) + M(t, s) = - \int_0^1 M^{-1}(t, u)M(u, s)du
\]

(4.15)

\[
= - \int_0^1 M(t, u)M^{-1}(u, s)du.
\]

Proceeding very much as in (4.12) one can show with the aid of (4.15) and (4.14) that the transformation (0.1) carries \(C\) onto \(C\) in a one to one manner.

**Bibliography**


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