LIKELIHOOD RATIOS FOR DIFFUSION PROCESSES
WITH SHIFTED MEAN VALUES

BY

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Let the stochastic process \( x(t) \) be the solution of the diffusion equation

\[
x(t) = \int_0^t a(r, x(r)) \, dr + \int_0^t \sigma(r) \, dy(r), \quad 0 \leq t \leq 1,
\]

\( x(0) = 0, \)

where \( y(t) \) is Brownian motion and \( a, a_2, \) and \( \sigma \) are continuous real-valued functions satisfying

(1) \( 0 < \epsilon \leq \sigma(t) \leq \frac{1}{\epsilon} \)

and

(2) \[
\frac{\partial}{\partial x} a(t, x) = a_2(t, x) = \int_{-\infty}^{\infty} i \mu e^{i\mu x} A(\mu, t) \, d\mu,
\]

\[
\int_{-\infty}^{\infty} (1 + |\mu|) | A(\mu, t) | \, d\mu \leq K < \infty.
\]

The existence and uniqueness of solutions to such equations is proved in [1, p. 277 ff]. Note that conditions (1) and (2) imply H1, H2, and H3 which are assumed there. Let \( F \) be the set of functions on the space of continuous paths from 0 to 1 of the form \( f(x) = f(x(t_1), \ldots, x(t_n)) \) where \( f \) is a bounded function on \( R^n \) with bounded second derivatives and \( (t_i) \) are points in \([0, 1]\). For each real-valued function \( m \) on \([0, 1]\) satisfying

(3) \[
m(t) = \int_0^t m'(s) \, ds, \quad \int_0^1 (m'(s))^2 \, ds < \infty,
\]

we define a group \( T_\alpha \) of transformations on \( F \) by

\[
T_\alpha f(x) = f(x(t_1) + \alpha m(t_1), \ldots, x(t_n) + \alpha m(t_n))
\]

and a set \( P_\alpha \) of probability measures by closing the functionals \( \int f \, dP_\alpha = \mathbb{E}(T_\alpha f) \)

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on $F$. (We will write $P$ for $P_0$.) It is not true in general that the $P_\alpha$ are mutually absolutely continuous nor even that the $T_\alpha$ can be extended to measurable transformations [3]. We shall prove that this is true under assumptions (1), (2), and (3) and find a formula for $\log (dP_\alpha/dP)$.

We define an operator $D$ on $F$ by

$$(Df)(x) = \frac{\partial}{\partial \alpha} (T_{\alpha}f)(x) \bigg|_{\alpha=0},$$

and a function $\phi_0$ in $L_2(P)$ by

$$\phi_0(x) = \int_0^1 \frac{m'(t) - m(t) \alpha_0(x(t), x(t))}{\sigma(t)} dy(t).$$

The existence of the stochastic integral $\phi_0$ follows from conditions (1), (2) and (3) [1, p. 426 ff].

**Lemma 1.** For any $f \in F$

$$\int \phi_0 T_\alpha f dP = \frac{\partial}{\partial \alpha} \int T_\alpha f dP.$$

**Proof.** Suppose $0 < t_1 < t_2 < \cdots < t_n < 1$ and define

$$f(\lambda, t) = \exp \left\{ i \sum_{j=1}^n \lambda_j x(t_j) + i\lambda x(t) \right\}$$

and

$$g(\lambda, t) = \int \phi_0 (\lambda, t) dP - i \left( \sum_{j=1}^n \lambda_j m(t_j) + \lambda m(t) \right) \int f(\lambda, t) dP$$

for all $\lambda$ and $t_n \leq t \leq 1$. The sample functions of the process $x(t)$ are almost all continuous and this plus the dominated convergence theorem implies that $g$ is continuous in the pair $(\lambda, t)$. Writing $d/dt$ for the right hand derivative we have

$$\frac{d}{dt} g(\lambda, t) = \lim_{\delta \to 0^+} \left[ \int \phi_0 (\lambda, t) (e^{i\delta x} - 1) \frac{\delta}{\delta} dP 

- i \left( \sum_{j=1}^n \lambda_j m(t_j) + \lambda m(t) \right) \int f(\lambda, t) (e^{i\delta x} - 1) \frac{\delta}{\delta} dP 

- i\lambda \frac{m(t + \delta) - m(t)}{\delta} \int f(\lambda, t) dP 

- i\lambda (m(t + \delta) - m(t)) \int f(\lambda, t) (e^{i\delta x} - 1) \frac{\delta}{\delta} dP \right].$$

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The first term is
\[
\lim_{\delta \to 0^+} \frac{1}{\delta} \int f(\lambda, t) \left[ \tilde{\lambda} \int_0^{t+\delta} a(r, x(r)) \, dr + \tilde{\lambda} \int_0^{t+\delta} \sigma(r) \, dy(r) \right] \, dP
\]

\[
= \tilde{\lambda} \int f(\lambda, t) a(t, x(t)) \, dP
\]

\[
+ \lim_{\delta \to 0^+} \frac{\tilde{\lambda}}{\delta} \int_0^{t+\delta} \left( \int f(\lambda, t) \left[ m'(\tau) - m(\tau) a_2(\tau, x(\tau)) \right] \, dP \right)
\]

\[
- \lim_{\delta \to 0^+} \frac{\lambda^2}{2\delta} \int_0^{t+\delta} \sigma^2(\tau) \, d\tau \int f(\lambda, t) \left[ \int_0^\tau \frac{m'(\tau) - m(\tau) a_2(\tau, x(\tau))}{\sigma(\tau)} \, dy(\tau) \right] \, dP
\]

using dominated convergence to get the first subterm and properties of the stochastic integral to get the other two. Further

\[
\lim_{\delta \to 0^+} \frac{\tilde{\lambda}}{\delta} \int_0^{t+\delta} \left( \int f(\lambda, t) \left[ m'(\tau) - m(\tau) a_2(\tau, x(\tau)) \right] \, dP \right)
\]

\[
= \lim_{\delta \to 0^+} \frac{\tilde{\lambda}}{\delta} \left( m(t + \delta) - m(t) \right) \int f(\lambda, t) \, dP - \tilde{\lambda} m(t) \int f(\lambda, t) a_2(t, x(t)) \, dP
\]

and

\[
- \lim_{\delta \to 0^+} \frac{\lambda^2}{2\delta} \int_0^{t+\delta} \sigma^2(\tau) \, d\tau \int f(\lambda, t) \left[ \int_0^\tau \frac{m'(\tau) - m(\tau) a_2(\tau, x(\tau))}{\sigma(\tau)} \, dy(\tau) \right] \, dP
\]

\[
= -\frac{\lambda^2\sigma^2(t)}{2} \int f(\lambda, t) \phi \, dP.
\]

A similar calculation shows that

\[
\lim_{\delta \to 0^+} \int f(\lambda, t) \left( e^{\frac{\tilde{\lambda} s}{\delta}} - 1 \right) \, dP = \tilde{\lambda} \int f(\lambda, t) a(t, x(t)) \, dP - \frac{\lambda^2\sigma^2(t)}{2} \int f(\lambda, t) \, dP.
\]

Incorporating these in the original formula gives

\[
\frac{d}{dt} g(\lambda, t) = -\frac{\lambda^2\sigma^2(t)}{2} g(\lambda, t) + \tilde{\lambda} \int f(\lambda, t) a(t, x(t)) \, dP
\]

\[
- \tilde{\lambda} m(t) \int f(\lambda, t) a_2(t, x(t)) \, dP
\]

\[
- \tilde{\lambda} \left( i \sum_{i=1}^{\tilde{\lambda}} \kappa m(t_i) + \tilde{\lambda} m(t) \right) \int f(\lambda, t) a(t, x(t)) \, dP.
\]
Using the Fourier transforms of $a$ and $a_2$ and interchanging $d\mu$ and $dP$ integrations gives
\[
\frac{d}{dt} g(\lambda, t) = i\lambda \int_{-\infty}^{\infty} g(\lambda + \mu, t) A(\mu, t) d\mu - \frac{\lambda^2 \sigma^2(t)}{2} g(\lambda, t).
\]
Since $|g(\lambda, t)| \leq B + C|\lambda|$
\[
\frac{d}{dt} \left( |g(\lambda, t)|^2 \right) = 2 \text{Re} \lambda \overline{g(\lambda, t)} \int_{-\infty}^{\infty} g(\lambda + \mu, t) A(\mu, t) d\mu - \frac{\lambda^2 \sigma^2(t)}{2} |g(\lambda, t)|^2 
\]
\[
\leq (B_1 |\lambda| + C_1 \lambda |\lambda|) |g(\lambda, t)| - \frac{\lambda^2 \varepsilon^2}{2} |g(\lambda, t)|^2.
\]
Suppose now $n = 0$ so that $|g(\lambda, 0)| = 0$. If, for fixed $\lambda$, $|g(\lambda, t)|$ takes its maximum at $t(\lambda) > 0$ and not before, then every interval to the left of $t(\lambda)$ contains a point where the right hand derivative of $|g(\lambda, t)|^2$ is non-negative; in particular this must be so for the $t$ where $|g(\lambda, t)|$ is a minimum for this interval. At each of these points $|g(\lambda, t)| = B + C|\lambda|^{-1}$ from the above inequality, and since max $|g(\lambda, t)|$ is a limit of these this inequality holds at all points of the interval. This implies that $g(\lambda, t)$ is bounded so that $h(t) = \sup_{x} |g(x, t)|$ is a bounded measurable function and
\[
\frac{d}{dt} \left( |g(\lambda, t)|^2 \right) \leq B_1 |\lambda g(\lambda, t)| |h(t)| - \frac{\varepsilon^2}{2} |\lambda g(\lambda, t)|^2.
\]
Now by a similar argument $|\lambda g(\lambda, t)| \leq B_2 |h(t)| \leq B_3 |\lambda|^{2/3}$ so that $d |g(\lambda, t)|^2 / dt \leq B_4 |h(t)|$. It also follows from this that $h$ is continuous since $|h(t) - h(t_0)| \leq 2B_4/|\lambda_0| + \left| \sup_{|\lambda| \leq \lambda_0} |g(\lambda, t)| - \sup_{|\lambda| \leq \lambda_0} |g(\lambda, t_0)| \right|$ and this can be made arbitrarily small by choosing first $|\lambda_0|$ large enough and then $t$ close enough to $t_0$ since the function $\sup_{|\lambda| \leq \lambda_0} |g(\lambda, t)|$ is continuous. Now it is easily shown that $|g(\lambda, t)|^2 \leq C_1 \int_{t}^{h(t)} ds$ so that $h^2(t) \leq C_1 \int_{t}^{h(t)} ds$ and hence that $h = g = 0$. If we have proved that $g(\lambda, t) = 0$ for $n \leq N$, then $g(\lambda, t_N) = 0$ and the argument above can be used on the interval $[t_N, 1]$ yielding an inductive proof that $g(\lambda, t) = 0$.

The above argument proves the lemma for $f(x) = f(x(t_1), \ldots, x(t_n))$ when $f = \exp \left\{ i \sum_j \lambda_j x_j \right\}$ and it follows from this, for any $a$ and $b$, that
\[
\int_{a}^{b} d\alpha f_0 T_\alpha dP = \int(T_\alpha f - T_\alpha f) dP
\]
for such $f$. An application of Fubini’s theorem extends this relation to all $f$ whose Fourier transforms are bounded and compactly supported, and hence to the algebra generated by these functions and the function 1 for which the lemma holds trivially. By the Stone-Weierstrass theorem such functions are uniformly dense in the algebra of functions continuous on $R^n$ and at $\infty$, so the integrated relation holds for all such functions. Since any $f$ in $F$ can be approximated by a uniformly bounded set $(f_n)$ of continuous functions of compact support converging at every (finite) point to $f$ this relation holds for all of $F$. The proof is now completed by
differentiating this relation with respect to \( b \) which can be done since \( T_\alpha f \) is \( L_2 \) continuous for \( f \) in \( F \).

Lemma 1 also holds for the conditional expectation of \( \phi_0 \) on the field generated by the \( x(t) \) for \( t \) in \([0, 1]\) which we shall call \( \phi \). For every \( f \) in \( F \) and \( \alpha \geq 0 \) the operator \( V_f(\alpha) \) defined by

\[
V_f(\alpha)g = \exp\left\{ \frac{1}{2} \int_0^\alpha T_{-\beta} d\beta \right\} T_{-\alpha} g
\]

takes \( F \) into itself and satisfies \( V_f(\alpha) V_f(\beta) = V_f(\alpha + \beta) \) and \( V_f(0) = I \). It is easily shown using Lemma 1 that

\[
\frac{\partial}{\partial \alpha} \int V_f(\alpha) g dP = \int \left( \frac{f}{2} - \phi \right) V_f(\alpha) g dP.
\]

**Lemma 2.** For any \( N \) there exists a sequence \((f_n)\) from \( F \) converging to \( \min(\phi, N) = \phi_N \) in \( L_2(P) \) and satisfying \( \sup f_n \leq N \). For any such sequence \((f_n)\), any \( g \) in \( F \), and any \( \alpha \geq 0 \) the sequence \( V_{f_n}(\alpha)g \) has a unique limit \( D_N(\alpha)T_{-\alpha}g \) in \( L_2(P) \). This convergence for fixed \( g \) is uniform in \( \alpha \) on every finite interval.

**Proof.** Since \( F \) is dense in \( L_2(P) \) there is a sequence \((g_n)\), \( g_n(x) = \delta_n(x(t_1), \ldots, x(t_j)) \) from \( F \) converging to \( \phi_N \). Then \( \min(g_n, N) \) also converges to \( \phi_N \) and the desired sequence can be gotten by convoluting \( \min(g_n, N) \) with a sequence \((c_n)\) of functions having bounded second derivatives and supports contained in sufficiently small neighborhoods of 0. It will be sufficient to prove the remainder of the lemma for \( g = 1 \). Setting \( V_f(\alpha)1 = Df(\alpha) \), we have

\[
\frac{\partial}{\partial \alpha} \int |Df_\alpha(\alpha) - Df_\alpha(\alpha)|^2 dP
\]

\[
= \frac{\partial}{\partial \alpha} \int [Df_\alpha(\alpha) - 2Df_\alpha(\alpha) + Df_\alpha(\alpha)] dP
\]

\[
= \int [(f_\alpha - \phi)Df_\alpha(\alpha) - (f_\alpha + f_m + 2\phi)Df_\alpha(\alpha) + (f_m + \phi)Df_\alpha(\alpha)] dP
\]

\[
= \int [(f_\alpha - \phi_\alpha) + (\phi_\alpha - \phi)][Df_\alpha(\alpha) - Df_\alpha(\alpha)]^2
\]

\[
+ \int (f_m - f_\alpha)(Df_\alpha(\alpha) - Df_\alpha(\alpha)) dP
\]

\[
\leq \left[ \|f_\alpha - \phi_\alpha\| + \|f_m - f_\alpha\| \right] 4e^{N_\alpha}
\]

which implies Lemma 2.

**Lemma 3.** \( \|D_N(\alpha)T_{-\alpha}g\| \leq \|g\| \) for any \( g \) in \( F \). If \( V_N(\alpha) \) is the extension of
this operator to $L_2(\mathbb{P})$ then $V_N(\alpha)$ is a strongly continuous semigroup satisfying $\|V_N(\alpha)\| \leq 1$.

Proof.

$$\frac{\partial}{\partial \alpha} \int V_{\alpha}(\alpha) g d\mathbb{P} = \int (f_{\alpha} - \phi) (V_{\alpha}(\alpha)(g^n)) d\mathbb{P} \leq \|f_{\alpha} - \phi\| e^{N\alpha} \sup |g(x)|$$

and this implies the first assertion. It is easily shown that $V_{\alpha}(\alpha) g$ is strongly continuous for any $g$ in $F$ and this plus uniform convergence in $\alpha$ implies the strong continuity of $V_N(\alpha) g$ for $g$ in $F$, but since $F$ is dense in $L_2(\mathbb{P})$ and $\|V_N(\alpha)\| \leq 1$ this is sufficient to prove the strong continuity of $V_N(\alpha)$. Finally for $g$ in $F$

$$\|V_N(\alpha) V_N(\beta) g - V_N(\alpha + \beta) g\| \leq \|V_N(\alpha) (V_N(\beta) g - V_{\alpha}(\beta) g)\| + \|V_N(\alpha) - V_{\alpha}(\alpha)\| V_N(\beta) g + \|V_N(\alpha + \beta) g - V_N(\alpha + \beta) g\| + \|V_{\alpha}(\alpha + \beta) g - V_N(\alpha + \beta) g\|,$$

which can be made arbitrarily small and this trivially implies the semigroup equality for all elements of $L_2(\mathbb{P})$.

On $F$ define $A_N = (1/2) \phi_N f - Df$ and $\hat{A} f = (1/2) \phi f - Df$, and let $A_N$ and $A$ be their respective closures.

**Lemma 4.** $A_N$ is the generator of $V_N(\alpha)$ and for any $\lambda > 0$, $f$ in $F$ and finite $a$ and $b$

$$(\lambda - A_N) \int_a^b e^{-\lambda} V_N(\alpha) f d\alpha = e^{-\lambda} V_N(\alpha) f - e^{-\lambda} V_N(b) f.$$

Proof. $\int_a^b e^{-\lambda} V_N(\alpha) g d\alpha$ is in $F$ and converges to $\int_a^b e^{-\lambda} V_N(\alpha) g d\alpha$ for all $g$ in $F$. We have

$$(\lambda - A_N) \int_a^b e^{-\alpha} V_N(\alpha) g d\alpha = \frac{1}{2} (f_{\alpha} - \phi_N) \int_a^b e^{-\alpha} V_N(\alpha) g d\alpha + e^{-\alpha} V_N(\alpha) g - e^{-\lambda} V_N(b) g \rightarrow e^{-\alpha} V_N(\alpha) g - e^{-\lambda} V_N(b) g$$

since

$$\| (f_{\alpha} - \phi_N) \int_a^b e^{-\alpha} V_N(\alpha) g d\alpha \| \leq \| f_{\alpha} - \phi_N \| \int_a^b e^{-\alpha} \sup |g(x)| d\alpha \rightarrow 0.$$

Let $A$ be the generator of $V_N(\alpha)$ and suppose that $\lambda > N$, then as $\alpha \rightarrow 0$ and
\[
\lambda \to \infty \text{ in the above we get } (\lambda - A_N)(\lambda - \overline{A})^{-1}g = g \text{ for all } g \text{ in } F \text{ and hence for all } g \text{ and this implies that } A_N \text{ contains } \overline{A}. \text{ Finally, for } g \text{ in } F
\]

\[
V_N(\alpha)g = g + \lim_{n \to \infty} \int_0^\alpha V_n(\beta)[(1/2)f_n g - Dg]d\beta
\]

\[
= g + \lim_{n \to \infty} \int_0^\alpha V_n(\beta)[(1/2)\phi_N g - Dg]d\beta
\]

\[
= g + \int_0^\alpha V_N(\beta)A_N g d\beta
\]

so \[
\overline{A}g = \lim_{n \to \infty} \frac{(V_n(\epsilon)g - g)}{\epsilon} = A_N g,
\]

which shows that \( \overline{A} \) contains \( A_N \) and completes the proof.

**Lemma 5.** \( V_N(\alpha) \) converges strongly to a strongly continuous semigroup \( V(\alpha) \) with generator \( A \).

**Proof.** It is easily seen that \( V_N(\alpha)1 \) is a nondecreasing set of functions and since \( \int V_N(\alpha)1dP \leq \| V_N(\alpha)1 \| \leq 1 \) it converges almost everywhere. This implies that \( V_N(\alpha)f \) converges for all \( f \) in \( F \) and this plus the uniform boundedness of \( \| V_N(\alpha) \| \) implies that all \( V_N(\alpha)f \) converge. If \( \lim_{N \to \infty} V_N(\alpha)f = V(\alpha)f \), then \( \| V(\alpha)f \| \leq \| f \| \) and \( \| V(\alpha+\beta)f - V(\alpha)V(\beta)f \| = \| V(\alpha)f - V_N(\alpha+\beta)f \| + \| V_N(\alpha)(V_N(\beta) - V(\beta))f \| + \| (V_N(\alpha) - V(\alpha))V(\beta)f \| \) can be made arbitrarily small. For \( f \) in \( F \)

\[
V(\alpha)f = f + \lim_{N \to \infty} \int_0^\alpha V_N(\beta)A_N f d\beta
\]

\[
= f + \lim_{N \to \infty} \int_0^\alpha V_N(\beta)A f d\beta
\]

\[
= f + \int_0^\alpha V(\beta)A f d\beta
\]

which shows that \( V(\alpha)f \) is strongly continuous for \( f \) in \( F \) and hence for all \( f \). This equation also shows, as in the proof of Lemma 4, that the generator of \( V(\alpha) \) contains \( A \).

Now if \( K_N \) is any sequence which converges to \( \infty \), \( \int_0^{K_N} e^{-\alpha} V_N(\alpha)f d\alpha \) converges to \( \int_0^\infty e^{-\alpha} V(\alpha)f d\alpha \). The proof of the lemma will be complete if we can show that \( (\lambda - A) \int_0^{K_N} e^{-\alpha} V(\alpha)f d\alpha = f \) for all \( f \) and this will be implied if we can show it for \( f \) in \( F \).

\[
(\lambda - A) \int_0^{K_N} e^{-\alpha} V_N(\alpha)f d\alpha
\]

\[
= \frac{1}{2} (\phi_N - \phi) \int_0^{K_N} e^{-\alpha} V_N(\alpha)f d\alpha - e^{-K_N} V_N(K_N)f + f.
\]
Since the second term goes to 0 it will be sufficient to show that \( \| \phi_N - \phi \| \int_0^K e^{-N} \sup |f| d\alpha \) goes to 0 for some \( K_N \) converging to \( \infty \). If \( \phi(0)_N \) is the original \( \phi_0 \) chopped at \( N \) then the conditional expectation of \( \phi_N \) on the field generated by the \( x(t) \)'s is less than or equal to \( \phi_N \). We have \( \phi_0 = \psi + \eta(1) \) where \( \psi = \int_0^t (m(s)/\sigma(s)) dy(s) \) and \( \eta(t) = -\int_0^t (m(s)a_2(s, x(s))/\sigma(s)) dy(s) \). Since \( 0 \leq \phi_0 - (\phi(0)_N) = \psi - \psi_N + \eta(1) - \eta_N(1) \) the two pieces can be handled separately. \( \psi \) is a Gaussian random variable so \( \| \psi - \psi_N \| \leq e^{-BN^k} \) for some \( B > 0 \). Also setting \( F(t) = -m(t)a_2(t, x(t))/\sigma(t) \) and defining \( \xi(t) \) to be 1 if \( \int_0^t F(s) dy(s) \leq N \) and 0 otherwise we have

\[
\eta(t) - \eta_N(t) = \int_0^t F(s) \xi(s) dy(s).
\]

Hence, using \( | F(t) | \leq C \),

\[
P(\eta(t) \geq N + 1) \leq E(| \eta(t) - \eta_N(t) |^2)
= \int_0^t E(F^2(s) \xi(s)) ds \leq C^2 \int_0^t P(\eta(s) \geq N) ds
\]

and this relation plus \( P(\eta(t) \geq 0) \leq 1 \) yields by induction \( P(\eta(t) \geq N) \leq (1)^N / N! \). This gives

\[
\| \eta(1) - \eta_N(1) \|^2 \leq \sum_{k=0}^N \frac{A^k}{k!} (k + 1 - N)^2 \leq \frac{A^N e^A}{(N - 2)!}
\]

and by Stirling's formula this goes to 0 like \( e^{BN} \log N \) for some \( D \).

**Theorem.** Under assumptions (1), (2), and (3):

(i) The measures \( P_a \) are mutually absolutely continuous.

(ii) \( (T_a) \) can be extended to a group of measurable linear transformations on all measurable functions preserving bounds and satisfying \( T_a(fg) = (T_a(f)T_a(g)) \).

(iii) \( \log dP_a/dP = \int_0^t T_a d\beta \) for any measurable version of \( T_a \).

(iv) \( V(\alpha)f = (dP_a/dP)^{1/2} T_{-\alpha}f \) is a strongly continuous unitary group with generator \( A \).

**Proof.** Since \( A \) generates a semigroup the range of \( iA - iI \) is all of \( L_a(P) \). A similar argument using \( T_a = T_{-a} \) and \( \phi = -\phi \) will prove that the range of \( iA + iI \) is all of \( L_a(P) \). These facts plus the symmetry of \( iA \) show that \( iA \) is self adjoint and this proves (iv). The proof of (i) follows from this for if \( (f_n) \) is a sequence from \( F \) decreasing to 0 almost everywhere with respect to \( P_a \) then \( T_a f_n \) decreases to 0 almost everywhere with respect to \( P \) and \( \int |f_n|^2 dP = \int T_a f_n \int |f|^2 dP = \int |D(\beta - \alpha)^2| T_a f_n \int |D| dP = 0 \) by the dominated convergence theorem. The extension of \( T_a \) to all functions measurable on the extended field is now straightforward. To prove (iii) assuming \( \alpha > 0 \), choose
a uniformly bounded sequence \((f_n, M)\) from \(F\) with \(\lim_{n \to \infty} f_n, M = \phi_{N, M}\) almost everywhere where

\[
\phi_{N, M}(x) = \begin{cases} 
N & \text{if } \phi(x) > N, \\
\phi(x) & \text{if } -M \leq \phi(x) \leq N, \\
-M & \text{if } \phi(x) < -M.
\end{cases}
\]

Now \(\int_0^\infty r^{-d}f_n, M d\beta\) converges on \(n\) to \(\int_0^\infty r^{-d}\phi_{N, M} d\beta\) in \(L_1(P)\) so a subsequence converges almost everywhere. \(n\) can be chosen as a function of \(M\) to make the sequence \(f_n(M), d\beta\) converge to \(\phi_n\) in \(L_1(P)\) and

\[
\left\| \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}f_n(M, M d\beta) \right\} - \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}\phi_{N, M} d\beta \right\} \right\| \to 0 \quad \text{as} \quad M \to \infty.
\]

Thus

\[
D_N(\alpha) = \lim \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}f_n(M, M d\beta) \right\} = \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}\phi_{N} d\beta \right\}
\]

and

\[
D(\alpha) = \left[ \frac{dp_a}{dp} \right]^{1/2} = \lim \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}\phi_{N} d\beta \right\} = \exp\left\{ \frac{1}{2} \int_0^\infty r^{-d}\phi d\beta \right\}.
\]

The restriction in the above theorem that \(\sigma\) not depend on \(x(t)\) seems essential for without it one is led intuitively to the formula

\[
\phi(x) = \int_0^1 \frac{m(t)\sigma_2(t, x(t))y(t)}{\sigma(t)} dy(t) + \int_0^1 \frac{m'(t) - m(t)\sigma_2(t, x(t))}{\sigma(t)} dy(t)
\]

and the first integral is rather strongly "divergent."

The technique used in this paper was suggested by the solution to the corresponding problem where \(x(t)\) is a Gaussian stochastic process. There, if \(R(s, t) = E(x(s)x(t))\) is the autocorrelation function for \(x(t)\) and if \(m(s) = \int_0^s R(s, t) dF(t)\) the infinitesimal generator again has the form \(Af = (1/2)\phi f - Df\) where now \(\phi(x) = \int_0^s x(t) dF(t)\) [2]. We hope in future papers to apply this technique to other similar problems.

**References**


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