MAXIMAL COMMUTATIVE ALGEBRAS OF LINEAR TRANSFORMATIONS‡

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The Problem; a Summary of the Main Results. In this paper $K_n$ shall denote the algebra of $n$ by $n$ matrices over a field $K$ or, equivalently, the algebra of linear transformations over an $n$-dimensional vector space. We attack the problem of determining conditions which imply that a commutative subalgebra of $K_n$ is maximal commutative by seeking conditions on a commutative ring $R$ with identity element and on a unital right $R$-module $M$ satisfying the A.C.C. and the D.C.C. which imply:

$$(P_1) \quad \text{Hom}_R(M, M) = R^*$$

where $R^*$ denotes the set of endomorphisms $\{a_n | x \rightarrow xa \text{ for } x \in M, a \in R\}$. The latter problem shall be referred to as the Centralizer Problem.

The following definition, due to E. Snapper [13, p. 125], is a prerequisite to one of the known results on the Centralizer Problem:

**Definition.** An $R$-module $M$ is said to be completely indecomposable provided that (1) $R$ is a commutative ring with identity, (2) $M$ satisfies the D.C.C. and the A.C.C., and (3) every submodule of $M$ is indecomposable. (We prove in Theorem 1.5 that, if such a module is faithful, the chain conditions hold in $R$, also.)

An equivalent form [13, p. 127, Remark 1.2] of the preceding definition replaces (3) by (3') : $M$ contains a unique irreducible submodule.

Two instances in which $(P_1)$ holds are:

$(P_2)$ $M$ is a cyclic $R$-module for a commutative ring $R$ with identity.

$(P_3)$ $M$ is a completely indecomposable module.

Let $(P_2)$ hold and let $E \subseteq \text{Hom}_R(M, M)$. If $M$ can be generated by $x \in M$ then for some $r \in R$, $E(x) = xr$ and it is easily seen that the mapping $m \rightarrow mr$, $m \in M$, is identical with $E$. The nontrivial result "$(P_2) \Rightarrow (P_1)$" is Snapper's [13, p. 129, Theorem 3.1]. The result intersects this paper only in a special instance, that in which $M$ is a finite dimensional vector space over a field $K$ and $R$ is a commutative subalgebra of $K_n$ containing the identity transformation. For this case the truth of Snapper's theorem is made apparent in §3 (Theorem 3.4 and Remark).

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I. Schur [12] proved that the maximal dimension of a maximal commutative subalgebra of $K_n$ is $[n^2/4] + 1$, provided $K$ is an algebraically closed field. This result was generalized to an arbitrary field by N. Jacobson [8]. Also in [8], it is proved that if $K$ is not an imperfect field of characteristic 2 then (up to conjugacy) exactly two algebras have the maximal dimension when $n$ is odd and greater than one, and exactly one when $n$ is even. Suprunenko [14] proved that if $K$ is algebraically closed and if $n > 6$ there exist infinitely many nonisomorphic maximal commutative subalgebras of $K_n$. Work on the problem has been published recently by B. Charles [3] and H. S. Allen [1].

A summary of the main results of this paper follows, beginning with the

Notation. If a function $(s_1, s_2) \rightarrow s_1 s_2$ is defined on the Cartesian product of two sets $S_1$ and $S_2$ into a third set with zero element, $\text{Ann} (S_1, S_2)$ shall denote $\{x | x \in S_2 \text{ and } (s_1, x) \rightarrow 0 \text{ for every } s_1 \in S_1\}$. A similar definition holds for $\text{Ann} (S_2, S_1)$.

In §1 the Centralizer Problem as stated above is reduced to the case in which $M$ is an indecomposable module with chain conditions (Theorem 1.1). It follows (Theorem 1.5) that $R/P$ is a field, where $P = \text{rad} (\text{Ann} (M, R))$, and that $MP^e = 0$ for some integer $e > 0$. With these restrictions on the ring, it is proved in the main theorem (§2, Theorem 2.7) that an $R$-module $M$ satisfies $(P_1)$ and is indecomposable provided $MP \neq 0$ and $M/MP$ has a finite $(R/P)$-basis $x_1 + MP, \ldots, x_k + MP, k > 1$, such that for $i = 1, \ldots, k$:

$(C_1) \quad \text{Ann}(P, M) \subseteq x_i P$;

$(C_2) \quad P = \text{Ann}(x_i, R) + \bigcap_{i \neq i'} \text{Ann}(x_i, R)$.

Because of the Chinese Remainder Condition $(C_2)$, such a module is referred to as an $R$-module $[C]$. Example 4.1 in §4 is an $R$-module $[C]$ which does not satisfy $(P_2)$ or $(P_3)$ and Example 4.2 is an $R$-module $[C]$ which satisfies neither chain condition. It is an inference of Theorem 2.3 and motivation as well for the definition of $R$-module $[C]$, that $M$ is one provided $R/P$ is a field, $MP$ is finitely generated and $\neq 0$, $1 < (R/P)\text{-dim } (M/MP) < \infty$, $MP^2 = 0$, and $(P_1)$ holds.

The application of $(P_2)$, $(P_3)$, and Theorem 2.7 to commutative algebras of linear transformations, with maximal radical $P$, acting on an $n$-dimensional vector space over a field $K$ is considered in §§3 and 4. For the class of maximal commutative subalgebras of $K_n$ so obtained it is shown in Theorem 3.6 that, if $R/P \cong K$, then $\dim R - \dim M = (k - 1)(d - 1)$, where $k$ and $d$ are the dimensions of $M/MP$ and of $M/(\text{Ann} (P, M))$, respectively. Example 4.3, a maximal commutative subalgebra of $K_6$ with $R/P \cong K$, fails to satisfy Theorem 3.6, proving that the sufficient conditions considered here do not combine to make a necessary condition for $(P_1)$.

1. Reduction of the problem.
Theorem 1.1. Let $R$ be a commutative ring with identity and let $M$ be an $R$-module which is a direct sum $M_1 \oplus \cdots \oplus M_k$ of $k$ submodules. For $i=1, 2, \cdots, k$, let $E_i$ denote the projection of $M$ onto $M_i$. Then (P$_1$) is equivalent to

\begin{equation}
\text{for } i=1, 2, \cdots, k, \text{ there is an element } e_i \in R \text{ such that } E_i(m) = me_i \text{ for all } m \in M
\end{equation}

and

\begin{equation}
\text{Hom}_R(M_i, M_i) = R^* = R_i^* = \text{Hom}_R(M_i, M_i) \text{ for each } i, \text{ where } R_i \text{ denotes } e_i R (1 \leq j \leq k) \text{ and where } R^* \text{ is viewed as acting on } M_i.
\end{equation}

Proof. From the definitions we have $\text{Hom}_R(M_i, M_i) \subseteq \text{Hom}_R(M_i, M_i)$ and $R_i^* \subseteq R^*$, and we prove the reverse inclusions, assuming (1.1). If $E \in \text{Hom}_R(M_i, M_i)$ and $m \in M_i$ then $E(m) \in M_i$, so that $E(m)e_i = E(m)$. Now for $r \in R$, $E(mr) = E(me, r) = E(m)e, r = E(m)r$, proving $E \in \text{Hom}_R(M_i, M_i)$. If $f \in R^*$, let $r \in R$ satisfy $f(m) = mr$, $m \in M_i$. From $m = me_i$ we have $f(m) = me_r$, proving $f \in R^*$. Thus $R^* = R_i^*$ and $\text{Hom}_R(M_i, M_i) = \text{Hom}_R(M_i, M_i)$, assuming (1.1). The theorem is now reduced to the equivalence of (P$_1$) with (1.1) and

\begin{equation}
\text{Hom}_R(M_i, M_i) = R^*.
\end{equation}

Assuming (1.1) and (1.3), let $H \in \text{Hom}_R(M, M)$. It is easily verified that $H_i(= H_{[M_i]} \in \text{Hom}_R(M_i, M_i), 1 \leq i \leq k$. By (1.3) there exist elements $b_i \in R$, $1 \leq i \leq k$, such that $H_i(m) = mb_i$, $m \in M_i$. If $c = \sum_i e_i b_i$ and $m = \sum_i m_i$ is an element of $M$, then $H(m) = H(\sum m_i) = \sum H_i(m_i) = \sum H_i(m,e_i) = \sum m_i e_i b_i = \sum me_i b_i = m \sum e_i b_i = mc$, proving (P$_1$). If (P$_1$) holds then (1.1) is immediate, since it is easily verified that $E_i \in \text{Hom}_R(M, M)$. Let $G \in \text{Hom}_R(M_i, M_i)$ and define $H$ by $H(m) = G[E_i(m)]$, $m \in M$. Then $H = G \cdot E_i \in \text{Hom}_R(M, M)$, and by (P$_1$) there is an element $h \in R$ such that $H(m) = mh$, $m \in M$. For $m \in M_i$, $mh = H(m) = G[E_i(m)] = G(m)$, proving (1.3). Q.E.D.

Remark. If $M$ satisfies the D.C.C., $M$ is a direct sum of a finite number of indecomposable submodules [15, p. 169, Theorem 28]. The Centralizer Problem for such a module is reduced by Theorem 1.1 to the case in which $M$ is indecomposable. The further reduction of the problem to modules for a completely primary, commutative ring assumes both chain conditions for $M$, and is the topic next investigated.

The terms prime ideal, primary ideal, and rad $J$ for an ideal $J$ shall have the conventional definitions as given in [11, pp. 9–13].

Theorem 1.2. If $R$ is a commutative ring and $M$ is an indecomposable $R$-module satisfying both chain conditions, then $\text{Ann}(M, R)$ is a primary ideal and $\text{rad}(\text{Ann}(M, R))$ is a prime ideal.
Proof. It is well known [6, p. 174] that the radical of a primary ideal is a prime ideal. To prove that Ann\((M, R)\) is primary, we observe first that, since the indecomposable module \(M\) satisfies the chain conditions, every non-nilpotent \(R\)-endomorphism of \(M\) is an automorphism (Fitting's Lemma [6, pp. 155–156]). If \(Q = \text{Ann}\((M, R)\) were not primary, there would exist in \(R\) elements \(s\) and \(t\) with \(st \in Q\) and \(s, t \in Q\), for all positive integers \(n\). Now \(s^n \in Q\) implies that \(M_s^\infty \neq 0\) and \(s^n \in Q\) implies that the kernel of the map \(E: m \rightarrow mt, m \in M\), contains the nonzero submodule \(Ms\). Consequently, \(E\) is not an automorphism and must be nilpotent. But \(t^n \in Q\) for \(n = 1, 2, \cdots\) contradicts nilpotency.

**Theorem 1.3.** If \(R\) is a commutative ring with identity, then the D.C.C. holds for ideals if, and only if,

1. the A.C.C. holds for ideals;
2. every prime ideal different from \(R\) is maximal.

A proof of Theorem 1.3 appears in [15, p. 203, Theorem 2].

**Theorem 1.4.** If \(R\) is a commutative ring with identity and \(M\) is an \(R\)-module which satisfies the A.C.C., then \(R/\text{Ann}(M, R)\) satisfies the A.C.C. for ideals.

This result appears in [4, p. 245, Theorem 4]. The proof uses the fact that if \(x \in R\) the module \(R/\text{Ann}(x, R)\) is \(R\)-isomorphic to the submodule \(xR\) of \(M\) and hence satisfies the A.C.C. Then if \(\{x_1, x_2, \cdots, x_n\}\) is a set of generators of \(M, R/\text{Ann}(M, R) = R/(\bigwedge_{i=1}^n \text{Ann}(x_i, R))\) satisfies the A.C.C. as a result of the theorem [4, p. 242, Theorem 3]: Let \(G\) be a group with operators \(R\) and let \(G_1, G_2, \cdots, G_n\) be normal \(R\)-subgroups of \(G\). If the A.C.C. holds for \(R\)-subgroups of \(G/G_i, 1 \leq i \leq n\), then the A.C.C. holds for \(R\)-subgroups of \(G/(G_1 \cap \cdots \cap G_n)\).

**Definition.** A commutative ring \(R\) is said to be completely primary provided that \(R\) contains an identity element and that \(R/J\) is a field, where \(J\) is the Jacobson radical of \(R\). (If \(R\) satisfies the D.C.C. for ideals, \(J = \text{rad} 0 [9, pp. 38–39]\), so that the second requirement reads: \(R/\text{rad} 0\) is a field. This form of the definition will be used when applicable without further comment.)

**Definition.** Let \(R\) be a commutative ring and let \(J\) be an ideal of \(R\). If \(M\) is an \(R\)-module, then the exponent of \(J\) relative to \(M\) is defined to be

\[
\begin{cases} 
\text{the positive integer } e \text{ (if it exists) such that } J^e \text{ (but not } J^{e-1} \text{) } \subseteq \text{Ann}(M, R) \cap J^n; \\
\infty, \text{ otherwise.}
\end{cases}
\]

**Theorem 1.5.** If \(M\) is a faithful indecomposable \(R\)-module having composition length \(h < \infty\), where \(R\) is a commutative ring with identity, then

\( (*) \) The convention \(J^* = R\) (for ideals \(J\) of \(R\)) is used.
1. R satisfies both chain conditions for ideals,
2. R is completely primary,
3. $e \leq h$ where $e$ is the exponent of $P[=\text{rad } 0]$.

**Proof.** For $i = 1, \ldots, h$ let $B_i = \{ r | r \in R$ and $M_{i-1}r \subseteq M_i \}$, where $M = M_0 \supset M_1 \supset \cdots \supset M_h = \{0\}$ is a composition series of $M$. If $x \in M_{i-1}, x \in M_i$, the irreducibility of $M_{i-1}/M_i$ implies $(x + M_i)^{}R = M_{i-1}/M_i$. Thus the mapping that takes $r$ onto $(x + M_i)^{}r, r \in R$, is an $R$-homomorphism of $R$ onto $M_{i-1}/M_i$. Its kernel is $\{ r | (x + M_i)^{}r \subseteq M_i \} = \{ r | M_{i-1} \subseteq M_i \} = B_i$. The isomorphism of $R/B_i$ with irreducible module $M_{i-1}/M_i$ proves that $B_i$ is a maximal proper ideal. Thus $R/B_i$ is a field for $i = 1, 2, \ldots, h$, so that the ideal $P$ of nilpotent elements is contained in $\bigcap_{i=1}^h B_i$. Since $M$ is indecomposable and satisfies both chain conditions by hypothesis, Theorem 1.2 asserts that $P$ is a prime ideal. From the definition of the ideals $B_i$ we see that $\prod_{i=1}^h B_i$ annihilates each element of $M$. Thus $\prod_{i=1}^h B_i \subseteq \{0\} \subseteq P$ and by the primality of $P$ we have $B_i \subseteq P$ for some $i$. $P$, then, is maximal and from $P \subseteq B_i$ for all $i$ we have $P = B_i (1 \leq i \leq h)$. Thus $P^h = \prod_{i=1}^h B_i = 0$, proving (3). By Theorem 1.4 $R$ satisfies the A.C.C. for ideals. By Theorem 1.3 $R$ will also satisfy the D.C.C. if every prime ideal different from $R$ is maximal. If $P \neq R$ is a prime ideal, then $Q = \text{rad } (1 + \text{rad } 0) = P$, so that $Q$ is the maximal ideal $P$. Thus $R$ satisfies the chain conditions, which is conclusion (1). Since $\text{rad } 0$ is a maximal ideal and $R$ satisfies both chain conditions, Theorem 1.2 asserts that $R$ is completely primary.

**Theorem 1.6.** Let $M$ be an $R$-module for a commutative ring $R$ with identity such that $\text{Hom}_R(M, M) = R^*$. Assume also that $R$ has a proper ideal $P$ which contains every proper ideal containing $\text{Ann}(M, R)$. Then $M$ is indecomposable.

**Proof.** If $M_1$ and $M_2$ are nonzero submodules of $M$ such that $M = M_1 \oplus M_2$, we have for $i = 1, 2, \text{Ann}(M, R) \subseteq \text{Ann}(M_i, R) \subseteq R$ (since $1 \notin \text{Ann}(M_i, R)$), whence $P \supseteq \text{Ann}(M_i, R)$. If $E$ is the projection of $M$ onto $M_1, E \in \text{Hom}_R(M, M)$, so that by hypothesis there exists an element $b \in R$ such that $E(m) = mb$ for all $m \in M$. Clearly $0 = M_1(1 - b) = M_2b$. Thus the false statement

$$1 = (1 - b) + b \in \text{Ann}(M_1, R) + \text{Ann}(M_2, R) \subseteq P$$

is implied, proving that $M$ is indecomposable.

**Corollary.** Let $M$ be an $R$-module for a commutative ring $R$ with identity such that $P = \text{rad}(\text{Ann}(M, R))$ is maximal in $R$. Then, if $\text{Hom}_R(M, M) = R^*$, $M$ is indecomposable.

**Proof.** If $Q$ is a proper ideal containing $\text{Ann}(M, R), \text{rad } Q$ is a proper ideal since $1 \notin \text{rad } Q$. Now from $P = \text{rad}(\text{Ann}(M, R)) \subseteq Q, P$ maximal, we have $P = \text{rad } Q$. Consequently, $Q \subseteq P; P$ contains every ideal containing $\text{Ann}(M, R)$. Since the hypotheses of the theorem are satisfied, $M$ is indecomposable.
DEFINITION. An element \( u \) of an \( R \)-module \( M \) is said to be a nongenerator of \( M \) if a subset \( S \) of \( M \) generates \( M \) whenever \( S \cup \{ u \} \) generates \( M \).

REMARK. The set \( \phi \) of nongenerators of \( M \) is the intersection of the maximal proper \( R \)-submodules (\( \phi = M \) if there are no maximal proper submodules). This characterization of \( \phi \) is proved for \( R \)-modules as it is proved for groups [10, p. 217]. Evidently \( \phi \) is a submodule of \( M \).

THEOREM 1.7. Let \( M \neq 0 \) be an \( R \)-module for a commutative ring \( R \) with identity such that \( P = \text{rad}(\text{Ann}(M, R)) \) is a maximal ideal with finite exponent \( e \). Then \( MP \) is the submodule \( \phi \) of nongenerators of \( M \).

Proof. \( M \neq MP \), for otherwise \( M = MP = \cdots = MP^e = \{0\} \) contradicting \( M \neq 0 \). If \( x \in MP \) let \( \{x_\alpha + MP\} \) for ordinals \( \alpha = 1, 2, \cdots \) be an \((R/P)\)-basis of \( M/MP \) with \( x = x_1 \). If \( Y \) denotes \( MP + \sum_{\alpha \in \Gamma} x_\alpha R, x \notin Y \) (since the \( x_\alpha \) are independent mod \( MP \)). Clearly the proper submodule \( Y \) is a maximal proper submodule. Since \( x \notin Y \), \( x \) does not belong to the intersection \( \phi \) of the maximal submodules. Thus the complement of \( \phi \) contains the complement of \( MP \). To prove \( \phi \subseteq MP \) we need only prove \( M_i \supset MP \) for every maximal proper submodule \( M_i \). Since \( M_i \supset M = MP \) and \( M_i \supset \{0\} = MP \), there exists a positive integer \( k \) such that \( MP^k \subseteq M_1 \) and \( MP^{k+1} \subseteq M_1 \). Let \( y \) be an element of \( MP^k, y \in M_1 \). Since \( M/M_1 \) is irreducible \( y + M_1 \) generates \( M/M_1 \). If \( x \in M_1 \), \( x + M_1 = (y + M_1)r \) for some \( r \in R \); equivalently, \( x = yr \) modulo \( M_1 \). Then for all \( p \in P \), \( xp = yp \) modulo \( M_1 \) and, since \( yp \subseteq MP^{k+1} \subseteq M_1 \), we have \( xp \subseteq M_1 \). Now \( xp \subseteq M_1 \) for \( x \in M_1 \) and \( M_1 \subseteq MP \) imply \( MP \subseteq M_1 \). Thus \( MP \subseteq \phi \), \( MP = \phi \).

COROLLARY. As an additional hypothesis let \( MP \) be finitely generated; then

1. \( M = \sum_{i \in S} x_i R \) and
2. \( MP = \sum_{i \in S} x_i P \) where \( S \) is a set of indices and \( \{ x_i + MP \} \) is an \((R/P)\)-basis of \( M/MP \). The \((R/P)\)-dimension of \( M/MP \) will be finite \((k, \text{say}) \) if \( M \) satisfies the A.C.C. and we will have

3. \( M = \sum_{i=1}^k x_i R \)
4. \( MP = \sum_{i=1}^k x_i P \).

Proof. Let \( MP = \sum_{i=1}^n y_i R \). By the theorem the \( y_i \) are nongenerators of \( M \) and we conclude that

\( M = \sum_{i=1}^n x_i R + \sum_{i=1}^n y_i R = \sum_{i=1}^n x_i R + \sum_{i=1}^{n-1} y_i R = \cdots = \sum_{i=1}^n x_i R + y_i R = \sum_{i=1}^n x_i R \). If \( z = \sum_{i=1}^n x_i T_i \subseteq MP \) then, by the independence mod \( MP \) of the \( x_i \), \( T_i \in P \) for \( \alpha = 1, 2, \cdots, n \). Thus \( MP \subseteq \sum x_i P \) and the reverse inequality is trivial. This completes the proof of (1) and (2).

If there is an infinite set of elements \( x_i, i = 1, 2, \cdots \), which are \((R/P)\)-independent mod \( MP \), a strictly ascending sequence \( x_i R + MP \subseteq x_i R + x_{i+1} R + MP \subseteq \cdots \) is obtained; thus the assumption of the A.C.C. for \( M \) implies a finite dimension for \( M/MP \). The A.C.C. also implies that \( MP \) is finitely generated, so that statements (1) and (2) hold and imply (3) and (4).

THEOREM 1.8. Let \( M \) be a faithful \( R \)-module satisfying the A.C.C. for \( R \)-modules, where \( R \) is a commutative ring with identity. Let \( P \) be an ideal having
finite exponent $e$ such that $R/P$ is a field. Then $M$ is cyclic if, and only if, the $(R/P)$-dimension of $M/MP$ is unity.

Proof. Since $M$ satisfies the A.C.C., the $(R/P)$-dimension of $M/MP$ is finite by the corollary to Theorem 1.7. If $k = (R/P)$-dim $M/MP$ and $x_1, \ldots, x_k$ are $(R/P)$-independent mod $MP$, then by the corollary to Theorem 1.7 $M = \sum x_iR$; if $k = 1$, $M = x_1R$. If, conversely, $M = xR$ for some $x \in M$, the mapping $r \mapsto xr, r \in R$, is an isomorphism of $R$ onto $M$, since $M$ is faithful. Since by hypothesis $P$ is a maximal ideal of $R$, the isomorphic image $xP$ of $P$ is a maximal submodule of $M$. As in the proof of Theorem 1.7 we have $MP \neq M$ from $e < \infty$, and from $xP \subseteq MP \subseteq M$ we conclude that $MP$ is the maximal submodule $xP$. By the $R$-isomorphism $r \mapsto xr, r \in R$, $\dim M/MP = \dim M/xP = \dim (R/P) = 1$.

2. General results on the Centralizer Problem. Investigation of $\text{Hom}_R(M, M) = R^*$, where $R$ is a commutative ring with identity and $M$ is an indecomposable $R$-module with both chain conditions, replaces the problem for the more general module $M$ with chain conditions (Theorem 1.1 and Remark). It is clear that the radical $P$ of $\text{Ann}(M, R)$ is a maximal ideal since, by Theorem 1.5, $P$ is maximal among the ideals containing $\text{Ann}(M, R)$. By the same theorem the exponent $e$ of $P$ is finite, and $R/\text{Ann}(M, R)$ satisfies both chain conditions. Since $M$ satisfies the A.C.C. the $(R/P)$-space $M/MP$ has finite dimension (corollary to Theorem 1.7). The A.C.C. for $M$ also implies that the $(R/P)$-space $\text{Ann}(P, M)$ has finite dimension and that $MP$ is finitely generated.

The following lemma holds for an arbitrary ring $R$ of operators. Its proof is straightforward.

Lemma 2.1. Let a group $M$ with operators $R$ contain a set of elements $\{x_1, \ldots, x_k\}$ such that $M = \sum x_iR$. For a fixed $y \in M$ and $j \in \{1, \ldots, k\}$ let $E$ be the mapping defined by $E(\sum x_i) = yr$. Then, if $E$ is well-defined, $E \in \text{Hom}_R(M, M)$.

Lemma 2.2. Let $R$ be a commutative ring with identity element and maximal ideal $P$. Let $M$ be an indecomposable $R$-module such that $\text{Ann}(M, R) \subseteq P$ and $MP \neq 0$. Assume that $M = \sum x_iR$ holds for any set $\{x_i\}$ such that $\{x_i + MP\}$ is an $(R/P)$-basis of $M/MP$. Then $\text{Ann}(P, M) \subseteq MP$.

Proof. The result is obvious if $MP = M$. Let $MP \neq M$ and suppose the conclusion false. If $x \in MP$ is an element of $\text{Ann}(P, M)$, let $x + MP$ be the first element of a basis $\{x_i + MP\}$ of $M/MP$. If $\dim (M/MP) = 1$ we have $M = xR$ by hypothesis. Then $MP = xRP \subseteq xP \subseteq \text{Ann}(P, M) \cdot P = \{0\}$, contrary to hypothesis. If $\dim (M/MP) > 1$, $x_iR$ and $\sum_{i \geq 1} x_iR$ are nonzero submodules. If $0 = \sum x_i b_i, b_i \in R$, then, by the independence of the $x_i, b_i \in P$ for each $i$. Consequently, $x_i b_1 \in \text{Ann}(P, M) \cdot P = \{0\}$ and $\sum_{i \geq 1} x_i b_i = 0$. Thus the
Theorem 2.3. Let $R$ be a commutative ring with identity and let $M$ be an $R$-module satisfying $\text{Hom}_R(M, M) = R^*$. Let $P$ denote $\text{rad}(\text{Ann}(M, R))$ and assume further that $R/P$ is a field, that $P$ has finite exponent $e > 1$, that $MP$ is finitely generated, and that the dimension $k$ of the $(R/P)$-space $M/MP$ is finite. Then the following conclusions hold:

1. $M$ is indecomposable;
2. $\text{Ann}(P, M) \subseteq MP$;
3. if $x \in MP$, $\text{Ann}(P, M) \subseteq xP$;
4. $\text{Ann}(MP, R) \subseteq \text{Ann}(x_i, R) + \sum_{j \neq i} \text{Ann}(x_j, R)$, for any set $\{x_1, \ldots, x_k\}$ of $k$ elements $(R/P)$-independent mod $MP$;
4'. using the notation of (4), $P = \text{Ann}(x_i, R) + \sum_{j} \text{Ann}(x_j, R)$, for any set $\{x_1, \ldots, x_k\}$ of $k$ elements $(R/P)$-independent mod $MP$;
5. $(R/P)\dim [\text{Ann}(MP, R)/\text{Ann}(M, R)] = kd$, where $d = (R/P)\dim \text{Ann}(P, M)$ is assumed finite.

Proof. Since $\text{Hom}_R(M, M) = R^*$ and $\text{rad}(\text{Ann}(M, R))$ is maximal in $R$, the corollary to Theorem 1.6 asserts the indecomposability of $M$, which is conclusion (1). $MP$ is assumed to be finitely generated and $e$ is assumed finite in order that $M = \sum x_i R$ hold for any set $\{x_1, \ldots, x_k\}$ of $k$ elements $(R/P)$-independent mod $MP$ (corollary to Theorem 1.7). We also have $MP \neq 0$ from $e > 1$ and the conclusion of Lemma 2.2, $\text{Ann}(P, M) \subseteq MP$, follows. Thus conclusion (2) is proved.

If (3) is false, let $x \in MP$ and $y \in \text{Ann}(P, M)$ be such that $y \notin xP$. Let $x = x_1, x_2, \ldots, x_k$ be $k$ elements $(R/P)$-independent mod $MP$. Let $E$ be the mapping defined by $E(\sum x_i b_i) = y b_i$, $b_i \in R$. Then $E$ is well defined; for if $0 = \sum x_i b_i$, then, by the independence of the $x_i$, $b_i \in P$ for each $i$ and $y b_i \in \text{Ann}(P, M) \cdot P = \{0\}$. By Lemma 2.1 $E \in \text{Hom}_R(M, M)$ and there is by hypothesis an element $h \in R$ such that $E(m) = mh$ for all $m \in M$. If $h \notin P$ the coset $h + P$ has an inverse $g + P$ for some $g \in R$; thus $gh = 1 + t$ for some $t \in P$. From

$$x_1 + x_it = x_1 gh = E(x_1 g) = yg \in \text{Ann}(P, M) \subseteq MP$$

and $x_it \in MP$, we have $x_i \in MP$ contrary to the choice of $x_i$. We have $h \in P$, whence $y = E(x_1) = x_1 h \in x_1 P$, contrary to the choice of $y$ and $x_1[= x]$, which proves (3).

In the remainder of the proof $\{x_1, \ldots, x_k\}$ denotes a set of $k$ elements independent mod $MP$.

Toward proving (4) let $i \in \{1, 2, \ldots, k\}$ and let $q \in \text{Ann}(MP, R)$. Let $E$ be the mapping defined by

$$E\left(\sum_{j=1}^{k} x_j b_j\right) = x_j b_j q,$$

$b_j \in R$. 

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$E$, then, is well-defined since $E$ takes $0 = \sum x_i b_i$ onto an element $x_i b_i q \in MP \cdot Ann(M, P) = \{0\}$.

By Lemma 2.1, $E \in \text{Hom}_R(M, M)$ and by hypothesis there exists an element $h \in R$ such that $E(m) = mh$ for all $m \in M$. In particular, $x_j h = E(x_i) = x_i q$ and $x_j h = E(x_i) = 0$ if $j \neq i$ or, equivalently, $h \in \bigcap_{i \neq k} Ann(x_i, R)$ and $q - h \in Ann(x_i, R)$. From $q = (q - h) + h$, we have (4).

If $e = 2$, $P \subseteq Ann(M, P)$ and we have $P \subseteq Ann(x_i, R) \cap \bigcap_{i \neq k} Ann(x_i, R)$ (1 ≤ $i$ ≤ $k$) by conclusion (4). The reverse inequality holds, since the $(R/P)$-independence of the $x_i$ implies $Ann(x_i, R) \subseteq P$, $s = 1, \ldots, k$, proving (4').

If $y_1, y_2, \ldots, y_d$ are $d$ $(R/P)$-independent elements of $Ann(P, M)$ and $k_0$ mappings $E_{ij}, 1 \leq i \leq k, 1 \leq j \leq d$, are defined by

$$E_{ij} \left( \sum_{i=1}^{k} x_i b_i \right) = y^i_j b_i,$$

the $E_{ij}$ take $0 = \sum x_i b_i$ onto $y^i_j b_i \in Ann(P, M) \cdot P = \{0\}$. By Lemma 2.1 $E_{ij} \in \text{Hom}_R(M, M)$, and by hypothesis there exist $k_0$ elements $u_{ij} \in R$ such that $E_{ij}(m) = mu_{ij}$ for all $m \in M$. We observe that $Mu_{ij} \subseteq \sum_{i=1}^{d} y_i R = Ann(P, M)$. Now for $m \in M$ and $p \in P$, $mpu_{ij} \in Ann(P, M) \cdot P = \{0\}$, so that $u_{ij} \in Ann(M, P)$ for all $i$ and $j$. To prove the independence of the elements $u_{ij}$ we suppose that $\sum_{i=1}^{s} u_{ij} r_{ij} = 0$ with the $r_{ij} \in R$. Then, for fixed $s$, $x_s (\sum_{i=1}^{d} u_{ij} r_{ij}) = 0$ implies $\sum_{i=1}^{d} E_{ij}(x_i) r_{ij} = \sum_{i=1}^{d} y^i_j r_{ij} = 0$, whence $\{r_{11}, \ldots, r_{sd}\} \subseteq P$ by the independence of the $y_j$. This proves the $(R/P)$-independence of the $u_{ij}$. To prove that the $k_0$ element $u_{ij}$ generate $Ann(M, P)$ modulo $Ann(M, P)$, we let $h \in Ann(M, P)$ and, for $i = 1, 2, \ldots, k$, have $x_i h = \{0\}$ or, equivalently, $x_i h \in Ann(P, M)$. Thus $x_i h = \sum_{i=1}^{d} y^i_j b_i$ with $t_{ij} \in R$. For any element $\sum_{i=1}^{d} x_i b_i$ of $M$ we have:

$$\sum_{i=1}^{k} x_i b_i h = \sum_{i,j} y^i_j t_{ij} = \sum_{i,j} (x_i b_i t_{ij}) = \sum_{i=1}^{k} (x_i b_i) \left( \sum_{i,j} t_{ij} u_{ij} \right)$$

whence $h - \sum_{i,j} t_{ij} u_{ij} \in Ann(M, P)$. Thus the $(R/P)$-dimension of $Ann(M, P)/Ann(M, P)$ is $kd$. Q.E.D.

**Corollary.** If $R$ is a commutative ring with identity and $M \neq \{0\}$ is an indecomposable module satisfying both chain conditions and $\text{Hom}_R(M, M) = R^*$, then either the conclusions (2) through (5) of the theorem hold or $Ann(M, R)$ is its own radical $P$. If $P = Ann(M, R)$, $M$ is a 1-dimensional $(R/P)$-space.

**Proof.** Except for $e > 1$ the hypotheses of Theorem 2.3 were deduced from the hypotheses of this corollary in §1 and listed as opening observations of §2. Thus we have conclusions (2) through (5) or $e = 1$ (equivalently, $P = Ann(M, R)$). A vector space over a field $F \cong (R/P)$ with dimension $\geq 2$ has nonscalar linear transformations, that is, transformations not in $R^*$. Thus the $(R/P)$-dimension of $M \neq \{0\}$ is unity in the case $P = Ann(M, R)$.

According to conclusion (4') of Theorem 2.3, a necessary condition for $\text{Hom}_R(M, M) = R^*$ (when it is assumed that $e = 2$, $k \geq 2$) is that $P$ satisfy a
Chinese Remainder Condition with respect to a set of ideals. For \( k \geq 2 \) and arbitrary exponent \( e > 1 \), conclusions (3) and (4') combine to make a sufficient condition. To facilitate the discussion we make the

**Definition of R-module \([C]\).** Let \( M \) be an \( R \)-module for a commutative ring \( R \) with identity such that \( R/P \) is a field where \( P = \text{rad}(\text{Ann}(M, R)) \). \( M \) shall be called an \( R \)-module \([C]\)—for Chinese Remainder Condition—provided the exponent \( e \) of \( P \) and the \((R/P)\)-dimension \( k \) of \( M/MP \) are finite and greater than one; and provided further that there exist \( k \) elements \( x_1, \ldots, x_k \) \((R/P)\)-independent modulo \( MP \) such that \( M = \sum_{i=1}^{k} x_i R \) and that for \( i = 1, 2, \ldots, k \)

\[
\begin{align*}
\text{(C1)} & \quad \text{Ann}(P, M) \subseteq x_i P; \\
\text{(C2)} & \quad P = \text{Ann}(x_i, R) + \bigcap_{s \neq i} \text{Ann}(x_s, R).
\end{align*}
\]

The existence of such modules is demonstrated by examples (Examples 4.1 and 4.2) in §4. The module of Example 4.1 is neither cyclic nor completely indecomposable, so that the class of modules satisfying \( \text{Hom}_R(M, M) = R^* \) is properly extended by the main theorem (Theorem 2.7), which establishes that property for \( R \)-modules \([C]\). The notation of the definition will be assumed without further comment wherever \( R \)-modules \([C]\) arise. We shall also use equivalent forms of \((C_2)\) as stated in the

**Lemma 2.4.** \((C_2)\) in the definition of \( R \)-module \([C]\) is equivalent to each of

\[
\begin{align*}
\text{(C_2')} & \quad \text{for } p \in P, \ i \in \{1, 2, \ldots, k\}, \ \text{there is an element } p' \in P \text{ such that } p = p' \mod \text{Ann}(x_i, R) \text{ and } p' = 0 \mod \bigcap_{s \neq i} \text{Ann}(x_s, R), \\
\text{(C_2'')} & \quad \text{if, for } i = 1, \ldots, k, \ b_i \in P, \ \text{the } k \text{ equations } x_i b_i = x_i b_i \text{ have a solution } b \in P.
\end{align*}
\]

**Proof.** Let \( i \in \{1, \ldots, k\} \) and \( p, p' \in P \) and first assume \((C_2')\). Setting \( b_i = p \) and \( b_s = 0 \) for \( s \neq i \) we obtain an element \( b \) such that \( x_i b = x_i p \) (equivalently \( p - b \in \text{Ann}(x_i, R) \)) and \( x_i b = 0 \) if \( s \neq i \) (equivalently \( b \in \bigcap_{s \neq i} \text{Ann}(x_s, R) \)). Since \( p = (p - b) + b \) we have \((C_2)\). Assuming \((C_2)\), \( i \) fixed, and \( p \in P \) there exist elements \( g' \in \text{Ann}(x_i, R) \) and \( p' \in \bigcap_{s \neq i} \text{Ann}(x_s, R) \) such that \( p = g' + p' \). Since \( x_i(p - p') = x_i g' = 0, p = p' \mod \text{Ann}(x_i, R) \), so that \( p' \) satisfies the requirement of \((C_2')\). If \((C_2')\) is assumed and \( b_i \in P \) for \( 1 \leq i \leq k \), then elements \( g_i \) exist such that \( b_i = g_i \mod \text{Ann}(x_i, R) \) for each \( i \), while, for \( s \neq i \), \( x_i g_s = 0 \). Then, if \( b = \sum_{i=1}^{k} b_i \), we have for each \( j \):

\[
x_j b = \sum_{i=1}^{k} x_j g_i = x_j g_i = x_j b_i,
\]

proving \((C_2'')\).

**Lemma 2.5.** If \( M \) is a \( R \)-module \([C]\), then \( x_i P \neq 0 \) for \( i = 1, \ldots, k \).

**Proof.** By the definition of exponent \( MP^{e-1} \neq 0 \). From \((C_1)\) and \( MP^* = 0 \) we have \( x_i P \supseteq \text{Ann}(P, M) \supseteq MP^{e-1} \).

**Lemma 2.6.** If \( M \) is an \( R \)-module \([C]\) then, for \( j = 1, \ldots, k \), \( x_j P = \text{Ann}(\text{Ann}(x_j, R), MP) \).

**Proof.** The integer \( j \) is a constant in the proof. If \( h \in \text{Ann}(\text{Ann}(x_j, R), MP) \) we write \( h = \sum a_i x_i n_i \) with \( n_i \in P \) (since \( MP = \sum x_i P \)). Let \( i \neq j \) be a fixed
integer and let $p \in P$. There exists an element $p' \in P$ satisfying $x_ip = x_ip'$ and for $s \neq i$, $x_ip' = 0$ (Lemma 2.4, (C2')); in particular $x_ip = 0$. By definition of $h$, $hp' = 0$ and we have

$$0 = hp' = \sum_{i=1}^{k} x_in_i p' = x_in_i p' = x_in_i p.$$  

For $i \neq j$ we have proved $x_i n_i \subseteq \text{Ann}(P, M)$. Combining this with (C1) we have $x_i n_i \subseteq x_j P$ for $i \neq j$ (obviously for $i = j$), so that $h \in x_j P$. We have $\text{Ann}(\text{Ann}(x_j, R), MP) \subseteq x_j P$ and the reverse inequality is trivial.

**Theorem 2.7.** If $M$ is an $R$-module $\left[ C \right]$, $\text{Hom}_R(M, M) = \mathbb{R}^*$ and $M$ is indecomposable.

**Proof.** If $E \in \text{Hom}_R(M, M)$ we take advantage of $M = \sum_1^k x_i R$ and write $E(x_i) = \sum_1^k x_i a_{ij}, a_{ij} \in R, i = 1, \ldots, k$. We claim that

$$0 = \sum_1^k x_i a_{ij} = a_{ii}.$$  

If $i$ and $j \neq i$ are fixed integers there exists by Lemma 2.5 an element $q \in P$ such that $x_i q \neq 0$. By (C1), Lemma 2.4, $q$ may be chosen so that $x_i q = 0$ if $j \neq i$. In particular $x_i q = 0$ and we have:

$$0 = E(x_i q) = \left( \sum_{i=1}^{k} x_i a_{ij} \right) q = x_i a_{ij} q.$$  

If $a_{ij} \in P$ there is an element $b$ such that $a_{ij} = 1 + P$; thus $a_{ij} b = 1 + f, f \in P$. From (2.2) follows $x_i q + x_i qf = x_i q a_{ij} b = 0$, whence $x_i q = -x_i qf$. Now the choice of $q$ is contradicted by $0 = (-1)^* x_i q f = -x_i q f = x_i q$; we have $a_{ij} \in P$ as claimed.

Next we prove

$$0 = \sum_{i=1}^{k} x_i a_{ij} = a_{ii} - a_{11}.$$  

By Lemma 2.5 and (C2'), Lemma 2.4, there is an element $q \in P$ such that $x_i q$ is a nonzero element of $\text{Ann}(P, M)$ and $x_i q = 0$ if $s \neq i$. Since $x_i q \in \text{Ann}(P, M) \subseteq x_i P$ by (C1), there is an element $t \in P$ such that $x_i q = -x_i t$ and we choose $t$ so that $x_i t = 0$ if $s \neq 1$. We have

$$0 = E(0) = E(x_i q + x_i t) = \sum_{j=1}^{k} x_j a_{ij} q$$  

$$+ \sum_{j=1}^{k} x_j a_{ij} t = x_i a_{ii} q + x_i a_{11} t = x_i (a_{ii} - a_{11}) q.$$  

If $a_{ii} - a_{11} \in P$, let $b \in R$ and $f \in P$ satisfy $(a_{ii} - a_{11}) b = 1 + f$. From (2.4) we have $x_i q + x_i qf = x_i (a_{ii} - a_{11}) q = 0$, whence $x_i q = -x_i qf = (-1)^* x_i q f = 0$. Since the choice of $q$ has been contradicted, $a_{ii} - a_{11} \in P$, proving (2.3).
We denote $a_{ii}$ by $a$ and $x_i(a_{ii} - a) + \sum_{j \neq i} x_j a_{ij}$ by $z_i$, so that $E(x_i) = \sum x_j a_{ij} = x_a + z_i$, $i = 1, \ldots, k$. As a consequence of (2.1) and (2.3) $z_i \in MP$. Defining $T$ by $T(m) = E(m) - ma$ for $m \in M$ we have $T(x_i) = z_i \in MP$; also,

(2.5) \[ E(m) = ma + T(m) \]

and

(2.6) \[ T \in \text{Hom}_R(M, M). \]

If $f \in \text{Ann}(x_i, R)$, $0 = T(0) = T(x_i f) = z_i f$ (by (2.6)), so that $z_i = T(x_i) \in \text{Ann}(\text{Ann}(x_i, R), MP)$ which, by Lemma 2.6, equals $x_i P$. Consequently, there exist elements $s_i \in P$ such that $T(x_i) = x_i s_i$ for $i = 1, 2, \ldots, k$. Taking advantage of $(C_2')$, Lemma 2.4, there is an element $s \in P$ such that $x_i s = x_i s_i$ and we have $T(x_i) = x_i s_i = x_i s$, $i = 1, 2, \ldots, k$. Since $M = \sum x_i R$, $T(m) = ms$ for $m \in M$ and $E(m) = m(a + s)$ by (2.5). Thus $E \in R^*$.

Since $P$ is a maximal ideal and $\text{Hom}_R(M, M) = R^*$, the indecomposability of $M$ is implied by the corollary to Theorem 1.6.

In view of the omission of the chain conditions in the hypotheses of Theorem 2.7 it appears that the theorem has wider application than to the stated problem, which concerns direct summands of modules with chain conditions. Example 4.2 in §4 is an $R$-module $[C]$ which satisfies neither the A.C.C. nor the D.C.C.

**Theorem 2.8.** If $M$ is an $R$-module $[C]$ and if $x$ denotes the coset $x + \text{Ann}(P, M)$, then

(i) $M/\text{Ann}(P, M) = x_1 R \oplus \cdots \oplus x_k R$;

(ii) $MP/\text{Ann}(P, M) = x_1 P \oplus \cdots \oplus x_k P$.

**Proof.** From the definition of $R$-module $[C]$ $M = \sum x_i R$; $MP = \sum x_i P$ is deduced as in the proof of the corollary, Theorem 1.7. Consequently (i) and (ii) will follow if we prove for each $i$

(2.7) \[ x_i P \cap \left( \sum_{j \neq i} x_j P \right) = \text{Ann}(P, M) \]

and

(2.8) \[ x_i R \cap \left( \sum_{j \neq i} x_j R \right) = \text{Ann}(P, M). \]

The identities

$\text{Ann}(J_1 + J_2, M) = \text{Ann}(J_1, M) \cap \text{Ann}(J_2, M)$,

$\text{Ann}(J_1 \cap J_2, M) \supseteq \text{Ann}(J_1, M) + \text{Ann}(J_2, M)$

(which hold for ideals $J_1$ and $J_2$), applied to $(C_2)$, yield
\[
\text{Ann}(P, M) = [\text{Ann}(\text{Ann}(x_i, R), M)] \cap [\text{Ann} \left( \bigcap_{s \neq i} \text{Ann}(x_s, R), M \right)]
\]
(2.9)

\[
\supseteq [\text{Ann}(\text{Ann}(x_i, R), M)] \cap \left[ \left( \sum_{s \neq i} \text{Ann}(\text{Ann}(x_s, R), M) \right) \right].
\]

From (C_i) and \( MP = \sum x_iP \) we have \( \text{Ann}(P, M) \subseteq MP \), so that
(2.10)
\[
\text{Ann}(P, M) = \text{Ann}(P, M) \cap MP.
\]

Applying (2.10) to (2.9), we have
\[
\text{Ann}(P, M) \supseteq [\text{Ann}(\text{Ann}(x_i, R), M) \cap MP]
\]
(2.11)
\[
\cap \left[ \left( \sum_{s \neq i} \text{Ann}(\text{Ann}(x_s, R), M) \right) \cap MP \right]
\]
\[
\supseteq \text{Ann}(\text{Ann}(x_i, R), MP)
\]
\[
\cap \left[ \sum_{s \neq i} (\text{Ann}(\text{Ann}(x_s, R), MP)) \right].
\]

By Lemma 2.6 \( \text{Ann}(\text{Ann}(x_j, R), MP) \) may be replaced in (2.11) by \( x_jP \), \( 1 \leq j \leq k \), giving
\[
\text{Ann}(P, M) \supseteq x_jP \cap \left( \sum_{s \neq i} x_sP \right).
\]

The reverse inequality follows from \( x_jP \supseteq \text{Ann}(P, M) \), \( 1 \leq j \leq k \), and (2.7) is proved. From the independence mod \( MP \) of the \( x_j \) follows \( x_jR \cap (\sum_{s \neq i} x_sR) \subseteq MP \). Thus if \( x_jb_i = \sum_{s \neq i} x_s b_s \) is an element of \( x_jR \cap (\sum_{s \neq i} x_sR) \), we have \( b_s \subseteq P \) for \( s = 1, 2, \ldots, k \). It follows that the left member of (2.8) is a subset of the left member of (2.7) and the reverse inequality between left members of (2.7) and (2.8) is trivial. (2.8) is proved by comparison of its left and right members with those of (2.7).

3. Maximal commutative, completely primary algebras. In the examples in §4 the module \( M \) is a vector space and \( R \) is a ring of linear transformations. Some theorems prerequisite to these examples is the concern of §3. Let \( K \) be a field and let \( K_n \) denote the full set of \( n \) by \( n \) matrices over \( K \) or, equivalently, the algebra \( L(M, M) \) of linear transformations of an \( n \)-dimensional vector space \( M \) into itself. Let \( R \) be a subalgebra of \( K_n \) containing the identity matrix \( I_n \). The \( n \)-dimensional space on which \( K_n \) acts shall be referred to as the representation space of \( K_n \) and of \( R \).

Let \( M' \) be a vector space such that \( M \) and \( M' \) are dual according to a non-degenerate bilinear form [7, pp. 140–141]:
\[
(m, m') \to g(m, m'), \quad m \in M, \quad m' \in M', \quad g(m, m') \in K.
\]

We shall write \( (m, m') \) for \( g(m, m') \). Evidently \( M' \) and \( M \) have the same dimension [7, p. 141]. By the definition of nondegenerate [7, p. 140] we have
If \( r \in R \) and \( m' \in M' \) the mapping \( m \mapsto (mr, m') \) is a linear functional; consequently, \([7, \text{p. 141}]\) there exists a unique element \( v' \in M' \) such that \( (m, v') = (mr, m') \). It can be verified that the map \( m' \mapsto v' \) determined in this way by \( r \) is a linear transformation of \( M' \) into itself. More briefly, we say that to each \( r \in R \)

\[
(3.3) \quad (m, m'r') = (mr, m') , \quad m \in M , m' \in M'
\]
determines a linear transformation \( r' \) (the adjoint of \( r \)) which acts on \( M' \). Let \( r_1 \) and \( r_2 \) be elements of \( R \) such that, for all \( m \in M \) and \( m' \in M' \), \( (m_1, m') = (mr_1, m') \). Then by \((3.2)\) \( mr_1 = mr_2 \) for each \( m \), and \( r_1 = r_2 \) since \( M \) is faithful. Thus the map \( r \mapsto r' \) is one-to-one. It is well known that \( R \) and \( R' = \{ r' \mid r \in R \} \) and (3.3) define \( r' \) are anti-isomorphic rings, that \( M' \) is a right \( R' \)-module, and (assuming dual bases for \( M \) and \( M' \)) that the matrix form of \( r \) is the transpose of that of \( r \). \( P \) and \( P' \) shall stand respectively for \( \text{rad} \, R \) and \( \text{rad} \, R' \). It is also evident from the anti-isomorphism that \( P \) and \( P' \) have the same exponent, which we denote by \( e \).

Except in Theorems 3.1 and 3.2 commutativity is assumed for \( R \) and the anti-isomorphism is an isomorphism. Except as stated specifically \( R/\text{P} \) is not assumed to be a field. \( R \) is assumed to possess the identity transformation \( \mathbb{I} \) throughout \( \S 3 \).

Theorem 3.1. (i) The ring \( R \) is commutative \([\text{maximal commutative}]\) if, and only if, the ring \( R' \) of adjoints is so; (ii) if \( M_1 \) is an \( R \)-module, \( \text{Ann}(M_1, M') \) is an \( R' \)-module; (iii) if \( M_1 \) and \( M_2 \) are submodules of \( M \) such that \( M = M_1 \oplus M_2 \), then \( M' = \text{Ann}(M_1, M') \oplus \text{Ann}(M_2, M') \).

Proof. (i) is deduced from the anti-isomorphism of \( R \) with \( R' \) and (iii) holds for submodules since it holds more generally for subspaces \([5, \text{p. 31}]\). To obtain (ii) we let \( r' \in R' \) and \( x' \in \text{Ann}(M_1, M') \); then for all \( m \in M_1 \) we have \( (m, x'r') = (mx', x') = 0 \), since \( mr \in M_1 \). Thus \( x'r' \in \text{Ann}(M_1, M') \). Q.E.D.

From the anti-isomorphism it is clear that an element \( p \in R \) is nilpotent if, and only if, its adjoint \( p' \) is nilpotent. If \( p \) is a product of \( s \) nilpotent elements \( 0 < s \leq e \), then by the anti-isomorphism the adjoint element \( p' \) is also; similarly for sums of such products. It follows that for \( s = 0, 1, \cdots, e \) \( p \in P^s \) if, and only if, its adjoint element \( p' \in P'^s \).

Theorem 3.2. Let \( M, M', R, R', P \) and \( P' \) meet the general specifications of this section; for \( s = 0, 1, \cdots, e \) let \( Y_s \) denote \([\text{Ann}(P^s, M) \) and let \( Y'_s \) denote \( \text{Ann}(P'^s, M') \). Then

\[
(3.4) \quad \text{Ann}(Y_s, M') = M'P'^s ;
\]

\((*)\) The convention \( J^s = \mathbb{R} \) (for ideals \( J \) or \( R \)) is used.
(3.5) \[ \text{Ann}(MP^s, M') = Y'_s; \]

(3.6) \[ \text{Ann}(M'P^s, M) = Y'_s; \]

and

(3.7) \[ \text{Ann}(Y'_s, M) = MP^s \]

hold for \( s = 0, 1, \ldots, e. \)

**Proof.** By the symmetry enjoyed by \( M \) and \( M' \) only (3.5) and (3.7) require proof. If \( z' \in Y'_s = \text{Ann}(P'^s, M'), m \in M, \) and \( \pi \in P^s, \) then by the observations preceding this theorem \( \pi' \in P^s \) and \( (m\pi, z') = (m, z'\pi') = (m, 0) = 0. \)

We infer that \( z' \) annihilates every element \( \sum m_i \pi_i \) with \( \pi_i \in P^s. \) Thus \( Y'_s \subseteq \text{Ann}(MP^s, M'). \) Conversely, if \( m' \in \text{Ann}(MP^s, M') \) and \( q' \in P^s, \) then \( q' \in P^s \) and, for all \( m \in M, (m, m'q') = (mq, m') = 0. \) By (3.1) \( m'q' = 0 \) and \( m' \in \text{Ann}(P'^s, M') = Y'_s, \) proving that \( \text{Ann}(MP^s, M') \subseteq Y'_s \) and completing the proof of (3.5).

We apply (3.5) and the identity \([5, \text{p. 27}]\)

\[ M_1 = \text{Ann}(\text{Ann}(M_1, M'), M) \]

which holds for subspaces \( M_1 \) of \( M \) to obtain

\[ MP^s = \text{Ann}(\text{Ann}(MP^s, M'), M) = \text{Ann}(Y'_s, M) \]

which is (3.7). Q.E.D.

Evidently the subalgebra \( R \) is not assumed to be commutative in Theorem 3.2. If \( R/P \) is a division ring its dimension over \( K \) is less than \( n^2. \) Thus its dimension over its center (which contains \( \{kI_n + P \mid k \in K\} \) is finite and the division ring \( R/P \) is commutative.

**Corollary.** Assume now that \( R/P [\cong R'/P'] \) is a field. Using the notation of the theorem

(3.8) \[ K-\text{dim}(MP^{s-1}/MP^s) = K-\text{dim}(Y'_s/Y'_{s-1}), \ s = 1, 2, \ldots, e; \]

(3.9) \[ (R/P)-\text{dim}(M/MP) = (R'/P')-\text{dim}(\text{Ann}(P', M')). \]

**Proof.** By (3.5) we have \( Y'_s = \text{Ann}(MP^s, M'). \) To this we apply a well-known result \([5, \text{p. 31}]\): If \( M_1 \) is a subspace of vector space \( M \) then \( M/M_1 \) and \( \text{Ann}(M_1, M') \) are dual vector spaces. Thus \( Y'_s \) and \( M/MP^s \) are dual spaces and have the same dimension; similar statements hold for \( Y'_{s-1} \) and \( M/MP^{s-1}. \) Since \( \dim Y'_s/Y'_{s-1} = \dim Y'_s - \dim Y'_{s-1} \) and \( \dim MP^{s-1}/MP^s = \dim M/MP^s - \dim M/MP^{s-1}, \) we have (3.8). Assume now that (3.9) is false: \( (R/P)-\text{dim} M/MP \neq (R'/P')-\text{dim}(\text{Ann}(P', M')). \) Let \( t \) denote \( (R/P):K = (R'/P'):K. \) From \( K-\text{dim} M/MP = t \cdot (R/P)-\text{dim} M/MP; \) and \( K-\text{dim}(\text{Ann}(P', M')) = t \cdot (R'/P')-\text{dim}(\text{Ann}(P', M')) \) follows \( K-\text{dim}(M/MP) \neq K-\text{dim}(\text{Ann}(P', M')), \) a contradiction of the case \( s = 1 \) of (3.8). Q.E.D.

**Remark.** If \( A \) is an arbitrary ring, an \( A \)-module \( V \) is said to be completely
reducible if it can be expressed as a sum of irreducible submodules. A completely reducible module can be expressed as a direct sum of a subfamily of the family of irreducible modules [9, p. 61, Corollary 2 to Theorem 1] the cardinality of which is an invariant [9, p. 62, Theorem 3]. Concerning the maximal completely reducible submodule of an $A$-module $W$ we have from [2, pp. 103–104, Theorems 9.4A and 9.4C]:

**Theorem A.** If $A$ is a ring satisfying the D.C.C. for right ideals and if $W$ is an $A$-module, the maximal completely reducible submodule of $W$ is $\text{Ann}(Q, W)$ where $Q = \text{rad} \, A$.

In the ensuing discussion and theorems in §3 we consider the subalgebra $R$ of $K_*$ to be commutative. We observe that the vector space $M$ satisfies both chain conditions as an $R$-module since each submodule is a vector space, and $R$ satisfies the chain conditions for ideals by similar reasoning.

**Theorem 3.3.** $M$ is completely indecomposable as an $R$-module if, and only if, $P$ is a maximal ideal and the $(R/P)$-dimension of $\text{Ann}(P, M)$ is unity.

**Proof.** $P$ is a maximal ideal assuming either of the conditions of the Theorem. For Theorem 1.5 implies the maximality of $P$ when $M$ (which satisfies the chain conditions) is indecomposable. Since $R$ satisfies the D.C.C., the completely reducible submodule of $M$ is $\text{Ann}(P, M)$ by Theorem A in the preceding remark. $M$ has at least one irreducible submodule (by the D.C.C.); thus, for some positive integer $d$, $\text{Ann}(P, M) = \sum_{i=1}^{d} M_i$, an irreducible submodule for $i = 1, 2, \ldots, d$. We write $M_i = x_iR$, $x_i \in M_i$, since an irreducible module is cyclic [9, p. 6, Proposition 1]. Since $x_i \in \text{Ann}(P, M)$, $x_iP = 0$ and we have

$$\text{Ann}(P, M) = \sum_{i=1}^{d} \oplus x_iR \cong \sum_{i=1}^{d} \oplus x_iF$$

where $F \cong R/P$. Thus $d = (R/P)-\text{dim}(\text{Ann}(P, M)) = 1$ if, and only if, $M$ has exactly one irreducible submodule. Since $R$ is a commutative ring with identity and $M$ satisfies the chain conditions, the complete indecomposability of $M$ is, by definition, equivalent to its having exactly one irreducible submodule. Q.E.D.

**Theorem 3.4.** $M$ is a cyclic, indecomposable $R$-module if, and only if, $M'$ is a completely indecomposable $R'$-module.

**Proof.** By (iii), Theorem 3.1, it follows that $M$ is indecomposable if, and only if, $M'$ is so; consequently, the indecomposability of both modules follows from either of the conditions of this theorem. Since the modules satisfy the chain conditions Theorem 1.5 assures the maximality of $P [\cong P']$. By Theorem 3.2 and corollary $\text{Ann}(P', M')$ and $M/MP$ have the same $(R/P)$-dimension $t$. Since the modules are indecomposable and the radical is maximal
under either of the conditions of this Theorem, \( t = 1 \) is equivalent to "\( M \) is cyclic" and to "\( M' \) is completely indecomposable" (Theorems 1.8 and 3.3), proving the theorem.

**Remark.** If \( M \) is completely indecomposable for \( R \), \( M' \) is cyclic for \( R' \) by Theorem 3.4 and \( R' \) (hence \( R \) also) is maximal commutative in \( K_n \). We have Snapper's Theorem: \( \text{Hom}_R(M, M) = R^* \) if \( M \) is completely indecomposable" for the case in which \( M \) is an \( n \)-dimensional vector space and \( R \) a subalgebra of \( K_n \).

**Definition of a class \( \mathfrak{N}_n \) of maximal commutative subalgebras.** Let \( R \) be commutative and completely primary. The following conditions each of which implies that \( \text{Hom}_R(M, M) = R^* \)

(i) \( M \) is \( R \)-cyclic,

(ii) \( M \) is an \( R \)-module \([C]\),

together with their analogues for the \( R' \)-module \( M' \), is a set of conditions each of which is sufficient to insure that the completely primary, commutative algebra \( R \) is a maximal commutative subalgebra of \( K_n \) (Theorem 3.1). The class of maximal commutative, completely primary algebras so obtained will be designated as the class \( \mathfrak{N}_n \).

**Remark.** Among the conditions known to imply \( \text{Hom}_R(M, M) = R^* \) is the condition

(iii) \( M \) is completely indecomposable in the sense of Snapper.

Since by Theorem 3.4 (iii) holds for \( M[M'] \) only if (i) holds for \( M'[M] \), it is not necessary to include (iii) in the definition of the class \( \mathfrak{N}_n \).

**Theorem 3.5.** If \( M \) is a cyclic or completely indecomposable \( R \)-module, then \( K \)-dim \( R = n \).

**Proof.** If \( M = xR, x \in M \), the mapping \( r \rightarrow xr \) of \( R \) onto \( M \) is an \( R \)-isomorphism, since \( M \) is faithful. The isomorphism is a \( K \)-isomorphism since \( KI_n \subseteq R \), and we conclude that \( R \) and \( M \) have the same dimensionality. If \( M \) is completely indecomposable for \( R \) then \( M' \) is \( R' \)-cyclic (Theorem 3.4) and we have \( \text{dim } R = \text{dim } R' = n \) from the first case of this theorem.

**Remark.** If \( R \) is completely primary and if \( R/P \cong K \) then, by Theorem 3.2 and corollary and Theorem 1.8, \( K \)-dim(Ann(\( P, M) = K \)-dim(\( M'/M'$P' \)

=1 if \( M' \) is \( R' \)-cyclic, and \( K \)-dim \( M/MP \) =1 if \( M \) is \( R \)-cyclic. In either case, if \( k = K \)-dim(\( M/MP \) and \( d = K \)-dim(Ann(\( P, M) \)), \( (k - 1)(d - 1) = 0 \). By Theorem 3.5, \( K \)-dim \( M = K \)-dim \( R \) when \( M \) or \( M' \) is cyclic, and the formula

(3.10) \[ K \text{-dim } R - K \text{-dim } M = (k - 1)(d - 1) \]

holds.

**Theorem 3.6.** If \( R \subseteq \mathfrak{N}_n \) is such that \( R/P \cong K \) and if \( k \) and \( d \) denote the dimensions of \( M/MP \) and \( \text{Ann}(P, M) \) respectively, then (3.10) holds.

**Proof.** Because of the preceding remark (3.10) needs proof only for the
cases in which $M$ is an $R$-module $[C]$ or $M'$ is an $R'$-module $[C]$. If $k'$ denotes dim $M'/M'P'$ and $d'$ denotes dim($\text{Ann}(P', M')$), $d = k'$ and $d' = k$ by (3.9), corollary to Theorem 3.2. Since dim $M = \text{dim } M'$ and dim $R = \text{dim } R'$, (3.10) holds if, and only if, dim $R' - \text{dim } M' = (k' - 1)(d' - 1)$. Thus it is sufficient to prove (3.10) assuming that $M$ is an $R$-module $[C]$. By hypothesis the dimension of $R/P$ is unity. By Theorem 2.7, $\text{Hom}_R(M, M) = R^*$, and we have from conclusion (4), Theorem 2.3: dim($\text{Ann}(MP, R)$) = $k d$. Thus dim $R = 1 + k d + \text{dim}\{P/\text{Ann}(MP, R)\}$, and (3.10) is equivalent to $1 + k d + \text{dim}\{P/\text{Ann}(MP, R)\} - n = (k - 1)(d - 1)$, or

\begin{equation}
(3.11) \dim\{P/\text{Ann}(MP, R)\} = n - k - d.
\end{equation}

By definitions of $k$ and $d$ the right member of (3.11) is the dimension of $MP/\text{Ann}(P, M)$. We prove (3.11) by showing that

\begin{equation}
(3.12) P/\text{Ann}(MP, R) \cong MP/\text{Ann}(P, M).
\end{equation}

From the definition of "$R$-module $[C]$" $M = \sum x_i R, x_i \in M$, and by Theorem 2.8

\begin{equation}
(3.13) MP/\text{Ann}(P, M) \cong \hat{x}_1 P \oplus \cdots \oplus \hat{x}_k P \quad \text{where } \hat{v} \text{ denotes } v + \text{Ann}(P, M).
\end{equation}

We let $x = \sum x_i$ and consider the mapping $E$ of $P$ into $MP/\text{Ann}(P, M)$ defined by

\begin{equation*}
E(\varphi) = \hat{x} \varphi, \quad \varphi \in P.
\end{equation*}

If $p_1, p_2, \cdots, p_k$ are elements of $P$, then by $(C_1''')$, Lemma 2.4, there exists an element $\varphi \in P$ such that $\varphi \equiv p_i \text{ mod } \text{Ann}(x_i, R)$, $1 \leq i \leq k$; we have $x \varphi = \sum x_i \varphi = \sum x_i p_i$. Considering (3.13) $E$ is a mapping onto $MP/\text{Ann}(P, M)$.

Since $K \subseteq R$ the $R$-homomorphism $E$ is a $K$-homomorphism. Now $\ker E = \{ \varphi \mid \hat{x} \varphi = 0 \} = \{ \varphi \mid \sum x_i \varphi \in \text{Ann}(P, M) \} = \{ \varphi \mid x_i \varphi \in \text{Ann}(P, M), 1 \leq i \leq k \}$. The last equality comes from the direct sum in (3.13). Thus $\ker E = \{ \varphi \mid mp \in \text{Ann}(P, M) \text{ for all } m \}$, since $M = \sum x_i R$, and we have $\ker E = \text{Ann}(MP, R)$, proving (3.12) and the theorem.

**Remark.** Since $R/P$ is a finite extension of a field isomorphic to $K$, $K$ and $R/P$ will be isomorphic if $K$ is algebraically closed. Thus Theorem 3.6 is applicable to all algebras in $\mathfrak{M}_\kappa$ provided the field $K$ is algebraically closed.

4. **Examples. Scope of the class $\mathfrak{M}_\kappa$.** Demonstration that a commutative, completely primary subalgebra $R$ of $K_\kappa$ is maximal commutative is usually greatly simplified when the representation module of $R$ (or of the algebra $R'$ of adjoints of $R$) is an $R$-module $[C]$ or is cyclic. That these conditions need not exist for a maximal commutative subalgebra is made clear in Example 4.3. First we present an example of an algebra whose representation module is an $R$-module $[C]$ but is neither a cyclic nor a completely indecomposable $R$-module.
In what follows the unit matrices in $K_n$ will be denoted by $E_{ij}$ \((1 \leq i, j \leq n)\) and the identity matrix by $I_n$.

**Example 4.1.** Let $M$ be a 6-dimensional vector space $Ku_1 \oplus \cdots \oplus Ku_6$ acted upon by

$$R = \begin{bmatrix}
  a & b & c & 0 & d & e \\
  0 & a & b & 0 & 0 & 0 \\
  0 & 0 & a & 0 & 0 & 0 \\
  0 & 0 & f & a & g & h \\
  0 & 0 & 0 & 0 & a & 0 \\
  0 & 0 & 0 & 0 & 0 & a
\end{bmatrix}$$

$a, b, c, d, e, f, g, h$ arbitrary in $K$.

As a vector space $R$ has a basis consisting of $I_6$ and the set

$$S = \{ E_{12} + p_{23}, E_{14}, E_{16}, E_{45}, E_{46}, E_{65} \}.$$

The product $xy$, $(x, y) \in S \times S$, is zero unless $x = y = E_{12} + E_{23}$, and $(E_{12} + E_{23})^2 = E_{12}$. Thus, if $P$ denotes the space generated by $S$, $P^2 = 0$, and multiplication is closed and commutative in $P$ (and clearly is so in $R$). $P$ is an ideal since $P \supseteq PI_6$. Considering the dimensions of $P$ and $R$, $P$ is a nilpotent, maximal ideal, proving that $P$ is the radical of $R$. Since $MP = Ku_2 \oplus Ku_3 \oplus Ku_4 \oplus Ku_6$, $\dim M/MP = 2$, proving that $M$ is not $R$-cyclic (Theorem 1.8). For $i = 3, 5, 6$, we verify $Ku_i = u_i R$, and $u_i P = 0$ and conclude that $\text{Ann}(P, M)$ contains the independent irreducible $R$-modules $Ku_3$, $Ku_5$, and $Ku_6$. This proves that $M$ is not completely indecomposable (Theorem 3.3). If $v = \sum_{i} c_i u_i \in \text{Ann}(P, M)$ then $v(E_{12} + E_{23} + E_{45}) = c_1 u_2 + c_2 u_3 + c_4 u_6 = 0$ implies $c_1 = c_2 = c_4 = 0$, proving that $v \in Ku_3 \oplus Ku_5 \oplus Ku_6$. Thus $\text{Ann}(P, M) = Ku_3 \oplus Ku_5 \oplus Ku_6$.

Since $u_i \subseteq u_i R$ \((i = 1, 2, 3, 5, 6)\) and $u_4 \subseteq u_4 R$, $M = u_1 R + u_2 R$. We claim that \(\{ u_1 + MP, u_4 + MP \} \) is a basis of $M/MP$ in which the requirements (C) and (C2) for an $R$-module $[C]$ are satisfied. The equations $u_2 = u_1 E_{12} = u_4 E_{43}$, $u_4 = u_1 E_{16} = u_4 E_{45}$, and $u_6 = u_1 E_{16} = u_4 E_{46}$ prove $\text{Ann}(P, M) \subseteq u_i P$ for $i = 1, 4$, which is (C). To obtain (C2): $P = \text{Ann}(u_1, R) + \text{Ann}(u_4, R)$ one verifies that $\text{Ann}(u_1, R) \supseteq KE_{43} \oplus KE_{46} \oplus KE_{45}$ and that $\text{Ann}(u_4, R) \supseteq K(E_{12} + E_{23}) \oplus KE_{13} \oplus KE_{15} \oplus KE_{16}$. $M$, then, is an $R$-module $[C]$ and $R$ is maximal commutative in $K_6$ by Theorem 2.7.

**Example 4.2.** (An $R$-module $[C]$ that satisfies neither the A.C.C. nor the D.C.C.) Let $M$ be an infinite-dimensional vector space $Ku_1 \oplus Ku_2 \oplus \cdots$ and let $L$ denote the ring of linear transformations on $M$ or, equivalently, of row-finite infinite matrices. The set $S = \{ E_{ij} \mid i = 1, 2; j = 3, 4, \ldots \}$ generates a zero subalgebra $P$ of $L$. Then the subalgebra $R$ generated by $S$ and the identity transformation is commutative (since $P$ is a zero algebra) and its radical is the nilpotent, maximal ideal $P$. Direct verification yields

\[(1) \quad M = u_1 R + u_2 R, \]

\[(2) \quad MP = \text{Ann}(P, M) = \sum_{i} u_i R \subseteq u_j P, \quad j = 1, 2.\]
By (2) the basis \( \{ u_1 + MP, u_2 + MP \} \) of \( M/MP \) satisfies (C1). \( \text{Ann}(u_1, R) \) contains \( \{ E_2 \mid j = 3, 4, \ldots \} \) and \( \text{Ann}(u_2, R) \) contains \( \{ E_3 \mid j = 3, 4, \ldots \} \), so that \( S \subseteq \text{Ann}(u_1, R) + \text{Ann}(u_2, R) \). Since \( \text{Ann}(u_1, R) \subseteq P \) by the \((R/P)\)-independence of \( \{ u_i \mid i = 1, 2 \} \) and since \( S \) generates \( P \), we have

\[
(C_2): \quad P = \text{Ann}(u_1, R) + \text{Ann}(u_2, R) .
\]

\( M \), then, is an \( R \)-module \([C]\), and \( R \) is maximal commutative in \( L \). The A.C.C. and the D.C.C. fail to hold in \( M \) since \( \text{Ann}(P, M) \) contains the infinite set of independent submodules \( \{ Ku_i \mid i = 3, 4, \ldots \} \).

The remainder of this paper will be devoted to showing that the class of maximal commutative, completely primary subalgebras of \( K_n \) is not exhausted by the class \( \mathcal{M}_n \) defined in §3. The counter-example (Example 4.3) is preceded by two theorems.

**Theorem 4.1.** Let \( K \) be a field and for \( i = 1, 2 \), let \( n_i \) and \( e_i \) be positive integers. For \( i = 1, 2 \), let \( R_i \) be a commutative, completely primary subalgebra of \( K_{n_i} \) with radical \( P_i \) such that \( e_i \) is the exponent of \( P_i \) and \( R_i/P_i \cong K \). Let \( R \) denote the Kronecker product ring \( R_1 \otimes_K R_2 \) and let \( P \) denote \( \text{rad} R \). Then \( P = (R_1 \otimes_K R_2) + (P_1 \otimes_K R_2) \), \( e_1 + e_2 - 1 \) is the exponent \( e \) of \( P \), and \( R/P \cong K \). Thus \( R \) is completely primary.

**Proof.** If \( Q \) denotes \( (R_1 \otimes P_2) + (P_1 \otimes R_2) \), \( Q \) is clearly an ideal in \( R \). If, for \( i = 1, 2, \ldots, e_1 + e_2 - 1 \), \( w_i \in g_i \otimes h_i \subseteq Q \), either \( e_1 \) of the \( g_i \) belong to \( P_1 \) or \( e_2 \) of the \( h_i \) belong to \( P_2 \), so that \( \prod w_i^{i+e_i-1} = 0 \); the same conclusion is reached if each \( w_i \) has the form \( \sum g_{ij} \otimes h_{ij} \), proving that \( Q^{i+e_i-1} = 0 \). Let \( m_i \) denote \( R_i, i = 1, 2 \), and let \( \{ u_1 = u_{11}, u_2, \ldots, u_{1m_1} \} \) and \( \{ v_1 = v_{11}, v_2, \ldots, v_{1m_1} \} \) be bases of \( R_1 \) and \( R_2 \), respectively, such that \( u_i \in P_1 \) and \( v_i \in P_2 \) for \( i > 1 \). Then \( I_{n_1} \) and the set \( \{ m_1 m_2 - 1 \text{ independent elements } \{ u_i \otimes v_j \mid i + j > 1 \} \} \) form a basis of \( R \). One verifies \( L \subseteq Q \), and (since \( Q \) is nilpotent) \( I_{n_1} \otimes Q \). Thus \( Q = m_1 m_2 - 1 \) and \( Q \) is a nilpotent, maximal ideal in the \((m_1 m_2)\)-dimensional algebra \( R \). \( P \), then, is necessarily \( Q \) and has exponent \( e \leq e_1 + e_2 - 1 \). To prove \( e = e_1 + e_2 - 1 \), we observe that if \( s \) and \( t \) are nonzero elements of \( P_1^{e_1-1} \) and \( P_2^{e_2-1} \), respectively, then \( s \otimes t \in (s \otimes I_{n_1}) \cdot (I_{n_1} \otimes t) \) is a nonzero element of \( P^{e_1+e_2-2} \). The isomorphism between \( R/P \) and \( K \) is clear since \( \dim P = \dim R - 1 \).

**Notation.** If \( R \) is a ring and \( S \) is a subset of \( R \), we denote by \( R^S \) the set of elements of \( R \) that commute with every element of \( S \).

**Theorem 4.2.** Let \( A \) and \( B \) be rings with identity element whose centers contain a field \( K \). Let \( U \subseteq A \) and \( V \subseteq B \) be rings containing \( K \). Then \( (A \otimes_K B)_{U \otimes_K V} = A^U \otimes_K B^V \).

A proof of Theorem 4.2 appears in [2, p. 68, Lemma 7.3B].

**Corollary.** If \( U [V] \) is a maximal commutative subalgebra of \( K_{n_1} [K_{n_2}] \) then \( U \otimes_K V \) is isomorphic to a maximal commutative subalgebra of \( K_{n_1 n_2} \).
Proof. It is proved elsewhere [7, p. 226, Theorem 6] that $K_n \otimes K_n \cong K_{n,n}$ as algebras. We shall have the result, then, if $U \otimes V$ is maximal commutative in $K_n \otimes K_n$. By hypothesis $U = K^U_m$ and $V = K^V_m$, and by the theorem $(K_n \otimes K_n)(U \otimes V) = K^U_m \otimes K^V_m = U \otimes V$.

Toward constructing Example 4.3, let $R_1$ denote the subalgebra of $K_3$ which consists of matrices of the form

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

where $a$, $b$, and $c$ are arbitrary elements of $K$. $R_1$ as a vector space over $K$ is generated by $I_3$, $E_{13}$, and $E_{12}$ and, since $E^2_{12} = E^3_{13} = E_{12}E_{13} = E_{13}E_{12} = 0$, multiplication in $R_1$ is closed and commutative. Since the nilpotent, maximal ideal $KE_{12} \oplus KE_{13}$ is the radical $P_1$ of $R_1$, $R_1/P_1$ is isomorphic to $K$. If $M = Ku_1 \oplus Ku_2 \oplus Ku_3$ denotes the representation module of $R_1$ then, since $M[=\mu_1R_1]$ is cyclic, $R_1$ is a maximal commutative subalgebra of $K_3$. The algebra $R_2$ of adjoints of $R_1$ is likewise a maximal commutative, completely primary subalgebra of $K_3$ with $R_2/P_2 \cong K$. $R_3$ is the algebra of all matrices in the form

$$\begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix}$$

where $a$, $b$, and $c$ are elements of $K$.

Example 4.3. (A completely primary, maximal commutative subalgebra $R$ of $K_3$ which is not in the class $\mathfrak{M}_9$). Let $R = R_1 \oplus R_2$. By Theorems 4.1 and 4.2 with corollary, $R$ is a maximal commutative, completely primary subalgebra of $K_9$ with $R/P \cong K$. We shall see that $R$ does not satisfy the conclusion of Theorem 3.6 and consequently is not in the class $\mathfrak{M}_9$. The matrix representation of $R$ is

$$R = \begin{pmatrix} a & 0 & 0 & \cdots & d & 0 & 0 & g & 0 & 0 \\ b & a & 0 & \cdots & e & d & 0 & h & g & 0 \\ c & 0 & a & \cdots & f & 0 & d & k & 0 & g \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & 0 & \cdots & b & a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & a & 0 & 0 \\ 0 & \cdots & b & a & 0 & \cdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & a & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & c & 0 & a & \end{pmatrix}$$

$a, b, c, d, e, f, g, h, k \in K$. 

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The matrix (4.1) conforms with the schema of the Kronecker product of two matrices appearing in [7, p. 213]. By the same schema the matrix (4.1) with $a = 0$ reveals the 8-dimensional algebra $P_1 \otimes R_3 + R_4 \otimes P_2$ which by Theorem 4.1 is $P \equiv \text{rad } R$ and has exponent 3 $[\text{exponent of } P_1 + \text{exponent of } P_2 - 1]$.

Translated into matrix units $P$ has the basis

$$\{ E_{21} + E_{34} + E_{47}, E_{21} + E_{34} + E_{47}, E_{14} + E_{28} + E_{37}, \}
E_{17} + E_{28} + E_{37}, E_{24}, E_{34}, E_{37}, E_{37}. \}$$

According to Theorem 3.6, if $R$ is in the class $\mathfrak{M}_6$ defined in §3, the following equation must be satisfied

$$(4.3) \quad \dim R - \dim M = (k - 1)(d - 1)$$

where $M = Kw_1 \oplus \cdots \oplus Kw_9$ is the space on which $R$ acts, $k = \dim (M/MP)$, and $d = \dim (\text{Ann}(P, M))$. Examination of the submodules $MP$, $p$ a generator (see (4.2)) of $P$, discloses that $MP = Kw_1 \oplus Kw_4 \oplus Kw_5 \oplus Kw_6 \oplus Kw_7 \oplus Kw_9$, whence $k = 2$. The list (4.2) discloses also that $Kw_4 \oplus Kw_7 \subseteq \text{Ann}(P, M)$. If $x = \sum c_i w_i \in \text{Ann}(P, M)$, the equations $0 = x(E_{21} + E_{34} + E_{47}) = c_1 w_1 + c_4 w_4 + c_7 w_7$, $0 = x(E_{21} + E_{34} + E_{47}) = c_2 w_1 + c_5 w_4 + c_8 w_7$, and $0 = x(E_{14} + E_{28} + E_{37}) = c_3 w_1 + c_6 w_4 + c_9 w_7$, imply that $c_1 = c_2 = c_3 = c_5 = c_6 = c_8 = c_9 = 0$, so that $x = c_4 w_4 + c_7 w_7$. Thus $\text{Ann}(P, M) = Kw_4 \oplus Kw_7$ and $\dim(\text{Ann}(P, M)) = 2$. Now the left-hand side of (4.3) is 0 and the right-hand side is 1, proving that $R$ is not in the class $\mathfrak{M}_6$. Thus it is demonstrated that the aggregate of the conditions discussed in this paper which imply that a commutative, completely primary algebra is maximal commutative in $K_\pi$ does not constitute a necessary condition.

It is worth mentioning that the failure to get a necessary condition occurred at exponent $e = 3$. According to conclusions (3) and (4') of Theorem 2.3 $M$ is necessarily an $R$-module $[C]$ if $\text{Hom}_R(M, M) = R^*$ under somewhat general hypotheses including $e = 2$.

**Bibliography**


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