ON SOME CLASSES OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES(1)

BY

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1. **Introduction.** In a recent paper [6] of B. Sz.-Nagy and the author a new unified treatment was given to some classical results of function theory centering around the Pick-Nevanlinna interpolation problem and Loewner's theorems on monotone matrix functions. In the present paper we wish to generalize these investigations to functions of several complex variables with the help of the Hilbert space method developed in [6].

The main objects of study in [6] are the class H of functions analytic and having a non-negative real part in the unit disk, and the class N of functions f analytic in the upper half-plane, having there a non-negative imaginary part and such that $f(z)/z$ is bounded in every angular domain

$$C(\phi) = \{z \mid \phi \leqq \arg z \leqq \pi - \phi\} \qquad \left(0 < \phi \leqq \frac{\pi}{2}\right).$$

Normalizing the functions f in H by the condition $\operatorname{Im} f(0) = 0$ and extending their definition to the outside of the unit circle by the relation $f(z^{-1}) = -\bar{f}(z)$(2) we have the Riesz-Herglotz formula,

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{i\phi}}{1 - ze^{i\phi}}\, dm(\phi) \tag{1}$$

where m is a bounded positive measure on the unit circle. From this formula it is easy to see that the Taylor coefficients of f around the origin are the positive trigonometric moments of m. In other words this means that the part of f inside the unit circle is the complex Fourier transform of the restriction to positive integers of a positive definite function on the additive group I of integers. Similarly, the part of f outside the circle is the Fourier transform of the "negative half" of the same positive definite function.

The class N has similar properties. Extending $f \in N$ to the lower half-plane by $f(\bar{z}) = \bar{f}(z)$ we have the formula due to R. Nevanlinna:

$$f(z) = \int_{-\infty}^{\infty} \frac{z}{1 - \lambda z}\, dm(\lambda) \tag{2}$$

with m a bounded positive measure on the real line. Using (2) one can show that f admits an asymptotic development around 0 whose coefficients are the

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(2) $\bar{f}(z)$ means the complex conjugate of $f(z)$.

positive moments of m. It is also easy to see that f can be obtained from a positive definite function on the real line R by taking the complex Fourier transform of its restrictions to the positive and negative half-axis.

In the Pick-Nevanlinna problem we are given a set S in the interior of the unit disk or the upper half-plane, and a function f defined on S. The question is under what conditions can f be extended to a function of the class H resp. N. It turns out that a necessary and sufficient condition for the class H is that the function k_H defined on $S \times S$ by

$$k_H(s, t) = \frac{f(s) + \bar{f}(t)}{1 - s\bar{t}} \tag{3}$$

be positive definite. In the case of N we assume that for some $0 < \phi \leqq \pi/2$ there is a sequence $\{\sigma_n\}$ tending to 0 in $S \cap C(\phi)$; a necessary and sufficient condition then is that $f(\sigma_n)/\sigma_n$ be bounded and the function

$$k_N(s, t) = \frac{f(s) - \bar{f}(t)}{s - \bar{t}} \tag{4}$$

be positive definite. (As usual, given any set X, we say that k defined on $X \times X$ is positive definite if

$$\sum_m \sum_n k(x_m, x_n)\alpha_m \bar{\alpha}_n \geqq 0$$

holds for any finite set of points $x_1, \cdots, x_M$ in X and any complex numbers $\alpha_1, \cdots, \alpha_M$.)

In [6] we used the following well-known lemma, which will also be used in the present paper.

LEMMA 1. *Every positive definite function k defined on the set $X \times X$ can be represented as an inner product*

$$k(x, y) = (\epsilon_x, \epsilon_y),$$

where the ϵ_x $(x \in X)$ are vectors in a Hilbert space $\mathfrak{H}$. It can be assumed that the vectors ϵ_x $(x \in X)$ span $\mathfrak{H}$.

With the help of Lemma 1 the proof of the above results is based on a Hilbert space argument, the main point being the simple fact that every isometric operator in $\mathfrak{H}$ can be dilated to a unitary operator in an enlarged space.

A limiting case of the Pick-Nevanlinna problem is the following. Let S be the interval $(-1, 1)$ and f a real-valued function on S. Under what conditions can f be extended continuously to a function in N? K. Loewner [4] proved that a necessary and sufficient condition is $f \in C^1$ and the positive definiteness of (4). In [3] and [6] this was also proved in a simple way by our Hilbert space method. The main interest of this theorem is that, as Loewner has

shown [4; 5], this class of real-valued functions on $(-1, 1)$ is just the class of monotone matrix functions, i.e., such that for any two Hermitian operators A, B with spectrum in $(-1, 1)$ the relation $A \geqq B$ implies $f(A) \geqq f(B)$.

All the above considerations admit generalizations to functions of n variables. In order to avoid great notational complications we consider only the case of two variables; this will already reveal all the essential features and the generalizations to any number of variables will be apparent.

We define the generalized class H as follows:

DEFINITION 1. *The class H_2 is the class of functions f of two complex variables z_1, z_2 defined and holomorphic for all $|z_1|$, $|z_2| \neq 1$ (including ∞) and satisfying the conditions*

(a) $f(\bar{z}_1^{-1}, \bar{z}_2^{-1}) = \bar{f}(z_1, z_2)$ *for all* $|z_1|$, $|z_2| \neq 1$,

(b) $f(z_1, z_2) - f(\bar{z}_1^{-1}, z_2) - f(z_1, \bar{z}_2^{-1}) + f(\bar{z}_1^{-1}, \bar{z}_2^{-1}) \geqq 0$ *for* $|z_1|$, $|z_2| < 1$,

(c) $f(z_1, 0) + f(z_1, \infty) = 0$, $f(0, z_2) + f(\infty, z_2) = 0$ *for all* $|z_1| \neq 1$ *and* $|z_2| \neq 1$.

Condition (c) here is only a normalizing condition; (a) and (b) determine H_2 up to the addition of functions depending on only one of the variables. It turns out as a by-product of our investigations (Theorem 1), and it is also easy to show directly that every function f in H_2 admits the integral representation (14) generalizing the Riesz-Herglotz formula. f is now defined on four disconnected domains, and from (14) it follows that the Taylor coefficients of f at $(0, 0)$, $(0, \infty)$, $(\infty, 0)$ and (∞, ∞) are the double trigonometric moments of m. Hence the four pieces of f are the complex Fourier transforms of the restrictions of a positive definite function on $I \times I$ to the four quadrants of $I \times I$.

The analogue of the class N is defined as follows.

DEFINITION 2. *The class N_2 is the class of functions f of two complex variables z_1, z_2, defined and holomorphic for all* $\operatorname{Im} z_1$, $\operatorname{Im} z_2 \neq 0$, *and satisfying the conditions*

(a) $f(\bar{z}_1, \bar{z}_2) = \bar{f}(z_1, z_2)$ *for all* $\operatorname{Im} z_1$, $\operatorname{Im} z_2 \neq 0$,

(b) $f(z_1, z_2) - f(\bar{z}_1, z_2) - f(z_1, \bar{z}_2) + f(\bar{z}_1, \bar{z}_2) = 2 \operatorname{Re}[f(z_1, z_2) - f(\bar{z}_1, z_2)] \leqq 0$ *for* $\operatorname{Im} z_1$, $\operatorname{Im} z_2 > 0$,

(c) *for all* ϕ $(0 < \phi \leqq \pi/2)$ *there exists a constant* $M(\phi)$ *such that* $|f(z_1, z_2)/z_1 z_2| \leqq M(\phi)$ *for all* z_1, z_2 *in* $C(\phi)$.

We show (Theorem 2) that every function f in N_2 admits the integral representation (30) generalizing (2). From (30) it follows again that f has an asymptotic expansion whose coefficients are the double moments of a bounded positive measure on the plane, and it also follows in full analogy to the case of N and H_2 that f can be obtained by Fourier transformation from a positive definite function on $R \times R$.

The classes H_n and N_n can be defined analogously as H_2 and N_2; all our results generalize to these. It would also be possible to consider mixed classes, i.e., classes of functions obtained in the above way by Fourier transformation

from positive definite functions defined on the direct product of several copies of I and R.

In Theorems 1 and 2 we solve the analogue of the Pick-Nevanlinna problem for H_2 and N_2. As usual in this type of problem, the natural approach now is to assume that we are given a set S of points and a set of analytic functions of one variable z, $f(z, s)$ $(s \in S)$, and we look for conditions that this function be extendable to a function in H_2 or N_2. Similarly, for H_n and N_n we would assume f to be given on a set of $(n-1)$-dimensional hyperplanes.

The method of proof is similar to that in [6]; there is however an additional difficulty. We need some facts about the existence of commutative unitary dilations of commutative isometric operators and commutative selfadjoint dilations of formally commuting symmetric operators; these are proved in Lemmas 2, 3, and 4. These results may have some independent interest, and it seems that they can not be essentially improved since it was recently announced in [12] that an example of two formally commuting symmetric operators has been found which do not have commutative selfadjoint dilations.

In Theorem 3 we generalize Loewner's theorem (Theorem C in [6]) to two variables. In Theorem 4 we show that the functions considered here are precisely the monotone matrix functions of two variables, in complete analogy to the one-variable case.

Theorems 5, 6, and 7 are generalizations of the preceding results to functions whose values are bounded operators in a Hilbert space. These theorems turn out to be in close connection with some investigations initiated by B. Sz.-Nagy in the theory of dilations of Hilbert space operators [7]. In the last section we show that two theorems due to B. Sz.-Nagy [7] and S. Brehmer [1] can be derived from our Theorem 5. Using Theorem 7 we also give a generalization to double moments of a theorem of B. Sz.-Nagy [7] on a moment problem for selfadjoint operators.

2. Lemmas on dilations of operators.

LEMMA 2. *Let $\{U_\gamma\}_{\gamma\in\Gamma}$ be a unitary representation of a group Γ in the Hilbert space $\mathfrak{H}$, and let $V: \mathfrak{D} \to \mathfrak{R}$ be a mapping of a subspace $\mathfrak{D}$ of $\mathfrak{H}$ onto a subspace $\mathfrak{R}$. There exists a unitary dilation $\{\hat{U}_\gamma\}$ of the representation $\{U_\gamma\}$ and a unitary dilation $\hat{V}$ of V in some enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, such that $\hat{U}_\gamma \hat{V} = \hat{V} \hat{U}_\gamma$ for all $\gamma \in \Gamma$, if and only if*

$$\|U_\gamma V x + V U_\gamma y\| = \|x + y\| \tag{5}$$

holds for all $x \in \mathfrak{D}$, $y \in U_\gamma^{-1}\mathfrak{D}$, $\gamma \in \Gamma$.

Proof. The necessity of the condition is trivial. To prove its sufficiency, first define the mapping V' for finite sums $f = \sum_\gamma c_\gamma f_\gamma$ $(f_\gamma \in U_\gamma \mathfrak{D})$ by

$$V'f = \sum_\gamma c_\gamma U_\gamma V U_\gamma^{-1} f_\gamma. \tag{6}$$

We show that V' is isometric, this will also show that it is uniquely determined. We have

$$\|f\|^2 = \sum_{\gamma,\delta} c_\gamma \bar{c}_\delta (f_\gamma, f_\delta), \tag{7}$$

$$\|V'f\|^2 = \sum_{\gamma,\delta} c_\gamma \bar{c}_\delta (U_\gamma V U_\gamma^{-1} f_\gamma, U_\delta V U_\delta^{-1} f_\delta). \tag{8}$$

Using (5) we obtain

$$\begin{aligned}\|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x, y) &= \|x + y\|^2 = \|U_\gamma V x + V U_\gamma y\|^2 \\ &= \|U_\gamma V x\|^2 + \|V U_\gamma y\|^2 + 2\,\mathrm{Re}(U_\gamma V x, V U_\gamma y) \\ &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(U_\gamma V x, V U_\gamma y),\end{aligned}$$

hence $\mathrm{Re}(x, y) = \mathrm{Re}(U_\gamma V x, V U_\gamma y)$. Repeating this with ix instead of x, we have

$$(x, y) = (U_\gamma V x, V U_\gamma y) \qquad (x \in \mathfrak{D}, y \in U_\gamma^{-1}\mathfrak{D}, \gamma \in \Gamma). \tag{9}$$

With the aid of (9) we can show that the corresponding terms in (7) and (8) are equal:

$$\begin{aligned}(U_\gamma V U_\gamma^{-1} f_\gamma, U_\delta V U_\delta^{-1} f_\delta) &= (U_\delta^* U_\gamma V U_\gamma^{-1} f_\gamma, V U_\delta^{-1} f_\delta) \\ &= (U_\delta^* U_\gamma V x, V U_\delta^{-1} U_\gamma y) = (U_{\delta^{-1}\gamma} V x, V U_{\delta^{-1}\gamma} y)\end{aligned}$$

with $x = U_\gamma^{-1} f_\gamma \in \mathfrak{D}$, $y = U_\gamma^{-1} f_\delta \in U_\gamma^{-1} U_\delta \mathfrak{D} = U_{\gamma^{-1}\delta}\mathfrak{D}$, which by (9) is equal to

$$(x, y) = (U_\gamma^{-1} f_\gamma, U_\gamma^{-1} f_\delta) = (f_\gamma, f_\delta)$$

thus proving the isometry of V'.

Let $\overline{V}$ be the closure of V'; it is defined on the subspace $\mathfrak{L} = \{U_\gamma \mathfrak{D}\}_{\gamma \in \Gamma}$ which reduces all U_γ ($\gamma \in \Gamma$). We show that

$$U_\gamma \overline{V} f = \overline{V} U_\gamma f \qquad (\gamma \in \Gamma) \tag{10}$$

for all $f \in \mathfrak{L}$. By continuity, it suffices to consider elements of the form $f = \sum_\gamma c_\gamma f_\gamma$ ($f_\gamma \in U_\gamma \mathfrak{D}$). Now,

$$U_\delta \overline{V} f = \sum_\gamma c_\gamma U_\delta U_\gamma V U_\gamma^{-1} f_\gamma = \sum c_\gamma U_{\delta\gamma} V U_{\gamma^{-1}} f_\gamma,$$

$$U_\delta f = \sum_\gamma c_\gamma U_\delta f_\gamma$$

where $U_\delta f_\gamma \in U_\delta U_\gamma \mathfrak{D} = U_{\delta\gamma}\mathfrak{D}$. So, by definition of $\overline{V}$,

$$\overline{V} U_\delta f = \sum_\gamma c_\gamma U_{\delta\gamma} V U_{\delta\gamma}^{-1} U_\delta f_\gamma = \sum c_\gamma U_{\delta\gamma} V U_{\gamma^{-1}} f_\gamma,$$

which proves (10).

Let $\mathfrak{L}' = \overline{V}\mathfrak{L}$. By (10) $\mathfrak{L}'$ also reduces each U_γ ($\gamma \in \Gamma$). Denoting the projec-

tion of $\mathfrak{H}$ onto $\mathfrak{L}$ by P, and the projection onto $\mathfrak{L}'$ by P', P and P' will commute with each U_γ ($\gamma\in\Gamma$). From (10) we also have

$$U_\gamma \overline{V}^{-1}g = \overline{V}^{-1}U_\gamma g$$

for all $g\in\mathfrak{L}'$, $\gamma\in\Gamma$.

These remarks show that the operators

$$\hat{U}_\gamma = \begin{pmatrix} U_\gamma & 0 \\ 0 & U_\gamma \end{pmatrix}, \qquad \hat{V} = \begin{pmatrix} \overline{V}P & 1-P' \\ 1-P & \overline{V}^{-1}P' \end{pmatrix}$$

acting on $\mathfrak{H}\oplus\mathfrak{H}$, the Hilbert space of pairs $\{f, g\}$ ($f, g\in\mathfrak{H}$), commute for each γ in Γ. Imbedding the space $\mathfrak{H}$ into $\mathfrak{H}\oplus\mathfrak{H}$ by the identification $f\sim\{f, 0\}$, it is a trivial matter to verify that $\hat{U}_\gamma$, $\hat{V}$ are just the required dilations.

COROLLARY. *Let $U_1, \cdots, U_k$ be commuting unitary operators in $\mathfrak{H}$, and $V: \mathfrak{D}\to\mathfrak{R}$ an isometric mapping of the subspace $\mathfrak{D}$ onto the subspace $\mathfrak{R}$. For the existence of commuting unitary dilations $\hat{U}_1, \cdots, \hat{U}_k$, $\hat{V}$ in an enlarged Hilbert space $\hat{\mathfrak{H}}\supseteq\mathfrak{H}$, it is necessary and sufficient that*

$$\|U_1^{n_1}\cdots U_k^{n_k}Vx + VU_1^{n_1}\cdots U_k^{n_k}y\| = \|x+y\| \tag{11}$$

holds for all $x\in\mathfrak{D}$, $y\in U_1^{-n_1}\cdots U_k^{-n_k}\mathfrak{D}$, and all natural numbers $n_1, \cdots, n_k$.

Proof. The corollary is obvious if we show that (11) is necessarily fulfilled even if some (or all) of the n_i are negative. For this it is sufficient to see that for an arbitrary unitary U, $\|UVx+VUy\|=\|x+y\|$ for all $x\in\mathfrak{D}$, $y\in U^{-1}\mathfrak{D}$ implies $\|U^{-1}Vx+VU^{-1}y\|=\|x+y\|$ for all $x\in\mathfrak{D}$, $y\in U\mathfrak{D}$. The latter fact, however, is evident.

REMARK. In the case of only one unitary operator U condition (11) reduces to $\|U^nVx+VU^ny\|=\|x+y\|$ ($x\in\mathfrak{D}$, $y\in U^{-n}\mathfrak{D}$), for all natural numbers n. One might ask whether it is sufficient to assume this for $n=1$ only. The answer is negative even for a finite-dimensional $\mathfrak{H}$, as can be shown by simple examples.

LEMMA 3. *Let $U: \mathfrak{H}\to\mathfrak{R}$ be an isometric mapping of the Hilbert space $\mathfrak{H}$ onto a subspace $\mathfrak{R}$. Let $V: \mathfrak{D}\to\mathfrak{R}$ be an isometric mapping of a subspace $\mathfrak{D}$ of $\mathfrak{H}$ onto another subspace $\mathfrak{R}$. Suppose $U\mathfrak{D}\subseteq\mathfrak{D}$, and $UVf=VUf$ for all vectors f in $\mathfrak{D}$. Then there exist commuting unitary dilations $\hat{U}$, $\hat{V}$ of U and V in an enlarged Hilbert space $\hat{\mathfrak{H}}\supseteq\mathfrak{H}$.*

Proof. We may assume that U is unitary, since if it is not, we can dilate it to a unitary operator without changing the assumptions. So, by the corollary of Lemma 2, we only have to show that

$$\|\hat{U}^nVx + VU^ny\| = \|x+y\| \tag{12}$$

holds for all $x\in\mathfrak{D}$, $y\in U^{-n}\mathfrak{D}$, $n>0$.

$\mathfrak{D} \supseteq U\mathfrak{D}$ implies $U^{-n}\mathfrak{D} \supseteq \mathfrak{D}$. Hence any element y in $U^{-n}\mathfrak{D}$ can be written as $y = y' + y''$ with $y' \in \mathfrak{D}$, $y'' \perp \mathfrak{D}$. We have $U^n V y' = V U^n y'$, and hence

$$\|U^n V x + V U^n y\| = \|U^n V x + V U^n (y' + y'')\| = \|U^n V(x + y') + V U^n y''\|.$$

This expression must be shown to be equal to $\|x + y' + y''\|$. Since $x + y' \in \mathfrak{D}$, this reduces to proving (12) with the additional assumption $y \perp \mathfrak{D}$. This is done as follows.

$y \perp \mathfrak{D}$ implies $U^n y \perp U^n \mathfrak{D}$, whence $V U^n y \perp V U^n \mathfrak{D} = U^n V \mathfrak{D} = U^n \mathfrak{R}$. On the other hand we have $U^n V x \in U^n \mathfrak{R}$. Therefore

$$\|U^n V x + V U^n y\|^2 = \|U^n V x\|^2 + \|V U^n y\|^2 = \|x\|^2 + \|y\|^2$$

holds. By $y \perp \mathfrak{D}$ we also have $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, which completes the proof of our assertion. Lemma 3 is thereby proved.

LEMMA 4. *Let $\mathfrak{M}$ be a dense linear manifold in the Hilbert space $\mathfrak{H}$. Let A_0, B_0 be two symmetric operators defined on $\mathfrak{M}$; A and B their closures. Let A be selfadjoint, and let B be equal to the closure of its restriction to the manifold $(A + iI)\mathfrak{M}$. Suppose that the domain of both products AB and BA contains $\mathfrak{M}$, and $ABf = BAf$ holds for all f in $\mathfrak{M}$. Then in an enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$ there exist commuting selfadjoint dilations $\hat{A}$, $\hat{B}$ of A and B.*

Proof. Let $U = (A - iI)(A + iI)^{-1}$, $V = (B - iI)(B + iI)^{-1}$ be the Cayley-transforms of A and B. Since A is selfadjoint, U is unitary. V is an isometric mapping of the subspace

$$\mathfrak{D} = (B + iI)\mathfrak{D}_B = \overline{(B_0 + iI)\mathfrak{M}}$$

onto a subspace $\mathfrak{R}$. ($\mathfrak{D}_B$ denotes the domain of B.)

We want to prove that $UVf = VUf$ for all f in $\mathfrak{D}$. It suffices to show this for f in $\mathfrak{R}$. For $f \in \mathfrak{R}$, $g = (B + iI)f \in (A + iI)\mathfrak{M}$. Since $A + iI$ and $B - iI$ commute on $\mathfrak{M}$, we have

$$(B - iI)(A + iI)^{-1} g = (A + iI)^{-1}(B - iI) g,$$

and hence

$$\begin{aligned} UVf &= (A - iI)(A + iI)^{-1}(B - iI)(B + iI)^{-1} f \\ &= (A - iI)(B - iI)(A + iI)^{-1}(B + iI)^{-1} f. \end{aligned}$$

By the commutativity of A and B on $\mathfrak{M}$ we have further

$$UVf = (B - iI)(A - iI)(B + iI)^{-1}(A + iI)^{-1} f.$$

Now, $h = (A + iI)^{-1} f \in (B + iI)\mathfrak{M}$; therefore,

$$(A - iI)(B + iI)^{-1} h = (B + iI)^{-1}(A - iI) h,$$

and so, finally,

$$UVf = (B - iI)(B + iI)^{-1}(A - iI)(A + iI)^{-1}f = VUf,$$

proving the assertion.

Next we show that $U\mathfrak{D} \subseteq \mathfrak{D}$. Again it suffices to prove $U\mathfrak{N} \subseteq \mathfrak{D}$; this however is evident:

$$\begin{aligned} U\mathfrak{N} &= (A - iI)(A + iI)^{-1}(A + iI)(B + iI)\mathfrak{M} = (A - iI)(B + iI)\mathfrak{M} \\ &= (B + iI)(A - iI)\mathfrak{M} \subseteq (B + iI)\mathfrak{D}_B = \mathfrak{D}. \end{aligned}$$

By Lemma 3 there exist commuting unitary dilations $\hat{U}$, $\hat{V}$ of U, V in a larger space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$. We show that $\hat{U}$, $\hat{V}$ can be chosen so that neither of them has 1 as a proper value. In fact, U, V themselves do not have 1 as a proper value, since they are the Cayley transforms of densely defined operators. Now assume $\hat{V}h = h$. Then for every $f = (I - \hat{V})g$ $(g \in \hat{\mathfrak{H}})$ we have

$$(h, f) = (h, g) - (h, \hat{V}g) = (\hat{V}h, \hat{V}g) - h, \hat{V}g) = -((I - \hat{V})h, \hat{V}g) = 0;$$

hence h is orthogonal to the range of $I - \hat{V}$, and then also to the range of $I - V$, i.e., to the domain of B, and hence to $\mathfrak{H}$. So the subspace H of all $h \in \hat{\mathfrak{H}}$ with $\hat{V}h = h$ is orthogonal to $\mathfrak{H}$.

$\hat{V}h = h$ implies $\hat{V}^*h = h$, therefore H reduces $\hat{V}$. For h in H we have $\hat{U}h = \hat{U}\hat{V}h = \hat{V}\hat{U}h$, hence $\hat{U}h \in H$. Similarly $U^*h \in H$, so H reduces $\hat{U}$.

Now we can split off the subspace H from $\hat{\mathfrak{H}}$, and we are left with two commuting unitary dilations $\hat{U}'$, $\hat{V}'$ of U, V such that $\hat{V}'$ does not have the proper value 1. Repeating the above procedure with U in place of V, finally we obtain the dilations with the desired property.

The inverse of the Cayley transform now carries $\hat{U}$ and $\hat{V}$ into selfadjoint dilations of A and B, and these dilations will commute in the sense of the usual definition of commutativity for unbounded selfadjoint operators.

3. The main theorems.

THEOREM 1. *Let S be a set of points in the interior of the unit disk, and for every s in S let f_s be a function defined and holomorphic for all $|z| \neq 1$ (including ∞). For the existence of a function F such that $F(z_1, z_2) = g(z_1, z_2) + h(z_1) + k(z_2)$, for all $|z_1|$, $|z_2| \neq 1$, where $g \in H_2$, h is holomorphic and $h(z_1^{-1}) \equiv -h(z_1)$, and where $f_s(z) = F(z, s)$ for all $s \in S$, $|z| \neq 1$, it is necessary and sufficient that the function k defined for all $s, t \in S$; $|z|$, $|w| \neq 1$, by*

$$k(z, s; w, t) = \begin{cases} \dfrac{f_s(z) - f_s(\bar{w}^{-1}) + \bar{f}_t(w) - \bar{f}_t(\bar{z}^{-1})}{(1 - z\bar{w})(1 - s\bar{t})} & (w \neq \bar{z}^{-1}), \\[2ex] \dfrac{-zf_s'(z) - z^{-1}\bar{f}_t(\bar{z}^{-1})}{1 - s\bar{t}} & (w = \bar{z}^{-1}) \end{cases} \tag{13}$$

be positive definite.

Every function g in H_2 can be represented in the form

$$g(z_1, z_2) = \int_0^{2\pi}\int_0^{2\pi} \frac{1+z_1e^{i\phi}}{1-z_1e^{i\phi}}\,\frac{1+z_2e^{i\psi}}{1-z_2e^{i\psi}}\,dm(\phi, \psi), \tag{14}$$

where m is a bounded positive measure on the torus.

REMARK. The functions of one variable, h and k do not play an essential role here. By putting some normalizing conditions on the functions f_s we can make $h(z_1)\equiv k(z_2)\equiv 0$, i.e., $F\in H_2$. E.g., if $0\in S$, these conditions are $f_0(\bar{z}^{-1}) = -\bar{f}_0(z)$ for all $|z|\neq 1$ and $f_s(\infty) = -\bar{f}_s(0)$ for all $s\in S$.

Proof. To prove the necessity of the condition, let $F(z_1, z_2) = g(z_1, z_2) + h(z_1) + k(z_2)$, $g\in H_2$, h analytic and $h(\bar{z}_1^{-1}) = \bar{h}(z_1)$ for all $z_1\neq 1$. Let $s_1, \cdots, s_N$ be points in the interior of the unit disk, $z_1, \cdots, z_N$ any points not on the unit circle. We apply to g the Cauchy-Poisson formulas for two variables: If $0<r<1$, we have for $|z_1|, |z_2|<r$,

$$\begin{aligned} g(z_1, z_2) &= \frac{1}{2}\left[g(z_1, 0) + g(z_1, \infty) + g(0, z_2) + g(\infty, z_2)\right] \\ &\quad - \frac{1}{4}\left[g(0, 0) + g(0, \infty) + g(\infty, 0) + g(\infty, \infty)\right] \\ &\quad + \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{re^{i\phi}+z_1}{re^{i\phi}-z_1}\,\frac{re^{i\psi}+z_2}{re^{i\psi}-z_2}\,p(re^{i\phi}, re^{i\psi})d\sigma d\psi \end{aligned}$$

with $p(w_1, w_2) = g(w_1, w_2) - g(\bar{w}_1^{-1}, w_2) - g(w_1, \bar{w}_2^{-1}) + g(\bar{w}_1^{-1}, \bar{w}_2^{-1}) \geqq 0$. Since $g\in H_2$ the sum of the terms preceding the integral on the right-hand side is zero. Similarly, for $|z_1|>1/r$, $|z_2|<r$, the representation

$$g(z_1, z_2) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \frac{e^{i\phi}+rz_1}{e^{i\phi}-rz_1}\,\frac{re^{i\psi}+z_2}{re^{i\psi}-z_2}\,p(re^{i\phi}, re^{i\psi})d\phi d\psi$$

holds, and there are analogous formulas for the domains $|z_1|<r$, $|z_2|>1/r$ and $|z_1|, |z_2|>1/r$.

In the case where $z_1, \cdots, z_N$ are all inside the unit disk, we choose r such that $|s_k|, |z_k|<r<1$ for all $k=1, \cdots, N$. Then, for any complex numbers $\alpha_1, \cdots, \alpha_N$ we have

$$\begin{aligned} &\sum_m\sum_n \frac{F(z_m, s_m) - F(\bar{z}_n^{-1}, s_m) + \bar{F}(z_n, s_n) - \bar{F}(\bar{z}_m^{-1}, s_n)}{(r^2 - z_m\bar{z}_n)(r^2 - s_m\bar{s}_n)}\,\alpha_m\bar{\alpha}_n \\ &\qquad = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\sum_m \frac{\alpha_m}{(re^{i\phi}-z_m)(re^{i\psi}-s_m)}\right| p(re^{i\phi}, re^{i\psi})d\phi d\psi \geqq 0, \end{aligned}$$

and letting r tend to 1 we obtain

$$\sum_m\sum_n k(z_m, s_m; z_n, s_n)\alpha_m\bar{\alpha}_n \geqq 0, \tag{15}$$

which is the required result.

If some of the z_k are situated outside the unit circle, we choose r such that all s_k, z_k $(k=1, \cdots, N)$ are either inside the circle of radius r or outside the circle of radius $1/r$. A similar computation as before shows that (15) holds even in this general case for any $\alpha_1, \cdots, \alpha_N$.

In the proof of the sufficiency we shall use the more symmetric notation $f(z, s)=f_s(z)$. First we consider the case where $0\in S$.

Let $\mathfrak{H}$ be the Hilbert space of Lemma 1, spanned by the vectors ϵ_{zs} $(|z|\neq 1, s\in S)$ and such that

$$k(z, s; w, t) = (\epsilon_{zs}, \epsilon_{wt}). \tag{16}$$

ϵ_{zs} is here a strongly continuous function of z, since for $w\to z$, by the analyticity of $f(z, s)$ in z we have

$$\|\epsilon_{zs} - \epsilon_{wt}\|^2 = k(z, s; z, s) - k(z, s; w, s) - k(w, s; z, s) + k(w, s; w, s) \to 0.$$

For $z\neq 0, \infty$ we define

$$\epsilon'_{zs} = \frac{1}{z}(\epsilon_{zs} - \epsilon_{0s}). \tag{17}$$

Then an easy computation based on (13), (16), and (17) shows that for $z, w\neq 0, \infty$, and all $s, t\in S$,

$$(\epsilon'_{zs}, \epsilon'_{wt}) = (\epsilon_{zs}, \epsilon_{wt}).$$

By continuity, this holds also if $z=\bar{w}^{-1}$, since for $z, w\neq 0, \infty$ $\epsilon_{ws}\to\epsilon_{zs}$ implies $\epsilon'_{ws}\to\epsilon'_{zs}$.

Hence the operator U_0 defined by

$$U_0 \sum c_p\epsilon_{z_p s_p} = \sum_p c_p\epsilon'_{z_p s_p} \tag{18}$$

on the manifold $\mathfrak{M}$ of all finite complex linear combinations of the ϵ_{zs} $(|z|\neq 1, s\in S; z\neq 0, \infty)$ is well defined and isometric. By the strong continuity of ϵ_{zs}, $\mathfrak{M}$ is dense in $\mathfrak{H}$, hence the closure U of U_0 is an everywhere defined isometric operator.

Now we define

$$\epsilon''_{zs} = \frac{1}{s}(\epsilon_{zs} - \epsilon_{z0}) \tag{19}$$

for all $|z|\neq 1$, $s\in S$, $s\neq 0$.

For $s\neq 0$, $t\neq 0$ another simple computation yields

$$(\epsilon''_{zs}, \epsilon''_{wt}) = (\epsilon_{zs}, \epsilon_{wt}).$$

Hence the operator V_0 defined on the manifold $\mathfrak{D}_0$ of finite linear combinations of the ϵ_{zs} ($|z| \neq 1$, $s \in S$, $s \neq 0$) by

$$V_0 \sum c_p \epsilon_{z_p s_p} = \sum_p{}'' c_p \epsilon_{z_p s_p}$$

is isometric. Its closure, V, defined on the subspace $\mathfrak{D} = \tilde{\mathfrak{D}}_0$ is therefore also isometric.

By the definition of U we have $U\mathfrak{D}_0 \subseteq \mathfrak{D}_0$, hence also $U\mathfrak{D} \subseteq \mathfrak{D}$.

We show that $UVf = VUf$ holds for all f in $\mathfrak{D}$. Note that $\mathfrak{D}_0 \cap \mathfrak{M}$, which consists of all finite linear combinations of the ϵ_{zs} with $|z| \neq 1$, $s \in S$; $z \neq 0$, $z \neq \infty$, $s \neq 0$, is dense in $\mathfrak{D}$ by the continuity of ϵ_{zs}. Hence it is sufficient to show $UVf = VUf$ for $f \in \mathfrak{D}_0 \cap \mathfrak{M}$. By linearity, then it is even sufficient to consider $f = \epsilon_{zs}$ ($z = 0, \infty$; $s \neq 0$) only. But, in this case, we have

$$\begin{aligned} UV\epsilon_{zs} &= U \frac{1}{s} (\epsilon_{zs} - \epsilon_{z0}) = \frac{1}{zs} (\epsilon_{zs} - \epsilon_{0s} - \epsilon_{z0} + \epsilon_{00}) \\ &= V \frac{1}{z} (\epsilon_{zs} - \epsilon_{0s}) = VU\epsilon_{zs}, \end{aligned} \tag{20}$$

proving the assertion.

We have shown that for U, V all conditions of Lemma 3 are satisfied. Therefore, in a properly enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, U and V have commuting unitary dilations $\hat{U}$, $\hat{V}$.

By (17) and (19) we have

$$(I - zU)(I - sV)\epsilon_{zs} = \epsilon_{00}$$

for all $|z| \neq 1$, $s \in S$. This relation remains true for $\hat{U}$, $\hat{V}$ instead of U, V, and hence

$$\epsilon_{zs} = (I - z\hat{U})^{-1}(I - s\hat{V})^{-1}\epsilon_{00} \tag{21}$$

holds. Using the fact that $(I + z\hat{U})(I - z\hat{U})^{-1} = 2(I - z\hat{U})^{-1} - I$ and the analogous relation for $\hat{V}$, we obtain from (21) the equality

$$\frac{1}{4}\left(\frac{I + z\hat{U}}{I - z\hat{U}} \frac{I + s\hat{V}}{I - s\hat{V}} \epsilon_{00}, \epsilon_{00}\right)$$

$$= (\epsilon_{zs}, \epsilon_{00}) - \frac{1}{2}(\epsilon_{z0}, \epsilon_{00}) - \frac{1}{2}(\epsilon_{0s}, \epsilon_{00}) + \frac{1}{4}(\epsilon_{00}, \epsilon_{00}),$$

or, by (16) and (13),

$$f(z, s) = g(z, s) + h(z) + k(s), \tag{22}$$

with

$$g(z, s) = \frac{1}{4}\left(\frac{1 + z\hat{U}}{1 - z\hat{U}} \frac{1 + s\hat{V}}{1 - s\hat{V}} \epsilon_{00}, \epsilon_{00}\right), \tag{23}$$

$$h(z) = \frac{1}{2}[f(z, 0) + f(\bar{z}^{-1}, 0)], \tag{24}$$

$$k(s) = \frac{1}{2}[f(0, s) + f(\infty, s) - \operatorname{Re} f(0, 0) - \operatorname{Re} f(\infty, 0)]. \tag{25}$$

Now, the definition of g can be extended by the same formula to any value of the arguments not on the unit circle,

$$g(z_1, z_2) = \frac{1}{4}\left(\frac{I + z_1\hat{U}}{I - z_1\hat{U}} \frac{I + z_2\hat{V}}{I - z_2\hat{V}} \epsilon_{00}, \epsilon_{00}\right). \tag{26}$$

(In order not to crowd notations we use the same letter g for the extended function.) It is clear that, since $\hat{U}$, $\hat{V}$ are unitary, the function (26) is defined and analytic for all $|z_1|, |z_2| \neq 1$.

We show that $g \in H_2$. With the abbreviated notation

$$U(z) = \frac{I + zU}{I - zU}$$

for U unitary and $|z| \neq 1$, we have the simple relations

$$U(\bar{z}^{-1}) = -U(z^*), \tag{27}$$

$$U(z) - U(\bar{z}^{-1}) = (1 - |z|^2)(I - zU)^{*-1}(I - zU)^{-1} \geqq 0 \tag{28}$$

for $|z| < 1$. By (27) we have

$$g(z_1^{-1}, z_2^{-1}) = (U(\bar{z}_1^{-1})V(\bar{z}_2^{-1})\epsilon_{00}, \epsilon_{00}) = (\hat{U}(z_1)^*\hat{V}(z_2)^*\epsilon_{00}, \epsilon_{00}) = \bar{g}(z_1, z_2)$$

for all $|z_1|, |z_2| \neq 1$, which is property (a) in Definition 1.

By (33), and since the product of two commuting positive operators is positive, we have for $|z_1|, |z_2| < 1$,

$$\begin{aligned} g(z_1, z_2) - g(\bar{z}_1^{-1}, z_2) - g(z_1, \bar{z}_2^{-1}) + g(\bar{z}_1^{-1}, \bar{z}_2^{-1}) \\ = ((\hat{U}(z_1) - \hat{U}(\bar{z}_1^{-1}))(\hat{V}(z_2) - V(\bar{z}_2^{-1}))\epsilon_{00}, \epsilon_{00}) \geqq 0, \end{aligned}$$

which is property (b). Property (c) of Definition 1 is trivially satisfied.

We have established that $g \in H_2$. From (24) we see at once that h is analytic and $h(\bar{z}_1^{-1}) = \bar{h}(z_1)$. Thus the sufficiency of the condition of Theorem 1 is proved in the special case where $0 \in S$.

Our considerations also prove the Remark; the conditions there just make sure that $h(z_1) \equiv k(z_2) \equiv 0$, as is immediately shown by (24) and (25).

The case where $0 \notin S$ can be reduced to the former one. We pick an arbitrary point s_0 in S, and consider the mapping $s \to \sigma$ defined by

$$\sigma = l(s) = \frac{s - s_0}{1 - s\bar{s}_0}$$

which carries the unit disk into itself, and s_0 into 0. An easy computation shows that the function ϕ defined by $\phi(z, \sigma) = f(z, s)$ for all $|z| \neq 1$, $\sigma \in l(S)$, satisfies the condition of the theorem. Since $0 \in l(S)$, the function ϕ possesses an extension Φ of the desired kind. But then the function F defined for all $|z_1|$, $|z_2| \neq 1$ by $F(z_1, z_2) = \Phi(z_1, l(z_2))$ is an extension of F and a trivial computation, based on the identity $l(\bar{z}_2^{-1}) = \bar{l}(z_2)^{-1}$ shows that it fulfills the requirements of the theorem.

So the sufficiency of the condition is fully proved. Remains to prove the integral representation (14). To prove this, we note that by what we have proved until now, every function g in H_2 admits a representation (26) with some commutative unitary operators $\hat{U}$, $\hat{V}$ in some Hilbert space $\hat{\mathfrak{H}}$. Let the spectral resolutions of these operators be $\hat{U} = \int_0^{2\pi} e^{i\phi} dE_\phi$, $V = \int_0^{2\pi} e^{i\psi} dF_\psi$. The spectral families E_ϕ and F_ψ commute, and hence by elementary spectral theory

$$g(z_1, z_2) = \int_0^{2\pi} \int_0^{2\pi} \frac{1 + z_1 e^{i\phi}}{1 - z_1 e^{i\phi}} \frac{1 + z_2 e^{i\psi}}{1 - z_2 e^{i\psi}} dm(\phi, \psi),$$

with $m(\phi, \psi) = (1/4)(E_\phi F_\psi \epsilon_{00}, \epsilon_{00})$, which in fact determines a bounded positive measure on the torus.

THEOREM 2. *Let S be a subset of the upper half-plane such that for some $0 < \phi \leqq \pi/2$, 0 is an accumulation point of $S \cap C(\phi)$. For each s in S let f_s be an analytic function of z for all nonreal z. For the existence of a function g in N_2 such that $f_s(z) = g(z, s)$ for all $s \in S$,* Im $z \neq 0$, *it is necessary and sufficient that* (a) *the function k defined for all* Im z, Im $w \neq 0$, s, $t \in S$ *by*

$$k(z, s; w, t) = \begin{cases} \dfrac{f_s(z) - f_s(\bar{w}) + \bar{f}_t(w) - \bar{f}_t(\bar{z})}{(z - \bar{w})(s - \bar{t})} & (z \neq \bar{w}), \\[2ex] \dfrac{f_s'(z) - \bar{f}_t'(\bar{z})}{s - \bar{t}} & (z = \bar{w}) \end{cases} \tag{29}$$

be positive definite, and (b) *for at least one sequence $\{\sigma_n\}$, $\sigma_n \to 0$, $\sigma_n \in S \cap C(\phi)$, $|f_{\sigma_n}(z)/z\sigma_n| \leqq M$ hold for all z in $C(\phi)$ with a constant M independent of n.*

Every function g in N_2 can be represented in the form

$$g(z_1, z_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z_1}{1 - \lambda z_1} \frac{z_2}{1 - \mu z_2} dm(\lambda, \mu), \tag{30}$$

with a bounded positive measure m on the plane.

Proof. To prove the necessity of the conditions let $g \in N_2$. Consider the mapping $z \to \zeta$ defined by $\zeta = a(z) = (z - i)/(z + i)$ which carries the upper half-

plane onto the unit disk, and its inverse, $z=b(\zeta)=-i(\zeta+1)/(\zeta-1)$. On the basis of the relation $b(\bar{\zeta}^{-1})=\bar{b}(\zeta)$ it is easily seen that the function G defined by

$$G(\zeta_1, \zeta_2) = -g(b(\zeta_1), b(\zeta_2))$$

satisfies conditions (a) and (b) in Definition 1, hence up to addition of functions of one variable, belongs to H_2.

We have the relation

$$\frac{1}{4}(z+i)(w+i)(s+i)(t+i)k(z, s; w, t) = h(\zeta, \sigma; \eta, \tau)$$

with $\zeta=a(z)$; $\eta=a(w)$, $\sigma=a(s)$, $\tau=a(t)$, and

$$h(\zeta, \sigma; \eta, \tau) = \frac{G(\zeta, \sigma) - G(\bar{\eta}^{-1}, \sigma) + \bar{G}(\eta, \tau) - \bar{G}(\bar{\zeta}^{-1}, \tau)}{(1-\zeta\bar{\eta})(1-\sigma\bar{\tau})}$$

which is positive definite by Theorem 1. Hence for any points $s_1, \cdots, s_N$ in the upper-half plane, $z_1, \cdots, z_N$ not on the real line, and any complex numbers $\alpha_1, \cdots, \alpha_N$ we have

$$\sum_m \sum_n k(z_m, s_m; z_n, s_n)\alpha_m\bar{\alpha}_n = \sum_m \sum_n h(\zeta_m, \sigma_m; \zeta_n, \sigma_n)\beta_m\bar{\beta}_n \geqq 0$$

with

$$\zeta_j = a(z_j), \qquad \sigma_j = a(s_j), \qquad \beta_j = \frac{2\alpha_j}{(z_j+1)(s_j+i)},$$

which proves the necessity of the positive definiteness of (29). The necessity of condition (b) is evident.

To prove the sufficiency of the conditions, let $\mathfrak{H}$ be the Hilbert space constructed by Lemma 1 in which

$$k(z, s; w, t) = (\epsilon_{zs}, \epsilon_{wt}). \tag{31}$$

In the following we use the notation $f(z, s)=f_s(z)$. Let $t\in S$, and let $\{w_n\}$, $w_n\to 0$ be a sequence of points contained in some $C(\phi)$ $(\phi>0)$. Then, by (b), we have $f(w_n, t)\to 0$, and

$$\|\epsilon_{w_n t}\|^2 = k(w_n, t; w_n, t) = -\frac{\operatorname{Re}[f(w_n, t) - f(\bar{w}_n, t)]}{2 \operatorname{Im} w \cdot \operatorname{Im} t} \leqq \frac{M}{\sin\phi \cdot |\operatorname{Im} t|}.$$

Now, for every $\operatorname{Im} z\neq 0$, $s\in S$,

$$(\epsilon_{zs}, \epsilon_{w_n t}) = \frac{f(z, s) - f(\bar{w}_n, s) + \bar{f}(w_n, t) - \bar{f}(\bar{z}, t)}{(z-\bar{w}_n)(s-\bar{t})} \to \frac{f(z, s) - \bar{f}(\bar{z}, t)}{z(s-\bar{t})}$$

and since the ϵ_{zs} $(\operatorname{Im} z\neq 0,\ s\in S)$ span $\mathfrak{H}$, it follows that the sequence $\epsilon_{w_n t}$ converges weakly. Its limit will be denoted by ϵ_{0t}. We have

$$(\epsilon_{zs}, \epsilon_{0t}) = \frac{f(z, s) - \bar{f}(\bar{z}, t)}{z(s - \bar{t})} . \tag{32}$$

Now let w ($\operatorname{Im} w = 0$) be fixed, and $\{\sigma_n\}$ be the sequence mentioned in condition (b). Again by (b) we have $f(w, \sigma_n) \to 0$, and

$$\|\epsilon_{w\sigma_n}\|^2 = k(w, \sigma_n; w, \sigma_n) = -\frac{\operatorname{Re}[f(w, \sigma_n) - f(\bar{w}, \sigma_n)]}{2 \operatorname{Im} w \cdot \operatorname{Im} \sigma_n} \leqq \frac{M}{|\operatorname{Im} w| \sin \phi} .$$

Thus for every z ($\operatorname{Im} z \neq 0$), $s \in S$,

$$(\epsilon_{zs}, \epsilon_{w\sigma_n}) = \frac{f(z, s) - f(\bar{w}, s) + \bar{f}(w, \sigma_n) - \bar{f}(\bar{z}, \sigma_n)}{(z - \bar{w})(s - \bar{\sigma}_n)} \to \frac{f(z, s) - f(\bar{w}, s)}{(z - \bar{w})s} .$$

(This is correct even if $z = \bar{w}$, interpreting it as a differential quotient.) This shows that $\epsilon_{w\sigma_n}$ converges weakly to a limit, which we shall denote by ϵ_{w0}, and we have

$$(\epsilon_{zs}, \epsilon_{w0}) = \frac{f(z, s) - f(\bar{w}, s)}{(z - \bar{w})s} \tag{33}$$

(and $(\epsilon_{zs}, \epsilon_{\bar{z}0}) = (1/s)f'(z, s)$).

Finally let $\{\sigma_n\}$ be the sequence (b) and $\{z_m\}$ $z_m \to 0$ some sequence contained in $C(\phi)$. By (b) we have $f(z_m, \sigma_n) \to 0$ for $m, n \to \infty$, and

$$\|\epsilon_{z_m\sigma_n}\|^2 = k(z_m, \sigma_n; z_m, \sigma_n) = -\frac{\operatorname{Re}[f(z_m, \sigma_n) - f(\bar{z}_m, \sigma_n)]}{2 \operatorname{Im} z_m \cdot \operatorname{Im} \sigma_n} \leqq \frac{M}{\sin^2 \phi} . \tag{34}$$

Therefore, for all $\operatorname{Im} z \neq 0$, $s \in S$,

$$(\epsilon_{zs}, \epsilon_{z_m\sigma_n}) = \frac{f(z, s) - f(\bar{z}_m, s) + \bar{f}(z_m, \sigma_n) - \bar{f}(\bar{z}, \sigma_n)}{(z - \bar{z}_m)(s - \bar{\sigma}_n)} \to \frac{f(z, s)}{zs}$$

and so $\epsilon_{z_m\sigma_n}$ converges weakly to a vector ϵ_{00} such that for ($z, s \neq 0$),

$$(\epsilon_{zs}, \epsilon_{00}) = \frac{f(z, s)}{zs} . \tag{35}$$

We now show that for any fixed s in S, ϵ_{zs} ($\operatorname{Im} z \neq 0$) is a weakly continuous function of z. In fact, for $w \to z$, $\|\epsilon_{zs} - \epsilon_{ws}\|^2 = k(z, s; z, s) - k(z, s; w, s) - k(w, s; z, s) + k(w, s; w, s) \to 0$ by the analyticity of f, and so even strong continuity holds.

We also show that $\epsilon_{0\sigma_n} \to \epsilon_{00}$ and $\epsilon_{z_n 0} \to \epsilon_{00}$ ($z_n \to 0$, $z_n \in C(\phi)$, $\phi > 0$) in the weak sense. In fact, by (34) we have

$$\|\epsilon_{0\sigma_n}\|^2, \|\epsilon_{z_m 0}\|^2 \leqq \frac{M}{\sin^2 \phi} ,$$

and by (32) and (33)

$$(\epsilon_{zs}, \epsilon_{0\sigma_n}) \to (\epsilon_{zs}, \epsilon_{00}),$$
$$(\epsilon_{zs}, \epsilon_{z_m 0}) \to (\epsilon_{zs}, \epsilon_{00}),$$

for all $\operatorname{Im} z \neq 0$, $s \in S$.

Now, for all $\operatorname{Im} z \neq 0$, $s \in S$, we define

$$\epsilon'_{zs} = \frac{1}{z}(\epsilon_{zs} - \epsilon_{0s}). \tag{36}$$

For $\operatorname{Im} w \neq 0$, $z \neq \bar{w}$, $t \in S$ we have by (29),

$$(\epsilon'_{zs}, \epsilon_{wt}) = \frac{1}{z}(\epsilon_{zs} - \epsilon_{0s}, \epsilon_{wt}) = \frac{1}{z}[k(z, s; w, t) - k(0, s; w, t)]$$
$$= \frac{1}{z - \bar{w}} \cdot \frac{1}{s - \bar{t}} \left(\frac{f(z, s)}{z} - \frac{f(z, t)}{z} - \frac{f(w, s)}{\bar{w}} + \frac{f(w, t)}{\bar{w}} \right).$$

Interchanging z, s with w, t in this formula changes its value to its complex conjugate. This proves without further computation the relation

$$(\epsilon'_{zs}, \epsilon_{wt}) = (\epsilon_{zs}, \epsilon'_{wt}).$$

This is true by continuity also for $z = \bar{w}$; we have to note only that like ϵ_{zs}, ϵ'_{zs} is also a continuous function of z.

Let $\mathfrak{M}$ be the manifold of all finite complex linear combinations $\phi = \sum_p c_p \epsilon_{z_p s_p}$ ($\operatorname{Im} z_p \neq 0$, $s_p \in S$). Define

$$\phi' = \sum_p c_p \epsilon'_{z_p s_p}.$$

Then, taking any other element ψ of $\mathfrak{M}$,

$$\psi = \sum_q d_q \epsilon_{w_q t_q}, \qquad \psi' = \sum_q d_q \epsilon'_{w_q t_q},$$

we have

$$(\phi', \psi) = \sum_p \sum_q c_p \bar{d}_q (\epsilon'_{z_p s_p}, \epsilon_{w_q t_q}) = \sum_p \sum_q c_p \bar{d}_q (\epsilon_{z_p s_p}, \epsilon'_{w_q t_q}) = (\phi, \psi').$$

$\mathfrak{M}$ being dense in $\mathfrak{H}$ this shows that $\phi = 0$ implies $(\phi', \psi) = 0$ for all $\psi \in \mathfrak{M}$, and hence $\phi' = 0$. So the operator A_0 defined by

$$A_0 \phi = \phi' \tag{37}$$

is uniquely determined and symmetric. Let A be its closure.

We note that

$$\lim_{\sigma_n \to 0} A \epsilon_{z\sigma_n} = \lim_{\sigma_n \to 0} \frac{1}{z}(\epsilon_{z\sigma_n} - \epsilon_{0\sigma_n}) = \frac{1}{z}(\epsilon_{z0} - \epsilon_{00})$$

holds in the weak sense. Hence

$$A\epsilon_{z0} = \frac{1}{z}(\epsilon_{z0} - \epsilon_{00}). \tag{38}$$

Next we show that A is selfadjoint. We have to show only that the range of $A+iI$ and $A-iI$ is dense in $\mathfrak{H}$. Now, for $z\neq 0$, i, $-i$; $s\in S$ it follows immediately from (36) and (37) that

$$\epsilon_{zs} = (A+iI)\frac{z\epsilon_{zs} - i\epsilon_{is}}{1+iz} = (A-iI)\frac{z\epsilon_{zs} + i\epsilon_{-is}}{1-iz}.$$

Since the ϵ_{zs} ($\operatorname{Im} z\neq 0$, $s\in S$, $z\neq i, -i$) span $\mathfrak{H}$, the assertion is proved.

Now define

$$\epsilon''_{zs} = \frac{1}{s}(\epsilon_{zs} - \epsilon_{z0}) \tag{39}$$

for $\operatorname{Im} z\neq 0$, $s\in S$. A computation based on (29) shows analogously as before that

$$(\epsilon''_{zs}, \epsilon_{wt}) = (\epsilon_{zs}, \epsilon''_{wt}).$$

Similarly as in the case of (37) we can show that the operator B_0 defined on $\mathfrak{M}$ by

$$B_0 \sum_p c_p\epsilon_{z_p s_p} = \sum_p c_p\epsilon''_{z_p s_p} \tag{40}$$

is well defined and symmetric. Let B denote its closure.

We note that for any $s\in S$ and $z_n\to 0$ ($z_n\in C(\phi)$, $\phi>0$),

$$\lim_{z_n\to 0} B\epsilon_{z_n s} = \lim_{z_n\to 0}\frac{1}{s}(\epsilon_{z_n s} - \epsilon_{z_n 0}) = \frac{1}{s}(\epsilon_{0s} - \epsilon_{00}).$$

Hence, B being a closed operator,

$$B\epsilon_{0s} = \frac{1}{s}(\epsilon_{0s} - \epsilon_{00}). \tag{41}$$

Now we show that B is equal to the closure of its restriction to $(A+iI)\mathfrak{M}$. We have seen that $(A+iI)\mathfrak{M}$ contains all ϵ_{zs} with $z\neq i$ ($\operatorname{Im} z\neq 0$, $s\in S$). But $\epsilon_{is} = \lim_{z\to i}\epsilon_{zs}$ and

$$\lim_{z\to i} B\epsilon_{zs} = \lim_{z\to i}\frac{1}{s}(\epsilon_{zs} - \epsilon_{z0}) = \frac{1}{s}(\epsilon_{is} - \epsilon_{i0}) = B\epsilon_{is}.$$

Hence the closure of the restriction of B to $(A+iI)\mathfrak{M}$ contains the closure of its restriction to $\mathfrak{M}$, and so must be equal to B.

For every ϵ_{zs} ($\operatorname{Im} z \neq 0$, $s \in S$) we have

$$AB\epsilon_{zs} = A\frac{1}{s}(\epsilon_{zs} - \epsilon_{z0}) = \frac{1}{zs}(\epsilon_{zs} - \epsilon_{0s} - \epsilon_{z0} + \epsilon_{00})$$

$$= B\frac{1}{z}(\epsilon_{zs} - \epsilon_{0s}) = BA\epsilon_{zs},$$

which by linearity implies $AB\phi = BA\phi$ for all ϕ in $\mathfrak{M}$.

By these considerations we have verified that the conditions of Lemma 4 are satisfied for A and B. Therefore, A, B have commuting selfadjoint dilations in an enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, which we denote by $\hat{A}$, $\hat{B}$.

Now we note that by the definition of A, B we have

$$\epsilon_{00} = (I - zA)(I - sB)\epsilon_{zs}$$

for all $\operatorname{Im} z \neq 0$, $s \in S$. This relation remains true with $\hat{A}$, $\hat{B}$ instead of A, B, and then, since the resolvent of a selfadjoint operator is defined for any nonreal number, we obtain

$$\epsilon_{zs} = (I - z\hat{A})^{-1}(I - s\hat{B})^{-1}\epsilon_{00}.$$

Substituting this into (40) the result is

$$f(z, s) = zs(\epsilon_{zs}, \epsilon_{00}) = zs((I - z\hat{A})^{-1}(I - s\hat{B})^{-1}\epsilon_{00}, \epsilon_{00}).$$

It is clear now, that the definition of f can be extended to that of a function g defined and analytic for any nonreal value of its arguments by the relation

$$g(z_1, z_2) = z_1z_2((I - z_1\hat{A})^{-1}(I - z_2\hat{B})^{-1}\epsilon_{00}, \epsilon_{00}). \tag{42}$$

We show that $g \in N_2$. For this purpose we introduce the notation

$$\eta_{z_1z_2} = (I - z_1\hat{A})^{-1}(I - z_2\hat{B})^{-1}\epsilon_{00}.$$

Then we have

$$\begin{aligned} g(z_1, z_2) &= z_1z_2(\eta_{z_1z_2}, (I - z_1A)(I - z_2\hat{B})\eta_{z_1z_2}) \\ &= z_1z_2\|\eta_{z_1z_2}\|^2 - z_2|z_1|^2(\eta_{z_1z_2}, A\eta_{z_1z_2}) \\ &\quad - z_1|z_2|^2(\eta_{z_1z_2}, \hat{B}\eta_{z_1z_2}) + |z_1z_2|^2(\eta_{z_1z_2}, A\hat{B}\eta_{z_1z_2}), \end{aligned}$$

and hence,

$$\operatorname{Re}[g(\bar{z}_1, z_2) - g(z_1, z_2)] = -\|\eta_{z_1z_2}\|^2 \operatorname{Im} z_1 \cdot \operatorname{Im} z_2,$$

which yields (b) in Definition 2. The relation $g(\bar{z}_1, \bar{z}_2) = \bar{g}(z_1, z_2)$ is obvious. Finally for z_1, z_2 in $C(\phi)$ $(0 < \phi \leqq \pi/2)$ we also have

$$\frac{g(z_1, z_2)}{z_1 z_2} = \left| ((I - z_1 A)^{-1}(I - z_2 B)^{-1}\epsilon_{00}, \epsilon_{00}) \right|$$

$$\leqq \|(I - z_1 A)^{-1}\| \cdot \|(I - z_2 B)^{-1}\| \cdot \|\epsilon_{00}\|^2 \leqq \frac{\|\epsilon_{00}\|^2}{\sin^2 \phi}$$

because of the elementary inequality

$$\|(I - zA)^{-1}\| \leqq \frac{1}{|\sin \arg z|} \tag{43}$$

valid for any selfadjoint A and nonreal z.

This finishes the proof of the sufficiency of our conditions. Remains to prove the integral representation (30).

By what we have already proved, any function g in H_2 can be represented in the form (42) with commuting selfadjoint operators $\hat{A}$, $\hat{B}$ in some Hilbert space $\hat{\mathfrak{H}}$. Let the spectral resolutions of $\hat{A}$, $\hat{B}$ be

$$\hat{A} = \int_{-\infty}^{\infty} \lambda dE_\lambda, \qquad \hat{B} = \int_{-\infty}^{\infty} \mu dF_\mu.$$

E_λ and F_μ commute for all λ, μ, hence by spectral theory we have

$$g(z_1, z_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z_1}{1 - \lambda z_1} \frac{z_2}{1 - \mu z_2} dm(\lambda, \mu),$$

with $m(\lambda, \mu) = (E_\lambda F_\mu \epsilon_{00}, \epsilon_{00})$, which is a positive bounded measure on the plane.

Conversely, it is also immediate that a function admitting such a representation belongs to N_2.

In the proof of Theorem 3 we shall need the following lemma.

LEMMA 5. *Let A be a symmetric operator in the Hilbert space $\mathfrak{H}$, and let $\{\mathfrak{H}_\gamma\}_{\gamma\in\Gamma}$ be a family of subspaces of $\mathfrak{H}$ spanning $\mathfrak{H}$, with each $\mathfrak{H}_\gamma$ invariant under A. If the restriction A_γ of A to $\mathfrak{H}_\gamma$ is densely defined and bounded for all γ in Γ, then the closure $\overline{A}$ of A is selfadjoint. If A is bounded by the same number M for all γ in Γ, then we also have $\|\overline{A}\| \leqq M$.*

Proof. Since A is densely defined and bounded, its closure is selfadjoint in $\mathfrak{H}_\gamma$. Hence, as is well known, the range $\mathfrak{R}_\gamma^+$ of $A_\gamma + iI$ (and of $A_\gamma - iI$) is dense in $\mathfrak{H}_\gamma$ for all $\gamma \in \Gamma$. Now, the range $\mathfrak{R}^+$ of $A + iI$ and $\mathfrak{R}^-$ of $A - iI$ are linear manifolds in $\mathfrak{H}$, containing all $\mathfrak{R}_\gamma$. Any vector orthogonal to $\mathfrak{R}^+$ or $\mathfrak{R}^-$ must be orthogonal to each $\mathfrak{R}_\gamma$, hence orthogonal to each $\mathfrak{H}_\gamma$, and hence equal to zero. So $\mathfrak{R}^+$ and $\mathfrak{R}^-$ are dense in $\mathfrak{H}$, which implies that the closure of A is selfadjoint.

Now assume $\|A_\gamma\| \leqq M$ for all $\gamma \in \Gamma$, and let $\overline{A} = \int_{-\infty}^{\infty} \lambda dE_\lambda$ be the spectral resolution of $\overline{A}$. Let $\Delta = [\lambda_1, \lambda_2]$ be any interval, disjoint from $[-M, M]$.

With $E(\Delta)=E_{\lambda_2}-E_{\lambda_1}$ we have $E(\Delta)f_\gamma=0$ for all f in any one of the spaces $\mathfrak{H}_\gamma$. Since such f_γ span $\mathfrak{H}$, $E(\Delta)=0$ follows, and hence we have $\|\overline{A}\|\leqq M$ as stated.

THEOREM 3. *Let f be a real-valued function of two real variables x_1, x_2 in the interval $(-1, 1)$. In order that f can be extended to a function $g\in N_2$, defined for all* Im z_1, Im $z_2\neq 0$ *and continuous at the points x_1, x_2 in $(-1, 1)$, it is necessary and sufficient that (α) $f(x_1, 0)\equiv f(0, x_2)\equiv 0$, ($\beta$) the first partial derivatives and the mixed second partial derivative of f exist and be continuous, (γ) the function k defined for all x_1, x_2, y_1, y_2 in $(-1, 1)$ by*

$$k(x_1, x_2; y_1, y_2) = \frac{f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \tag{44}$$

be positive definite. For $x_1=y_1$ or $x_2=y_2$ or both, k here is interpreted as a differential quotient. The function g can be represented in the form

$$g(z_1, z_2) = \int_{-1}^{1}\int_{-1}^{1} \frac{z_1}{1-\lambda z_1}\,\frac{z_2}{1-\mu z_2}\,dm(\lambda, \mu) \tag{45}$$

with a positive bounded measure m on the square $[-1, 1]\times[-1, 1]$.

Proof. To prove the necessity of the conditions assume that f possesses a continuous extension g in N_2. Then f is necessarily analytic, and hence infinitely differentiable. By Theorem 2 we have

$$\sum_m\sum_n \frac{g(\bar{z}_m, s_m) - g(z_n, s_m) + \bar{g}(\bar{z}_n, s_n) - \bar{g}(\bar{z}_m, s_n)}{(z_m - \bar{z}_n)(s_m - \bar{s}_n)}\,\alpha_m\bar{\alpha}_n \geqq 0 \tag{46}$$

for any $z_1, \cdots, z_N$; $s_1, \cdots, s_N$ in the upper half-plane, and any numbers $\alpha_1, \cdots, \alpha_N$. Letting these points converge to the real points $x_1, \cdots, x_N$; $u_1, \cdots, u_N$ in $(-1, 1)$ we obtain

$$\sum_m\sum_n k(x_m, u_m; x_n, u_n)\alpha_m\bar{\alpha}_n \geqq 0,$$

which proves the necessity of (γ). The necessity of (α) is trivial.

To prove the sufficiency of the conditions, we construct by Lemma 1 the Hilbert space $\mathfrak{H}$, spanned by the vectors $\epsilon_{x_1x_2}$ ($x_1, x_2\in(-1, 1)$) and such that

$$k(x_1, x_2; y_1, y_2) = (\epsilon_{x_1x_2}, \epsilon_{y_1y_2}).$$

We define

$$\epsilon'_{x_1x_2} = \frac{1}{x_1}(\epsilon_{x_1x_2} - \epsilon_{0x_2}) \tag{47}$$

for $x_1\neq 0$. Then, for $x_1, y_1\neq 0$, $x_1\neq y_1$, $x_2\neq y_2$, a simple computation based on (44) gives

$$(\epsilon'_{x_1x_2}, \epsilon_{y_1y_2}) = (\epsilon_{x_1x_2}, \epsilon'_{y_1y_2}). \tag{48}$$

To show that this holds also if $x_1=y_1$ or $x_2=y_2$ we point out that $\epsilon_{x_1x_2}$ is a strongly continuous function of the two variables x_1, x_2. In fact, by (44), we have

$$\|\epsilon_{x_1x_2}-\epsilon_{y_1y_2}\|^2 = f''_{x_1x_2}(x_1, x_2) - 2\frac{f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} + f''_{x_1x_2}(y_1, y_2),$$

which tends to zero by (β) if $y_1\to x_1$ and $y_2\to x_2$.

Let $\mathfrak{M}_1$ be the manifold of all finite linear combinations of the $\epsilon_{x_1x_2}$ $(x_1\neq 0)$. $\mathfrak{M}_1$ is dense in $\mathfrak{H}$.

Let

$$\phi = \sum_p c_p\epsilon_{x_1^{(p)}x_2^{(p)}}, \qquad \psi = \sum_q d_q\epsilon_{y_1^{(q)}y_2^{(q)}}$$

be two elements of $\mathfrak{M}_1$. Writing

$$\phi' = \sum_p c_p\epsilon'_{x_1^{(p)}x_2^{(p)}}, \qquad \psi' = \sum_q d_q\epsilon'_{y_1^{(q)}y_2^{(q)}}$$

we obtain

$$\begin{aligned}(\phi, \psi) &= \sum_p\sum_q c_p\bar{d}_q(\epsilon'_{x_1^{(p)}x_2^{(p)}}, \epsilon_{y_1^{(q)}y_2^{(q)}})\\ &= \sum_p\sum_q c_p\bar{d}_p(\epsilon_{x_1^{(p)}x_2^{(p)}}, \epsilon'_{y_1^{(q)}y_2^{(q)}}) = (\phi, \psi').\end{aligned}$$

If $\phi=0$, hence we have $(\phi, \psi)=0$ for all ψ in $\mathfrak{M}_1$, and $\mathfrak{M}_1$ being dense in $\mathfrak{H}$, $\phi'=0$.

From the above it follows that the operator A_0 defined by

$$A_0\phi = \phi'$$

is uniquely defined and symmetric. We denote its closure by A_1.

For fixed $x_2\in(-1, 1)$ let $\mathfrak{H}_{x_2}$ denote the subspace of $\mathfrak{H}$ which is spanned by all $\epsilon_{x_1x_2}$ with x_1 in $(-1, 1)$. By the definition of A_0, every $\mathfrak{H}_{x_2}$ is invariant under A_0. By the continuity of $\epsilon_{x_1x_1}$ we see that $\mathfrak{M}\cap\mathfrak{H}_{x_2}$ is dense in $\mathfrak{H}_{x_2}$. Now, $\mathfrak{H}_{x_2}$ is exactly the Hilbert space corresponding to the function of one variable $F(x)=f'_{x_2}(x, x_2)$ in the sense of Theorem C [6]. In fact, the inner product of two vectors $\epsilon_{x_1x_2}$, $\epsilon_{y_1x_2}$ in $\mathfrak{H}_{x_2}$ is

$$(\epsilon_{x_1x_2}, \epsilon_{y_1x_2}) = k(x_1, x_2; y_1, x_2) = \frac{F(x_1) - F(y_1)}{x_1 - y_1}.$$

A_{x_2}, the restriction of A_0 to $\mathfrak{H}_{x_2}$, is the operator constructed in the proof of Theorem C [8]. Hence, by this theorem, $\|A_{x_2}\|\leqq 1$ for all x_2 in $(-1, 1)$. Since the subspaces $\mathfrak{H}_{x_2}$ $(x_2\in(-1, 1))$ span $\mathfrak{H}$, we can apply Lemma 5, and obtain that A_1 is selfadjoint and $\|A_1\|\leqq 1$.

Having a complete symmetry with respect to the variables x_1 and x_2, we can define an operator A_2 first by setting

$$A_2\epsilon_{x_1x_2} = \frac{1}{x_2}(\epsilon_{x_1x_2} - \epsilon_{x_10}) \tag{49}$$

for $x_2 \neq 0$, then extending its definition to the manifold $\mathfrak{M}_2$ of all finite linear combinations of such $\epsilon_{x_1x_2}$ by linearity, and taking closure. In the same way as above, it follows that A_2 is selfadjoint and $\|A_2\| \leqq 1$.

For $x_1, x_2 \neq 0$ we have by (47) and (49)

$$A_1A_2\epsilon_{x_1x_2} = A_2A_1\epsilon_{x_1x_2}.$$

Hence A_1 and A_2 commute on the manifold of finite linear combinations of the $\epsilon_{x_1x_2}$ $(x_1, x_2 \neq 0)$, i.e., on $\mathfrak{M}_1 \cap \mathfrak{M}_2$. This manifold is dense in $\mathfrak{H}$; therefore, A_1 and A_2 commute everywhere in $\mathfrak{H}$.

By (47) and (49) we have

$$(I - x_2A_2)(I - x_1A_1)\epsilon_{x_1x_2} = (I - x_2A_2)\epsilon_{0x_2} = \epsilon_{00}.$$

Since $\|A_1\|, \|A_2\| \leqq 1$, the operators $(I - x_1A_1)^{-1}$, $(I - x_2A_2)^{-1}$ exist for all x_1, x_2 in $(-1, 1)$. Therefore

$$\epsilon_{x_1x_2} = (I - x_1A_1)^{-1}(I - x_2A_2)^{-1}\epsilon_{00},$$

and, for $x_1, x_2 \neq 0$,

$$\frac{f(x_1, x_2)}{x_1x_2} = k(x_1, x_2; 0, 0) = (\epsilon_{x_1x_2}, \epsilon_{00}) = ((I - x_1A_1)^{-1}(I - x_2A_2)^{-1}\epsilon_{00}, \epsilon_{00}).$$

Hence for any x_1, x_2 in $(-1, 1)$,

$$f(x_1, x_2) = x_1x_2((I - x_1A_1)^{-1}(I - x_2A_2)^{-1}\epsilon_{00}, \epsilon_{00}). \tag{50}$$

This representation essentially finishes the proof. Clearly the function

$$g(z_1, z_2) = z_1z_2((I - z_1A_1)^{-1}(I - z_2A_2)^{-1}\epsilon_{00}, \epsilon_{00})$$

defined for all nonreal z_1, z_2 is an analytic continuation of f, and in the proof of Theorem 2 we have seen that it belongs to N_2.

Using the spectral resolutions $A_1 = \int_{-1}^{1}\lambda dE_\lambda$, $A_2 = \int_{-1}^{1}\mu dF_\mu$, we obtain the integral representation (45) with $m(\lambda, \mu) = (E_\lambda F_\mu \epsilon_{00}, \epsilon_{00})$, which is in fact a bounded positive measure on the square $[-1, 1] \times [-1, 1]$.

4. **Monotone matrix functions of two variables.** In this section we show that the class of functions of two real variables considered in Theorem 3 is identical with the class of monotone functions of two matrix variables, in analogy to the one-variable case. In order to avoid lengthy computations which are of no interest for the main course of our investigation, we are going to assume throughout some differentiability conditions about the functions we are going to consider.

Let f be a real-valued function of two real variables x, u in $(-1, 1)$. Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ with spectrum contained in $(-1, 1)$; let $A=\int_{-1}^{1} x dE_x$ be its spectral resolution. Similarly, let $B=\int_{-1}^{1} u dF_u$ be a selfadjoint operator in another Hilbert space $\mathfrak{K}$. Then $E_x \otimes F_u$ determines a two-parameter spectral family in the tensor product space $\mathfrak{H} \otimes \mathfrak{K}$. By $f(A, B)$ we understand the operator

$$f(A, B) = \int_{-1}^{1}\int_{-1}^{1} f(x, u) dE_x \otimes dF_u$$

acting on the space $\mathfrak{H} \otimes \mathfrak{K}$. If, in particular, $\mathfrak{H}$ and $\mathfrak{K}$ are of finite dimension m and n respectively, and A, B are given by diagonal matrices with the proper values $x_1, \cdots, x_m$ and $u_1, \cdots, u_n$, then $f(A, B)$ is determined by the matrix $\operatorname{diag}\{f(x_1, u_1), f(x_2, u_1), \cdots, f(x_m, u_n)\}$ in the Kronecker product space.

We shall say that f is a monotone matrix function, if for any $\mathfrak{H}$ and $\mathfrak{K}$, and for any selfadjoint operators A, A' in $\mathfrak{H}$, B, B' in $\mathfrak{K}$ whose spectrum is contained in $(-1, 1)$ and for which $A' \geqq A$, $B' \geqq B$ holds, we have

$$f(A', B') - f(A', B) - f(A, B') + f(A, B) \geqq 0.$$

THEOREM 4. *Let f be a real-valued function of the two real variables x, u in $(-1, 1)$. Assume that (α) $f(x, 0)=f(0, u)=0$ for all x, u, and (β) the first partial derivatives and the mixed second partial derivative of f exist and are continuous. Then, f is a monotone matrix function of two variables if and only if it is analytic, and can be continued analytically for all nonreal values of the two variables to a function belonging to N_2.*

REMARK. Since the addition of functions of one variable to f does not change its monotone character, the condition (α) is merely an unessential normalizing condition. It will also be clear from the proof, that we could study monotone matrix functions defined on any rectangle of the plane without any essential change.

Proof. Assume f is a monotone matrix function. We take an arbitrary finite sequence of numbers such that $x_1 < x_1' < x_2 < x_2' < \cdots < x_m < x_m'$. By a theorem of Loewner [5, Theorem 5], cf. also [4], there exist real symmetric m-by-m matrices $A=\operatorname{diag}(x_1, \cdots, x_m)$ and $A'=S^* \operatorname{diag}(x_1', \cdots, x_m')S$, S real orthogonal, satisfying the relation $A' > A$. $S=(\sigma_{ik})_{i,k=1}^{m}$ can be chosen so that

$$\sigma_{ik} = \frac{\gamma_i' \gamma_k}{x_i' - x_k} \qquad (i, k = 1, \cdots, m) \tag{51}$$

with some positive numbers γ_i, γ_i' $(i=1, \cdots, m)$.

Taking another arbitrary sequence of the form $u_1 < u_1' < \cdots < u_n < u_n'$

we similarly have $B' > B$ with real symmetric matrices $B = \operatorname{diag}(u_1, \cdots, u_n)$, $B' = T^* \operatorname{diag}(u_1', \cdots, u_n')T$. $T = (\tau_{jl})_{j,l=1}^n$ is such that

$$\tau_{jl} = \frac{\delta_j' \delta_l}{u_j' - u_l} \qquad (j, l = 1, \cdots, n) \tag{52}$$

and the δ_j, δ_j' $(j = 1, \cdots, n)$ are all positive.

Now, by the monotonicity of f, the matrix

$$\begin{aligned} D &= f(A', B') - f(A', B) - f(A, B') + f(A, B) \\ &= (S \otimes T)^* \operatorname{diag}(f(x_1', u_1'), \cdots, f(x_m', u_n'))(S \otimes T) \\ &\quad - (S \otimes I)^* \operatorname{diag}(f(x_1', u_1), \cdots, f(x_m', u_n))(S \otimes I) \\ &\quad - (I \otimes T)^* \operatorname{diag}(f(x_1, u_1'), \cdots, f(x_m, u_n'))(I \otimes T) \\ &\quad + \operatorname{diag}(f(x_1, u_1), \cdots, f(x_m, u_n)) \end{aligned}$$

is non-negative definite. Hence its determinant is non-negative, and, also making use of (51), (52), we have

$$\begin{aligned} \det D &= \det(S \otimes T)D \\ &= \det[\operatorname{diag}(f(x_1', u_1'), \cdots)(S \otimes T) \\ &\quad - (I \otimes T) \operatorname{diag}(f(x_1', u_1), \cdots)(S \otimes I) \\ &\quad - (S \otimes I) \operatorname{diag}(f(x_1, u_1'), \cdots)(I \otimes T) \\ &\quad + (S \otimes T) \operatorname{diag}(f(x_1, u_1), \cdots)] \\ &= \det[(f(x_1', u_j')\sigma_{ik}\tau_{jl} - \tau_{jl}f(x_i', u_l)\sigma_{ik} - \sigma_{ik}f(x_k, u_j')\tau_{jl} + \sigma_{ik}\tau_{jl}f(x_k, u_l)]_{i,j;k,l} \\ &= \det\left|\frac{f(x_i', u_j') - f(x_i', u_l) - f(x_k, u_j') + f(x_k, u_l)}{(x_i' - x_k)(u_j' - u_l)} \gamma_i' \gamma_k \delta_j' \delta_l\right|_{i,j;k,l} \geqq 0. \end{aligned}$$

Dividing the last inequality by the positive numbers γ_i', γ_k, δ_j', δ_l, we obtain

$$\det\left|\frac{f(x_i', u_j') - f(x_i', u_l) - f(x_k, u_j') + f(x_k, u_l)}{(x_i' - x_k)(u_j' - u_l)}\right|_{i,j;k,l} \geqq 0.$$

Now let us make x_i' tend to x_i for each $i = 1, \cdots, m$ and each u_j' tend to u_j $(j = 1, \cdots, n)$. By the differentiability conditions on f, our expression will have a limit, and

$$\det\left|\frac{f(x_i, u_j) - f(x_i, u_l) - f(x_k, u_j) + f(x_k, u_l)}{(x_i - x_k)(u_j - u_l)}\right|_{i,j;k,l} \geqq 0 \tag{53}$$

will hold. (The elements of (53) for $x_i = x_k$ or $u_j = u_l$ are differential quotients.)

Next we note that since the system $x_1, \cdots, x_m; u_1, \cdots, u_n$ could be chosen arbitrarily, (53) holds for any subsystem of the fixed system under

consideration. From this it follows by the usual algebraic reasoning (diagonalization and Sylvester's inertia theorem) that the matrix

$$\left(\frac{f(x_i, u_j) - f(x_i, u_l) - f(x_k, u_j) + f(x_k, u_l)}{(x_i - x_k)(u_j - u_l)}\right)_{i,j;k,l}$$

is non-negative definite.

Since the system $x_1, \cdots, x_m; u_1, \cdots, u_n$ can be chosen arbitrarily, this means that for f the conditions of Theorem 3 are satisfied, and so the necessity of the condition follows from Theorem 3.

To prove the converse, we use Theorem 3 again, and obtain the representation

$$f(x, u) = \int_{-1}^{1}\int_{-1}^{1} \frac{x}{1 - \lambda x} \cdot \frac{u}{1 - \mu u}\, dm(\lambda, \mu) \tag{54}$$

for f, with a bounded positive measure m.

We note that the function $f_\lambda(x) = x/(1-\lambda x)$ is a monotone matrix function of one variable in $(-1, 1)$ for any fixed value of the parameter λ such that $|\lambda| \leqq 1$. Hence, the product of two functions of this type

$$f_{\lambda\mu}(x, u) = f_\lambda(x) f_\mu(u) \tag{55}$$

is monotone in two variables. In fact, $A' \geqq A$, $B' \geqq B$ implies

$$\begin{aligned} f_{\lambda\mu}(A', B') &- f_{\lambda\mu}(A', B) - f_{\lambda\mu}(A, B') + f_{\lambda\mu}(A, B) \\ &= (f_\lambda(A') - f_\lambda(A)) \otimes (f_\mu(B') - f_\mu(B)) \geqq 0. \end{aligned}$$

Now we make the obvious remarks that positive linear combinations of monotone matrix functions are also monotone, and that the uniform limit of a sequence of monotone matrix functions again has the same property. A function representable in the form (54), however, is on every compact subset the uniform limit of positive linear combinations of functions of the type (55), whence the statement of the theorem follows.

5. **Operator-valued functions.** In this section we show that Theorems 1, 2, 3 can be generalized to functions whose values are bounded operators in a Hilbert space $\mathfrak{H}$. The emphasis now will be laid on the generalizations of the integral representation of the classes H_2, N_2, which seems to be the most interesting feature here. We also restrict ourselves to a slightly less general situation than in the case of numerical functions.

We use the concepts of strong and weak positive definiteness, which were defined in [6] as follows. The operator-valued function K defined on a set $X \times X$ is called *strongly positive definite* if for any finite system of points $x_1, \cdots, x_N$ in X and any vectors $h_1, \cdots, h_N$ in $\mathfrak{H}$ the inequality

$$\sum_m \sum_n (K(x_m, x_n)h_m, h_n) \geqq 0$$

holds. $K(x, y)$ is called *weakly positive definite* if this condition is required only for systems of vectors $h_n = \alpha_n h$ $(n = 1, \cdots, N)$ with fixed h and arbitrary complex numbers $\alpha_1, \cdots, \alpha_N$; in other words if

$$\sum_m \sum_n K(x_m, x_n)\alpha_m \bar{\alpha}_n \geqq 0$$

in the operator sense. Strong positive definiteness implies weak positive definiteness; the converse is in general not true.

We use the following notation which was introduced in [7]: Let $\mathfrak{H}$ be a subspace of the Hilbert space $\hat{\mathfrak{H}}$. We say that the operator T in $\mathfrak{H}$ is a projection of the operator $\hat{T}$ in $\hat{\mathfrak{H}}$, denoted $T = \mathrm{pr}\ \hat{T}$, if for every h in $\mathfrak{H}$, Th is the orthogonal projection of $\hat{T}h$ onto $\mathfrak{H}$.

LEMMA 6. *Let X be any set, F a function on X whose values are bounded operators in a Hilbert space $\mathfrak{H}$. Assume that for every vector $h \in \mathfrak{H}$ and every x in X a representation*

$$(F(x)h, h) = \iint_{S \times T} u(x, s, t)dm(s, t; h) \tag{56}$$

holds, where S, T are real intervals, u is a continuous function of s and t in $S \times T$, and m is a bounded positive measure on $S \times T$, which is uniquely determined by h. Assume further that

$$(h, h) = \iint_{S \times T} dm(s, t; h) \tag{57}$$

for all h in $\mathfrak{H}$. Then there exist commutative selfadjoint operators A, B in a larger Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$ such that their spectrum lies in S, T respectively, and

$$F(x) = \mathrm{pr}\ u(x, A, B)$$

for all x in X.

Proof. Let h, h' be two elements of $\mathfrak{H}$. From (56) by polarization we have

$$(F(x)h, h') = \iint_{S \times T} u(x, s, t)dm(s, t; h, h'), \tag{58}$$

with

$$m(s, t; h, h') = \frac{1}{4}\left[m(s, t; h + h') - m(s, t; h - h') + im(s, t; h + ih') - im(s, t; h - ih')\right].$$

$m(s, t; h, h')$ is a function of bounded variation of the two variables s, t. Normalizing it, say, by continuity from the left and by setting it equal to 0 in a corner of the rectangle $S \times T$, it is uniquely determined by h, h'.

It follows that for any fixed s, t in $S\times T$, $m(s, t; h, h')$ is a bilinear form in h, $h'\in\mathfrak{H}$, and by (57)

$$m(s, t; h, h) = m(s, t; h) \leqq \iint_{S\times T} dm(s, t; h) = (h, h).$$

Therefore, we have a representation

$$m(s, t; h, h') = (D(s, t)h, h')$$

for all h, h' in $\mathfrak{H}$ with a selfadjoint operator $D(s, t)$ bounded by 1.

Since the measure determined by $m(s, t; h) = (D(s, t)h, h)$ is positive for all fixed h in $\mathfrak{H}$, $D(s, t)$ is a generalized spectral family in the sense of Naimark, and hence, by Naimark's theorem there exists an ordinary spectral family $E(s, t)$ in an enlarged Hilbert space such that

$$D(s, t) = \operatorname{pr} E(s, t)$$

for all s, t.

Now denoting the right endpoint of S and T by s_0 resp. t_0, $E_s = E(s, t_0)$, $F_t = E(s_0, t)$ are one-parameter spectral families, and E_s, F_t commute for all s, t. Furthermore, we have $E(s, t) = E_sF_t$ for all s, t. Defining the operators A, B by

$$A = \int_S s dE_s, \qquad B = \int_T t dF_t,$$

we have, by (58),

$$(F(x)h, h') = \iint_{S\times T} u(x, s, t) d(E_sF_th, h') = (u(s, A, B)h, h'),$$

from which the equality

$$F(x) = \operatorname{pr} u(x, A, B)$$

follows, finishing the proof of the lemma.

THEOREM 5. *Let S be a set of points in the unit disk, containing* 0 *and having an accumulation point inside the unit disk. For every fixed s in S let $F(z, s)$ be a function whose values are bounded operators in a Hilbert space $\mathfrak{H}$ and which is defined and analytic in z for all $|z| \neq 1$. $F(z, s)$ admits a representation*

$$F(z, s) = \operatorname{pr} \frac{1 + zU}{1 - zU} \frac{1 + sV}{1 - sV} \tag{59}$$

with commuting unitary operators U, V in an enlarged Hilbert space $\hat{\mathfrak{H}}\supseteq\mathfrak{H}$, if and only if ($\alpha$) $F(0, 0) = I$, (β) $F(\bar{z}^{-1}, 0) = -F(z, 0)^*$ *for all* $|z| \neq 1$, (γ) $F(\infty, s) = -F(0, s)$ *for all* $s\in S$, *and* (δ) *the function K defined for all* $|z|, |w| \neq 1$, $s, t\in S$ *by*

$$K(z, s; w, t) = \frac{F(z, s) - F(\bar{w}^{-1}, s) + F(w, t)^* - F(\bar{z}^{-1}, t)^*}{(1 - z\bar{w})(1 - s\bar{t})}, \tag{60}$$

is weakly positive definite.

Proof. To prove the necessity of the conditions we assume that $F(z, s)$ is of the form (59). Making use of the simple relation

$$\frac{1}{1 - z\bar{w}}\left[\frac{U + zI}{U - zI} + \frac{U^* + wI}{U^* - \bar{w}I}\right] = 2(U^* - \bar{w}I)^{-1}(U - zI)^{-1}$$

and the analogous relation for V, we obtain

$$\sum_m \sum_n (K(z_m, s_m; z_n, s_n)h_m, h_n) = 4\left\| \sum_m (U - z_mI)^{-1}(V - s_mI)^{-1}h_m \right\|^2 \geqq 0$$

for any $|z_m| \neq 1$, $s_m \in S$, $h_m \in \mathfrak{H}$ $(m = 1, \cdots, N)$, which shows the positive definiteness of K even in the strong sense. The necessity of the other conditions is obvious.

To prove the sufficiency of our conditions we note that for every fixed h in $\mathfrak{H}$ the function $(F(z, s)h, h)$ satisfies the conditions of Theorem 1 and of the Remark after Theorem 1. Hence we have a representation

$$(F(z, s)h, h) = \int_0^{2\pi} \int_0^{2\pi} \frac{1 + ze^{i\phi}}{1 - ze^{i\phi}} \frac{1 + se^{i\psi}}{1 - se^{i\psi}} dm(\phi, \psi; h)$$

with a bounded positive measure $m(\phi, \psi; h)$. We also have

$$(h, h) = (F(0, 0)h, h) = \int_0^{2\pi} \int_0^{2\pi} dm(\phi, \psi; h)$$

for all $h \in \mathfrak{H}$.

Since S has an accumulation point inside the unit disk, the holomorphic extension of $(F(z, s)h, h)$ is uniquely determined for any fixed h. This function then admits a Taylor expansion

$$(F(z, s)h, h) = \sum_{p,q} c_{pq} z^p s^q$$

with

$$c_{pq} = \int_0^{2\pi} \int_0^{2\pi} e^{i(\phi+\psi)} dm(\phi, \psi; h) \qquad (p, q = 1, 2, \cdots).$$

Now, the function determines its Taylor coefficients c_{pq} uniquely, the c_{pq}, in turn being the double trigonometric moments of a bounded measure, determine the measure $m(\phi, \psi; h)$ uniquely.

So Lemma 6 can be applied. If A, B are commuting selfadjoint operators,

$U=e^{iA}$, $V=e^{iB}$ are commuting unitary operators, and hence by Lemma 6 we obtain

$$F(z, s) = \mathrm{pr}\frac{I+zU}{I-zU}\frac{I+sV}{I-sV}$$

which was to be proved.

COROLLARY. *The operator-valued function $F(z_1, z_2)$, defined and analytic for all $|z_1|, |z_2| \neq 1$, can be represented in the form*

$$F(z_1, z_2) = \mathrm{pr}\frac{I+z_1U}{I-z_1U}\frac{I+z_2V}{I-z_2V}$$

with commuting unitary operators U, V in some enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, if and only if ($\alpha$) $F(0, 0)=I$, $F(z_1, 0)+F(z_1, \infty) \equiv F(0, z_2)+F(\infty, z_2) \equiv 0$ *for all* $|z_1|, |z_2| \neq 1$, (β) $F(\bar{z}_1^{-1}, \bar{z}_2^{-1}) = F(z_1, z_2)^*$ *for all* $|z_1|, |z_2| \neq 1$, (γ) *the operator*

$$M(z_1, z_2) = F(z_1, z_2) - F(\bar{z}_1^{-1}, z_2) - F(z_1, \bar{z}_2^{-1}) + F(\bar{z}_1^{-1}, \bar{z}_2^{-1})$$

is positive for all $|z_1|, |z_2| < 1$.

Proof. The conditions imply that the function $f_h(z_1, z_2) = (F(z_1, z_2)h, h)$ belongs to the class H_2 for all fixed h in $\mathfrak{H}$. By Theorem 1 the function $k_h(z, s; w, t)$ constructed from f_h according to (13) is positive definite for all $|z|, |w| \neq 1$ and all s,t in the unit disk. However, we have $k_h(z, s; w, t) = (K(z, s; w, t)h, h)$, hence K is weakly positive definite. The conclusion now follows from Theorem 5.

The converse can be verified immediately.

For the sake of completeness we mention here the following theorem.

THEOREM 5′. *If K is strongly positive definite, then Theorem 5 holds even if S does not have an accumulation point in the unit disk.*

We have seen that the condition of strong positive definiteness is necessary. The sufficiency of this condition can be proved by an analogous construction as in the proof of Theorem A** in [6]. The steps of the proof then follow the proof of our Theorem 1. On the basis of these remarks the proof is obvious and is omitted.

THEOREM 6. *Let S be a subset of the upper half-plane, such that for some $0<\phi\leqq\pi/2$, $S\cap C(\phi)$ has 0 as an accumulation point. Assume further that S has an accumulation point interior to the upper half-plane. For all $s\in S$ let $F(z, s)$ be a function whose values are bounded operators in a Hilbert space $\mathfrak{H}$, and which is defined and analytic for all* $\mathrm{Im}\, z \neq 0$. *$F(z, s)$ admits a representation*

$$F(z, s) = \mathrm{pr}\, z(I-zA)^{-1}s(I-sB)^{-1} \tag{61}$$

with commuting selfadjoint operators A, B in a larger Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$ if and only if (α) *the function K defined by*

$$(62)\qquad K(z, s; w, t) = \begin{cases} \dfrac{F(z, s) - F(\bar{w}, s) + F(w, t)^* - F(\bar{z}, t)^*}{(z - \bar{w})(s - \bar{t})} & (z \neq \bar{w}), \\[2ex] \dfrac{F'(z, s) - F'(\bar{z}, t)^*}{s - \bar{t}} & (z = \bar{w}) \end{cases}$$

is weakly positive definite, and (β) *there is at least one sequence* $\{\sigma_n\}$, $\sigma_n \to 0$, $\sigma_n \in S \cap C(\phi)$ $(0 < \phi \leqq \pi/2)$, *such that* $F(z_n, \sigma_n)/z_n\sigma_n \to I$ *in the sense of weak convergence for every sequence* $\{z_n\}$, $z_n \to 0$, $z_n \in C(\phi)$.

Proof. Assume that $F(z, s)$ is of the form (61). By the relation

$$\frac{1}{z - \bar{w}}\left[z(I - zA)^{-1} - \bar{w}(I - \bar{w}A)^{-1}\right] = (I - \bar{w}A)^{-1}(I - zA)^{-1}$$

valid for all nonreal z, w, and from the analogous relation for B, we obtain

$$\sum_m \sum_n (K(z_m, s_m; z_n, s_n)h_m, h_n) = \left\| \sum_m (I - s_mA)^{-1}(I - z_mB)^{-1}h_m \right\|^2 \geqq 0,$$

which shows that K is positive definite even in the strong sense. This proves the necessity of (α). (β) follows immediately from the identity

$$(I - zA)^{-1}(I - sB)^{-1} - I = zsA(I - zA)^{-1}B(I - sB)^{-1} + zA(I - zA)^{-1} + sB(I - sB)^{-1}.$$

To prove the sufficiency of the conditions we note that the function $(F(z, s)h, h)$ satisfies the conditions of Theorem 2 for all h in $\mathfrak{H}$. So we have

$$(F(z, s)h, h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z}{1 - \lambda z}\,\frac{s}{1 - \mu s}\, dm(\lambda, \mu; h)$$

for all $h \in \mathfrak{H}$, with a bounded positive measure $m(\lambda, \mu; h)$. We also have

$$(h, h) = \lim_{n \to \infty} \sigma_n^{-1}z_n^{-1}(F(z_n, \sigma_n)h, h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dm(\lambda, \mu; h)$$

by the dominated convergence theorem of Lebesgue.

Since S is assumed to have an accumulation point inside the upper half-plane, the analytic extension of the function $(F(z, s)h, h)$ is uniquely determined by h. This analytic function, in turn, determines uniquely the measure $m(\lambda, \mu; h)$ by the inversion formula of Stieltjes which is easily seen to hold also for functions of two variables.

Lemma 6 now yields the representation (61).

REMARKS. 1. Similarly as in the corollary of Theorem 5, we see that the class of operator-valued functions defined for all $\operatorname{Im} z$, $\operatorname{Im} s \neq 0$ and representable in the form (61) is the exact analogue of the class N_2, and can also be defined in a similar way.

2. In analogy to Theorem 5′ we can prove that Theorem 6 holds even without assuming that S has an accumulation point inside the upper half-plane, if K is strongly positive definite. The proof of this statement can be carried out analogously to that of Theorem B** in [6], by following the steps of our proof of Theorem 2.

THEOREM 7. *Let $F(x_1, x_2)$ be a function defined for all x_1, x_2 in $(-1, 1)$ whose values are bounded selfadjoint operators in a Hilbert space $\mathfrak{H}$. $F(x_1, x_2)$ admits a representation*

$$F(x_1, x_2) = \operatorname{pr} x_1(I - x_1A_1)^{-1}x_2(I - x_2A_2)^{-1} \tag{63}$$

with commutative selfadjoint operators A_1, A_2 bounded by 1 in a larger Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, if and only if ($\alpha$) the first weak partial derivatives and the mixed second weak partial derivative of F exist and are continuous, (β) $F_{x_1x_2}(0, 0) = I$, $F(x_1, 0) = F(0, x_2) = 0$ for all x_1, x_2, (γ) the function

$$K(x_1, x_2; y_1, y_2) = \frac{F(x_1, x_2) - F(x_1, y_2) - F(y_1, x_2) + F(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)} \tag{64}$$

defined for x_1, x_2, y_1, y_2 in $(-1, 1)$ is weakly positive definite.

Proof. Assuming that F is of the form (63) conditions (α) and (β) are obviously satisfied. To prove (γ) we use the identity

$$\frac{1}{x - y}\left[x(I - xA)^{-1} - y(I - yA)^{-1}\right] = (I - yA)^{-1}(I - xA)^{-1}$$

valid for any selfadjoint A, $\|A\| \leqq 1$ and all x, y in $(-1, 1)$. For any $x_1, \cdots, x_N$; $u_1, \cdots, u_N$ in $(-1, 1)$, $h_1, \cdots, h_N$ in $\mathfrak{H}$ we have

$$\sum_m \sum_n (K(x_m, u_m; x_n, u_n)h_m, h_n) = \left| \sum_m (I - x_mA_1)^{-1}(I - u_mA_2)^{-1}h_m \right|^2 \geqq 0,$$

which proves that K is positive definite even in the strong sense.

To prove the sufficiency of the conditions, we note that the function $(F(x_1, x_2)h, h)$ satisfies the conditions of Theorem 3 for every fixed h in $\mathfrak{H}$. Hence the integral representation

$$(F(x_1, x_2)h, h) = \int_{-1}^{1}\int_{-1}^{1} \frac{x_1}{1 - \lambda x_1} \frac{x_2}{1 - \mu x_2}\, dm(\lambda, \mu; h)$$

holds with a bounded positive measure $m(\lambda, \mu; h)$. This function has the convergent Taylor development

$$(F(x_1, x_2)h, h) = \sum_{m,n=0}^{\infty} x_1^{m+1} x_2^{n+1} \int_{-1}^{1}\int_{-1}^{1} \lambda^m \mu^n\, dm(\lambda, \mu; h).$$

The Taylor coefficients are uniquely determined by the function; being the double Hausdorff moments of the measure $m(\lambda, \mu; h)$, they determine this measure uniquely.

We also note that

$$(h, h) = (F''_{z_1z_2}(0, 0)h, h) = \int_{-1}^{1}\int_{-1}^{1} dm(\lambda, \mu; h)$$

holds for all h in $\mathfrak{H}$. Applying Lemma 6, we obtain the representation (63) with the commuting selfadjoint operators A_1, A_2 whose spectrum lies in $[-1, 1]$, i.e., which are bounded by 1.

6. **Applications to dilation problems.** Finally we wish to indicate the connections of the results in the preceding section with some previous studies on dilations of Hilbert space operators. Theorems 8 and 9 are known, we merely show that they can be derived from our Theorem 5. Theorem 10 is a generalization of a result of B. Sz.-Nagy [7]; it turns out that it is a consequence of our Theorem 7.

THEOREM 8 (B. SZ.-NAGY [7]). *Let T_1, T_2 be doubly commuting*(3) *contractions in a Hilbert space $\mathfrak{H}$. In a larger Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$ there exist commutative unitary operators U_1, U_2 such that*

$$T_1^m T_2^n = \text{pr } U_1^m U_2^n \tag{65}$$

holds for all $m, n \geqq 0$.

Proof. For $|z| < 1$ we define

$$H_i(z) = (I + zT_i)(I - zT_i)^{-1} \qquad (i = 1, 2),$$

which is an analytic function of z. Now we define the function F by the relations $F(z_1, z_2) = H_1(z_1)H_2(z_2)$, $F(\bar{z}_1^{-1}, z_2) = -H_1(z_1)^*H_2(z_2)$, and $F(\bar{z}_1^{-1}, \bar{z}_2^{-1}) = F(z_1, z_2)^*$. Thus F is defined for all $|z_1|, |z_2| \neq 1$, is analytic, and clearly satisfies conditions (α), (β) of the corollary of Theorem 5. To show that it satisfies condition (γ) too, we have to prove that

$$M(z_1, z_2) = H_1(z_1)H_2(z_2) + H_1(z_1)^*H_2(z_2) + H_2(z_2)^*H_1(z_1) + H_2(z_2)^*H_1(z_1)^* \tag{66}$$

is a positive operator for all $|z_1|, |z_2| < 1$.

By the double commutativity of T_1, T_2 we have

$$M(z_1, z_2) = (H_1(z_1) + H_1(z_1)^*)(H_2(z_2) + H_2(z_2)^*).$$

The operators $H_i(z_i) + H_i(z_i)^* = 2 \text{ Re } H_i(z_i)$ are positive; let in fact, f be any element of $\mathfrak{H}$, and denote $g = (I - z_iT_i)^{-1}f$. Then we have for $|z_i| < 1$,

(3) I.e. T_1 commutes with both T_2 and T_2^*.

$$(\operatorname{Re} H_i(z_i)f, f) = \operatorname{Re}((I + z_iT_i)(I - z_iT_i)^{-1}f, f)$$
$$= \operatorname{Re}((I + z_iT_i)g, (I - z_iT_i)g) = \|g\|^2 - |z_i|^2\|T_ig\|^2 \geqq 0.$$

Hence $M(z_1, z_2)$, being the product of two commutative positive operators, is itself positive if $|z_1|, |z_2| < 1$.

Now, by the corollary of Theorem 5 we have

$$F(z_1, z_2) = \frac{1 + z_1T_1}{1 - z_1T_1}\,\frac{1 + z_2T_2}{1 - z_2T_2} = \operatorname{pr}\frac{1 + z_1U_1}{1 - z_1U_1}\,\frac{1 + z_2U_2}{1 - z_2U_2}$$

with commuting unitary operators U_1, U_2 in an enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$. Taking the power series development of both sides of this equation and comparing coefficients we obtain the relations (65).

THEOREM 9 (S. BREHMER [1]). *Let T_1, T_2 be commutative contractions in a Hilbert space $\mathfrak{H}$, such that $\|T_1\|^2 + \|T_2\|^2 \leqq 1$. Then there exist commutative unitary operators U_1, U_2 in a larger Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$, such that*

$$T_1^m T_2^n = U_1^m U_2^n$$

for all $m, n \geqq 0$.

Proof. We define F in the same way as in the proof of Theorem 8. The conditions (α), (β) of the corollary of Theorem 5 are again trivial; we have to use a different argument only to prove that (66) is a positive operator.

We use the abbreviated notations

$$S_i = z_iT_i, \qquad I + R_i = (I - S_i)^{-1} \qquad (i = 1, 2).$$

We have the relations

$$S_i(I + R_i) = R_i, \qquad H_i(z_i) = I + 2R_i.$$

Now consider the following identities, used also by Brehmer.

$$\begin{aligned} M(z_1, z_2) &= (I + R_2^*)(I + R_1^*)(I + R_1)(I + R_2) - (I + R_2^*)R_1^*R_1(I + R_2) \\ &\quad - (I + R_1^*)R_2^*R_2(I + R_1) + R_2^*R_1^*R_1R_2 \\ &= (I + R_2^*)(I + R_1^*)(I + R_1)(I + R_2) \\ &\quad - (I + R_2^*)(I + R_1^*)S_1^*S_1(I + R_1)(I + R_2) \\ &\quad - (I + R_2^*)(I + R_1^*)S_2^*S_2(I + R_1)(I + R_2) \\ &\quad + (I + R_2^*)(I + R_1^*)S_2^*S_1^*S_1S_2(I + R_1)(I + R_2). \end{aligned}$$

Hence with $A = (I - S_1)(I - S_2)$ we have

$$M' = \frac{1}{4}A^*M(z_1, z_2)A = I - S_1^*S_1 - S_2^*S_2 + S_2^*S_1^*S_1S_2,$$

and therefore, for any f in $\mathfrak{H}$,

$$(M'f, f) = \|f\|^2 - \|S_1 f\|^2 - \|S_2 f\|^2 + \|S_1 S_2 f\|^2 \geqq 0,$$

whence also $M(z_1, z_2) \geqq 0$.

The positivity of (66) being established, the same argument as in Theorem 8 finishes the proof.

THEOREM 10. *Let T_{rs} $(r, s = 0, 1, \cdots)$ be a double sequence of bounded selfadjoint operators in a Hilbert space $\mathfrak{H}$ such that* (α) *for every real polynomial $p(\lambda, \mu) = \sum_{r,s=0}^{N} a_{rs}\lambda^r\mu^s$ which is non-negative for all λ, μ in $(-1, 1)$, the operator $\sum_{r,s=0}^{N} a_{rs}T_{rs}$ is non-negative,* (β) $T_{00} = I$.

Then there exist commutative selfadjoint operators A_1, A_2 bounded by 1 in an enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$ such that

$$T_{rs} = \text{pr } A_1^r A_2^s \qquad (r, s = 0, 1, \cdots). \tag{67}$$

Proof. The mapping of positive polynomials $p(\lambda, \mu)$ into the set of bounded selfadjoint operators defined by $T(p) = \sum_{r,s=0}^{N} a_{rs}T_{rs}$ is by assumption linear and of positive type. So, by a standard argument, it can be extended, preserving these properties, to all bounded continuous functions f of λ, μ in $(-1, 1)$. In particular, if $f(\lambda, \mu) = \sum_{r,s=0}^{\infty} a_{rs}\lambda^r\mu^s$ is a power series convergent in an interval properly containing $(-1, 1)$, and having a non-negative sum everywhere in $(-1, 1)$, then $T(f) = \sum_{r,s=0}^{\infty} a_{rs}T_{rs}$ converges and is a non-negative operator.

Now we consider the function $f_{\lambda\mu}$ defined by

$$f_{\lambda\mu}(x_1, x_2) = \frac{x_1}{1 - \lambda x_1}\,\frac{x_2}{1 - \mu x_2} = x_1 x_2 \sum_{r,s=0}^{\infty} (\lambda x_1)^r (\mu x_2)^s \tag{68}$$

with fixed λ, μ in $(-1, 1)$. $f_{\lambda\mu}$ is of the type considered in Theorem 3, hence for any finite system of points $u_1, \cdots, u_N$; $v_1, \cdots, v_N$ in $(-1, 1)$ and any complex numbers $\alpha_1, \cdots, \alpha_N$ we have

$$p(\lambda, \mu) = \sum_m \sum_n \frac{f_{\lambda\mu}(u_m, v_m) - f_{\lambda\mu}(u_m, v_n) - f_{\lambda\mu}(u_n, v_m) + f_{\lambda\mu}(u_n, v_n)}{(u_m - u_n)(v_m - v_n)}\,\alpha_m\bar{\alpha}_n > 0 \tag{69}$$

(which could also be verified directly).

For u_j, v_j, α_j $(j = 1, \cdots, N)$ fixed, $p(\lambda, \mu)$ is therefore a positive function of λ, μ, admitting a Taylor development converging in an interval larger than $(-1, 1)$ as can be seen by substituting the series (68) into (69).

Hence $T(p)$ is a non-negative operator. From (68) and (69) however it is seen that

$$T(p) = \sum_m \sum_n \frac{F(u_m, v_m) - F(u_m, v_n) - F(u_n, v_m) + F(u_n, v_n)}{(u_m - u_n)(v_m - v_n)}\,\alpha_m\bar{\alpha}_n \geqq 0$$

with

$$F(x_1, x_2) = x_1 x_2 \sum_{r,s=0}^{\infty} T_{rs} x_1^r x_2^s.$$

Since this holds for arbitrary u_j, v_j, α_j $(j=1, \cdots, N)$, it follows that the operator-valued function F satisfies the condition of weak positive definiteness in Theorem 7. The other conditions being trivially satisfied, we have by Theorem 7,

$$F(x_1, x_2) = \text{pr}\, x_1(I - x_1 A_1)^{-1} x_2 (I - x_2 A_2)^{-1} \tag{70}$$

with commutative selfadjoint operators A_1, A_2, bounded by 1 in an enlarged Hilbert space $\hat{\mathfrak{H}} \supseteq \mathfrak{H}$.

Now developing both sides of (70) into power series

$$\sum_{r,s=0}^{\infty} T_{rs} x_1^{r+1} x_2^{s+1} = \sum_{r,s=0}^{\infty} \text{pr}\, A_1^r A_2^s x_1^{r+1} x_2^{s+1}$$

and comparing coefficients we obtain the desired result.

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