FUNCTIONS SATISFYING THE MEAN VALUE PROPERTY(1)

BY

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Introduction. In the preceding paper [8] we proved that if all harmonic functions in n-dimensional bounded domain D satisfy the mean value property (MVP) with respect to some point P and a given density function μ (volume-density, surface-density, etc.) then, under some simple assumptions on μ, D must necessarily be a ball with center P. In the present paper we are interested in studying the MVP from a complementary point of view, namely, we are interested in finding conditions on μ ($\mu \geqq 0$) under which the MVP holds at most for a finite number of linearly independent functions. The MVP is meant to be:

$$u(x) = \int_K u(x + ty)d\mu(y) \tag{0.1}$$

for all $x \in D$, $t > 0$ and sufficiently small, K being the support of μ.

In the case of two dimensions it is known [2; 7; 10] that if μ is a homogeneous distribution on the vertices, sides or area of a regular n-gon then the functions having the MVP (0.1) are precisely the harmonic polynomials of degree $\leqq n$ having zero nth derivatives in each of the directions of the radii of the polygons. There is also a similar characterization in 3-dimensions with a homogeneous distribution on the vertices of the regular solids [3].

In §1 we prove (Theorem 1) that the MVP (0.1) is equivalent to the system of partial differential equations

$$\sum_{|\nu|=j} A_\nu D_\nu u = 0 \qquad (1 \leqq j < \infty) \tag{0.2}$$

where the coefficients A_ν are the usual moments (with respect to the origin) of the measure μ. Flatto [6] has recently derived, by a different method, a different set of equations which are also equivalent to the MVP (0.1). Following his argument, it is shown (Theorem 2) that the system (0.2) has a finite dimensional solution-space if and only if the system of algebraic equations

$$\sum_{|\nu|=j} A_\nu z_\nu = 0 \qquad (1 \leqq j < \infty) \tag{0.3}$$

Received by the editors April 3, 1961.

(1) Prepared under Contract Nonr 710(16) (NR 044 004) between the Office of Naval Research and the University of Minnesota.

have no common complex root $\neq 0$. Such a reduction was still previously noticed by Aronszajn and P. Lax [1].

In §2 we consider the case of discrete measures, i.e., $\mu(Q_i)=\mu_i>0$ on a finite set of points Q_i $(1\leqq i\leqq N)$, and $\mu=0$ elsewhere. The solution-space of (0.1) is proved to be finite dimensional and, in case all the μ_i are equal, it consists of polynomials of degrees $\leqq C_2^N$. In the even more specialized case that the Q_i lie in the centers of the $2n$ faces of the unit cube, the degrees are $\leqq n^2$ and this inequality is sharp. Most of these results depend on some algebraic lemma which is also proved in §2.

In §3 we consider general measures and establish sufficient conditions for the finite dimensionality of the solution-space by a method of projection into 2-dimensional planes. These conditions are rather sharp, but not easy to verify.

1. The differential equations. Let μ be a non-negative Borel measure in n-dimensional Euclidean space R^n having compact support K lying in the unit sphere which is not contained in any $(n-1)$-dimensional hyperplane. Let $u(x)$ be a real, bounded and measurable function defined in a domain D in R^n. We are interested in characterizing functions which satisfy the mean value property (MVP)

$$u(x) = \int_K u(x+ty)d\mu(y) \qquad \text{for all } x \in D, \tag{1.1}$$

where t varies in the interval $0<t<\text{dist.}(x, S)$, S being the boundary of D. It will further be assumed that the total measure of μ is 1 (i.e., $\int_K d\mu=1$). In the sequel we use the following notation: $\nu=(\nu_1, \cdots, \nu_j)$, $|\nu|=j$, the ν_i vary from 1 to n; $D_\nu=D_{\nu_1}\cdots D_{\nu_j}$ where $D_{\nu_i}=\partial/\partial x_{\nu_i}$; $\xi_\nu=\xi_{\nu_1}\cdots\xi_{\nu_j}$. We shall prove:

THEOREM 1. *If u and μ satisfy the foregoing assumptions then u is real analytic in D and satisfies the system of partial differential equations*

$$\sum_{|\nu|=j} A_\nu D_\nu u = 0, \qquad j = 1, 2, 3, \cdots, \tag{1.2}$$

where

$$A_\nu = \int_K y_\nu d\mu(y). \tag{1.3}$$

Conversely, if u is infinitely differentiable in D and satisfies the differential equations (1.2), *and if μ satisfies the foregoing assumptions, then u is analytic and satisfies the* MVP (1.1).

REMARK. If the support of μ is contained in some $(n-1)$-dimensional hyperplane, then by a proper transformation we can reduce the problem to that of $n-1$ dimensions.

Proof. We first assume that (1.1) holds and proceed to establish the analyticity of u and (1.2). Clearly it is enough to consider x varying only in a ball B whose closure $\overline{B}$ is contained in D. Let dist.$(B, S)=2\epsilon$ and let B' be a ball concentric with B such that dist.$(B', S)=\epsilon$. We define a function $u'(x)$ as follows:

$$u'(x) = \begin{cases} u(x) & \text{for } x \text{ in } B', \\ 0 & \text{for } x \text{ in } CB' \end{cases}$$

where CX denotes the complement of X.

Clearly u' is bounded and measurable in R^n and its support is compact. Let $v(x)$ be any C^∞ function in R^n with support in B. Then, for $x \in B$, $0<t<\epsilon$, $u(x+ty)=u'(x+ty)$. Hence

$$v(x)u'(x) = v(x)\int_K u'(x+ty)d\mu(y). \tag{1.4}$$

Since $v(x)=0$ in CB, (1.4) is satisfied also for $x \in CB$; hence for all $x \in R^n$ (provided $0<t<\epsilon$). Integrating both sides of (1.4) on R^n we get

$$\int_{R^n} v(x)u'(x)dx = \int_{R^n} v(x)\left[\int_K u(x+ty)d\mu(y)\right]dx. \tag{1.5}$$

Denoting by $\mathfrak{F}w(\xi)$ the Fourier transform of the function $w(x)$, at the point ξ, and using Parseval's formula, we obtain from (1.5),

$$\int_{R^n} \mathfrak{F}v(\xi)\mathfrak{F}u'(\xi)d\xi = \int_{R^n} \mathfrak{F}v(\xi)\mathfrak{F}\left[\int_K u'(x+ty)d\mu(y)\right](\xi)d\xi. \tag{1.6}$$

Now,

$$\mathfrak{F}\left[\int_K u'(x+ty)d\mu(y)\right](\xi) = \int_{R^n} e^{-ix\cdot\xi}\left[\int_K u'(x+ty)d\mu(y)\right]dx$$

$$= \int_K\left[\int_{R^n} e^{-ix\cdot\xi}u'(x+ty)dx\right]d\mu(y) = \int_K [\mathfrak{F}u'(x+ty)](\xi)d\mu(y).$$

Substituting this in (1.6) and noting that

$$[\mathfrak{F}u'(x+ty)](\xi) = e^{i\xi\cdot ty}[\mathfrak{F}u'(x)](\xi)$$

we get

$$\int_{R^n} \mathfrak{F}v(\xi)\mathfrak{F}u'(\xi)d\xi = \int_{R^n} \mathfrak{F}v(\xi)\mathfrak{F}u'(\xi)\left[\int_K e^{it(\xi\cdot y)}d\mu(y)\right]d\xi. \tag{1.7}$$

We claim that the right side of (1.7) is C^∞ in t and that each t-derivative can be calculated by differentiation under the integral sign. Indeed, applying

any derivative $\partial^q/\partial t^q$ inside the integral sign, the resulting integrand is bounded by

$$\text{const}(1 + |\xi|^q)\,|\mathfrak{F}(v)(\xi)|\,|\mathfrak{F}u'(\xi)|,$$

where the constant is independent of t if $0<t<1$. Since v has compact support and is C^∞,

$$\|(1 + |\xi|^q)\mathfrak{F}v(\xi)\|_{L^1} < \infty.$$

Since also $\sup|\mathfrak{F}u'(\xi)| < \infty$ in R^n, the conditions for differentiability under the integral sign are satisfied.

Differentiating both sides of (1.7) with respect to t j times, we obtain, upon comparing for $t=0$,

$$\int_{R^n} \mathfrak{F}v(\xi)\mathfrak{F}u'(\xi)\left[\int_K (\xi\cdot y)^j d\mu(y)\right]d\xi = 0 \qquad (j = 1, 2, 3, \cdots). \tag{1.8}$$

Noting that

$$\int_K (\xi\cdot y)^j d\mu(y) = \int_K (\xi_1 y_1 + \cdots + \xi_n y_n)^j d\mu(y)$$
$$= \int_K \left(\sum_{|\nu|=j} \xi_\nu y_\nu\right)^j d\mu(y) = \sum_{|\nu|=j} A_\nu \xi_\nu,$$

(1.8) becomes

$$\int_{R^n}\left[\sum_{|\nu|=j} A_\nu \xi_\nu \mathfrak{F}v(\xi)\mathfrak{F}u'(\xi)\right]d\xi = 0.$$

Hence, using Parseval's formula,

$$\int_{R^n}\left[\sum_{|\nu|=j} A_\nu D_\nu v(x)\right]u'(x)dx = 0 \qquad (j = 1, 2, 3, \cdots). \tag{1.9}$$

Since these relations hold for all $v\in C^\infty$ with support in B, we conclude, from (1.9) with $j=2$, that u is a weak solution of

$$\sum_{i,j=1}^{n} A_{ij}D_{ij}u = 0 \qquad \left(D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\right) \tag{1.10}$$

in B. Equation (1.10), however, is of elliptic type since

$$\sum A_{ij}\xi_i\xi_j = \sum \int_K y_i\xi_i y_j\xi_j d\mu(y) = \int_K (y\cdot\xi)^2 d\mu(y) > 0$$

for any nonzero and real ξ, since the support of μ is not contained in any $(n-1)$-dimensional hyperplane.

Now, a weak solution of the elliptic equation (1.10) is necessarily twice differentiable and even real analytic. Hence u is an analytic function satisfying the equations (1.2).

(Note that the equations (1.2) for j even are all of elliptic type.)

It remains to prove the second part of the theorem, namely, that every C^∞ solution of (1.2) is analytic and satisfies the MVP (1.1). Since (1.2) for $j=2$ is an elliptic equation, u is analytic. We can then develop $u(x+ty)$ into a power series in ty for every fixed x, y varying in K and t sufficiently small (depending on x). The right side of (1.1) becomes

$$\sum_{j=0} t^j \frac{\partial^j u(x)}{\partial x_{\nu_1} \cdots \partial x_{\nu_j}} \int_K y_{\nu_1} \cdots y_{\nu_j} d\mu(y).$$

Using the equations (1.2) with the definitions (1.3) of the coefficients A_ν, the last series is found to be $u(x)$, which proves (1.1).

REMARK. Using a different method, Flatto [6] derived a result similar to Theorem 1. His system of equations is different from ours and the coefficients of the equations are of more complicated nature; they involve the whole sequence of harmonic polynomials. For the problems to be discussed in §2, our system is much more suitable.

There is a very simple connection between the study of the solution-space of the system (1.2) and the existence of nontrivial complex roots for the system of algebraic equations

$$\sum_{|\nu|=j} A_\nu z_\nu = 0 \qquad (j = 1, 2, 3, \cdots), \tag{1.11}$$

where $z_\nu = z_{\nu_1} \cdots z_{\nu_j}$, $1 \leqq \nu_i \leqq n$, the z_k being complex numbers. This connection was first pointed out by Aronszajn and P. Lax [1] and very recently by Flatto [6]. Since this connection will be used in the sequel we describe it here in detail.

THEOREM 2. *A necessary and sufficient condition that the system of differential equations* (1.2) *have at most a finite number of linearly independent solutions is that the system of algebraic equations* (1.11) *has* $z=0$ *as its only common root.*

Proof. The proof given below is that of Flatto. Suppose first that $z \neq 0$ is a common root of (1.11). Then the solution-space of (1.2) is not finite dimensional since for any integer $m \geqq 0$, $\mathrm{Re}\{(z \cdot x)^m\}$ is a solution of (1.2).

Conversely, let us assume that the equations (1.11) have no common root $z \neq 0$. It will suffice to show that every solution of (1.2) is a polynomial of degree $\leqq k$, for some k. Now, by [9, p. 18] the set of polynomials $\sum A_\nu z_\nu$ is finitely generated. Hence there exists a finite number of them which have no common root $z \neq 0$. We then can apply to x_1 (with respect to the above set of polynomials) the Hilbert's Nullstellensatz [9, p. 6] and conclude that for some q, x^q is a linear combination with polynomial coefficients of the set

of the generating polynomials. Substituting in the polynomial identity thus obtained D_{ν_i} for x_{ν_i}, we conclude that if u is a solution of (1.2), then $\partial^q u/\partial x_1^q = 0$. Proceeding similarly for the other variables $x_2, \cdots, x_n$, we conclude that there exists p such that $\partial^p u/\partial x_i^p = 0$ for $i = 1, 2, \cdots, n$. Hence, u is necessarily a polynomial of degree $< np$.

REMARK 1. From Theorem 2 it follows that *if there exists a nonpolynomial solution u of* (1.2) *then there exist infinitely many linearly independent solutions of* (1.2). This result can also be established directly by developing u locally into a series of harmonic polynomials H_m and observing then that the H_m must also satisfy the MVP in the open set where we have developed u; hence the H_m satisfy (1.2) throughout R^n. *The converse of the above italicized statement is also true*, as is easily seen. Hence:

The solutions of (1.2) (*or* (1.1)) *form a finite dimensional space if and only if the solution-space consists only of polynomials.*

REMARK 2. It will be useful later on to use the equations (1.11) written in the form

$$\int_K (z_1y_1 + z_2y_2 + \cdots + z_ny_n)^j d\mu(y) = 0 \qquad (j = 1, 2, 3, \cdots). \tag{1.12}$$

2. **The case of discrete measures.** In this section we shall study the solution-space of (1.1) or (1.2) in the special case that μ is a discrete measure. We shall prove that the solutions must be polynomials of degree $\leqq k$, and for homogeneous measures we shall give a sharp bound on k. We first need an auxiliary result, of intrinsic interest, from algebra of polynomials.

ALGEBRAIC LEMMA. *Let $R[x_1, \cdots, x_n]$ be the ring of polynomials in $x_1, \cdots, x_n$ with real coefficients, and let σ be the ideal in $R[x_1, \cdots, x_n]$ generated by all the elementary symmetric polynomials*

$$\sigma_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \quad \textit{where} \quad 1 \leqq i_j \leqq n.$$

Then every monomial $x_1^{l_1}x_2^{l_2} \cdots x_n^{l_n}$ $(l_i \geqq 0)$ of degree $l = l_1 + \cdots + l_n$ where $l > C_2^n$ belongs to σ. On the other hand there exists a monomial of degree C_2^n which does not belong to σ.

Proof. We shall need the auxiliary formulas:

$$x_2^1 x_3^2 x_4^3 \cdots x_k^{k-1} x_{k+1}^{k+1} x_{k+2}^{k+1} \cdots x_n^{k+1} \equiv 0 \pmod{\sigma}. \tag{2.1$_k$}$$

The proof is by induction on k. The case $k = 1$ is meant to be

$$x_2^2 x_3^2 x_4^2 \cdots x_n^2 \equiv 0 \pmod{\sigma}. \tag{2.1$_1$}$$

Writing the left side of $(2.2)_1$ as

$$(x_2x_3x_4 \cdots x_n)(x_2x_3x_4 \cdots x_n)$$

and noting that $x_2x_3x_4 \cdots x_n$ is one term in $\sigma_{n-1}(x)$ and that all the other terms contain x_1 as a factor, the proof of $(2.1)_1$ follows (recall that all the σ_i are $\equiv 0 \pmod{\sigma}$).

Assuming now that $(2.1)_j$ has already been established for all $j \leqq k$, we proceed to prove $(2.1)_{k+1}$. The left side of $(2.1)_{k+1}$ can be written as

$$(\overset{1}{x_2}\overset{2}{x_3}\overset{3}{x_4} \cdots \overset{k-1}{x_k}\overset{k}{x_{k+1}}\overset{k+1}{x_{k+2}}\overset{k+1}{x_{k+3}} \cdots \overset{k+1}{x_n})(x_{k+2}x_{k+3} \cdots x_n).$$

The second factor is one term in $\sigma_{n-k-1}(x)$, and each of the remaining terms contains at least one x_j with $1 \leqq j \leqq k+1$. But then the product of the first factor by x_j must vanish by $(2.1)_{j-1}$ if $2 \leqq j \leqq k+1$, or by $x_1x_2 \cdots x_n \equiv 0 \pmod{\sigma}$ if $j=1$.

The next identity needed is concerned with expressing x_1^p in a suitable way, mod σ.

$$x_1^p = x_1^{p-1}x_1 \equiv -x_1^{p-1}(x_2 + x_3 + \cdots + x_n) \pmod{\sigma}$$

since $x_2+x_3+ \cdots +x_n$ completes x_1 to the symmetric polynomial $\sigma_1(x)$. In a similar way we obtain

$$\begin{aligned} -x_1^{p-1}(x_2 + x_3 + \cdots + x_n) &= -x_1^{p-2}(x_1x_2 + x_1x_3 + \cdots + x_1x_n) \\ &= x_1^{p-2}(x_2x_3 + x_2x_4 + \cdots + x_2x_n + \cdots + x_{n-1}x_n). \end{aligned}$$

Proceeding step by step we arrive at

$$(2.2)_p \qquad x_1^p \equiv (-1)^p(x_2x_3 \cdots x_{p+1} + \cdots + x_{n-p+1} \cdots x_n) \pmod{\sigma}$$

provided $1 \leqq p \leqq n-1$. For $p=n$ we get

$$(2.2)_n \qquad x_1^n \equiv 0 \pmod{\sigma}.$$

We are now ready to establish identities which imply that every monomial of degree $> C_2^n$ belongs to σ. The identities are:

$$(2.3)_k \quad x_1^{n-1}x_2^{n-2} \cdots x_k^{n-k} \equiv \pm\, \overset{1}{x_2}\overset{2}{x_3}\overset{3}{x_4} \cdots \overset{k-2}{x_{k-1}}\overset{k-1}{x_k}\overset{k}{x_{k+1}}\overset{k}{x_{k+2}} \cdots \overset{k}{x_n} \pmod{\sigma}.$$

The sign $\pm$ means: either $+$ or $-$. $(2.3)_k$ for $k=1$ follows from $(2.2)_p$ with $p=n-1$. Hence we can proceed by induction on k. Assuming $(2.3)_j$ for all $j \leqq k$ we shall prove $(2.3)_{k+1}$, i.e., we shall prove that

$$(2.3)_{k+1} \qquad \begin{aligned} &x_1^{n-1}x_2^{n-2} \cdots x_k^{n-k}x_{k+1}^{n-k-1} \\ &\equiv \pm\, \overset{1}{x_2}\overset{2}{x_3}\overset{3}{x_4} \cdots \overset{k-2}{x_{k-1}}\overset{k-1}{x_k}\overset{k}{x_{k+1}}\overset{k+1}{x_{k+2}}\overset{k+1}{x_{k+3}} \cdots \overset{k+1}{x_n} \pmod{\sigma}. \end{aligned}$$

We use the following convention: Whenever we write

$$f_1 \equiv g_1 \pmod{\sigma},$$
$$\uparrow [f_2 \equiv g_2] \pmod{\sigma},$$

the second congruence is understood to mean that

$$f_1 f_2 \equiv g_1 g_2 \pmod{\sigma}.$$

Now, if we write $(2.3)_k$ as $f_1 \equiv g_1 \pmod{\sigma}$, then we have

$$\uparrow [x_{k+1}^{n-k-1} \equiv x_{k+2} x_{k+3} \cdots x_n] \pmod{\sigma}. \tag{2.4}$$

Indeed, from the formulas $(2.2)_p$ with x_1 replaced by x_{k+1} and with $p = n-k+1$ we see that each term, with the exception of $x_{k+2}x_{k+3} \cdots x_n$ contains at least one x_j with $1 \leqq j \leqq k+1$. But each such term, upon multiplication by the right side of $(2.3)_k$ gives zero (mod σ) in view of $(2.1)_{j-1}$. Now multiplying each side of (2.4) by the corresponding side of $(2.3)_k$, we obtain $(2.3)_{k+1}$.

Having established $(2.3)_k$, we multiply both sides by x_k and obtain

$$x_1^{n-1} x_2^{n-2} \cdots x_{k-1}^{n-k+1} x_k^{n-k+1} \equiv 0 \pmod{\sigma}. \tag{2.5$_k$}$$

since the right side of $(2.3)_k$ multiplied by x_k gives zero in view of $(2.1)_{k-1}$.

Suppose now that $M(x)$ is any monomial which does not belong to σ. We want to prove that its degree is $\leqq C_2^n$. Without loss of generality we can take

$$M(x) = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad \text{where} \quad a_1 \geqq a_2 \geqq \cdots a_n \geqq 0.$$

If $a_1 \geqq n$ then $M(x) \equiv 0 \pmod{\sigma}$ by $(2.2)_n$, thus contradicting our assumption on $M(x)$. Hence $a_1 \leqq n-1$. Next, if $a_2 \geqq n-1$ then $M(x) \equiv 0 \pmod{\sigma}$ by $(2.5)_2$. Hence $a_2 \leqq n-2$. Proceeding in this manner, and using $(2.5)_k$ with $k = 3, 4, \cdots, n-1$ we find that $a_i \leqq n-i$ for $i = 1, 2, \cdots, n-1$. Hence, the degree of $M(x)$ is at most

$$(n-1) + (n-2) + \cdots + 2 + 1 = C_2^n.$$

It remains to prove that there exists a monomial of degree C_2^n which is not in σ. To prove it we recall that the ideal σ is also the ideal generated by the sequence of symmetric polynomials

$$\tau_p(x) = x_1^p + x_2^p + \cdots + x_n^p.$$

Suppose now that every monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ of degree C_2^n belongs to σ, then

$$x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = \sum_{p=1}^{N} \alpha_p(x) \tau_p(x)$$

for some polynomials $\alpha_1, \cdots, \alpha_N$. Substituting in this ideality $D_i = \partial/\partial x_i$

for the variable x_i and applying both sides to a function u, we conclude that if a function u satisfies

$$\tau_p(D)u \equiv \frac{\partial^p u}{\partial x_1^p} + \frac{\partial^p u}{\partial x_2^p} + \cdots + \frac{\partial^p u}{\partial x_n^p} = 0 \qquad (1 \leqq p < \infty) \tag{2.6}$$

then it is necessarily a polynomial of degree $< C_2^n$. We are going to derive a contradiction by exhibiting a polynomial of degree C_2^n which is a solution of (2.6), and thus completing the proof the lemma.

The polynomial is

$$h(x) \equiv \sum (-1)^I x_{i_1}^0 x_{i_2}^1 x_{i_3}^2 \cdots x_{i_k}^{k-1} \cdots x_{i_n}^{n-1}$$

where summation is extended over all the $n!$ permutations, and I denotes the number of inversions in the permutation $(i_1, i_2, \cdots, i_n)$. In applying $\tau_p(D)$ to $h(x)$ we collect together all those resulting terms which have for their coefficient

$$\pm \beta_k \equiv \pm (k-1)(k-1-2) \cdots (k-1-p).$$

Call each sum, thus obtained, $\sum_k$. We shall prove that $\sum_k = 0$ for any k (for $k-1 \leqq p$ this is trivial).

The general term in $\sum_k$ is

$$\beta_k(-1)^I x_{i_1}^0 x_{i_2}^1 \cdots x_{i_{k-1}}^{k-2} x_{i_k}^{k-1-p} x_{i_{k+1}}^k \cdots x_{i_n}^{n-1}.$$

The term corresponding to the permutation

$$(i_1, i_2, \cdots, i_{k-p-1}, i_k, i_{k-p+1}, \cdots, i_{k-1}, i_{k-p}, i_{k+1}, \cdots, i_n)$$

(whose number of inversions is denoted by I') gives the same term as before, except for sign: $(-1)^I$ is replaced by $(-1)^{I'}$. Since $I'-I$ is odd, the two terms cancel each other. Thus, taking summation over all even permutations and joining to each term the term corresponding to it (as above) which belongs to an odd permutation we get $\sum_k = 0$. This completes the proof of the lemma.

We shall use the lemma in characterizing the solution-space of (1.2) (or (1.1)), in case μ is a discrete measure. We first consider the case where we have $2n$ points Q_j situated at the centers of the $2n$ hyper-faces of the cube $-1 \leqq x_i \leqq 1$, that is,

$$Q_i = (q_{i1}, q_{i2}, \cdots, q_{in}), \quad Q_{-i} = (-q_{i1}, -q_{i2}, \cdots, -q_{in}) \text{ and } q_{ij} = \delta_{ij}.$$

We define

$$\mu(Q_i) = \mu(Q_{-i}) = \frac{1}{2n} \qquad \text{for } 1 \leqq i \leqq n, \tag{2.7}$$

$$\mu = 0 \text{ elsewhere.}$$

The algebraic equations (1.12) become

$$z_1^{2j} + z_2^{2j} + \cdots + z_n^{2j} = 0 \qquad (1 \leqq j < \infty) \tag{2.8}$$

and the differential equations (1.2) become

$$\frac{\partial^{2j} u}{\partial x_1^{2j}} + \frac{\partial^{2j} u}{\partial x_2^{2j}} + \cdots + \frac{\partial^{2j} u}{\partial x_n^{2j}} = 0 \qquad (1 \leqq j < \infty). \tag{2.9}$$

Every monomial $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ of degree $> C_2^n$ belongs to σ and hence is a finite linear combination of the polynomials $\tau_p(x)$ with polynomial coefficients, say $\sum \alpha_p(x)\tau_p(x)$. Replacing x_i by y_i^2 and then y_i by $\partial/\partial x_i$ and applying to u, we conclude that if u is a solution of (2.9) then

$$\frac{\partial^{2b} u}{\partial x_1^{2b_1} \cdots \partial x_n^{2b_n}} = 0 \quad \text{whenever} \quad b = b_1 + \cdots + b_n \geqq C_2^n + 1.$$

We now observe that any derivative

$$\frac{\partial^c u}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} \text{ vanishes if } c = c_1 + \cdots + c_n = n^2 + 1.$$

Indeed, not all the c_i can be odd since otherwise we get that $2p+n=n^2+1$ for some integer p, which is clearly false. Hence, we can write

$$\frac{\partial^c u}{\partial x_1^{c_1} \cdots \partial x_n^{c_n}} = D^{c-2b} \frac{\partial^{2b} u}{\partial x_1^{2b_1} \cdots \partial x_n^{2b_n}}$$

where $c-2b \leqq n-1$ and D^j means a certain jth derivative. Since

$$2b \geqq n^2 + 1 - (n-1) \geqq 2\left[\left(\frac{n}{2}\right) + 1\right],$$

it follows that the right side vanishes. Hence every (n^2+1)th derivative of u vanishes. Consequently, u is a polynomial of degree n^2.

On the other hand there exist polynomials of degree n^2 which are solutions of (2.9). One such polynomial is

$$\sum (-1)^I x_{i_1}^3 x_{i_2}^5 x_{i_3} \cdots x_{i_n}^{2n-1}.$$

The proof that this polynomial satisfies the equations (2.9) is similar to an analogous proof at the end of the proof of the algebraic lemma; details are therefore omitted.

We have thus proved:

THEOREM 3. *If u satisfies the* MVP (1.1) *with respect to the discrete measure μ defined in* (2.7) *then u is a harmonic polynomial of degree $\leqq n^2$. There exist polynomials of degree n^2 satisfying the* MVP (1.1).

Consider now the case of *arbitrary* N points $Q_i=(q_{i1}, q_{i2}, \cdots, q_{in})$, $1 \leqq i \leqq N$ and the measure

$$\mu(Q_i) = \frac{1}{N} \text{ for } 1 \leqq i \leqq N, \qquad \mu = 0 \text{ elsewhere.} \tag{2.10}$$

We shall prove:

THEOREM 4. *If u satisfies the* MVP (1.1) *with respect to the measure μ defined in* (2.10) *then u is a polynomial of degree $\leqq C_2^N$.*

Proof. The algebraic equations (1.12) become

$$T_j(z) \equiv \sum_{i=1}^{N} (q_{i1}z_1 + q_{i2}z_2 + \cdots + q_{in}z_n)^j = 0 \quad (1 \leqq j < \infty). \tag{2.11}$$

Since the support of the measure is always assumed not to be in any hyperplane, the rank of the matrix (q_{ik}) is n. We now perform the linear transformation

$$\xi_k = q_{k_1}z_1 + q_{k_2}z_2 + \cdots + q_{k_n}z_n \qquad (1 \leqq k \leqq N) \tag{2.12}$$

considering the z's as variables. Equations (2.11) then reduce to

$$\xi_1^j + \xi_2^j + \cdots + \xi_N^j = 0 \qquad (1 \leqq j < \infty).$$

By the algebraic lemma, every monomial of degree $> C_2^N$ in the ξ's, say $\xi_1^{c_1} \cdots \xi_n^{c_n}$ can be written in the form

$$\xi_1^{c_1} \cdots \xi_N^{c_n} = \sum \gamma_p(\xi)\tau_p(\xi).$$

Solving, from (2.12), $z_1, \cdots, z_n$ in terms of the ξ's, and using the equality $\tau_p(\xi) = T_p(z)$ we conclude that every monomial of degree $> C_2^N$ in the z's is a finite linear combination of the $T_p(z)$. Substituting in each such identity $\partial/\partial x_i$ for z_i and recalling the connection between the coefficients of the algebraic equations and the differential equations, we conclude that every solution of the differential system is a polynomial of degree $\leqq C_2^N$.

We shall now consider the case of a general discrete measure on the points Q_i, that is,

$$\mu(Q_i) = \mu_i > 0 \quad \text{for} \quad 1 \leqq i \leqq N, \ \sum \mu_i = 1, \quad \text{and} \quad \mu = 0 \text{ elsewhere.} \tag{2.13}$$

We shall prove:

THEOREM 5. *There exists a k depending on the μ_i such that every function satisfying the* MVP (1.1) *with respect to the measure defined in* (2.13) *is a polynomial of degree $\leqq k$.*

Proof. Using Theorem 2 and the transformation (2.12) it follows that it is sufficient to prove that the equations

$$\mu_1\xi_1^j + \mu_2\xi_2^j + \cdots + \mu_N\xi_N^j = 0 \qquad (1 \leqq j < \infty) \tag{2.14}$$

have no nontrivial common complex root. The proof is very elementary. Indeed, if $\xi_1, \xi_2, \cdots, \xi_N$ is a nonzero solution then we write $\xi_k = |\xi_k| \exp\{i\Theta_k\}$ for some Θ_k and substitute in (2.14). Denoting by $\xi_a, \xi_b, \cdots, \xi_c$ those ξ_k whose absolute value is equal to $\max_l |\xi_l|$, we find that the function

$$f(s) \equiv \mu_a e^{is\Theta_a} + \mu_b e^{is\Theta_b} + \cdots + \mu_c e^{is\Theta_c} \qquad (i = (-1)^{1/2})$$

tends to zero as $s \to \infty$ along the positive integers. Since $f(s)$ is almost periodic it follows, from well known theorems, that $f(s) \equiv 0$. This is however impossible if all the μ's are positive.

REMARK. Theorem 5 is false if some of the point-masses are negative. Thus, if for the points Q_i in (2.7) all the μ_i, μ_{-i} are 1 except for μ_1 and μ_{-1} which are taken to be -1, we get the algebraic equations

$$z_1^{2j} - z_2^{2j} - \cdots - z_n^{2j} = 0 \qquad (1 \leqq j < \infty)$$

which have common nontrivial roots.

Added in proof. A. Garsia [2] has extended Theorem 4 by appropriately extending the algebraic lemma, and thus proved that Theorem 5 holds with $k = C_N^2$.

3. **The case of arbitrary measures.** In this section we shall give some sufficient conditions on a general measure μ which imply that the solution-space of (1.2) (or (1.1)) is finite dimensional. These conditions are rather directly expressed in terms of the measure but it is not easy to verify them. In the case of discrete measures these conditions can be verified, thus giving another proof to Theorem 5.

We denote by R_{ij} linear transformations of the form

$$\begin{aligned} x_k' &= a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n && \text{for } k = i, j, \\ x_l' &= x_l && \text{for all } l,\ 1 \leqq l \leqq n,\ l \neq i,\ l \neq j. \end{aligned}$$

We call it a *regular 2-affine rotation* if the transformation is regular. R_{ij} maps K onto a set K_{ij}. We denote by $\overline{K}_{ij}$ the orthogonal projection of K_{ij} on the (x_i, x_j)-plane and by $\bar{\mu}_{ij}$ the integral of μ along sets of points in K having the same projection in the (x_i, x_j)-plane.

Given a measure μ^* with compact support K^* and with $\int d\mu^* = 1$, we say that μ^* is a *normal distribution* (with respect to the origin) if for any harmonic polynomial v,

$$v(0) = \int_{K^*} v(ty)\,d\mu^*(y) \qquad \text{for all } 0 < t < \infty.$$

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[2] *Editorial note.* See *A note on the mean value property*, immediately following this paper.

This definition is essentially the one introduced by Choquet and Deny [5] in the case of two dimensions.

In [8] we gave some simple conditions on μ^* which guarantee that if K^* is not a sphere or a ball, then μ^* is not a normal distribution. We shall need the assumption:

(A) For any regular 2-affine rotation R_{ij}, the resulting (projection) measure $\bar{\mu}_{ij}$ is a non-normal distribution.

We shall prove:

THEOREM 6. *If μ satisfies the assumption* (A) *and the assumptions of Theorem* 1, *then there exists at most a finite number of linearly independent functions satisfying the* MVP (1.1).

All the functions are polynomials by Remark 1 at the end of §1.

Proof. In view of Theorems 1, 2 and (1.12), it is sufficient to prove that there exists no complex root $z \neq 0$ of the system of algebraic equations:

$$\int_K (z_1y_1 + z_2y_2 + \cdots + z_ny_n)^j d\mu(y) = 0 \qquad (1 \leqq j < \infty). \tag{3.1}$$

Assuming that there exists a root $z \neq 0$, we shall derive a contradiction. Since the left side of (3.1) for $j=2$ is a positive form in the z_i, considered as variables, z cannot be a constant multiple of a real vector. Hence the linear functions

$$z_{k1}y_1 + z_{k2}y_2 + \cdots + z_{kn}y_n \qquad (k = 1, 2) \tag{3.2}$$

where $z_{1j} = \mathrm{Re}\{z_j\}$, $z_{2j} = \mathrm{Im}\{z_j\}$ are linearly independent. It follows that for some i, j the vectors (z_{ki}, z_{kj}) for $k=1, 2$, are linearly independent. Hence the transformation

$$\begin{aligned} y_i' &= z_{11}y_1 + z_{12}y_2 + \cdots + z_{1n}y_n, \\ y_j' &= z_{21}y_1 + z_{22}y_2 + \cdots + z_{2n}y_n, \\ y_l' &= y_l \qquad\qquad\qquad\qquad\qquad\qquad \text{for } l \neq i,\ l \neq j \end{aligned}$$

is a regular 2-affine rotation R_{ij}. Equations (3.1) then reduce to

$$\int_{\bar{K}_{ij}} (y_i' + (-1)^{1/2}y_j')^h d\bar{\mu}_{ij} = 0 \qquad (1 \leqq h < \infty). \tag{3.3}$$

Since the monomials $\zeta^h \equiv (y_i' + (-1)^{1/2}y_j')^h$ generate the family of polynomials in the complex variable ζ which vanish at $\zeta = 0$, and since $\int d\bar{\mu}_{ij} = 1$, we conclude that for any polynomial $f(\zeta)$,

$$\int_{\bar{K}_{ij}} f(\zeta) d\bar{\mu}_{ij} = 0.$$

Hence the same is true for all harmonic polynomials, thus contradicting the assumption (A).

REMARK 1. For discrete measure μ, the assumption (A) is satisfied as follows from [8, §4]. Hence Theorem 5 yields the assertion of Theorem 4.

REMARK 2. The assumption (A) is essential. Indeed, let $d\bar{\mu}=\lambda dx_1dx_2$ be a normal distribution of a measure whose support K lies in the 2-dimensional (x_1, x_2)-plane. Defining $\lambda(x_1, x_2, \cdots, x_n)=\lambda(x_1, x_2)$ we can define a new measure $d\mu$ in the cylinder CK based on K by $d\mu=\lambda dx_1dx_2 \cdots dx_n$. The assumption (A) is violated with respect to R_{12} and all harmonic polynomials $u(x_1, x_2)$ satisfy the MVP (1.1).

REMARK 3. It was proved for $n=2$ by Choquet and Deny [5] that if the measure is not a normal distribution then the space of functions satisfying the MVP (1.1) is finite dimensional. This result also follows easily from the work of Flatto [6], as well as from Theorem 6. Brodel [4] stated this fact in case of unit line-density on the boundary or unit area-density in the interior of a convex domain (in the plane) but his proof seems to be incomplete.

REMARK 4. The problem of proving the finite dimensionality of the space of functions satisfying the MVP (1.1) (or (1.2)) for such simple bodies as convex polyhedra, ellipsoids, etc. with the ordinary Euclidean measure does not seem to be easy. In this paper we have considered two general methods: the direct attack of the equivalent algebraic equations and the projection method of the present section. Both methods, however, lead to rather involved technical problems in trying to apply them even to very simple bodies.

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