ON A CLASS OF NONFLEXIBLE ALGEBRAS(*)

BY
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1. Introduction. An algebra $A$ over a field $F$ of characteristic not two will belong to the class $\mathcal{A}$ if

I. The elements of $A$ satisfy a nontrivial identity of the form

$$
\alpha_1(xy)y + \alpha_2(yx)x + \alpha_3(yx) + \alpha_4(xy) + \alpha_5(xy)x + \alpha_6(yx)x
$$

for fixed $\alpha_i \in F$.

II. There is an algebra $B$ over $F$ such that $B$ satisfies (1), $B$ has an identity element, and $B$ is nonflexible; that is, there are elements $x$ and $y$ in $B$ such that $x(yx) \neq (xy)x$.

These conditions are similar to those used by Albert to define almost left alternative algebras [5]. Albert's paper led to the study of algebras of ($\gamma, \delta$) type by Kleinfeld and Kokoris [9; 11; 12]. Kokoris has shown that any simple finite dimensional algebra of characteristic prime to 30 of ($\gamma, \delta$) type is either alternative or has an identity element which is an absolutely primitive idempotent[7]. Our alteration of Albert's conditions yields a new class of simple power-associative algebras. We note that property II seems more natural in light of Oehmke's results [13] and the remark that most of the well-known nonassociative algebras (Jordan, noncommutative Jordan, Lie, alternative, associative) satisfy the flexible identity $x(yx) = (xy)x$.

In §2 it is shown that if $F$ is algebraically closed then any algebra $A$ over $F$ belonging to $\mathcal{A}$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ where $A(\mu)$ satisfies one of the following identities:

(i) $(xy)x - x(yx) = (xy)x - (yx)x$,
(ii) $x(xy) + (yx)x = 2(xy)x$,
(iii) $x(xy) + (yx)x = (xy)x + (yx)x$,
(iv) $x(yz + zy) + (yz + zy)x = (xy + yx)x + z(xy + yx)$.

The remainder of the paper is devoted to the study of algebras which satisfy some one of these four identities. We find that any power-associative

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(*) For results on rings of ($\gamma, \delta$) type see a forthcoming paper of Kleinfeld to appear in the Canadian Journal of Mathematics.
ring $A$ which satisfies (i) has an idempotent decomposition as $A = A_{11} + A_{10} + A_{01} + A_{00}$ where the $A_{ij}$ are defined just as in the associative case. Using this decomposition, we prove in §3 that any simple power-associative ring of characteristic not two which satisfies (i) and has an idempotent $e$ such that $A_{10} + A_{01} \neq 0$ is an associative ring. Examples of simple power-associative algebras satisfying (i) which are not flexible are constructed.

In §4 we are able to use Oehmke's methods [13] to prove that any semi-simple strictly power-associative algebra $A$ over a field $F$ of characteristic prime to 6 which satisfies (ii) has an identity and is the direct sum of simple algebras. The main result on algebras satisfying (ii) is that any simple strictly power-associative algebra of characteristic prime to 6 of degree $t > 2$ is flexible so that the results of [13] yield the result that $A$ is one of the following:

(a) a commutative Jordan algebra;
(b) a quasi-associative algebra;
(c) an algebra of degree 1 or 2.

Finally, examples of simple power-associative nonflexible algebras which satisfy either (iii) or (iv) are constructed.

As a matter of terminology, by an algebra we shall always mean a finite dimensional vector space on which there is a multiplication defined which satisfies both distributive laws. The radical of a power-associative ring is the maximal nil ideal and any ring with zero radical is said to be semi-simple. If $A$ is any power-associative ring of characteristic not two in which the equation $2x = a$ has a solution for all $a \in A$, then $A$ has an attached ring $A(\pm)$ which is the same additive group as $A$ but the multiplication $x \circ y$ of $A(\pm)$ is defined by $2x \circ y = xy + yx$. Then $A$ has a decomposition with respect to an idempotent $e$ as

$$A = Ae(2) + Ae(1) + Ae(0)$$

where $x \in Ae(\lambda)$ if and only if $2ex = \lambda x$, $\lambda = 0, 1, 2$ [3].

2. The Class $\mathfrak{A}$. By Property II there is an algebra $B$ in $\mathfrak{A}$ with elements $x$ and $y$ such that $xy \neq yx$. Then, by a series of substitutions of the elements 1, $x$, $y$ in (1), we find the following relations:

$$a_1 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 = 0,$$
$$a_1 + a_4 + a_5 + a_6 = a_2 + a_3 + a_5 + a_7 = 0,$$
$$a_1 + a_2 + a_3 + a_5 = a_4 + a_5 + a_7 + a_8 = 0,$$
$$a_1 + a_2 + a_4 + a_5 = a_3 + a_6 + a_7 + a_8 = 0.$$

Combining these we have $a_8 = a_2$, $a_7 = a_1$, $a_6 = -(a_1 + a_3 + a_5)$, and $a_5 = -(a_1 + a_2 + a_4)$.

For an arbitrary ring $A$ we define $R_x$ to be the mapping $a \rightarrow ax$ and $L_x$ is called a right multiplication of $A$. Similarly $L_x$ is defined as the mapping $a \rightarrow xa$ and is called a left multiplication of $A$. 

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Now using these relations and rewriting (1) in terms of right and left multiplications, we obtain
\begin{equation}
\alpha_1(R_xR_y + L_xL_y) + \alpha_2(R_yR_x + L_yL_x) + \alpha_3R_xL_y + \alpha_4R_yL_x
- (\alpha_1 + \alpha_2 + \alpha_3)L_yR_x - (\alpha_1 + \alpha_2 + \alpha_3)L_vRx = 0.
\end{equation}

If we interchange $x$ and $z$ in (1), we find
\begin{equation}
\alpha_1(L_xR_y + R_xL_y) + \alpha_2(L_zR_y + L_zR_x) + \alpha_3L_xRx + \alpha_4L_zL_y
- (\alpha_1 + \alpha_2 + \alpha_3)L_yRx = 0.
\end{equation}

Finally, setting $y = x$ in (2) yields
\begin{equation}
(\alpha_1 + \alpha_2)(R_x^2 + L_x^2 - 2L_xR_x) + (\alpha_3 + \alpha_4)(R_xL_x - L_xR_x) = 0.
\end{equation}

Suppose $\alpha_1 + \alpha_2 = 0$. Then from (4) we must have $(\alpha_3 + \alpha_4)(R_xL_x - L_xR_x) = 0$, but property II implies there is an $x \in \mathbb{B}$ such that $R_xL_x - L_xR_x \neq 0$, so that $\alpha_3 + \alpha_4 = 0$. Suppose also that $\alpha_1 = \alpha_2 = 0$. Substitution of these values in (3) along with the condition that not all the $\alpha_i$ are zero yields
\begin{equation}
R_{xy} - R_{xR_y} = -(L_yR_x - L_xR_y).
\end{equation}

Suppose $A$ and $B$ are algebras over a field $F$ such that $A$ and $B$ are isomorphic as vector spaces. We may then consider $A$ and $B$ as the same vector space and we shall say that $A$ is quasi-equivalent in $F$ to $B = A(\mu)$ if there is a $\mu \in F$, $\mu \neq 1/2$, such that the product $x \circ y$ in $B$ is given in terms of the product $xy$ of $A$ by $x \circ y = \mu xy + (1 - \mu)yx$ [3]. The multiplications $R'_x$ and $L'_x$ of $A(\mu)$ are given in terms of the multiplications $R_x$ and $L_x$ of $A$ by $R'_x = \mu R_x + (1 - \mu)L_x$ and $L'_x = (1 - \mu)R_x + \mu L_x$.

In the remainder of this section we suppose $A$ to belong to the class $\mathcal{Y}$ and $F$ to be an algebraically closed field.

Lemma 2.1. Suppose $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \neq \pm \alpha_3$. Then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ satisfying identity (5).

Proof. If $\alpha_1 = 0$ the result is immediate. If $\alpha_1 \neq 0$ and $\beta = -\alpha_3/\alpha_1 \neq \pm 1$, then (3) can be written as
\begin{equation}
L_{xy} - L_xR_y - R_xL_y + R_{yx} = \beta(R_{xy} - R_xR_y - L_xL_y + L_{yx}).
\end{equation}

We note that $R'_{xy} - R'_xR'_y - L'_xL'_y + L'_{yx} = 0$ in $A(\mu)$ is equivalent to $\mu^2R_{xy} + \mu(1 - \mu)L_{xy} + \mu(1 - \mu)R_{yx} + (1 - \mu)R_{xy} = 0$ in $A(\mu)$. Simplifying we have
\begin{equation}
(2\mu^2 - 2\mu + 1)(R_{xy} - R_xR_y - L_xL_y + L_{yx}) = -2\mu(1 - \mu)(L_{xy} - L_xL_y - R_xL_y + R_{yx}).
\end{equation}
Now consider the equation in $\mu$, $2\mu^2 - 2\mu + 1 = -2\mu(1-\mu)\beta$ or $2(1-\beta)\mu^2 - 2(1-\beta)\mu + 1 = 0$. Since $\beta \neq 1$ and $F$ is algebraically closed, there is a $\mu$ in $F$ satisfying this equation. Suppose $\mu = 1/2$. Then the above equation becomes $1 - \beta - 2(1-\beta) + 2 = 0$. Hence, $\beta = -1$ contrary to assumption. Then substitution of this value of $\mu$ in (6) yields (7) and the proof is complete.

Let $\alpha_1 = -\alpha_3$ and $\alpha_1 + \alpha_3 = 0$. Then (6) becomes $L_{y-y} - L_xR_y - R_xL_y + R_yz = R_{y-y} - R_yR_y - L_xL_y + L_yz$ or

$$L_{y-y} - R_{y-y} = L_xR_y - R_xR_y - L_xL_y + R_yL_y. \quad (8)$$

If $\alpha_1 = \alpha_3$ and $\alpha_1 + \alpha_3 = 0$, we see that (6) becomes

$$R_{y+y} + L_{y+y} = (R_x + L_x)(R_y + L_y) \quad (9)$$

(by symmetry) = $(R_y + L_y)(R_x + L_x)$.

This is exactly the condition that $A$ be associative-admissible.

**Lemma 2.2.** If $\alpha_1 + \alpha_2 \neq 0$, then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ satisfying either

$$R_x^2 + L_x^2 = 2L_xR_x \quad (10)$$

or

$$R_x^2 + L_x^2 = L_xR_x + R_xL_x. \quad (11)$$

**Proof.** Since $\alpha_1 + \alpha_2 \neq 0$, we may write (4) in the form

$$R_x^2 + L_x^2 - 2L_xR_x = \beta(L_xR_x - R_xL_x) \quad \text{where} \quad \beta = \frac{\alpha_1 + \alpha_3}{\alpha_1 + \alpha_3}. \quad (12)$$

Then as before, $(R_x')^2 + (L_x')^2 - 2L_x'R_x' = 0$ in $A(\mu)$ is equivalent to $\mu R_x + (1 - \mu) L_x^2 + \mu L_x + (1 - \mu) R_x^2 - 2(\mu L_x + (1 - \mu) R_x)(\mu R_x + (1 - \mu) L_x) = (4\mu^2 - 4\mu + 1)(R_x^2 + L_x^2 - 2L_xR_x) - (4\mu^2 + 6\mu + 2)(R_xL_x - L_xR_x) = 0$ in $A$. Now examine the equation $4\mu^2 - 4\mu + 1)\beta = -(4\mu^2 - 6\mu + 2)$ or $2(\mu - 1)((2 + 2\beta)\mu - (2+\beta)) = 0$. This has a solution $\mu \neq 1/2$ provided $\beta \neq -1$. Thus the theorem is valid except possibly when $\beta = -1$. But then (12) becomes $R_x^2 + L_x^2 = L_xR_x + R_xL_x$.

Linearization of (10) gives us

$$R_xR_y + R_yR_x + L_xL_y + L_yL_x = 2L_xR_y + 2L_yR_x \quad (13)$$

which is equivalent to (10) since the characteristic of $F$ is not two. Similarly (11) is equivalent to

$$R_xR_y + R_yR_x + L_xL_y + L_yL_x = R_xL_y + R_yL_x + L_xR_y + L_yR_x \quad (14)$$

provided the characteristic of $F$ is not two. We note that setting $y = x$ in (8) yields (11). Combining these remarks with Lemmas 2.1 and 2.2 we state

**Theorem 2.1.** Let $A$ belong to the class $\mathfrak{A}$ and suppose also that $F$ is algebraically closed. Then $A$ is quasi-equivalent in $F$ to an algebra $A(\mu)$ where $A(\mu)$
satisfies one of the identities (5), (9), (10), (11); each of which is a particular determination of (1).

3. The identity $R_{xy} - R_xR_y = -(L_{xy} - L_xL_y)$. In the following we shall be concerned with a ring $A$ of characteristic not two such that $A$ satisfies identity (5). Set $(x, y, z) = (xy)z - x(yz)$. $(x, y, z)$ is called an associator. Then (5), when applied to an element $z$ of $A$, becomes in terms of associators

$$(x, y, z) = (z, y, x).$$

We now write the linearization of $xx^2 = x^x$ [3] in terms of associators as

$$(x, y, z) + (x, z, y) + (y, x, z) + (y, z, x) + (z, x, y) + (z, y, x) = 0.$$  

Using (15) in (16) we obtain

$$(x, y, z) + (y, z, x) + (z, x, y) = 0.$$  

**Theorem 3.1.** If $A$ is a ring of characteristic zero satisfying (15) and $xx^2 = x^x$, then $A$ is power-associative.

**Proof.** By [1, Lemma 4], we need only show that $x^2x = x^x$ for all $x \in A$. Set $y = x$ and $z = x^2$ in (15) and (17). Then $(x, x, x^2) = (x^2, x, x)$ and $(x, x, x^2) + (x, x^2, x) + (x, x, x^2) = 0$. Thus $2x^2x^2 = x^2x + xx^2$ and $2x^2x - 2xx^2 = 0$, so that $x^2x^2 = x^x = xx^2$ and the result follows.

The following example shows that $xx^2 = x^x$ is actually necessary to guarantee power-associativity. Let $A$ be the algebra with a basis of the three elements $e$, $u$, and $v$ over a field $F$ of characteristic zero where multiplication is defined by $uv = e$, $eu = u$, $ve = v$, $e^2 = e$, and all other products zero. Then $A$ satisfies (5), but we observe that $(u + v)^2(u + v) = u$ and $(u + v)(u + v)^2 = v$.

We now write the flexible identity $x(yx) = (xy)x$ in its linearized form as

$$(x, y, z) + (y, z, x) + (z, x, y) = 0.$$  

The flexible identity and (18) are equivalent provided $A$ has characteristic different from 2.

We note that (15) along with (18) implies associativity while (15) and (17) together yield $2(x, y, x) = 2(y, x, x) = -(x, y, x)$. Hence, for rings of characteristic not two satisfying (15) and $xx^2 = x^x$, the notions of being associative, flexible, right alternative, and left alternative are equivalent.

Hereafter, whenever we refer to a substitution or a permutation of certain elements in an identity given in terms of the right and left multiplications of the elements $x$, $y$, $xy$, and $yx$, we shall consider these multiplications as acting on the element $z$. Also, we shall often make use of these identities without writing them in terms of right and left multiplications.

A ring $A$ is said to be Jordan-admissible if the attached ring $A^{(+)}$ is a Jordan ring. A ring $A$ is called Lie-admissible if the ring $A^{(-)}$, where $A^{(-)}$ is the same additive group as $A$ but $A^{(-)}$ has the multiplication $[x, y] = xy - yx$, is a Lie ring.
Theorem 3.2. A is both Jordan-admissible and Lie-admissible.

Proof. Permuting $x, y, z$ in (5) we obtain

(19) \[ R_yz - R_yR_z - L_yL_z + L_{yz} = 0 \]

and

(20) \[ R_zL_y - R_zR_y - L_zR_y + L_{zy} = 0. \]

Subtracting (5) from (19) and then adding (20) to the resulting right hand member we have $L_{zy} - R_yz - yz = (R_y - L_y)(R_z - L_z) - (R_z - L_z)(R_y - L_y)$, so that $A$ is Lie-admissible [3].

Now set $y = x^2$ in (5). Then

(21) \[ R_{(x)_2} + L_{(x)_2} = R_xR_x + L_xL_x, \]

and $y = x^2$ in (19) yields

(22) \[ R_{(x)_2} + L_{(x)_2} = R_{x_2}R_x + L_{x_2}L_x. \]

We also set $y = x^2$ in (20) to obtain

(23) \[ R_xL_{x_2} - R_{x_2}L_x - L_{x_2}R_x + L_{x_2}L_x = 0. \]

Subtracting (22) from (21) and then adding (23) we obtain

\[(R_x + L_x)(R_{x_2} + L_{x_2}) - (R_{x_2} + L_{x_2})(R_x + L_x) = 0\]

and the theorem is proved [3].

In the remainder of this section we suppose that $A$ is a power-associative ring.

Lemma 3.1. If $e$ is any idempotent of $A$, then $(e, e, x) = (x, e, e) = (e, x, e) = 0$ for all $x \in A$.

Proof. If we set $A = A_2(2) + A_1(1) + A_0(0)$, we then see that it is sufficient to prove the proposition for $x \in A_2(1)$. Thus, we suppose in the following that $x \in A_2(1)$. Substitution of $y = z = e$ in (15) yields

(24) \[ ex + xe = e(ex) + xe, \]

and since $x = ex + xe = e(ex) + (xe)e + e(xe) + (ex)e$ we have

(25) \[ (ex)e + e(xe) = 0. \]

Set $ex = x_2 + x_1 + x_0$ and $xe = -x_2 + x_1 - x_0$ where $x_i \in A_i(i)$. Then by substitution in

(26) \[ ex = e(ex + xe) = e(ex) + e(xe) = e(ex) - (ex)e \]

we find $x_2e = -(x_2 + x_0)/2$, so that $ex_1 = (x_2 + x_0)/2 + x_1$. But then $x_1 = ex_1 + x_1e$.

(1) This decomposition of power-associative rings is basic to our development so that we refer the reader to [2, Chapter I, Theorem 3].
= e(ex) + x = 0.

Hence, ex = 0 and x = 0, so that by consideration of (24), (25), and (26), we have e(ex) = 0 and the result follows.

**Theorem 3.3.** If e is any idempotent of A, then every x ∈ A may be written uniquely as x = x_{11} + x_{10} + x_{01} + x_{00} where x_{ij} A_{ij} = \{ a : ea = ia ; ae = ja \}, i, j = 0, 1.

**Proof.** The theorem follows immediately from Lemma 3.1 just as in the associative case.

Let x and y ∈ A_{11}. Then from (15) e(xy) + (yx)e = (ex)y + y(xe) = xy + yx. If xy = a_{11} + a_{10} + a_{01} + a_{00}, then yx = b_{11} - a_{10} - a_{01} - a_{00} so that a_{10} = a_{01} = 0, and hence, a_{10} = a_{01} = 0. Therefore A_{11} ⊆ A_{11} + A_{00}. If x, y ∈ A_{00}, then e(xy) + (yx)e = (ex)y + y(xe) = 0, and, as above, we find A_{00} ⊆ A_{11} + A_{00}. If x ∈ A_{11}, y ∈ A_{10}, then xy + yx = (xe)y + y(ex) = x(ye) + (ye)x = xy. Thus yx = 0. But then e(xy) + (yx)e = (ex)y + y(xe) or e(xy) = xy and e(yx) + (xy)e = (ey)x + x(ye) = 0, so that (xy)e = 0. Hence A_{11}A_{11} ⊆ A_{11} and A_{10}A_{11} = 0. Replacing y ∈ A_{01} by y ∈ A_{01}, we find that A_{11}A_{11} = 0 and A_{01}A_{11} ⊆ A_{11}. Then replacing x ∈ A_{11} by x ∈ A_{00}, we have A_{00}A_{11} = 0, A_{10}A_{00} ⊆ A_{11}, and A_{00}A_{00} ⊆ A_{11}. Suppose x, y ∈ A_{11}. Then yx = (xy)e = (ex)y + y(ex) = xy and e(xy) + (yx)e = (ex)y + y(xe) = x(ye) + (ye)x = xy. Hence (xy)e = xy = 0. Likewise A_{11} = 0. Now suppose x ∈ A_{11}, y ∈ A_{01}. Then e(xy) + (yx)e = (ex)y + y(xe) = xy and e(xy) + (yx)e = (ex)y + y(xe) = xy. If we set xy = a_{11} + a_{10} + a_{01} + a_{00} and yx = b_{11} - a_{10} - a_{01} - a_{00} and substitute in the above equations, we find a_{11} + a_{10} + b_{11} - a_{01} - a_{00} = 0, hence b_{11} + a_{10} = a_{10} = 0 so that xy = xy. Combining these results we state

**Theorem 3.4.** Suppose e is any idempotent of the ring A and A = A_{11} + A_{10} + A_{01} + A_{00} where the A_{ij} are defined as in Theorem 3.3. If i and j are distinct and i, j, k, m = 0, 1, then A_{ij}A_{ij} + A_{ij}A_{im} = 0, and if either k ≠ j, or j ≠ m then also A_{ij}A_{jm} ⊆ A_{km}.

**Corollary 3.1.** L = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10} is an ideal of A.

**Proof.** By the preceding theorem it is sufficient to show that A_{ij}A_{ij}A_{ij} + (A_{ij}A_{ij})A_{ij} ⊆ L for i ≠ j. But if x ∈ A_{ij}, y ∈ A_{ij}, z ∈ A_{ij}, then x(yz) = (xy)z + z(yx) = (xy)z. Then (xy)z ∈ A_{ij}A_{ij} ⊆ L. Likewise (A_{ij}A_{ij})A_{ij} ⊆ L.

**Corollary 2.** Let A be simple and e an idempotent of A. Then either A_{10} + A_{01} = 0 or A_{11} = A_{10}A_{01} = A_{00} = A_{01}A_{10}.

**Proof.** The result is immediate from Corollary 3.1. We now state the main result of this section.

**Theorem 3.5.** If A is a simple power-associative ring of characteristic not two possessing an idempotent e such that A_{10} + A_{01} ≠ 0 and satisfying identity (21), then A is associative.
Proof. Let \( x \in A_{ij}, y \in A_{jk}, z \in A_{mk} \). Then \( x(yz) = 0 \) unless \( s = m \) and \( j = r \). Similarly \( (xy)z = 0 \) unless \( s = m \) and \( j = r \). Hence, it is sufficient to consider products of the form \( x(yz) \) where \( x \in A_{ij}, y \in A_{jn}, \) and \( z \in A_{nk} \).

**Case 1.** Suppose \( i = j = s = m \). We may represent \( x \) as \( x = \sum x_{in}x_{ni} \) where \( i \neq n \) since we may use Corollary 3.2. Then, by repeated applications of identity (15) and Theorem 3.4 we find

\[
(xy)z = \sum ((x_{in}x_{ni})y)z = \sum (x_{in}(x_{ni}y))z = \sum x_{in}((x_{ni}y)z) = x_{in}(x_{ni}(yz)) = x_{in}(x_{ni}(yz)) = x_{in}(x_{ni}(yz)) = x(yz).
\]

**Case 2.** Let \( i = j \). Then \( x(yz) = (xy)z + z(yx) - (zy)x \) and the product is associative unless \( i = m = s \) since otherwise the last two terms of the right hand member are zero. But, if \( i = m = s \), Case 1 applies so that the result holds in this instance.

**Case 3.** Suppose \( i = s \). Proceed as in Case 2.

**Case 4.** Suppose \( s = m \). Proceed as in Case 2.

Thus we have reduced the proof to the case where \( x, z \in A_{ij}, y \in A_{ji} \), with \( i \neq j \). Substitution of \( x, y, \) and \( z \) in (16) along with Theorem 3.4 yields

\[
(27) \quad x(yz) + z(yx) = (xy)z + (zy)x
\]

which with (15) implies \( x(yz) = (xy)z \) and the proof is complete.

**Remark.** If \( A \) is a semi-simple strictly power-associative algebra of characteristic not two satisfying (15) and \( e \) is a principal idempotent of \( A \), then \( A_{10} + A_{01} = 0 \).

**Proof.** We observe that \( e \) is also a principal idempotent of \( A^{(+)} \) and \( A_{*}(1) = A_{10} + A_{01} \). Hence \( A_{10} + A_{01} + A_{00} \subseteq \text{radical of } A^{(+)} \) [10, Theorem 5]. Then we claim that the ideal \( L \) defined in Corollary 3.1 of Theorem 3.4 is contained in the radical of \( A^{(+)} \). For, if \( x \in A_{10}, y \in A_{01}, \) then \( 2x \circ y = xy + yx \subseteq \text{radical of } A^{(+)} \). But \( 2e(x \circ y) + 2(x \circ y)e = 2xy \subseteq \text{radical of } A^{(+)} \). Thus \( A_{10}A_{01} \subseteq \text{radical of } A^{(+)} \), so that \( L \) must be a nil ideal of \( A \). Therefore \( A_{10} + A_{01} = 0 \).

Theorem 3.5 implies that if we are to find any new simple power-associative algebras satisfying (15), they must have no idempotent \( e \) such that \( A_{10} + A_{01} \neq 0 \). The existence of such algebras is guaranteed by the following examples. First, let \( A \) be the 3-dimensional algebra over a field of characteristic \( p \neq 2 \) with a basis \( e, u, v \) where \( e \) is the identity element, \( uv = -vu = e \), and the remaining products are zero. Then \( A \) is a simple power-associative algebra satisfying (15) but \( A \) is not flexible for \( (uv)u = -u(vu) = u \). Suppose we set \( B \) equal to the supplementary sum of the two orthogonal subspaces \( A_{i}, i = 1, 2 \); where \( A_{i} \) has a basis \( e_{i}, u_{i}, v_{i} \) such that \( e_{i} \) is the unity of \( A_{i} \), \( u_{i} = v_{i} = 0 \), and \( u_{i}v_{i} = -v_{i}u_{i} = e_{1} + e_{2} = e \) the unity element of \( B \). Then \( B \) is a simple power-associative algebra of degree two satisfying (15) and, as before, \( B \) is not flexible. This construction is easily generalized to yield simple power-associative algebras of arbitrary degree \( n \) which satisfy (15) but not the flexible identity.
We now give an example of a semi-simple power-associative algebra $C$ which satisfies (15) but has no identity element and is not the direct sum of simple algebras. Consider $C$ equal to the supplementary sum of the algebra $A$ above and the subspace $A'$ with a basis $u_1$, $v_1$ where $AA' = A'A = 0$ and $u_1v_1 = -v_1u_1 = e$ and all other products zero. Then, since every proper ideal of $C$ contains $A$, we see that the nil radical of $C$ is 0.

One might hope for further results when $A$ is semi-simple with an idempotent $e$ such that $A_{10} + A_{e1} \neq 0$, but examination of the algebra $D = C \oplus F_2$, where $C$ is the algebra described just above and $F_2$ is the total matric algebra over $F$ of degree 2, shows there are such algebras which do not have an identity and are not the direct sum of simple algebras.

4. The identity $R_x^2 + L_x^2 = 2L_xR_x$. In this section we assume $A$ to be a ring having characteristic prime to 6 and having a solution to the equation $2x = a$ for all $a \in A$. Moreover, we shall assume that $A$ satisfies (13) or equivalently (10). We write (13) when applied to an element $z$ of $A$ as

$$(zz)y + (zy)x + y(xz) + x(zy) = 2(xz)y + 2(yz)x.$$  

**Theorem 4.1.** Suppose $A$ is a ring of characteristic prime to 30 such that $A$ satisfies (28) and $(x^2)^2 = (x^2x)x$ for all $x \in A$. Then $A$ is power-associative.

**Proof.** We define $x^n$ inductively by $x^{n+1} = xnx$. First we show that $xx^n = x^n x$ for all $x \in A$ and all positive integers $n$. Substitution of $x = y = z$ in (28) yields $x^2x + xx^2 = 2x^2x$. Hence $xx^2 = x^2x$. Suppose $xx^k = x^k x$. Then setting $x = z$ and $y = x^k$ in (28) yields $(x^k x)x + x(xx^k) = 2(xx^k)x$ so that $xx^{k+1} = x^{k+1}x$. Thus $xx^n = x^n x$ for all $x \in A$ and all positive integers $n$. The result then follows from [1, Lemma 4].

We now suppose that $A$ is a power-associative ring. We shall make frequent use of the identity $(x^2)^2 = (x^2x)x$ in its linearized form

$$\sum_{x,y,z} (xy + yx)(zw + zw) = \sum_{x,y,z} (xy + yx)w = \sum_{x,y,z} (xy + yx)w \quad (\text{symmetric in } x, y, z, w)$$

and we note that (29) is equivalent to $(x^2)^2 = (x^2x)x$ if $A$ has characteristic prime to 6. Let $e$ be an idempotent of $A$. Then from (28) we find

$$(xe)e + e(ex) = 2(ex)e \quad \text{for all } x \in A.$$  

Suppose $x \in A_e(1)$. Then by consideration of $A_+^{(+)}$ we may set $ex = x_2 + x/2 + x' + x_0$ and $xe = -x_2 + x/2 - x' - x_0$ where $x' \in A_e(1)$. Substitution in (30) yields $-x_2 + xe/2 - x'e - ex/2 + ex' = 2x_2 + xe + 2x'e$ so that we have $x_2 - x' - x_0 = ex' - 3x'e$. Hence $x_2 = 4(ex_2), x_0 = 4(ex_0)$, and $x' = (3x'e) - (ex')_1 = x'e + x'e$. Thus $(ex')_1 = (x'e) = x'$. If we now carry through the same argument with $x$ replaced by $x'$, we find that $(ex')_2 = (ex')_0 = 0$ and hence, $x_2 = x_0 = 0$. We note that $e(xe) = (ex)e = x/4$. Therefore, for any $x \in A_e(1)$, we have $ex = x/2 + x'$ where $x' \in A_e(1)$ and $ex' = x'e = x'/2$. 

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Suppose \( x, y \in A_\lambda(2) \). Then we set \( xy = a_2 + a_1 + a_0 \), \( yx = b_2 - a_1 - a_0 \). Setting \( y = e \), \( z = y \), (28) yields \( xy + (yx)e + e(xy) = 2yx + 2(xy)e \) so that \( ea_1 - a_1e = -2a_2 - 2a_1e \). Hence \( a_0 = 0 \) and \( 2a_1 = 3a_1e - ea_1 = 2a_1 + 2a_1a_0 \). But then \( 3a_1 = a_2 = 3a_1/4 \). For any \( a_1 \in A_\lambda(1) \) we have \( ea_1 = a_1/2 + a' \) where \( ea' = a'e = a'/2 \) so that \( a_1/2 - a' = 3a_1/4 \). Then \( -4a_1' = a_1 \) and therefore \( a_1 = a_1e \). Thus we finally have \( 4a_1' = a_1 = 0 \) and \( A_\lambda(2) \) is a subring of \( A \).

Let \( x, y \in A_\lambda(0) \). Then substitution of \( y = e \), \( z = y \) in (28) yields \( (yx)e + e(xy) = 2(xy)e \). We suppose \( xy = a_2 + a_1 + a_0 \) and \( yx = -a_2 - a_1 + b_0 \). Then \( -a_1e + ea_1 = 2a_0 + 2a_1e \). Hence \( a_2 = 0 \) and \( ea_1 = 3a_1e \). Then, just as for \( A_\lambda(2) \), we find \( a_1 = 0 \) and \( A_\lambda(0) \) is a subring of \( A \).

Suppose \( x \in A_\lambda(2) \), \( y \in A_\lambda(1) \). Then let \( xy = a_2 + a_1 + a_0 \), \( yx = b_2 + b_1 - a_0 \). Then (28) with \( x = e \), \( z = x \) becomes \( (yx)e + x(yx) + yx = 2(yx)e + 2xye \) and, hence, \( a_2 + eb_1 + b_1 + b_0 = 2be + a_1 + a_0 \). Thus \( a_0 = b_0 \) and \( a_1e + eb_1 = 2be + a_1 - b_1 \) or \( ea_1 = 2b_1 - 3be \). Another substitution of \( y = e \), \( z = y \) in (28) yields \( (yx)e + (yxe) + x(y) = 2(yx)e + 2(ey)x \). Now set \( ey = y'/2 + y' \) where \( ey' = y' = y'/2 \). Then \( b_1e + ea_1 + a_1/2 + xy' = a_2 + 2ae + b_1/2 + 2y'x \). Comparing components we find \( a_2 = (xy' - 3y'x)/4 = (xy')_2 \) and \( (y'x)_0 = (xy')_0 = 0 \). Yet another substitution of \( y = y' \), \( z = e \) in (28) yields \( y'x = y'x \) so that \( (xy')_2 = (y'x)_2 = 0 \). Thus \( a_0 = 0 \). Since \( ea_1 = 2b_1 - 3be \) a substitution of \( eb_1 = b_1/2 + b'_1 \) and \( ea_1 = a_1/2 + a'_1 \) gives the relation \( (b_1 - a_1)/2 = a'_1 - 3b'_1 \). Then \( e(b_1 - a_1) = (b_1 - a_1)e = (b_1 - a_1)/2 \). If \( ea_1 = 2b_1 - 3be \), then \( e(a_1) = 2eb_1 - 3(eb_1) \). But then we obtain \( (b_1 - a_1)/2 = (2a_1 - 4b'_1) \) so that \( a_1 = b'_1 \). Finally, \( eb_1 - a_1e = (b_1 - a_1)/2 + 2a'_1 = 0 \).

Let \( x \in A_\lambda(0) \), \( y \in A_\lambda(1) \). We set \( xy = a_2 + a_1 + a_0 \) and \( yx = b_2 + b_1 - a_0 \). Substitution of \( x = e \), \( z = e \) in (28) yields \( a_2 + a_0 + eb_1 = b_2 + 2be \). Then \( a_0 = b_0 \) and \( a_1 = 2b_0e - e_1 \). Set \( ey = y'/2 + y' \). Then, as before, \( y'x = y'x \) and substitution of \( z = e \) in (28) yields \( (xy - yx)/2 = -2xy' \) so that \( a_0 = 0 \). Now \( a_2 = 2b_1e - eb_1 \) implies \( e(a_1e) = 2e(b_1e) - e(eb_1) \). We shall use these relations in our later work.

Thus we may state

**Theorem 4.2.** Let \( e \) be an idempotent of a power-associative ring \( A \) which satisfies (28). Then \( A_\lambda(2) \) and \( A_\lambda(0) \) are orthogonal subrings of \( A \) and

\[
A_\lambda(\lambda) A_\lambda(1) + A_\lambda(1) A_\lambda(\lambda) \subseteq A_\lambda(2 - \lambda) + A_\lambda(1), \quad \lambda = 0, 2.
\]

Suppose \( A \) has an identity element \( 1 = e_1 + \cdots + e_t \) where \( e_t \) are pairwise orthogonal idempotents. Then \( A \) can be decomposed as the direct sum of the modules \( A_{ij} \) where for \( i = j \), \( A_{ij} = A_{ij}(2) \) and for \( i \neq j \), \( A_{ij} = A_{ij}(1) \cap A_{ij}(1) \) [6]. Let \( i, j, m, n \) be distinct. Then \( A_{ij} \circ A_{mj} \subseteq A_{im} \), \( A_{ij} \circ A_{ij} \subseteq A_{ii} + A_{jj} \), \( A_{ij} \circ A_{mn} = 0 \). But \( A_{mn} \subseteq A_{mn}(1) \), \( A_{ij} \subseteq A_{mn}(0) \) so that by our multiplication \( A_{ij} A_{mn} \subseteq A_{mn}(1) + A_{mn}(2) \). Since the \( A_{mn}(2) \) components of \( a_{ij}a_{mn} \) and \( a_{mn}a_{ij} \) are the same, \( A_{ij} A_{mn} \subseteq A_{mn}(1) \). Likewise \( A_{ij} A_{mn} \subseteq A_{mn}(1) \) for \( r = i, j, n \). Thus
Since $A_{mn} \subseteq A_{e}(0)$ we have $A_{ii}A_{mn} = A_{mn}A_{ii} = 0$. Again, these results will be of use to us in later developments.

A ring is said to be stable if for every idempotent $e$ of $A$ we have $A_{+(\mu)} A_{+(\mu)} A_{+(\mu)} \subseteq A_{+(\mu)}$ for $\mu = 0, 2$. By our multiplication it is readily seen that a ring $A$ satisfying (28) is stable if and only if $A_{+(\mu)}$ is stable.

We now suppose $A$ is strictly power-associative. Let $e$ be an idempotent of the ring $A$. Then for $x \in A_{+(\mu)}$ and $w \in A_{+(\mu)}$ we have $x \circ w = x_{1} + w_{0}$ where $w_{1} \in A_{+(\mu)}$ and $w_{0} \in A_{+(\mu)}$. Then the mapping $w \rightarrow w_{1}$ is an endomorphism of the module $A_{+(\mu)}$ determined by the element $x$ of $A_{+(\mu)}$. We denote this mapping by $S(x)$. By Albert's result $[6, \text{ Theorem } 1]$ the mapping $x \rightarrow 2S(x)$ of the ring $A_{+(\mu)}$ onto the special Jordan ring of endomorphisms $S(x)$ is a homomorphism with kernel $B_{e}$ where $B_{e}$ is the set of elements $x \in A_{+(\mu)}$ such that $x \circ w \in A_{+(\mu)}$ for all $w \in A_{+(\mu)}$. Certainly $B_{e}$ is an ideal of $A_{+(\mu)}$ and we shall show that $B_{e}$ is in fact an ideal of $A_{+(\mu)}$.

**Lemma 4.1.** If $x \in A_{+(\mu)}$ and $w \in A_{+(\mu)}$, $xw = a_{1} + a_{0}$, $wx = b_{1} + b_{0}$, then $a_{0} = b_{0}$ and $a_{1} e = b_{1} e$.

**Proof.** The result follows from our earlier remarks on such products.

**Lemma 4.2.** If $x \in A_{+(\mu)}$ such that $xw + wx \in A_{+(\mu)}$ for all $w \in A_{+(\mu)}$, then $xw = wx \in A_{+(\mu)}$.

**Proof.** If $xw + wx \in A_{+(\mu)}$, then by Lemma 4.1 we see that $a_{1} = - b_{1}$. From our earlier results $a_{1} - b_{1} = 4d_{1}$ where $e_{a_{1}} = a_{1}/2 + d_{1}$ and $e_{b_{1}} = b_{1}/2 - d_{1}$. Thus $2a_{1} = a_{1} - b_{1} = 4d_{1}$ and $e_{a_{1}} = e_{b_{1}} = a_{1}/2$. But then $d_{1} = 0$ so that $a_{1} = b_{1} = 0$ and $xw = wx = a_{0} \in A_{+(\mu)}$.

**Lemma 4.3.** $B_{e}$ is an ideal of $A_{+(\mu)}$.

**Proof.** Let $y$ be an arbitrary element of $A_{+(\mu)}$. We set $(xy)w = a_{1} + a_{0}$, $w(xy) = a_{1} + a_{0}$, $(yx)w = b_{1} + b_{0}$, $w(yx) = b_{1} + b_{0}$. By various substitutions of $x, y, w$ in (28) we find $(xy)w + (xw)y + w(xy) + y(wx) = 2(xy)w + 2(wx)y$ and $(yx)w + (yw)x + w(xy) + x(wy) = 2(xy)w + 2(wy)x$. Since $xw = wx \in A_{+(\mu)}$ and $x_{+(\mu)}(1) \subseteq A_{+(\mu)}(0)$, the first of these equations becomes $(xy)w + w(xy) = 2(xy)w$ and the second implies that $(yx)w + w(xy) = 2(xy)w \in A_{+(\mu)}(0)$. Hence, $a_{0} = b_{0}$, $a_{1} + b_{1}' = 2b_{1}$, and $b_{1} + a_{1}' = 2a_{1}$. Adding we find $a_{1}' + b_{1}' = a_{1} + b_{1}$ and then using Lemma 4.1 we obtain $ea_{1} + eb_{1} = ea_{1}' + eb_{1}' = a_{1}e + b_{1}e = a_{1}'e + b_{1}'e = (a_{1} + b_{1})/2 = (a_{1}' + b_{1}')/2$. Then $e_{a_{1} + b_{1}} = 2e_{b_{1}}$ or $e_{a_{1} + b_{1}} = 2e_{b_{1}} - e_{b_{1}} = 2e_{b_{1}} - e_{b_{1}}$. If $e_{a_{1} + b_{1}} = 2e_{b_{1}}$, then $e_{b_{1}} = b_{1}/2 - d_{1}$ so that $a_{1}/2 + d_{1} = b_{1}/2 - d_{1}$. Thus $(a_{1} - b_{1})/2 = -2d_{1}$ so that $e_{(a_{1} - b_{1})} = (a_{1} - b_{1})/2$. Then $e_{a_{1} + b_{1}} + e_{a_{1} - b_{1}} = (a_{1} + b_{1})/2 + (a_{1} - b_{1})/2 = a_{1}$. Hence $d_{1} = 0$ and $a_{1} = b_{1} = a_{1}' = b_{1}'$. We note that we have actually shown that $(xy)w = (yx)w = w(xy) = w(yx)$. Now set $e = e$ in (29). Then consideration of the preceding remark yields

2(xy)w + 2((xy)w)e + 2(yw)x + ((yw + wy)e)x - 4x((yw + wy) = 0. Then, using the properties of x, we have 2(xy)w + 2((xy)w)e \in A_4(0). Hence 2a_1 + 2a_2e = 3a_1 \in A_4(0) and, since the characteristic of A is not three, a_1 = 0. This completes the proof.

We now suppose A to be a simple ring with an identity 1 such that 1 = e_1 + e_2 + e_3 where the e_i are pairwise orthogonal idempotents.

**Lemma 4.4.** Let e be an idempotent of A such that e \neq 1 and suppose x \in A_4(2) with the property that xw + wx = 0 for all w \in A_4(1). Then x = 0.

**Proof.** By Lemmas 4.1 and 4.2 we see that (xw)_1 = (wx)_1 = 0 = (wxe)_0. Let C_e be the set of all such x \in A_4(2). We claim that C_e is an ideal of A and, since e \neq 1 and e is not in C_e, it will follow that C_e = 0. By the above remarks C_eA_e(1) = A_e(1)C_e = 0 and since A_e(2)A_e(0) = A_e(0)A_e(2) = 0 the proof will be complete if we show that A_e(2)C_e + C_eA_e(2) \subseteq C_e. Let y \in A_e(2). Then, as in the proof of Lemma 4.3, we find (xy)w = w(xy) = w(yx) = (yx)w. Again, as in the proof Lemma 4.3, a substitution of x, y, w, e = z in (29) yields 2(xy)w + 2((xy)w)e = 0. Thus ((xy)w)_0 = 0 and ((xy)w)\in A_4(0) which is impossible unless ((xy)w)_1 = 0. Therefore xy, yx \in C_e as was to be shown.

**Lemma 4.5.** Set g = e_1 + e_2. Then B_g = B_{e_1} + B_{e_2}.

**Proof.** From an earlier remark we may set A = A_{11} + A_{12} + A_{22} + A_{13} + A_{23} + A_{33} where A_{11} + A_{12} + A_{22} = A_{4}(2), A_{13} + A_{23} = A_{4}(1), and A_{33} = A_{4}(0). Suppose x_{11} + x_{12} + x_{22} \in B_g \subseteq A_4(2). Then x_{12} = 4(x_{11} + x_{12} + x_{22})o e_1 - 4(x_{11} + x_{12} + x_{22})o e_1 and since B_g is an ideal of A, x_{12} \in B_g. Then e_1(x_{11} + x_{22}) = x_{11} \in B_g so that x_{22} \in B_g also. Suppose a_{13} + a_{23} \in A_{4}(1). Then 2x_{12} o (a_{13} + a_{23}) = 2x_{12} o a_{13} + 2x_{12} o a_{23} \in A_{22} + A_{13} by our remarks on the multiplication of the A_{ij}. But x_{12} \in B_g so that 2x_{12} o (a_{13} + a_{23}) \in A_{4}(0) = A_{33}. Hence x_{12} o (a_{13} + a_{23}) = 0. Apply Lemma 4.4 to obtain x_{12} = 0 and thus B_g \subseteq A_{11} + A_{22} + A_{33}. For a_{12} \in A_{12} we find x_{11} o a_{12} \in A_{12} + A_{22}. But x_{11} \in B_g which is an ideal of A, so that x_{11} o a_{12} \in A_{22}. By the definition of B_g we have x_{11} o a_{12} \in A_{4}(0) = A_{33}. Thus if w \in A_{12} + A_{13} = A_{4}(1), then x_{11} o w \in A_{22} + A_{33} = A_{4}(0). Therefore x_{11} \in B_g and, in a similar manner, x_{22} \in B_g. Hence B_g \subseteq B_{e_1} + B_{e_2}. If x_{11} \in B_{e_1}, then x_{11} o (A_{12} + A_{13}) \subseteq A_{22} + A_{33} = A_{4}(0). But then x_{11} o A_{4}(1) = x_{11} o (A_{12} + A_{23}) = x_{11} o A_{12} \subseteq A_{33} = A_{4}(0). Hence B_{e_1} \subseteq B_g and likewise B_{e_2} \subseteq B_g. Thus B_{e_1} + B_{e_2} = B_g.

**Lemma 4.6.** B = B_{e_1} + B_{e_2} + B_{e_3} is an ideal of A.

**Proof.** Since B_{e_1} + B_{e_2} is an ideal of A_4(2), we have A_4(2)B_{e_1} + B_{e_1}A_4(2) \subseteq B_{e_1} + B_{e_2}. If h = e_1 + e_3, then A_4(2)B_{e_1} + B_{e_1}A_4(2) \subseteq B_{e_1} + B_{e_3}. Since A_{33} \subseteq A_4(0), B_{e_1}A_{33} = A_{33}B_{e_1} = 0. Thus AB_{e_1} + B_{e_1}A = (A_{11} + A_{12} + A_{22})B_{e_1} + B_{e_1}(A_{11} + A_{12} + A_{22}) + A_{33}B_{e_1} + A_{33}A_{33} \subseteq B_{e_1} + B_{e_2} + B_{e_3}. Interchanging subscripts we find AB_{e_2} + B_{e_2}A \subseteq B_{e_1} + B_{e_2} + B_{e_3} and AB_{e_3} + B_{e_3}A \subseteq B_{e_1} + B_{e_2} + B_{e_3}. Therefore B is an ideal of A.
Either \( B = A \) or \( B = 0 \) since we assumed \( A \) to be simple. \( B = A \) is impossible since \( B \subset A_{11} + A_{22} + A_{33} \). Thus we have \( B = 0 \) so that \( B_{i} = B_{ii} = 0 \). Hence

\[
A_{1}^{(+)1} = A_{11}^{(+)1} + A_{12}^{(+)1} + A_{22}^{(+)1},
\]

\[
A_{2}^{(+)1} = A_{11}^{(+)1} + A_{13}^{(+)1} + A_{23}^{(+)1}
\]

and \( A_{3}^{(+)1} = A_{22}^{(+)1} + A_{23}^{(+)1} + A_{33}^{(+)1} \)
are Jordan rings. We refer the reader to the proof of Theorem 1 in Albert's paper [6] and note that his combinatorial type proof will suffice to show that \( A^{(+)1} \) is a Jordan ring. We now state the following:

**Theorem 4.3.** If \( A \) is a simple strictly power-associative ring of characteristic prime to six satisfying (28) and possessing an identity element which is the sum of three pairwise orthogonal idempotents, then \( A \) is Jordan-admissible.

Now suppose that \( e \) is a principal idempotent of the strictly power-associative algebra \( A \) which has characteristic prime to six and satisfies (28). Then \( e \) is also a principal idempotent of \( A \) and by [10, Theorem 5] \( A_{1}(1) + A_{e}(0) \subset \text{radical of } A^{(+)1} \). Hereafter we shall denote the radical of \( A^{(+)1} \) by \( \text{Rad } A^{(+)1} \). We shall attempt to show that the ideals generated in \( A \) and \( A^{(+)1} \) by \( A_{1}(1) + A_{e}(0) \) are the same.

**Lemma 4.7.** If \( A_{1}(1) \subset \text{Rad } A^{(+)1} \) and \( z \) and \( w \) are elements of \( A_{1}(1) \), then \( zw \) and \( wz \) are in the \( \text{Rad } A^{(+)1} \). Also \( (zw)_{2} = 2(ex + w)_{2} \).

**Proof.** Substitution of \( x = z, y = w, s = e \) in (28) yields (ex)w + (wx) + (ew)z + (ze)w = 2(ex + w)z + (ze)w or equivalently

\[
(zw) - (ze)w + w(ze) + wz - (we)z + z(we) = 2(we)z + 2(ze)w.
\]

Another substitution of \( x = e, y = w \) yields (ze)w + w(ze) + (sw)e + e(sw) = 2(we)e + 2(esw)z or equivalently

\[
(zw) + wz + (sw)e + e(sw) = 2(wz)e + 2zw - 2(ze)w.
\]

Adding (31) and (32) we obtain

\[
2zw + (zw)e + e(sw) + z(we) = zw + 2(wz)e + 3(we)z.
\]

Set \( zw = a_{2} + a_{1} + a_{0}, wz = b_{2} - a_{1} + b_{0} \). Equating the \( A_{1}(2) \) components of (33) we obtain \( 2b_{2} + a_{2} + b_{2} + (sw)z_{2} = a_{2} + 2b_{2} + 3((we)z)_{2} \). Thus \( b_{2} = -(sw)z_{2} + 3((we)z)_{2} \). Let \( we = w' + w' \). Then \( b_{2} = -a_{2} - 2(\sw z')_{2} + 3b_{2} + 3(w'z)_{2} \) and so \( (a_{2} - b_{2})/2 = 3(w'z)_{2} - (sw')_{2} \). But, if we carry through the same argument with \( w \) replaced by \( w' \), we find \( (w'z)_{2} = (sw')_{2} \). Thus \( (a_{2} - b_{2})/2 = 2(wh)_{2} \). We note that 2(oww) = ((sw)z + (zw)) = (wx)z/2 + (sw)z/2 - (w'z) - (sw') so that \( 2(oww) = (b_{1} + a_{2} + a_{2} - 2(\sw)z_{2} = (b_{2} + a_{2})/2 - (a_{2} - b_{2})/2 = b_{2} \). But \( b_{2} = (sw')_{2} \). By consideration of the \( A_{1}(0) \) components of (33) we obtain \( 2b_{0} + (sw)z_{0} = a_{0} + 3((we)z)_{0} \). Hence \( 2b_{0} - a_{0} = 3b_{0}/2 - a_{0}/2 + 3(ww')z_{0} - (sw')_{0} \) or
\[(b_0-a_0)/2 = 3(w'z) = (sw')_0.\] As before \((w'z)_0 = (sw')_0.\) Then 2\((ew \circ z)_0 = ((ew)z + z(ew)) = (b_0 + a_0)/2 - 2(w'z)_0 = (b_0 + a_0)/2 = a_0 = (sw')_0.\) Then 2\(2(ew \circ z) = (a_0 + b_0)/2 + (a_0 + b_0)/2 - (b_0 - a_0)/2 - (a_2 - b_2)/2 = b_1 + a_0\) which is in the \(\text{Rad} A(+)\). Hence \(b_2\) and \(a_0\) are in \(\text{Rad} A(+)\) and likewise \(a_0\) and \(b_0\) are in \(\text{Rad} A(+)\) so that \(zw\) and \(wz\) are in \(\text{Rad} A(+)\).

**Lemma 4.8.** If \(x\) is in \(A_* (\mu)\) and \(w\) is in \(A_* (1)\) for \(\mu = 0, 2\) and if either \(x\) is in \(\text{Rad} A(+)\) or \(A_* (1) \subseteq \text{Rad} A(+)\), then \(xz\) and \(wz\) are in \(\text{Rad} A(+)\).

**Proof.** Set \(we = w/2 + w'\). Substitution of \(y = w, z = e\) in (28) yields
\[(34) \quad x(we) + w(xe) + (ew)x + (ex)w = 2(xe)w + 2(we)x.\]
Suppose \(x \in A_* (2).\) Then from (34) we have \(x(we) + wx + (ew)x + xw = 2xw + 2(we)x\) or \(xw - wx = x(we) + (ew)x - 2(we)x = x(we) + wx - 3wx/2 - 3w'x\) and hence, \((xw - wx)/2 = xw - 3w'x.\) But by the same argument with \(w\) replaced by \(w'\) we find \((xw - wx)/2 = wx - 3w'x\) and, hence, \(xw = wz\) is in \(\text{Rad} A(+)\). Hence, if \(x, z\) are in \(\text{Rad} A(+)\), then \(xw, wz\) are in \(\text{Rad} A(+)\).

**Lemma 4.9.** The ideal generated in \(A\) by \(A_* (1) + A_* (0)\) is contained in \(\text{Rad} A(+)\) if \(A_* (1) \subseteq \text{Rad} A(+)\).

**Proof.** We shall show that the ideal in \(A\) generated by \(A_* (1) + A_* (0)\) is contained in \(N + A_* (1) + A_* (0)\) where \(N\) is defined as the set of all finite linear combinations of elements of the form \((x_1 x_0)_2\) and \((x_1 x_0)_2.\) By our multiplication we see that it is sufficient to show that \(NA_* (2) + A_* (2)N \subseteq N.\) Set \(L = N + A_* (1) + A_* (0).\) Let \(y, z \in A_* (1)\) and \(x \in A_* (2).\) Then (28) reads \(x(ys) + y(xs) + (xs)x + (ys)y = 2(xs)y + 2(xy)x.\) Thus \(x(ys) + (xy)x - 2(ys)x \in L.\) Interchanging \(y\) and \(z\) we have \(x(zy) + z(zy) + (xy)y + (yx)z = 2(zy)z + 2(zy)x\) so that \(x(zy) + z(zy) = 2(zy)x \in L.\) Adding these two expressions we find \(x(ys + zy) - (ys + zy)x \in L.\) Now in (29) we substitute \(x, y, z = e\) obtaining \(2x(xy + yz) + (xy + yz)x + (xy + yz)z = 2(zy)z + 2(zy)x\) and hence \(x(y + z)x \in L.\) Thus from above \(x(ys + zy) - (ys + zy)x \in L\) and hence \(x(y + z)x \in L.\) Thus \(2x(xy + zy)z \in L\) and by Lemma 4.7, \(x(ys)_2 \in L.\)

Now let \(x \in A_* (2), y \in A_* (1),\) and \(z \in A_* (0).\) Set \(yz = a_2 + a_1, \) \(zy = a_4 + b_1.\) Then substitution in (28) yields \(x(ys) + (zy)x = 2(ys)x) or \(z + 2ax + ax = 2ax.\) Thus \(ax = xa_2.\) Substitution of \(x, y, z = e\) in (29) yields \(2x(ys + zy) + 2(2x(y + z)y) = (xy + yz)z + ((xy + yz)x) + 2(xy)z + ((xy + yz)x)z + ((xy + yz)x)e.\) By the restrictions on \(x, y, z\) and the assumption that \(A_* (1) \subseteq \text{Rad} A(+)\) we find that \(2x(ys + zy) - (ys + zy)x \in L.\) From above \(x(ys + zy) - (yz) + zy)x \in L\) and hence \(x(y + z)x \in L.\) Thus \(2x(xy + zy)z \in L\) and by Lemma 4.7, \(x(ys)_2 \in L.\)
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+ ((xy + yz)e)x. Hence 2x(yz + sy) + 2(yz + sy)x − (yz)x − ((xy + yz)x)e
− (yz + yz)e)x∈L so that 2x(yz + sy) − (yz)x∈L. Thus 2x(a_2 + a_1) − a_2x
= 3a_2x∈L. Since the characteristic of A is prime to six, x=a_2x∈L. Thus
the ideal generated by A_*(1) + A_*(0) is contained in N + A_*(1) + A_*(0). Therefore
this ideal must be equal to N + A_*(1) + A_*(0).

If e is a principal idempotent of A, then e is also a principal idempotent
of A_+ and, as we have stated earlier, A_+(1) + A_+(0) ⊆ Rad A_+ by the
preceding Lemma N + A_*(1) + A_*(0) is an ideal of A which is contained in
Rad A_+ Hence N + A_*(1) + A_*(0) is a nil ideal of A. If we now suppose A
to be semi-simple, then N + A_*(1) + A_*(0) = 0 so that A = A_*(2). Therefore e
is an identity for A and we state this as

THEOREM 4.4. Every semi-simple strictly power-associative algebra of char-
acteristic prime to six satisfying (28) has an identity element.

Suppose D is an ideal of a semi-simple strictly power-associative algebra
A of characteristic prime to six satisfying (28). If D ≠ 0, then since D is not
nil D must possess an idempotent e and we may suppose e to be principal.
Then D = D_1 + D_2 + D_3. Since e∈D, we must have A_2(e) and A_1(e)
contained in D. Hence D_1 = A_2(e) and D_2 = A_1(e) so that we may write
D = A_2(e) + D_1 + D_3. Let M be the radical of D. Then since e is principal,
D_1 + D_3 ⊆ M. In order that M be an ideal of A we see that it is sufficient
to show that A_2(e) + MA_1(e) ⊆ M. Let x∈A_1(e) and m = m_2 + m_1 + m_0 ∈ M.
Then xm = xm_1 + xm_0 where xm_1∈A_2(e) + A_1(e) and xm_0∈D_3 since D
is an ideal of A. Hence, it is sufficient to show that (xm_1)_2 is in A_+(e)
which is commutative. Setting x = y and z = m in (29) for A_+(e)
we find 8x^2 o m_1^2 + 16(x o m_1)^2 = 4(x^2 o m_1) o m_1 + 4(m_1^2 o x) o x + 8((x o m_1) o x)
 o m_1 + 8((x o m_1) o m_1) o x. Thus we have
4(x o m_1)^2 − 2((x o m_1) o m_1) o x
= (x^2 o m_1) o m_1 + (m_1^2 o x) o x + 2((x o m_1) o x) o m_1 + 2x^2 o m_1^2. Remembering
that m_1∈M∩D_1 = D_1(e) and D_2(e) + D_3(e) ⊆ M we obtain (x^2 o m_1) o m_1
∈ M, (m_1^2 o x) o x∈A_2(e)∩D = D_3(e) ⊆ M, 2((x o m_1) o x) o m_1∈D⊂M, and
 2x^2 o m_1^2∈A_2(e)∩D = D_3(e) ⊆ M. Thus the right-hand member of the
above equation is an element of M so that 2(x o m_1)^2 − (x o m_1) o m_1
 o m_1 ∈M. Then the D_2 component of this expression is in M. Set x o m_1 = b_2 + b_1.
Then 2(b_2 + b_1 o b_2 − (b_2 o m_1) o x − (b_1 o m_1) o x∈M and the D_2 component
is 2b_2^2 + 2(b_2 o m_1) o x∈M. Using [6, Identity 8] we obtain
((b_2 o m_1) o x)^2 = (b_2 o m_1) o x = (m_1 o x)^2 / 2 o b_2. But (m_1 o x)^2 / 2 o b_2 = b_2^2 / 2.
Therefore 3b_2^2 − 2(b_2^2) ∈ M. But (b_2^2) ∈ M since b_1∈D_1(e) ⊆ M. Thus since
the characteristic of A is not three, b_2^2 ∈ M. Every element of M is nilpotent
so that b_2 is nilpotent. Now consider the ideal of D generated by M and all
elements of the form b_2 = (xm_1)_2. If we can show that γ((xm_1)_2) is of the form
(xm_1)_2 for all γ∈A_2(e) = D_2(e), then this ideal will be a nil ideal of D containing
the maximal nil ideal M leaving b_2 = (xm_1)_2 ∈ M as the only remaining

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possibility. Substitution of \( x, y, m = z, \) and \( w = e \) in (29) yields 
\[ 2y(xm_1 + mx) + 2(xm + mx)y = 2(ym_1)x + ((ym_1 + m_1y)x)e + ((ym_1 + m_1y)e)x + (m_1y)x + (m_1y)e + ((mx + xm_1)y)e + ((mx + xm_1)e)y. \]

Considering the \( A_e(2) \) components and using \((y(xm_1))_2 = (y(m_1x))_2 = ((xm_1)y)_2 \) we find 
\[ 8(y(xm_1))_2 = 2((ym_1)x)_2 + 2((ym_1 + m_1y)x)_2 + 2((ym_1 + m_1y)e)x)_2 + 2((m_1y)x)_2 + (y(xm_1))_2 + 2(y(m_1x))_2. \]
Simplifying we obtain 
\[ 3y(xm_1)_2 = ((3ym_1 + 2m_1y + (ym_1 + m_1y)e)x)_2 \]
and we observe that 
\[ (3ym_1 + 2m_1y + (ym_1 + m_1y)e)_1 \in M \cap A_e(1). \]

Thus, since the characteristic of \( A \) is not three, \( y(xm_1)_2 \) is of the desired form and, hence, is nilpotent. Therefore \( M \) is a nil ideal of \( A \) which we had assumed to be semi-simple; therefore \( M = A_e(1) = D_e(0) = 0. \) Thus \( A = A_e(2) \oplus A_e(0) \) where \( A_e(2) \) and \( A_e(0) \) are semi-simple algebras with identity elements \( e \) and \( 1 - e \) respectively. Proceeding in the usual manner we may now state

**Theorem 4.5.** Every semi-simple strictly power-associative algebra \( A \) over a field \( F \) of characteristic prime to six and satisfying (28) has an identity and is the direct sum of simple algebras.

In the following we shall suppose \( A \) to be a strictly power-associative algebra over a field \( F \) of characteristic prime to six such that \( A \) is simple and satisfies (28). Moreover, we assume that \( A \) has an identity \( 1 \) which can be written as \( 1 = e_1 + e_2 \) where the \( e_i \) are pairwise orthogonal idempotents of \( A \) and that \( A \) is Jordan-admissible. \( A \) is simple over its center and simple over the algebraic closure \( K \) of its center. Thus \( A_K \) is a simple algebra which is Jordan-admissible and we may suppose \( F = K. \) Then we can set \( 1 = u + v \) where \( u \) is a primitive idempotent of \( A. \) Since \( A_e(2) \) is a Jordan algebra, we may use [8, Theorem A] to write \( A_u(2) = uK + N \) where \( N \) is the ideal of nilpotent elements of \( A_u(2). \) Suppose \( N \) is not an ideal of \( A_u(2). \) Then there are elements \( x, y \in N \) such that \( xy = u + n \) and \( yx = -u + n' \) where \( n, n' \in N. \)

Suppose \( w \in A_u(1). \) Then substitution of \( x = w, z = x \) in (28) yields 
\[ (xy)w + (zw)y + y(wx) + w(yx) = 2(yx)w + 2(wx)y. \]

Then \( uw + nw + (xw)y + y(wx) - uw + wn' = 2uw + 2n'w + 2(wx)y. \) Rearranging terms we find 
\[ 3uw - wu = 2n'w - nw - wn' + 2(wx)y + y(wx) - (xw)y. \]
Since \( A^{(4)} \) is a Jordan algebra, it is stable [2] and by our earlier remark, \( A \) is stable. Then applying Lemma 4.8 to each term of the right-hand member of the above relation we find that the right-hand member is in \( \text{Rad} A^{(4)} \) and, consequently, the left-hand member is also in \( \text{Rad} A^{(4)}. \)

Now we set \( uw = w'/2 + w' \) where \( w'u = uw' = w'/2. \) Since \( wU \) was an arbitrary element of \( A_u(1), \) \( 3uw - w'u = w' \in \text{Rad} A^{(4)}. \) But then \( 3uw - wu = 3w'/2 + 3w' - w/2 + w' \) so that \( w \in \text{Rad} A^{(4)}. \) Thus \( A_u(1) \subseteq \text{Rad} A^{(4)} \) and an application of Lemma 4.7 yields \( A_u(1) \oplus A_u(1)A_u(1) \subseteq \text{Rad} A^{(4)}. \) We claim \( A_u(1) + A_u(1)A_u(1) \) is, in fact, an ideal of \( A. \) Let \( x \in A_u(2) \) and \( y, z \in A_u(1). \) Substitution of \( y = e \) in (29) yields 
\[ 2x((wz + zw)x + w((x + xz)) + (zx + xz)w + z(wx + wx) + (xw + wx)z = 2(wx)z \]
+ 2(xz)w + ((xz + zx)e)x + ((xz + zx)w)e + ((zw + wz)e)x + ((zw + wz)x)e + ((zw + wz)x)z + ((zw + wz)x)z + 2(w/z)x + (zw)x + (zx)w.

Using the stability of A we find

x(wz) + (zw)x — 2(wz)x \in \text{Rad}(A(1))A(1).

Adding these two we have x(wz + wz) — (wz + wz)x \in \text{Rad}(A(1))A(1). Combining this with the above remark that 2x(wz + wz) — (wz + wz)x \in \text{Rad}(A(1))A(1), we find x(wz + wz) \in \text{Rad}(A(1))A(1). Replacing z by uz and applying Lemma 4.7, we see that x(zw) = x(uz \circ w) \in \text{Rad}(A(1))A(1). This result along with the stability of A makes \text{Rad}(A(1)) + \text{Rad}(A(1))A(1) closed under multiplication by A(2). We note that A(2) = A(0) and A(0) = A(2). Interchanging the roles of u and v, we find that A(1) + A(1)A(1)A(1) is closed under multiplication by A(0). Noting the stability of A it is clear that A(1) + A(1)A(1)A(1) is closed under multiplication by A(1). Hence, A(1) + A(1)A(1)A(1) is an ideal of A which is contained in \text{Rad}(A^{+}) and, since the simple algebra A contains an idempotent, we must have A(1) = 0. But then A = A(2) + A(1) which is impossible by the simplicity of A. Thus N is an ideal of A(2).

Now suppose 1 = e_1 + \cdots + e_i where the e_i are primitive pairwise orthogonal idempotents of A. Then for an arbitrary element x of A we may write

x = \sum k_i e_i + \sum x_{ij} + \sum x_i' \text{ where } k_i \in K, x_{ij} \in A_{ij} \text{ for } i \neq j, \text{ and } x_i' \in N_i \text{ where } N_i \text{ is the ideal of nilpotent elements of } A_i(2).

Setting \delta(x) = \sum k_i e_i defines a linear function on A to K. We show that the following conditions are satisfied by \delta:

1. \delta(xy) = \delta(yx);
2. \delta(x(yz)) = \delta((xy)z).

We shall first show that (1) is satisfied by \delta. Let x = \sum k_i e_i + \sum x_{ij} + \sum x_i' \text{ and } y = \sum k_i' e_i + \sum y_{ij} + \sum y_i' \text{. Since } N_i \text{ is an ideal of } A_i(2), x_i y_{ij} \in N_i. By the stability of A, for \text{ any } j, k \text{ arbitrary, we have } x_{ij} y_{ij} \in A_{ij}. Also } x_{ij} y_{ij} \in A_i(2) \text{ for } i, j, n \text{ distinct. Hence, } \delta(xy) - \delta(yx) = \delta(\sum x_{ij} y_{ij} - \sum y_{ij} x_{ij}), \text{ so that by the linear property of } \delta \text{ it is sufficient to show that } \delta(x_i y_{ij}) = \delta(y_i x_{ij}).

Set x_{ij} y_{ij} = x_{ij} y_{ij} = x_{ij} = a_1 + a_0 \text{ and } y_{ij} x_{ij} = y_{ij} x_{ij} = y_{ij} = b_1 - a_1 + b_0 \text{ where } a_1, b_1 \in A_{ij} \text{ and } a_0, b_0 \in A_{ij}. \text{ Using (28) we find } (xy) e_i + e_i (xy) + (x e_i) y + y (x e_i) = 2(xy) e_i + 2(x e_i) y. \text{ Set } x e_i = x e_i + 2(y) e_i \text{ where } e_i e_i = g_i = g_i = g_i / 2. \text{ Then we have from above } (xy) e_i + e_i (xy) + x e_i = 2(xy) e_i + 2(x e_i) y. \text{ Considering only the } A_{ii} \text{ and } A_{ij} \text{ components we obtain } a_1 + b_2 + a_2 / 2 + a_0 / 2 + g_2 + b_2 / 2 + b_0 / 2 - 2g_2 = 2b_2 + a_2 + a_2 - 2g_2 \text{ or } a_2 = b_2 / 2 + (b_0 - a_0) / 2 = (y g_2) + (y g_0) - 3(g y_2) - 3(g y_2) = -2(g y_2) - 2(g y_2). \text{ Thus } b_2 - a_0 / 2 = 2(y g_2) \text{ and } (a_0 - b_0) / 2 = 2(g y_2). \text{ Now } 2(y e_i) y = (y e_i) y + y (e_i y) = y (g y_2) + (g y_0). \text{ But by [6, Lemma 10], } 2(y e_i) y = y (e_i y) = y (g y_2) + (g y_0). \text{ Then, from above, we see that } \delta(b_2 - a_0) = \delta(b_2 - b_0) \text{ or } \delta(a_2) + \delta(a_0) = \delta(b_2) + \delta(b_0). \text{ Again by [6, Lemma 10] we obtain } x y + x y = k(e_i e_i) + n_i + n_i \text{ so that } \delta((g y_2) e_i = k e_i \text{ then } \delta((g y_2) e_i = \delta((y g_2) e_i). \text{ Finally, } \delta(xy) = \delta(a_2) + \delta(a_0) = \delta(b_2) + \delta(b_0) = \delta(xy).
We next consider condition (2). We must show that $\delta(x(yz)) = \delta((xy)z)$. We restate (28) as $x(yz) + y(xz) + (z; y)x + (zx)y = 2(xz)y + 2(ys)x$. Using $\delta(xy) = \delta(yx)$ we obtain $\delta(x(yz)) + \delta(y(xz)) = \delta((yz)x) - \delta((zx)y) = 0$. Hence $\delta(y(xz)) - \delta((yz)x) = \delta((zx)y) - \delta(x(yz))$ or $\delta[y(xz) - (yz)x] = \delta[(xz)y - x(yz)]$. Then

$$4\delta[x o (y o z) - (x o y) o z]$$

$$= \delta[x(yz) + y(xz) + (z; y)x + (zx)y - z(xy) - z(xy) - (yz)x]$$

$$= \delta[x(yz) - (xy)z] + \delta[x(yz) - (xz)y] + \delta[(yz)x - y(zx)] + \delta[(zx)y - (xy)z]$$

(by use of $\delta(xy) = \delta(yx)$ and $\delta(y(xz)) = \delta((xz)y - x(yz))$). Thus it is sufficient to show that (2) holds in $A^{(+)}$. Let the coefficient of $e_i$ in the decomposition of $x, y, z$ be respectively $\alpha_i, \beta_i, \gamma_i$. Then

$$2\delta[z o (x o y)] = 2(\sum \alpha_i \beta_j \gamma_k) + 2\delta[\sum (\gamma_i e_i + \gamma_j e_j) o (x_i o y_j)]$$

$$+ \delta[\sum (\alpha_i + \alpha_j) x_{ij} o y_{ij}] + \delta[\sum (\beta_i + \beta_j) z_{ij} o x_{ij}]$$

$$+ 2\delta[\sum z_{ik} o (x_i o y_i)]$$

From the proof of (1) we see that $\delta[e_i o (x_i o y_i)] = \delta(x_i o y_i)/2 = \delta[e_i o (x_i o y_i)]$. Hence, if we consider the above expression with $x$ and $z$ interchanged, we observe that it will suffice to show that $\delta[z_{ik} o (x_i o y_{jk})] = \delta[x_{ij} o (z_{ik} o y_{jk})]$ for $i, j, k$ distinct.

We now state the Jordan identity for $A^{(+)}$ in its linearized form.

$$\sum (x o y) o (w o z) = \sum ((x o y) o w) o z \quad \text{(symmetric in } x, y, z).$$

Set $x = x_{ij}$, $y = y_{jk}$, $w = z_{ik}$, and $z = e_i$. Then we find $(x_i o y_{jk}) o z_{ik} + (z_{ik} o y_{jk}) o x_{ij} = 2((x_i o y_{jk}) o z_{ik}) o e_i + (x_{ij} o z_{ik}) o y_{jk}$. Then interchanging $x$ and $z$, and $i$ and $k$ we have $(z_{ik} o y_{jk}) o x_{ij} = 2((z_{ik} o y_{jk}) o x_{ij}) o e_i + (z_{ik} o x_{ij}) o y_{jk}$. Subtracting we have $((x_i o y_{jk}) o z_{ik}) o e_i = ((z_{ik} o y_{jk}) o x_{ij}) o e_i$. Then, using the above remarks we obtain $\delta[(x_i o y_{jk}) o z_{ik}] o e_i] = \delta[(y_{jk} o z_{ik}) o e_i] - \delta[x_{ij} o (y_{jk} o z_{ik})]/2 = \delta[(x_{ij} o y_{jk}) o z_{ik}] / 2$. Therefore (2) holds. By (1) and (2) the set $N_i$ of all $x \in A$ such that $\delta(xy) = 0$ for all $y \in A$ is an ideal of $A$. Surely $A^{(+)} \subseteq N_i$ and, since $A$ is simple and $\delta(e_i) \neq 0$, $N_i = \text{Rad } A^{(+)} = 0$. Thus $A^{(+)}$ is a simple Jordan algebra [3, Chapter V, Theorem 8](4).

Theorem 4.6. If $A$ is a simple strictly power-associative algebra over field $K$ of characteristic $\neq 2$, 3 which satisfies (28) and such that $A^{(+)}$ is a Jordan algebra of degree $t > 1$, then $A^{(+)}$ is a simple Jordan algebra.

We now reproduce Albert’s argument [4] to show that under the hypotheses of the above theorem $A$ is flexible. Set $x = z$ in (2). Then $w = (xy + yx)x$.

(4) Albert’s result is for flexible algebras but holds equally well for this case since the $A_{ij}$ are orthogonal subalgebras of $A$. 

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— x(xy + yx) + x^2y - yx^2 = 0 and so wz = zw = 0 for all z ∈ A. Interchanging y and z and adding we obtain ((xy)x)z + (yx)x)z - (x(xy))z - (x(yx))z + (x^2y)z - (yx^2)z + ((xz)x)y + (zx)x)y - (x(xz))y + (zx)z)y - (xz^2)y = 0. Then applying δ and using properties (1) and (2) repeatedly we find 2δ[(xy)(xz)] = 2δ[(zx)(xy)]. Applying (2) to the left-hand member and (2) and (1) to the right-hand member we find δ[((xy)x)z] = δ[(x(xy))z] so that δ[((xy)x - x(yx))z] = 0 for all x, y, z ∈ A. Hence (xy)x - x(yx) ∈ Nδ and since A is simple we are finished.

**Theorem 4.7.** A simple strictly power-associative algebra A over a field K of characteristic ≠ 2, 3 which satisfies (28) is

(a) a commutative Jordan algebra;
(b) a quasi-associative algebra;
(c) an algebra of degree 2; or
(d) an algebra of degree 1.

**Proof.** The proof is immediate from the preceding remark, Theorems 4.3 and 4.6 and [13, Theorem 4.2].

5. The identities \( R_2^2 + L_2^2 = L_z R_z \) and \((R_z + L_z)(R_y + L_y)\) = \((R_x + L_x)(R_2 + L_2)\). We shall present in this section some examples of algebras satisfying (9) and (11). First consider the algebra A over a field F of characteristic zero with a basis \( e_1, a, \) and \( e_2 \) where the \( e_i \) are orthogonal idempotents such that \( e_1 + e_2 = 1, e_2 = ae_1 = 1 + a/2, e_2e_1 = e_1e_2 = -1 + a/2, \) and all the remaining products are zero. In order to show that A is power-associative we must show \( xx^2 = x^2x \) and \( (x^2x)x = (x^2x)x \) for all \( x ∈ A \). Set \( x = a_1e_1 + a_2a + a_2e_2 \). Then \( x^2 = a_1^2e_1 + a_2(a_1 + a_2)a + a_2e_2 \) and \( x^4 = a_1^4e_1 + a_2(a_1^2 + a_2^2)a + a_2e_2 \). We see that \( (x^2)^2 = a_1^4e_1 + a_2(a_1 + a_2)a + a_2e_2 \) and \( (x^2) = \alpha_1e_1 + \alpha_2(a_1 + a_2)a + a_2e_2 \). Thus \( \alpha_1e_1 + \alpha_2(a_1 + a_2)a + a_2e_2 \) = \( (x^2)^2 \). This along with [1, Lemma 4] implies that A is power-associative. We set \( x = a_1e_1 + a_2e_2 \) and \( y = \beta_1e_1 + \beta_2e_2 \). Then \( xy = (\alpha_1\beta_1 - \alpha_1\beta_1 - \alpha_1\beta - \alpha_2\beta + \beta_2)e_1 + (\alpha_1\beta_1 + \alpha_2\beta + \alpha_2\beta + \beta_2)e_2 \) and \( yx = (\alpha_1\beta_1 - \alpha_1\beta_1 - \alpha_2\beta + \alpha_2\beta)\). Hence \( xy - yx = 2(\alpha_1\beta_1 - \alpha_1\beta - \alpha_2\beta)e_1 + 2(\alpha_1\beta - \alpha_1\beta_1 + \alpha_2\beta - \alpha_2\beta)\). Thus \( xy - yx \) commutes with all \( z ∈ A \) so that \( z(xy - yx) = (xy - yx)z \) and (11) holds.

Suppose L is an ideal of A and \( x = a_1e_1 + a_2e_2 \). Then \( e_2x + xe_1 = 2a_1e_1 + a_2e_2 \). Thus \( 2a_1e_1 + a_2e_2 = 0 \). If \( a_1 \) or \( a_2 \) is not zero then by the orthogonality of the \( e_i \), either \( e_1 \) or \( e_2 \) is zero. But then \( e_2a + a_2e_1 = \frac{a}{2} \) implying that \( e_2a - \frac{a}{2} = 1 \). Thus \( L = A \). Suppose \( a_1 = a_2 = 0 \). Then \( \alpha \neq 0 \) so that \( \alpha \neq 0 \). Therefore A is a simple power-associative algebra satisfying (11) which is not flexible since \( e_1(ae_1) = -e_1/2 + a/4 + e_2/2 \) and \( (e_2a)e_1 = e_1/2 + a/4 - e_2/2 \).

We may construct new examples by setting A equal to the algebra over
a field $F$ of characteristic zero with a basis $e_1, \ldots, e_n$, $a_{ij}, i < j = 1, \ldots, n$, where $1 = e_1 + \cdots + e_n$, the $e_i$ are pairwise orthogonal idempotents, $e_i a_{ij} = a_{ij} e_i = 1 + a_{ij}/2$, $e_j a_{ij} = a_{ij} e_i = -1 + a_{ij}/2$, and all other products are zero. We note that the nonflexible examples given earlier which satisfy (5) also satisfy (11). This is not too surprising when we observe that any algebra $A$ which satisfies (5) and either (9) or (11) must satisfy all three.

Identities (9) and (11) are not strong enough conditions in themselves to enable us to obtain any significant results concerning algebras which satisfy either of them. It is not evident to us at this time what other conditions we might impose on these algebras.

**Bibliography**


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