A SUPERLINEAR STURM-LIOUVILLE PROBLEM(1)

BY

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I. This paper is intended as a companion work to a previous paper entitled *A sublinear Sturm-Liouville problem* [7]. The implication is clear: Whereas the previous paper was a study of a nonlinear boundary value problem involving a concave or subadditive function [3, p. 116], here we wish to complete the picture by studying the same sort of problem but with the opposite type of nonlinearity. It is interesting to see what becomes of the classical notions of point spectrum, eigenspace, etc., when one goes from the linear to the superlinear case.

Let us consider the following(2) problem for $0 \leqq x \leqq 1$:

$$(1) \quad \begin{aligned} &\text{D.E.} \quad (p(x)u')' + \lambda^{-1} f(x, u) = 0; \\ &\text{B.C.} \quad u(0) - ap(0)u'(0) = 0, \quad u(1) + bp(1)u'(1) = 0, \quad \lambda \text{ real}, \ a, b \geqq 0. \end{aligned}$$

We suppose that $p(x) > 0$ and $p'(x)$ are continuous, $0 \leqq x \leqq 1$, and that $f(x, u)$ is continuous in the strip $0 \leqq x \leqq 1$, $-\infty < u < +\infty$. In addition, $f(x, u)$ is assumed to conform to the following statement:

H-1: For fixed $x \in [0, 1]$, $f(x, u)$ is odd and strictly monotone in $u \in (-\infty, \infty)$. Derivatives $f_u > 0$ and f_{uu} exist with $f_{uu} \leqq 0$ for $u < 0$, $f_{uu} \geqq 0$ for $u > 0$. $\lim_{u \to 0} u^{-1} f(x, u) = f_u(x, 0) > 0$, this limit being uniform, $0 \leqq x \leqq 1$. The continuity of $f(x, u)$ in u is uniform for $x \in [0, 1]$.

We treat three subcases which go with the additional hypotheses:

H-2: $f(x, u) = 0(|u|)$ as $|u| \to \infty$ uniformly on $[0, 1]$.

H-3: $f(x, u) = 0(|u|^\alpha)$ as $|u| \to \infty$, $\alpha > 1$, uniformly on $[0, 1]$. (Power law behavior.)

H-4: Stronger nonlinear behavior, $f(x, u) = 0(R(|u|))$ as $|u| \to \infty$, uniformly on $[0, 1]$, where $R(u)$ is monotone increasing, non-negative, and continuous.

H-2 we shall always refer to as the asymptotically linear case; H-3 might be called the asymptotic power; let us call H-4 the "strongly superlinear" case.

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(2) We use the following abbreviations: D.E. for Differential Equation, B.C. for Boundary Condition, and I.C. for Initial Condition.

When dealing with case H-2, we can take advantage of a uniform Lipschitz condition. No such advantage pertains to cases H-3 or H-4. Hence some proofs are more involved than would be necessary were we to deal only with the asymptotically linear case.

The sublinear and superlinear problems of these two papers represent two extreme ideal situations. Problems usually are mixed. For example, $f(x, u)$ need not be odd in u as assumed in H-1; in this case, f could be superlinear on the right and sublinear on the left. This, together with problems involving functions f with fewer discernable properties, serve to illustrate the infinite variety present in nonlinear analysis as opposed to linear analysis. The sublinear and superlinear problems deserve attention theoretically for their characteristic behavior. Mixed cases can be dealt with when specific situations warrant the effort.

Problem (1) has an unbounded but closed operator with domain dense in those Banach spaces which it will be necessary to use, and a superlinear operator. The former is known to possess a completely continuous inverse. Therefore, problem (1) has the dual representation as an integral equation:

$$\lambda u(x) = \int_0^1 G(x, x')f(x', u(x'))dx' \tag{2}$$

where $G(x, x')$ is the continuous positive symmetric Green's function, and we assume the operations make sense, i.e., the spaces are chosen as in the next section.

The operator $\int_0^1 G(x, x')f(x', \cdot)dx'$ defined on an appropriate linear space is known as "Hammerstein operator." The results of this paper can be regarded as a set of results about the spectra of Hammerstein operators. An interesting question is that of the validity of these results for kernels G in (2) which are more general Green's functions or which are not Green's functions at all.

The author is again indebted to Professor I. I. Kolodner whose general approach [5] seems herein to be applicable to superlinear problems as well as to sublinear problems.

II. For case H-2, i.e., the asymptotically linear case, we have the uniform Lipschitz condition ($f_u > 0$ is bounded uniformly in $x \in [0, 1]$). Hence, Equation (2) is well-defined in *real* $L_2(0, 1)$ (the norm for L_p is designated by $\|\cdot\|_p$):

$$\left\| \int_0^1 G(x, x')f(x, u(x'))dx' \right\|_2 \leqq \text{const} \|f(x, u(x))\|_2 \leqq \text{const} \|u\|_2.$$

This is not the case if H-3 holds, however. For example, under H-3 with $\alpha \geqq 3$, the operator in (2) may not even be defined for the element $x^{-1/3} \in L_2(0, 1)$.

Under H-3, we have $|f(x, u)| \leqq c+d|u|^{\alpha}$ uniformly in $x \in [0, 1]$ with $\alpha > 1$. $c, d > 0$ are constants. Let $p-1$ be the first integer with $p-1 \geqq \alpha$. Then Equation (2) is well-defined on *real* $L_p(0, 1)$:

$$\left\| \int_0^1 G(x, x')f(x', u(x'))dx' \right\|_p \leqq \text{const } \|f(x, u(x))\|_q \leqq \text{const } \|c + d|u|^{\alpha}\|_q$$
$$\leqq \text{const}\{c + d(\|u\|_p)^{p-1}\}$$

where $(1/p)+(1/q)=1$, and we have used the Hölder inequality. Hence in H-3, *some* space L_p can be found in which the operator is defined. For H-4, however, we have $\lim_{u\to\infty} u^{-k}f(x, u) = \infty$ for every k. Clearly there is no integer p for which L_p is a suitable space.

In case H-4, we find that we must have recourse to the idea of Orlicz space [10, pp. 78–85]. M. A. Krasnoselskiĭ gives very general conditions under which one may find an Orlicz space in which a given problem is defined. Generally, the correct space $L_{\Phi}^{*}(0, 1)$ has for Φ a function more strongly nonlinear than f [6, pp. 355–360].

The requisite space, whether $L_2(0, 1)$, $L_p(0, 1)$, or $L_{\Phi}^{*}(0, 1)$ is a Banach space of functions defined on the x interval $[0, 1]$, and it will be convenient to denote it where necessary by $B(0, 1)$.

Any discussion of nonlinear problems involves various linearized problems. Here we introduce the linearized form of (2) "at the origin":

$$\mu l_0(x) = \int_0^1 G(x, x')f_u(x', 0)l_0(x')dx' \tag{3}$$

and the linearization of (2) "at infinity":

$$\gamma l_{\infty}(x) = \int_0^1 G(x, x')A(x')l_{\infty}(x')dx'. \tag{4}$$

Here, $A(x) = \lim_{u\to\infty} u^{-1}f(x, u)$. The latter exists only for H-2, and the operator is completely continuous in $L_2(0, 1)$. M. A. Krasnoselskiĭ gives conditions [6, p. 358] for the operator in (3) to be completely continuous in Orlicz space, which are certainly satisfied here. Hence, these linear bounded operators have pure discrete point spectra which accumulate only at the origin of the spectral line. Since the kernels are symmetrizable and under H-2 have continuous eigenfunctions in $L_2(0, 1)$, the spectrum is nonvoid under H-3 and H-4 where the underlying space is $B(0, 1)$. Let $\{\mu_n\}$ be the eigenvalue sequence of (3) with $\mu_n > \mu_{n+1} > 0$, $\lim_{n\to\infty} \mu_n = 0$, and $\{\gamma_n\}$ be the eigenvalue sequence for (4) with $\gamma_n > \gamma_{n+1} > 0$, $\lim_{n\to\infty} \gamma_n = 0$. The eigenvalues are positive, since the kernels in (3) and (4) are positive definite. For H-2 where (4) exists in $L_2(0, 1)$, we expect that $\gamma_n \geqq \mu_n$, $n = 1, 2, \cdots$ since for a superlinear problem, we clearly have $A(x) \geqq f_u(x, 0)$.

Let us now dispose of some preliminaries. In case H-2, linearized problem (4) makes sense in $L_2(0, 1)$ and the eigenvalue sequence $\{\gamma_n\}$ exists. Suppose $\lambda > \gamma_1$ where γ_1 is the highest eigenvalue. Then from (2), we have

$$|u| \leqq \lambda^{-1} \int_0^1 G(x, x') \, |f(x', u(x'))| \, dx' \leqq \lambda^{-1} \int_0^1 G(x, x') A(x') \, |u| \, dx'$$

since by H-1, H-2, $f(x, u)$ has maximum slope where $|u|$ is large. By putting $\bar{u} = A(x)^{1/2} u$ so as to symmetrize the operator, we get $\|\bar{u}\|_2 \leqq \lambda^{-1}\gamma_1 \|\bar{u}\|_2$ where $\|\bar{u}\|_2$ is the L_2 norm of $\bar{u}$. If $\lambda > \gamma_1$, then clearly $\|\bar{u}\|_2 = 0$ so that $u = 0$ almost everywhere. Since under H-3 or H-4, problem (4) is not defined, we cannot say under these circumstances that there exists only the trivial solution for any $\lambda > 0$. *Only under* H-2 *is the spectrum bounded.*

Now we study the range $\lambda < 0$. As described in H-1, $f(x, u)$ has the property that $f > 0$ when $u > 0$, $f < 0$ when $u < 0$. Hence the function $uf(x, u)$ is always positive for $u \neq 0$. Then if $\lambda < 0$ is an eigenvalue, i.e., if (2) has a nontrivial solution,

$$\int_0^1 \int_0^1 G(x, x') f(x', u(x')) f(x, u(x)) dx' dx = \lambda \int_0^1 u(x) f(x, u) dx$$

where

$$\int_0^1 uf(x, u) dx > 0 \quad \text{or} \quad u \sim 0.$$

Also $G(x, x') \geqq 0$ and is continuous on $0 \leqq x \leqq 1$, $0 \leqq x' \leqq 1$. By Mercer's theorem [8, p. 245], there exists a sequence $\{\phi_n(x)\}$ of functions continuous on $0 \leqq x \leqq 1$ and a sequence of numbers $\alpha_n > 0$ such that

$$G(x, x') = \sum_{n=0}^{\infty} \alpha_n \phi_n(x) \phi_n(x'),$$

the convergence being uniform. Hence,

$$\lambda \int_0^1 uf(x, u) dx = \sum_{n=0}^{\infty} \alpha_n \left| \int_0^1 \phi_n(x) f(x, u(x)) dx \right|^2 \geqq 0$$

which shows that if $\lambda < 0$, we have $u = 0$ almost everywhere. This argument is valid under either H-2, H-3, or H-4.

We are enabled to state a preliminary result:

THEOREM 1. *Under statements H-1 and either H-2, H-3, or H-4, problems (1) and (2) have only the trivial solution if $\lambda < 0$. Under H-1 and H-2, problems (1) and (2) have only the trivial solution if $\lambda > \gamma_1$ where γ_1 is the highest eigenvalue for (4).*

Those parts of the spectral line where $\lambda < 0$ and $\lambda > \gamma_1$ (the latter under

H-2) are included in the resolvent set of the operator in (2) in the sense that the "inhomogeneous case"

$$\lambda u(x) - \int_0^1 G(x, x')f(x', u(x'))dx' = g(x), \qquad g \in B(0, 1), \tag{5}$$

can be uniquely solved there. Under H-2, this follows for $\lambda > \gamma_1$ by direct iteration on (5) and for $\lambda < 0$ by the work of Hammerstein [9, pp. 202–213]. Under H-3 or H-4, uniqueness can be had for $\lambda < 0$ by modifying Hammerstein's argument as was done above in proving Theorem 1. For existence, we can modify Hammerstein's L_2 variational argument using ideas outlined by M. A. Krasnoselskiĭ [6, pp. 363, 364, 381, 382]. (The theorem in this reference at the top of p. 364 is directly applicable here. If $\lambda < 0$ the functional on p. 382, is, under H-1, of the type desired in order to assure existence of a critical point.)

III. Let us now study the spectral range $\lambda \geqq 0$ in considerably more detail. The chief tool is the initial value problem for ordinary differential equations.

Let us extend the functions $p(x), f(x, u)$ into the range $1 \leqq x < \infty$ so as to preserve properties H-1 and either H-2, H-3, or H-4 as the case may be, and also so that

$$\lim_{x \to \infty} f\left(x, q\int_0^x \frac{ds}{p(s)}\right) \neq 0 \tag{6a}$$

and

$$\lim_{x \to \infty} \frac{1}{p(x)} \int_0^x f(s, q)ds \neq 0 \tag{6b}$$

for every number $q \neq 0$. The extension is achieved in many ways; for example, periodic extension will suffice.

Consider now the problem:

$$\begin{aligned} &\text{D.E.} \quad (p(x)v')' + \lambda^{-1}f(x, v) = 0; \\ &\text{I.C.} \quad v(0, c) = ac, \\ &\qquad\quad v'(0, c) = c/p(0), \end{aligned} \tag{7}$$

where $c > 0$ is a parameter. This problem also can be put in integral equation form:

$$v(x, c) = c\phi(x) - \lambda^{-1} \int_0^x \int_s^x \frac{dt}{p(t)} f(s, v(s, c))ds, \tag{8}$$

where

$$\phi(x) = a + \int_0^x \frac{ds}{p(s)}\cdot$$

Under statement H-2, considerations of existence, uniqueness, continuously differentiable dependence of $v(x, c)$ on c and $\lambda > 0$, and the oscillatory nature of the solution in a phase plane are handled precisely as in the sublinear case [7, §2]. It is the uniform Lipschitz condition that makes this possible. Under H-3 or H-4, however, our approach is considerably more guarded.

In fact, let us define $p(x)v' = w$ and consider the direction field

$$(v', w') = (w/p(x), -\lambda^{-1}f(x, v)) \tag{9}$$

in the v-w phase plane. By standard existence theorems, there exists a unique solution "in the small" which gives a trajectory beginning at (ac, c) of quadrant I (Figure 1) and directed down and to the right in accordance with (9).

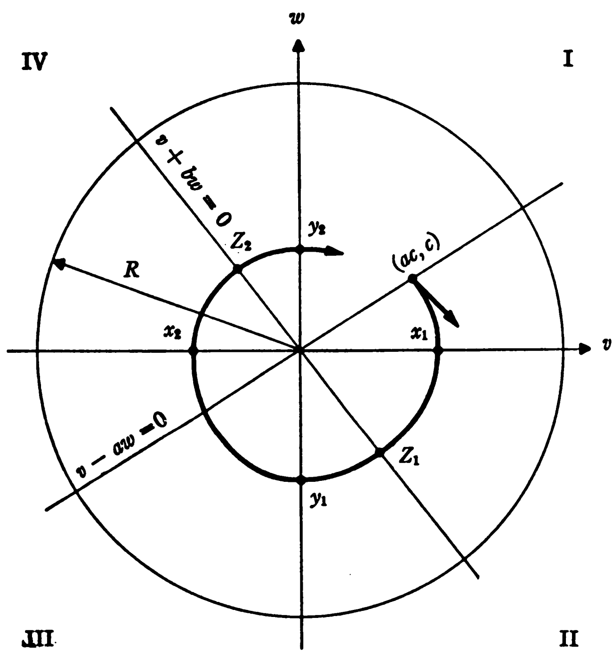


FIGURE 1

The points of the trajectory have cartesian coordinates $(v(x, c), w(x, c))$. Now draw a large circle (radius R) about the origin so as to include (ac, c) in the interior.

By a known theorem [2, p. 33, Theorem 6], the trajectory can be continued in quadrant I until either (i) it intersects the positive v axis or (ii) it leaves the interior of the circle of radius R. If the latter obtains, enlarge the circle to be of radius $2R$ and consider alternatives (i) and (ii) again. Repeating this process enough times, we see that we either continue the trajectory until it intersects the v axis or until it approaches a line $w=q>0$ in quadrant I. This latter eventuality contradicts (6a), however, so that the v-axis is eventually intersected. By (9), this intersection is orthogonal. Let the x-value of the intersection be x_1 with $(n-1)R \leqq v(x_1, c) < nR$.

Hence the trajectory enters quadrant II. Here, by (9), it is directed down and to the left. Draw a circle about the origin of radius $R_1 = nR$. By a known theorem [2, p. 33, Theorem 6], the trajectory may be continued in quadrant II until either (i) the curve intersects the negative w-axis or (ii) it leaves the interior of the circle of radius R_1. In the latter case, enlarge the circle to radius $2R_1$ and repeat consideration of (i) and (ii), and so on. Eventually the w-axis is intersected or the trajectory approaches a line $v=q>0$ in quadrant II. The latter possibility contradicts (6b), however, so the w-axis is indeed intersected, say at $x=y_1$, with $(n_1-1)R_1 < w(y_1, c) < n_1 R_1$. By (9), the intersection is orthogonal.

Hence the trajectory enters quadrant III. Put $R_2 = n_1 R_1$, draw a circle of radius R_2, and consider the continuation in quadrant III where it is directed up and to the left. Entirely analogous considerations apply for this and successive quadrants. It is finally seen that the integral curve continually winds about the origin of the phase plane. This process gives rise to an increasing sequence of positive numbers $x_1, y_1, x_2, y_2, \cdots$ which represent the x-values at which the trajectory intersects the axes of the phase plane.

This solution is unique because the continuation of the integral curve in each circumstance has been unique. Let $[0, X]$ be an arbitrary interval of the independent variable x over which the solution is defined. Then the solution $v(x, c)$, $w(x, c)$ depends continuously and differentiably on the parameters, c, $\lambda>0$ uniformly on $[0, X]$. This can be seen by several known theorems [1, Chapter 1, Theorems 7.1–7.5] using as domain D the interior of the circle about the origin of the phase plane of radius $2R_X$ where

$$R_X = \max_{0 \leqq x \leqq X} (v^2(x, c) + w^2(x, c))^{1/2}.$$

Because of the continually spiralling trajectory which corresponds to the solution $v(x, c)$, $w(x, c)$ which starts at the point (ac, c) on the line $v - aw = 0$ of the phase plane (which represents the left end condition of (1)), we see that there exists a discrete number of intersections with the line $v + bw = 0$ representing the right end condition of (1). Let us use the notation $x = Z_\nu(c, \lambda)$, $\nu = 1, 2, \cdots$ for the values of x corresponding to the intersections.

Similar though much less involved methods show that the linearized problem

$$(10) \qquad \begin{aligned} &\text{D.E.} \quad (p(x)h')' + \lambda^{-1} f_v(x, v)h = 0; \\ &\text{I.C.} \quad h(0, c) = a, \qquad f_v(x, v) > 0, \\ &\qquad\quad h'(0, c) = 1/p(0), \end{aligned}$$

also has an oscillatory solution $h(x, c)$, $k(x, c) = ph'$ represented by a spiralling trajectory in an h, k phase plane. Here, $h(x, c)$ depends on $c > 0$ through v. There are discretely many intersections of the trajectory with the line $h + bk = 0$. Let the corresponding x-values be denoted by $x = \beta_\nu(c, \lambda)$, $\nu = 1, 2, \cdots$.

For $\nu = 1, 2, \cdots$ and for fixed $\lambda > 0$, we consider $x = Z_\nu(c, \lambda)$ as functions of $c > 0$. That these functions are continuous in c, λ and differentiable in c may be seen using the ordinary implicit function theorem on the equation,

$$v(Z_\nu(c, \lambda), c) + p(Z_\nu(c, \lambda))w(Z_\nu(c, \lambda), c) = 0, \quad \nu = 1, 2, \cdots,$$

together with the facts about continuous dependence of $v(x, c)$ with respect to $c, \lambda > 0$. Much depends on the properties of the family of curves: $x = Z_\nu(c, \lambda)$, $\nu = 1, 2, \cdots$. By a succession of lemmas which will appear in §V, we have the following important property:

THEOREM 2. *Under statements* H-1 *and either* H-2, H-3, *or* H-4, $\partial Z_\nu(c, \lambda)/\partial c < 0$, $\nu = 1, 2, \cdots$.

This result is to be contrasted with that obtained in the sublinear case [7, Lemma 4]. In case of sublinearity, we had this derivative positive.

The problem of investigating the limits:

$$\lim_{c \to 0} Z_\nu(c, \lambda), \qquad \lim_{c \to \infty} Z_\nu(c, \lambda),$$

the former under statements H-1 and either H-2, H-3, or H-4, the latter restricted to H-1 and H-2, is so completely analogous to the corresponding problem in the sublinear case [7, §2, lines below (9)] that we merely state the results here:

THEOREM 3. *Under statements* H-1 *and either* H-2, H-3, *or* H-4, *we have* $\lim_{c \to 0} Z_\nu(c, \lambda) = \beta_\nu(0, \lambda)$, $\nu = 1, 2, \cdots$ *where the numbers* $\beta_\nu(0, \lambda)$ *represent, of course, the successive zeros of the function* $h(x, 0) + bk(x, 0)$.

The proof under H-2 is exactly the same as for the sublinear case. Under H-3 or H-4, however, the local Lipschitz condition valid in a neighborhood of the origin $v = 0$ must be used. Otherwise, the proof is the same.

For the other limit, observe statement H-2, which is interpreted to mean that $\lim_{u \to \infty} u^{-1} f(x, u) = A(x) > 0$. Let $h_\infty(x)$ be the solution of the "linearized problem at infinity":

$$\begin{aligned} &\text{D.E.} \quad (p(x)h_\infty')' + \lambda^{-1}A(x)h_\infty = 0; \\ (11) \qquad &\text{I.C.} \quad h_\infty(0) = a, \qquad k_\infty(x) = p(x)h_\infty'(x), \\ &\qquad\quad h_\infty'(0) = 1/p(0). \end{aligned}$$

Considerations exactly analogous to those of the sublinear case show the following:

THEOREM 4. *Under statements* H-1 *and* H-2, *we have* $\lim_{c\to\infty} Z_\nu(c, \lambda) = \beta_\nu(\infty, \lambda)$, $\nu = 1, 2, \cdots$. *Here, the numbers* $\beta_\nu(\infty, \lambda)$ *represent the successive zeros of the function* $h_\infty(x) + bk_\infty(x)$.

There is some novelty, however, in the investigation of the limit $\lim_{c\to\infty} Z_\nu(c, \lambda)$ in the case of stronger superlinearity. In fact, we can get the following result:

THEOREM 5. *Under statements* H-1 *and either* H-3 *or* H-4, $\lim_{c\to\infty} Z_\nu(c, \lambda) = 0$.

Proof. Since $y_\nu(c, \lambda) \geqq Z_\nu(c, \lambda) > 0$, $\nu = 1, 2, \cdots$, where y_ν represents the νth zero of the solution $v(x, c)$ of (7), it is sufficient to consider $\lim_{c\to\infty} y_\nu(c, \lambda)$. Now consider the following D.E.'s, the first of which obviously is the D.E. of (7):

$$\text{(i)} \qquad (p(x)v')' + \lambda^{-1}\frac{f(x, v)}{v} v = 0,$$

$$\text{(ii)} \qquad (\bar{p}\bar{v}')' + \lambda^{-1}\frac{\bar{f}(\bar{v})}{\bar{v}} \bar{v} = 0,$$

where

$$\bar{p} = \max_{0\leqq x\leqq X} p(x), \qquad \frac{\bar{f}(v)}{v} = \min_{0\leqq x\leqq\infty} \frac{f(x, v)}{v},$$

and $X > 0$ is arbitrary but big enough for our purposes. Using the Sturm Comparison Theorem [4, Chapter X], between two zeros $\bar{y}_{\nu-1}(c, \lambda)$, $\bar{y}_\nu(c, \lambda)$ of $\bar{v}(x, c)$, there exists at least one zero $y_\nu(c, \lambda)$ of $v(x, c)$. Therefore, we always have $\bar{y}_\nu(c, \lambda) \geqq y_\nu(c, \lambda)$ so that it is sufficient to prove that $\lim_{c\to\infty} \bar{y}_\nu(c, \lambda) = 0$ where $\bar{y}_\nu(c, \lambda)$ is the νth zero of the solution $\bar{v}(x, c)$ of the problem

$$\text{(12)} \qquad \begin{aligned} &\text{D.E.} \quad (\bar{p}\bar{v}')' + \lambda^{-1}\bar{f}(\bar{v}) = 0; \\ &\text{I.C.} \quad \bar{v}(0, c) = ac, \\ &\qquad\quad \bar{v}'(0, c) = c/\bar{p}. \end{aligned}$$

But since clearly $\bar{x}_{\nu+1}(c, \lambda) > \bar{y}_\nu(c, \lambda)$ where $\bar{x}_{\nu+1}(c, \lambda)$ is the $\nu+1$st zero of $\bar{w}(x, c) = \bar{p}\bar{v}'(x, c)$, it is sufficient to show that $\lim_{c\to\infty} \bar{x}_\nu(c, \lambda) = 0$, $\nu = 1, 2, 3, \cdots$. This we now proceed to do.

The D.E. in (12) is that of a conservative system, and the phase trajectories form closed loops about the origin of the phase plane given by the equation

$$\text{(13)} \qquad \bar{w}^2 + 2\lambda^{-1}\bar{p}F(\bar{v}) = c^2$$

where

$$F(v) = \int_0^v \bar{f}(v')dv'$$

is an even function. Let us define $\bar{v}_\nu = \bar{v}(\bar{x}_\nu, c)$, $\nu = 1, 2, \cdots$. Then by (13), these numbers are solutions of the equation $F(v) = c^2/2\lambda^{-1}\bar{p}$ which, however, has only two solutions, $\pm\bar{v}_1$, by the evenness of $F(v)$ and the known characteristics of $f(x, v)$ and therefore of $\bar{f}(v)$. Hence, $\bar{v}_\nu = (-1)^{\nu-1}\bar{v}_1$, $\nu = 1, 2, \cdots$.

From (13), we have (defining $\bar{x}_0 = 0$)

$$\bar{w} = \bar{p}\frac{d\bar{v}}{dx} = (-1)^{\nu-1}(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}, \quad \bar{x}_{\nu-1} < x < \bar{x}_\nu,\ \nu = 1, 2, \cdots$$

whence for $\bar{x}_\nu < x < \bar{x}_{\nu+1}$

$$x = \bar{p}\int_{ac}^{\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}} + (\nu - 1)\bar{p}\int_{-\bar{v}_1}^{+\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}}$$
$$+ \bar{p}\left(\int_{-\bar{v}_1}^{\bar{v}} \text{ or } \int_{\bar{v}}^{\bar{v}_1}\right)\frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}}$$

from which

$$\bar{x}_\nu(c, \lambda) = \bar{p}\int_{ac}^{\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}} + (\nu - 1)\bar{p}\int_{-\bar{v}_1}^{\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(v))^{1/2}}, \tag{14}$$
$$\nu = 1, 2, \cdots.$$

Let us consider integrals of the type

$$\int_\beta^{\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}}$$

with given lower limit $0 \leqq \beta \leqq \bar{v}_1$, and where we have $c^2 = 2\lambda^{-1}\bar{p}F(\bar{v}_1)$. We have

$$\int_\beta^{\bar{v}_1} \frac{d\bar{v}}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}))^{1/2}} \leqq \bar{v}_1\int_0^1 \frac{dy}{(c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}_1 y))^{1/2}} \leqq \frac{\bar{v}_1}{c}\int_0^1 \frac{dy}{(1 - y)^{1/2}} < \infty$$

since

$$c^2 - 2\lambda^{-1}\bar{p}F(\bar{v}_1 y) = c^2\left[1 - \frac{F(\bar{v}_1 y)}{F(\bar{v}_1)}\right] \geqq c^2(1 - y).$$

We have, therefore, $\bar{x}_\nu(c, \lambda) < K_\nu(\bar{v}_1/c)$ where K_ν is a constant depending (by (14)) upon ν. We know, however, that as $c \to \infty$, $\bar{v}_1 \to \infty$ since $\bar{v}_1$ is a solution of the equation $F(v) = c^2/2\lambda^{-1}\bar{p}$ with $F(v) = \int_0^v \bar{f}(v)dv$. Therefore,

$$\lim_{c\to\infty}\frac{\bar{v}_1}{c}=\lim_{c\to\infty}\frac{\bar{v}_1}{(2\lambda^{-1}\bar{p}F(\bar{v}_1))^{1/2}}=\lim_{\bar{v}_1\to\infty}\frac{\bar{v}_1}{(2\lambda^{-1}\bar{p}F(\bar{v}_1))^{1/2}}=0$$

by the superlinear behavior of $\bar{f}(v)$. Hence, $\lim_{c\to\infty} x_\nu(c,\lambda)=0$, and the theorem is proved.

Let us now collect the results of this section. There exists a sequence of functions $x=Z_\nu(c,\lambda)$, $\nu=1, 2, \cdots$, or curves of x vs. $c>0$ for fixed $\lambda>0$ the values of which represent the zeros of the function $v(x, c)+bp(x)v'(x, c)$ where $v(x, c)$ is the solution of (7). Stated geometrically, $Z_\nu(c, \lambda)$ represents the x-value of the intersection of the phase trajectory of Figure 1 with the line $v+bw=0$ on the phase plane.

These curves have the following properties:

(1) They are monotone decreasing, $dZ_\nu(c, \lambda)/dc<0$.

(2) $\lim_{c\to 0} Z_\nu(c, \lambda)=\beta_\nu(0, \lambda)$ where $\beta_\nu(0, \lambda)$ are the successive zeros of $h(x, 0)+bp(x)h'(x, 0)$, $h(x, 0)$ being the solution of (10) with $c=0$.

(3a) Under statements H-1 and H-2, $\lim_{c\to\infty} Z_\nu(c, \lambda)=\beta_\nu(\infty, \lambda)$, where $\beta_\nu(\infty, \lambda)$ are the successive zeros of $h_\infty(x)+bp(x)h'_\infty(x)$, $h_\infty(x)$ being the solution of (11).

(3b) Under statements H-1 and either H-3 or H-4, $\lim_{c\to\infty} Z_\nu(c, \lambda)=0$ for all $\nu=1, 2, 3, \cdots$.

As $\lambda>0$ is varied, the curves are displaced vertically in continuous fashion, the above properties being preserved.

IV. We now discuss the behavior of the function $Z_\nu(c, \lambda)$ as λ is varied. We recall that $\partial Z_\nu/\partial c<0$. We call to mind the eigenvalue sequences $\{\mu_n\}$, $\{\gamma_n\}$ of problems (3) and (4).

Suppose $\lambda=\mu_n$. By considering the D.E. problem equivalent to (3):

$$(15)\qquad \begin{aligned} &\text{D.E.}\quad (p(x)l')'+\lambda^{-1}f_u(x,0)l=0, && \lambda=\mu_n;\\ &\text{B.C.}\quad l(0)=ap(0)l'(0)=0, && l(1)+bp(1)l'(1)=0,\end{aligned}$$

together with problem (10) (with $v\equiv 0$), it is easy to see that $\lim_{c\to 0} Z_n(c, \mu_n)=\beta_n(0, \mu_n)=1$. For $m\neq n$ on the other hand, $\beta_m(0, \mu_n)<1$ for $m<n$, $\beta_m(0, \mu_n)>1$ for $m>n$. In other words, as $c\to 0$, the $Z_n(c, \mu_n)$ curve reaches the value $x=1$ at $c=0$; the $Z_m(c, \mu_n)$ curves for $m<n$ never do reach the value $x=1$; the $Z_m(c, \mu_n)$ curves for $m>n$ reach the value $x=1$ for positive values of c. This situation obtains, of course, for any integer $n=1, 2, 3, \cdots$.

Suppose λ is in an interval: $\mu_{n+1}<\lambda<\mu_n$. At the endpoints, we have $\beta_n(0, \mu_n)=1$, $\beta_{n+1}(0, \mu_{n+1})=1$ as above. By the continuity of $\beta_n(0, \lambda)$ as a function of $\lambda>0$, there exist some points λ_1 with $\mu_{n+1}<\lambda_1<\mu_n$, such that $\beta_n(0, \lambda_1)<1<\beta_{n+1}(0, \lambda_1)$. There is the question whether or not such points λ_1 comprise the entire interval (μ_{n+1}, μ_n). If not, there would be, by continuity, a value λ_* or a value λ_{**} with $\mu_{n+1}<\lambda_*$, $\lambda_{**}<\mu_n$, such that either $\beta_n(0, \lambda_*)=1$ or $\beta_{n+1}(0, \lambda_{**})=1$. In either case, we would conclude that λ_* or λ_{**} is an eigenvalue for problem (3). Since the sequence $\{\mu_n\}$ exhausts the eigenvalues of, (3), however, we have a contradiction. We have the result: If $\mu_{n+1} < \lambda < \mu_n$, then $\beta_n(0, \lambda) < 1 < \beta_{n+1}(0, \lambda)$.

In the asymptotically linear case where problem (4), the "linearized problem at infinity," has meaning, we can work similarly. If $\lambda = \gamma_n$, we have $\lim_{c\to\infty} Z_n(c, \gamma_n) = \beta_n(\infty, \gamma_n) = 1$, while if $m < n$, $\beta_m(\infty, \gamma_n) < 1$, if $m > n$, $\beta_m(\infty, \gamma_n) > 1$, $n = 1, 2, 3, \cdots$. If $\gamma_{n+1} < \lambda < \gamma_n$, then $\beta_n(\infty, \lambda) < 1 < \beta_{n+1}(\infty, \lambda)$. We collect these results in a theorem:

THEOREM 6. *Under statements* H-1 *and either* H-2, H-3, *or* H-4, $\mu_{n+1} < \lambda \leqq \mu_n$ *implies that* $\beta_n(0, \lambda) \leqq 1 < \beta_{n+1}(0, \lambda)$. *Under statements* H-1 *and* H-2 *only*, $\gamma_{n+1} < \lambda \leqq \gamma_n$ *implies that* $\beta_n(\infty, \lambda) \leqq 1 < \beta_{n+1}(\infty, \lambda)$; $n = 1, 2, 3, \cdots$.

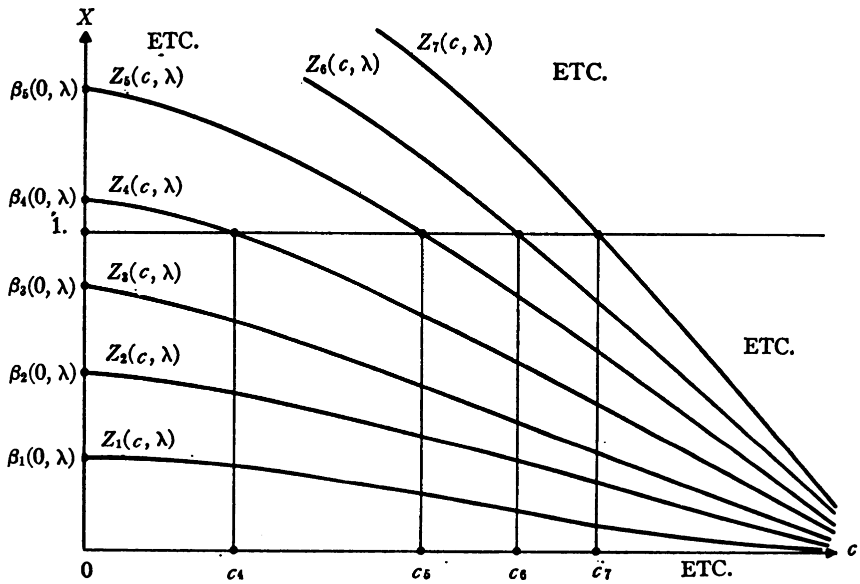


FIGURE 2

Thus under statements H-1 and either H-3 or H-4 where we have power law nonlinearity or stronger, the picture is as portrayed in Figure 2. Since by Theorem 5, all $Z_\nu(c, \lambda)$ curves go to zero as $c \to \infty$, there is always a denumerable infinity of intersections of the curves with the line $x = 1$ regardless of the position of the curves. Hence there is a denumerable infinity of c values which, when substituted in the initial conditions for problem (7), produces a denumerable infinity of solutions for problem (1) regardless of what positive value for λ we introduce in (1).

THEOREM 7. *Under statements* H-1 *and either* H-3 *or* H-4, *regardless of the value of* $\lambda > 0$, *problem* (1) *has a denumerable infinity of nontrivial solutions. If* $\mu_{n+1} < \lambda \leqq \mu_n$, *these solutions have respectively* n, $n+1$, $n+2$, *etc. zeros.*

COROLLARY. *If $f(x, u)$ has the properties described in statements* H-1 *and either* H-3 *or* H-4, *the spectrum of problems* (1) *and* (2) *embraces the whole positive* λ *axis.*

The statement about the zeros of the solutions can be seen by counting the number of times the trajectory of Figure 1 intersects the w-axis before the value of x reaches unity.

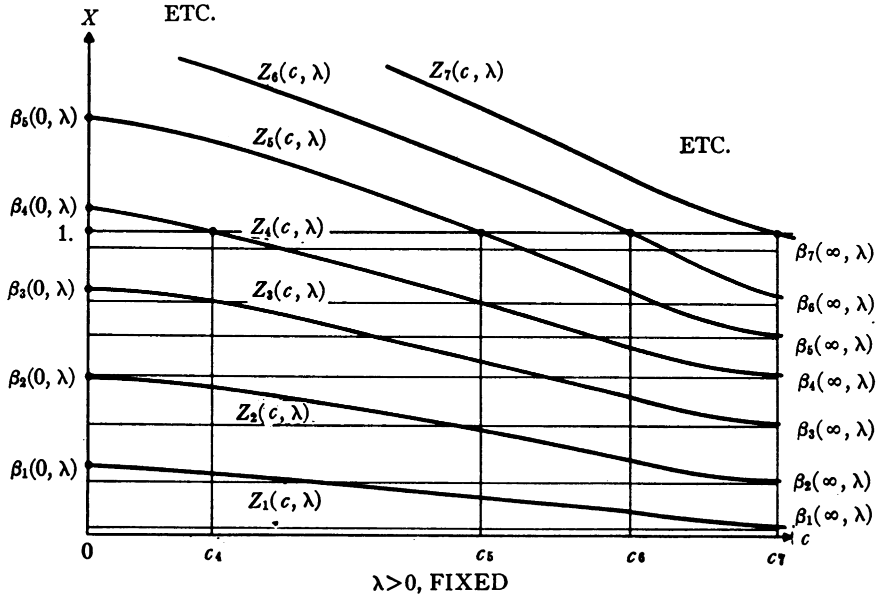


FIGURE 3

On the other hand, under statements H-1 and H-2, the case of asymptotic linearity, we know that the curves $Z_\nu(c, \lambda)$ are as portrayed in Figure 3. By Theorem 4, all curves approach positive levels determined by the numbers $\beta_\nu(\infty, \lambda)$. Hence for any value of $\lambda > 0$, there are only finitely many intersections with the line $x = 1$ producing only finitely many values of $c > 0$ which when substituted into the initial conditions of problem (7) produce only finitely many solutions for problem (1).

THEOREM 8. *Under statements* H-1 *and* H-2 *only and for any value* $\lambda > 0$, *problem* (1) *has only a finite number of nontrivial solutions. If* $\mu_{n+1} < \lambda \leqq \mu_n$, *these solutions have respectively* n, $n+1$, $n+2$, *etc. zeros.*

The number of solutions for given $\lambda > 0$ clearly depends on the relative values of the elements in the sequences $\{\beta_\nu(0, \lambda)\}$, $\{\beta_\nu(\infty, \lambda)\}$. The situation is similar to that of the sublinear problem [7, Theorems 3 and 4].

If it so happens that $\mu_n < \gamma_{n+1}$, $n = 1, 2, \cdots$, then given $\lambda > 0$, $\lambda < \gamma_1$, there always exists a number n such that $\mu_{n+1} < \lambda \leqq \mu_n < \gamma_{n+1}$. This means, in view of Theorem 6, $\beta_n(0, \lambda) \leqq 1 < \beta_{n+1}(0, \lambda)$ and $\beta_{n+1}(\infty, \lambda) < 1$. Some curve of Figure 3 is always intersected.

THEOREM 9. *Under statements* H-1 *and* H-2 *and under the supposition that* $\mu_n < \gamma_{n+1}$, $n = 1, 2, \cdots$, *where* $\{\mu_n\}$, $\{\gamma_n\}$ *are the eigenvalue sequences for* (3) *and* (4), *respectively, problem* (1) *has, for any* $\lambda > 0$, $\lambda < \gamma_1$, *at least one solution.*

COROLLARY. *Under the hypotheses of the theorem, the spectrum of* (1) *and* (2) *is the closed interval* $0 \leqq \lambda \leqq \gamma_1$.

If it so happens that the sequences $\{\mu_n\}$, $\{\gamma_n\}$ are interlaced: $\mu_{n+1} < \gamma_{n+1} \leqq \mu_n < \gamma_n \leqq \mu_{n-1} < \gamma_{n-1}$, $n = 2, 3, \cdots$, there will exist values $\lambda > 0$ for which $\mu_{n+1} < \lambda \leqq \mu_n$, $\gamma_{n+1} < \lambda \leqq \gamma_n$ hold simultaneously for some n. Then by Theorem 6, $\beta_n(0, \lambda) \leqq 1 < \beta_{n+1}(0, \lambda)$ and $\beta_n(\infty, \lambda) \leqq 1 < \beta_{n+1}(\infty, \lambda)$ simultaneously. There are no nontrivial solutions for problem (1) with such values λ. There are gaps in the spectrum:

THEOREM 10. *Under statements* H-1 *and* H-2 *only and under the supposition that the sequences* $\{\mu_n\}$, $\{\gamma_n\}$ *are interlaced, problem* (1) *has just one*(3) *nontrivial solution if* $\mu_n < \lambda < \gamma_n$, *and no nontrivial solutions if* $\gamma_n \leqq \lambda \leqq \mu_{n+1}$.

COROLLARY. *Under* H-1 *and* H-2 *and the interlacing condition, the spectrum of problems* (1) *and* (2) *consists of a collection of intervals* $[\mu_n, \gamma_n]$, $n = 1, 2, \cdots$.

If we regard a linear regular Sturm-Liouville problem to be the limit of asymptotically linear superlinear problems such that in the limit the two sequences $\{\mu_n\}$, $\{\gamma_n\}$ coalesce, we see that as the limit is approached, the spectral intervals of the corollary degenerate to the elements of the classical point spectrum.

V. We now turn to the proof of Theorem 2. This theorem, so important to all the consequences of this paper, so simple to state and to understand, turns out to be exceedingly intricate in the proof. We need several lemmas.

LEMMA 1. $\partial v(x, c)/\partial c = h(x, c)$, $\partial w(x, c)/\partial c = k(x, c)$, *where* v *is the solution of* (7), h *is the solution of* (10), *and we have defined* $w = pv'$, $k = ph'$.

Under H-1 and H-2, the proof of the lemma is identical with that offered in the sublinear case [7, §3, Lemma 1]. In the case of strong nonlinearities where H-2 is replaced by H-3 or H-4, the same proof can be carried out using the expedient of bounding the functions $v(x, c)$, $w(x, c)$ with a large circle in the phase plane over the interval $0 \leqq x \leqq X$. For the interior of the circle, one then has a Lipschitz condition K and a bound on $Q(x, v)$ which will serve in the proof.

LEMMA 2. *With regard to Figure* 1, *let there be given two trajectories, solution curves of* (7), *with initial points* (ac_1, c_1) *and* (ac_2, c_2), *respectively, where* $c_1 < c_2$.

(3) Apart from sign.

Then the trajectory described by the points $(v(x, c_1), w(x, c_1))$ *remains inside the trajectory described by the points* $(v(x, c_2), w(x, c_2))$ *for* $x>0$.

Proof. All possible initial points (ac, c) lie on the positive half line $v-aw=0$, $v\geqq 0$ of the phase plane and the variable $c\geqq 0$ can be regarded as a parametrization of the half line. Initial value problem (7), for fixed $x>0$, induces a continuous transformation T_x of this half line into some image set:

$$T_x\colon (ac, c) \Rightarrow (v(x, c), w(x, c)), \qquad x \text{ fixed, } c \text{ variable,}$$

which is again parametrized by c. Clearly, T_x carries the origin into the origin. It is known [1, p. 23] that T_x is a topological transformation so that the image of each compact subset of a line is a simple Jordan arc (no double points). Hence the image of the subset of points (ac, c) where $0\leqq c\leqq c_2$, under T_x, is a simple Jordan arc starting at the origin, intersecting all trajectories in the phase plane for $c\leqq c_2$ at the value x. The intersection with the trajectory $(v(x, c_1), w(x, c_1))$ is distinct from the intersection with $(v(x, c_2), w(x, c_2))$; otherwise there would be a double point. Since we can define T_x for any $x>0$, the lemma is proved.

Lemma 2 indicates that pairs of trajectories do not cross in the phase plane, at least not for neighboring values of x. This fact will be used in Lemma 3 to show that the lead (or lag) of the (h, k) trajectory, the solution curve for problem (10), (inscribed on an (h, k) plane considered superimposed on Figure 1) remains less than 90° with respect to the (v, w) trajectory with the same value of c. This information in turn is vital for the proof of Theorem 2.

We recall the definitions of $Z_\nu(c, \lambda)$, $\beta_\nu(c, \lambda)$, $x_\nu(c, \lambda)$, $y_\nu(c, \lambda)$. Let us list these and some other definitions for reference; $\nu=1, 2, \cdots$.

(1) $Z_\nu(c, \lambda)$: successive zeros of $v+bw$,
(2) $\beta_\nu(c, \lambda)$: successive zeros of $h+bk$,
(3) $y_\nu(c, \lambda)$: successive zeros of $v(x, c)$,
(4) $x_\nu(c, \lambda)$: successive zeros of $w(x, c)$,
(5) $\alpha_\nu(c, \lambda)$: successive zeros of $h(x, c)$.

For $\nu=0$, we arbitrarily define $y_0=x_0=\alpha_0=0$.

While we are listing information, let us put down the following sign information, which can be inferred by inspection of Figure 1; $\nu=0, 1, 2, \cdots$:

$$\begin{aligned}
&\operatorname{sgn} v(x, c) = (-1)^\nu, && y_\nu < x < y_{\nu+1},\\
&\operatorname{sgn} w(x, c) = (-1)^\nu, && x_\nu < x < x_{\nu+1},\\
&\operatorname{sgn} h(x, c) = (-1)^\nu, && \alpha_\nu < x < \alpha_{\nu+1},\\
&\operatorname{sgn} k(x, c) = (-1)^\nu, && x = \alpha_\nu,\\
&\operatorname{sgn}[v' + bw'] = (-1)^\nu, && x = Z_\nu.
\end{aligned} \tag{16}$$

In the last statement, the primes represent differentiation with respect to x, of course.

LEMMA 3. sgn $k(y_\nu(c,\lambda),c)=\operatorname{sgn} w(y_\nu(c,\lambda),c)=(-1)^\nu$.

Proof. Let $r(x,c)>0$ be the radial distance from the origin of the phase plane (Figure 1) to the endpoint (v,w) of the evolving trajectory: $r=(v^2+w^2)^{1/2}$. If we put $x=y_\nu(c,\lambda)$, then $r(y_\nu(c,\lambda),c)=(-1)^\nu w(y_\nu(c,\lambda),c)$, $\nu\geqq 1$, since $v(y_\nu,c)=0$. Also in view of Lemma 2, we have $(d/dc)r(y_\nu(c,\lambda),c)>0$. But

$$\begin{aligned}\frac{d}{dc}r(y_\nu(c,\lambda),c) &= (-1)^\nu\frac{d}{dc}w(y_\nu(c,\lambda),c)\\ &= (-1)^\nu\left[w'(y_\nu(c,\lambda),c)\frac{dy_\nu}{dc}+k(y_\nu(c,\lambda),c)\right]\\ &= (-1)^\nu k(y_\nu(c,\lambda),c),\end{aligned}$$

since $w'(y_\nu,c)=-\lambda^{-1}f(y_\nu,v(y_\nu,c))=0$. Hence $(-1)^\nu k(y_\nu(c,\lambda),c)>0$ which, together with the second item of (16), yields the result.

We now derive a useful relationship, using the D.E.'s

$$(pv')'+\lambda^{-1}f(x,v)=0, \tag{A}$$

$$(ph')'+\lambda^{-1}f_v(x,v)h=0. \tag{B}$$

Take (B) multiplied through by v and subtract (A) multiplied through by h. Upon integration over some arbitrary interval (α,β), we get

$$[hw-vk]\Big|_\alpha^\beta+\lambda^{-1}\int_\alpha^\beta\left[\frac{f(x,v)}{v}-f_v(x,v)\right]hv\,dx=0. \tag{17}$$

In the superlinear case, we have

$$\Theta(x,v)=\frac{f(x,v)}{v}-f_v(x,v)<0.$$

The corresponding quantity in the sublinear case was positive [7, Equation (14)].

Proof of Theorem 2. Identically in c, we have

$$v(Z_\nu(c,\lambda),c)+bw(Z_\nu(c,\lambda),c)\equiv 0.$$

Differentiating with respect to c:

$$\begin{aligned}&[v'(Z_\nu(c,\lambda),c)+bw'(Z_\nu(c,\lambda),c)]dZ_\nu/dc\\ &\qquad\qquad+h(Z_\nu(c,\lambda),c)+bk(Z_\nu(c,\lambda),c)=0.\end{aligned} \tag{18}$$

Negativity of dZ_ν/dc depends on whether or not $v'+bw'$ and $h+bk$ are of the same sign at $x=Z_\nu$.

By definition, $\alpha_0=y_0=0$. As an induction assumption, we put $\alpha_\nu\leqq y_\nu$, $\nu=0,1,\cdots,m-1,m$. For values of ν up to $\nu=m$, the (h,k) trajectory has not reached $x=\beta_{\nu+1}$ when the (v,w) trajectory gets to $x=y_\nu$, by Lemma 3. Hence $\alpha_m\leqq y_m<\beta_{m+1}<\alpha_{m+1}$.

In (17), we put successively, $\alpha=y_m$, $\beta=\beta_{m+1}$, and $\alpha=y_m$, $\beta=y_{m+1}$ to get

$$
\begin{aligned}
&-k(\beta_{m+1}, c)[v(\beta_{m+1}, c) + bw(\beta_{m+1}, c)] - (1 - \delta_{0m})h(y_m, c)w(y_m,c) \\
&\qquad + \lambda^{-1}\int_{y_m}^{\beta_{m+1}} hv\Theta\, dx = 0
\end{aligned}
\tag{19}
$$

and

$$
h(y_{m+1}, c)w(y_{m+1}, c) - (1 - \delta_{0m})h(y_m, c)w(y_m, c) + \lambda^{-1}\int_{y_m}^{y_{m+1}} hv\Theta\, dx = 0 \tag{20}
$$

where $\delta_{0m}=1$, $m=0$, and $\delta_{0m}=0$, $m>0$. The coefficient $1-\delta_{0m}$ introduced in (19) and (20) takes care of the special form assumed by these expressions for $m=0$.

By (16) and by the induction assumption, we have sgn $h = (-1)^m$ on (y_m, β_{m+1}). We assert that sgn $v=(-1)^m$ on the same interval. Suppose not; then we should have $y_m<y_{m+1}<\beta_{m+1}$ with sgn $v=(-1)^m$ on (y_m, y_{m+1}) (see (16)). In (20), the integral would be negative since $\Theta<0$. Also $h(y_m, c)w(y_m, c)>0$, since sgn $h(y_m, c)=(-1)^m$, sgn $w(y_m, c)=(-1)^m$. By (20), we would have $h(y_{m+1}, c)w(y_{m+1}, c)>0$. This couldn't be, however, since sgn $w(y_{m+1}, c)=(-1)^{m+1}$ while by the assumption sgn $h(y_{m+1}, c)=(-1)^m$. Hence sgn v $=$sgn $h=(-1)^m$ on (y_m, β_{m+1}).

Referring to (19), we see that the integral is negative. Also $h(y_m, c)w(y_m, c)>0$ as above. Evidently, therefore, $k(\beta_{m+1}, c)[v(\beta_{m+1}, c)+bw(\beta_{m+1}, c)]<0$. Obviously, sgn $k(\beta_{m+1}, c)=(-1)^{m+1}$ by inspection of Figure 1, which means that sgn$[v(\beta_{m+1}, c)+bw(\beta_{m+1}, c)]=(-1)^m$.

If we carry through this argument with $b=0$, there results sgn $v(\alpha_{m+1}, c)=(-1)^m$. Hence $\alpha_{m+1}<y_{m+1}$, which carries forth the induction assumption.

As products of this completed induction, we have sgn$[v(\beta_{\nu+1}, c)+bw(\beta_{\nu+1}, c)]=(-1)^\nu$, and with the help of Lemma 3, $\alpha_\nu\leqq y_\nu<\beta_{\nu+1}<\alpha_{\nu+1}<y_{\nu+1}<\beta_{\nu+2}<\cdots$, $\nu=0, 1, 2, \cdots$. Therefore, $\beta_{\nu+1}<Z_{\nu+1}$ and sgn$[h(Z_{\nu+1}, c)+bk(Z_{\nu+1}, c)]=(-1)^{\nu+1}$.

By (16), we have sgn$[v'(Z_\nu, c)+bw'(Z_\nu, c)]=(-1)^\nu$, and we now have sgn$[h(Z_\nu, c)+bk(Z_\nu, c)]=(-1)^\nu$, $\nu=0, 1, 2, \cdots$. The requisite quantities in (18) are therefore of the same sign, and the theorem is proved.

VI. The usual notion of a spectrum is defective in a nonlinear problem as the corollaries of §IV indicate. Something deeper is needed. The individual eigenfunctions are usually functions of the eigenvalues as well as of the independent variables.

Let us confine attention for the moment to the asymptotically linear case, with hypotheses H-1 and H-2. For $\mu_n\leqq\lambda<\gamma_n$, we see by the methods of Theorem 6 that $\beta_n(\infty, \lambda)<1\leqq\beta_n(0, \lambda)$. Hence, referring to Figure 3, the line $x=1$ has, very definitely, just one intersection with the curve $x=Z_n(c, \lambda)$, and just one(3) eigenfunction $u_n(x, \lambda)$ for problem (1) results. This eigenfunction is a function of λ on the interval $[\mu_n, \gamma_n)$. We see by inspection that, because of the continuity of $Z_n(c, \lambda)$ in its arguments, the monotonicity of $Z_n(c, \lambda)$

and the uniform continuity of the solution of (7) as a function of c, with respect to x, $u_n(x, \lambda)$ is continuous in λ, uniformly with respect to $x \in [0, 1]$. We call $u_n(x, \lambda)$ one "continuous branch" of eigensolutions for (1). The norm $r_n(\lambda) = \|u_n(x, \lambda)\|$ of the continuous branch $u_n(x, \lambda)$ defined on $[\mu_n, \gamma_n)$ is a single valued continuous function of λ which we shall call a "characteristic value function." Of course, $\|\cdot\|$ denotes the norm for the appropriate Banach space $B(0, 1)$ of the problem, which is $L_2(0, 1)$ under statement H-2 (see §II).

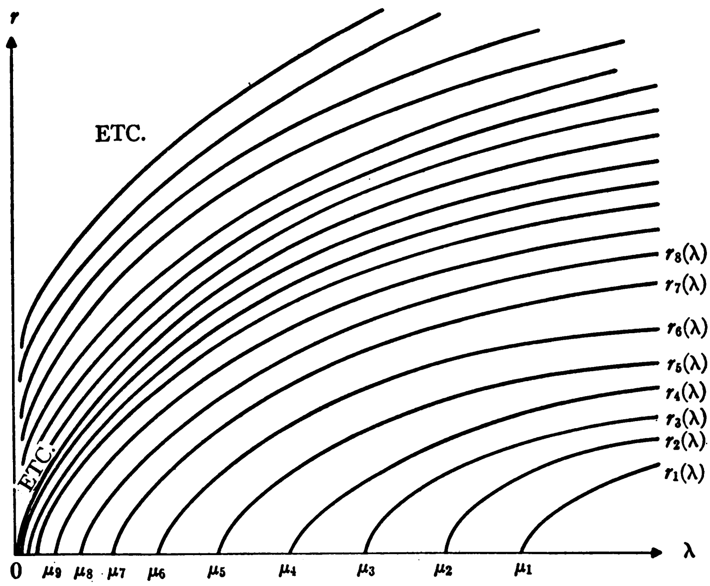
For $\lambda = \mu_n$, the left endpoint, $Z_n(0, \mu_n) = \beta_n(0, \mu_n) = 1$; hence, the intersection of $Z_n(c, \lambda)$ with $x = 1$ (Figure 3) occurs for $c = 0$. Therefore, $u_n(x, \mu_n) \equiv 0$ and $r_n(\mu_n) = 0$. On the other hand, for $\lambda = \gamma_n$, the right endpoint, $Z_n(\infty, \gamma_n) = 1$ so that $c = \infty$; by the asymptotic linearity property, we must have $r_n(\gamma_n) = \|u_n(x, \gamma_n)\| = \infty$. In the interior of the spectral interval $[\mu_n, \gamma_n)$, $r_n(\lambda)$ of course assumes intermediate positive values.

THEOREM 11. *Under hypotheses* H-1 *and* H-2, *the spectrum for problems* (1) *and* (2) *consists of a collection of intervals* $[\mu_n, \gamma_n]$, $n = 1, 2, \cdots$, *each with left endpoint a member of the sequence* $\{\mu_n\}$ *of eigenvalues for* (3) *and with right endpoint the corresponding member of the sequence* $\{\gamma_n\}$ *of eigenvalues for* (4). *Those intervals may or may not overlap. Associated with each spectral interval, and defined thereon, is a positive continuous "characteristic value function"* $r_n(\lambda)$. $r_n(\mu_n) = 0$, *and* $r_n(\lambda) \to \infty$ *as* $\lambda \to \gamma_n$, $\mu_n < \lambda < \gamma_n$. $r_n(\lambda)$ *gives the* L_2 *norm of that continuous branch* $u_n(x, \lambda)$ *of eigenfunctions which branches away from the trivial solution at the bifurcation point* $\lambda = \mu_n$. *The spectral intervals tend to the origin of the spectral line, and* $\lim_{n\to\infty} \text{meas}(\mu_n, \gamma_n) = 0$.

In cases of stronger nonlinearity, where statement H-3 or H-4 hold rather than H-2, it is clear on the basis of Theorem 6 that $\beta_n(0, \lambda) \geqq 1$ for $\mu_n \leqq \lambda < \infty$. Since $\lim_{c\to\infty} Z_n(c, \lambda) = 0$, $n = 1, 2, \cdots$, there is always a denumerable infinity of solutions. Referring to Figure 2 and considering one curve $Z_n(c, \lambda)$ and $\lambda \in [\mu_n, \infty)$, the line $x = 1$ has one intersection with the curve and one solution $u_n(x, \lambda)$ results which is a continuous branch. There is a positive continuous characteristic value function $r_n(\lambda)$, such that $r_n(\mu_n) = 0$, defined on the infinite interval $[\mu_n, \infty)$. These spectral intervals $[\mu_n, \infty)$, $n = 1, 2, \cdots$ can be regarded as "nested at infinity," since $\lim_{n\to\infty} \mu_n = 0$.

THEOREM 12. *Under hypotheses* H-1 *and either* H-3 *or* H-4, *the spectrum for problems* (1) *and* (2) *consists of a collection of nested half infinite intervals* $[\mu_n, \infty)$, $n = 1, 2, \cdots$, *each with left endpoint a member of the sequence* $\{\mu_n\}$ *of eigenvalues for* (3). *We have* $\lim_{n\to\infty} \mu_n = 0$. *Associated with each interval and defined thereon, is a positive continuous "characteristic value function"* $r_n(\lambda)$ *with* $r_n(\mu_n) = 0$. $r_n(\lambda)$ *gives the Banach space norm of that continuous branch* $u_n(x, \lambda)$ *of eigenfunctions which branches away from the trivial solution at the bifurcation point* $\lambda = \mu_n$.

Clearly, since $\lim_{n\to\infty} \mu_n = 0$, for *any* value $\lambda > 0$, we have a denumerable



CHARACTERISTIC VALUE FUNCTIONS (STRONG SUPER-LINEARITY) SUGGESTED APPEARANCE

FIGURE 4

infinity of values r_n representing the norms of all the eigenfunctions under H-3 or H-4. Figure 4 shows how the characteristic value curves of Theorem 12 might look.

The classical linear Sturm-Liouville problem is, of course, a special case of H-2, where the two sequences $\{\mu_n\}$ and $\{\gamma_n\}$ coalesce. Associated with $\lambda=\mu_n=\gamma_n$ is a linear subspace containing eigenfunctions of arbitrary norm. Hence the characteristic value curves $r_n(\lambda)$ are merely the vertical lines $\lambda=\mu_n$. Evidently, in a sequence of superlinear problems approaching linearity, $\gamma_n\to\mu_n$, and continuous branches of eigenfunctions $u_n(x, \lambda)$ become linear spaces in some sense which needs to be made precise.

The distinction of the superlinear problem as opposed to the linear problem is that the characteristic value curves lean over to the right, thus smearing the spectrum. (In case of strong superlinearity, the spectrum is smeared all the way to "infinity.") In the sublinear problem [7, Figure 5], the curves leaned to the left, also smearing the spectrum. Solution multiplicities of finite or infinite order result. Only in the linear problem is one afforded the luxuries of a point spectrum with possible uniqueness of the normalized eigenfunction, uniqueness of the trivial solution except on a spectral set of measure zero.

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