A DECOMPOSITION FOR COMPLETE NORMED
ABELIAN GROUPS WITH APPLICATIONS TO
SPACES OF ADDITIVE SET FUNCTIONS

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1. Introduction. The purpose of this paper is twofold. Our principal objective is to present a Lebesgue type decomposition Theorem (Theorem 2.3) for a generalized complete normed abelian group $G$, where generalized means (1) that the norm (\(\|x\|\)) of the nonzero elements $x$ of $G$ may be infinite (i.e. if \(x \in G\) and \(x \neq 0\), then \(0 < \|x\| \leq \infty\)) and (2) that only the subgroup of bounded elements $x$ (i.e. \(\{x; \|x\| < \infty\}\)) is required to be complete. In §3, we apply this decomposition theorem to the space of finitely additive set functions on an algebra $S$ of subsets of a set $X$ in order to generalize the Lebesgue decomposition for bounded and finitely set functions on $S$ (cf. [2]).

The basic form of our decomposition depends on what we call an admissible algebra $T$ of endomorphisms on $G$ (Definition 2.3). It will be seen that $T$ is a Boolean algebra of projection operators with a condition on the manner in which projection on disjoint subgroups effects the norm. It is this latter condition which will provide our principal analytic tool.

Throughout this paper, $G$ will denote a generalized complete normed abelian group.

2. Decompositions and examples. We shall develop the notion of an admissible algebra $T$ of endomorphisms on $G$ in two stages: the first algebraic and the second analytic.

Definition 2.1. A set $T$ of endomorphisms on $G$ is said to be an algebra of endomorphisms on $G$ if whenever each of $a$ and $b$ is an element of $T$, then

- \((1)\) $ab = ba \in T$ where $ab(x) = a(b(x))$ for $x \in G$,
- \((2)\) $aa = a$, and
- \((3)\) $a' = \epsilon - a \in T$ where $\epsilon(x) = x$ for $x \in G$.

Moreover, for each element $a$ of $T$ we let $P(a) = \{x \in G; a(x) = x\}$.

We shall see that the mapping $a \rightarrow P(a)$ is an isomorphism of $T$ onto a Boolean algebra of subgroups of $G$. We have, from (2), that \(\|a\| = \|a^*\| \leq \|a\|^*\) and, hence, if $a \neq 0$ then $\|a\| \geq 1$ (\(\|a\|\) may be infinite). Moreover, $T$ has the following properties:

\(\text{(i)}\) $0 \in T$ (\(aa' = 0\)),

\(\text{(ii)}\) $\epsilon \in T$ (\(\epsilon = 0^*\)), and

\(\text{(iii)}\) $a + b - ab = (a'b')' \in T$ [note that $a'b'(a + b - ab) = 0$ and $a'b' + (a + b - ab) = a'b' + (ab + ab') + b - ab = (a'b' + ab') + b = b' + b = \epsilon$].

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**Definition 2.2.** If $T$ is an algebra of endomorphisms on $G$ and each of $a$ and $b$ is an element of $T$, then $a \leq b$ means $ab = a$.

**Theorem 2.1.** If each of $a$ and $b$ is an element of an algebra $T$ of endomorphisms on $G$, then

(i) $0 \leq a \leq e$,
(ii) $a \leq b \iff ab = ab' = 0 \iff b' = a' b' \iff a' \geq b' \iff b = a + a' b \iff$ there exists an element $c$ of $T$ such that $a = bc$,
(iii) $ab = 0 \iff ab' = a \iff a' = b'$,
(iv) if $ab = 0$, $c \leq a$, and $d \leq b$, then $cd = 0$,
(v) $a \cap b = ab$ where $a \cap b = \sup \{ c \in T; c \leq a, c \leq b \}$,
(vi) $a \cup b = a + b - ab$ where $a \cup b = \inf \{ c \in T; c \geq a, c \geq b \}$,
(vii) $a \leq b \iff P(a) \subseteq P(b)$,
(viii) if $a \leq b$, then $P(b) = P(a) \oplus P(a'b)$, and
(ix) $P(ab) = P(a) \cap P(b)$.

**Proof.** Parts (i), (ii), (iii), (iv), (v), and (vi) follow readily from our definitions. (vii) If $a \leq b$, then $b'a = ab = 0$ and, hence, if $ax = x$, then $b'(x) = b'a(x) = 0$. (viii) It follows from (vii) that $P(a) \oplus P(a'b) \subseteq P(b)$. Suppose $x \in P(b)$. Then $x = b(x) = (a + a'b)(x) = ax + a'b(x)$; however, $a(x) \in P(a)(aa = a)$ and $a'b(x) \in P(a'b)$. Thus, $x \in P(a) \oplus P(a'b)$. (ix) We have, by (vii), that $P(ab) \subseteq P(a) \cap P(b)$. Suppose $x \in P(a) \cap P(b)$. Then $ab(x) = a(b(x)) = a(x) = x$ and, hence, $x \in P(ab)$.

We shall now introduce our analytic tool which we shall denote by Property A.

**Definition 2.3.** If $T$ is an algebra of endomorphisms on $G$, then $T$ is said to be an admissible algebra of endomorphisms on $G$, if $T$ has Property A: If $x \in G$, $\|x\| < \infty$, and $\delta > 0$, then there exists $\epsilon > 0$ such that if each of $a$ and $b$ is an element of $T$ and $\|a'b(x)\| > \delta$, then $\|(a + a'b)(x)\| > \|a(x)\| + \epsilon$.

**Remark.** We note that Property A is a condition only on the bounded elements of $G$. At the end of this section, we shall give examples to show (1) that $\epsilon$ may depend only on $\delta$ (Example 2.2 with $Q = 1$), (2) that $\epsilon$ may depend on $\delta$ and $\|x\|$ but not on $x$ (Example 2.2 with $Q > 1$), and (3) that $\epsilon$ may depend not only on $\delta$ and $\|x\|$ but also on $x$ (Example 2.4).

Henceforth $T$ shall denote an admissible algebra of endomorphisms on $G$.

**Theorem 2.2.** Suppose each of $a$ and $b$ is an element of $T$, then $a \leq b$ if and only if $\|a(x)\| \leq \|b(x)\|$ for all $x \in G$.

**Proof.** If $a \leq b$, then $b = a + a'b$ and, hence, if $x \in G$, then $\|b(x)\| = \|(a + a'b)(x)\| \geq \|a(x)\|$; in fact, inequality holds unless $\|a'b(x)\| = 0$. If $a \not\leq b$, then $ab' \neq 0$ and, hence, there exists an element $x$ of $G$ such that $\|ab'(x)\| \neq 0$. Thus, $a(ab'(x)) = ab'(x) \neq 0$ while $b(ab'(x)) = 0$.

**Corollary 2.2.1.** If $a$ is an element of $T$ and $a \neq 0$, then $\|a\| = 1$. 

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Proof. We have remarked earlier that \( \|a\| = \|a^n\| \leq \|a\|^n \) and, hence, \( \|a\| \leq 1 \). By Theorem 2.2, we have that \( \|a\| \leq \|e\| = 1 \). Thus, \( \|a\| = 1 \).

Remark. Later we shall give an example (Example 2.1) to show that the condition: \( a \leq b \) if and only if \( \|a(x)\| \leq \|b(x)\| \) for each \( x \in G \) is not sufficient to insure a decomposition. Property A is equivalent to: if \( x \in G, \|x\| < \infty \), and \( \delta > 0 \), then there exists \( \epsilon > 0 \) such that if each of \( a \) and \( b \) is an element of \( T \), \( ab = 0 \), and \( \|b(x)\| > \delta \), then \( \|ab(x)\| > \|a(x)\| + \epsilon \).

Lemma 2.3.1. If \( x \in G \), \( \{a_i\} \downarrow \) in \( T \), and \( \lim_i \|a_i(x)\| < \infty \), then \( \lim_i a_i(x) \) exists.

Proof. Let \( L = \lim_i \|a_i(x)\| \) and let \( \delta > 0 \). There exists a positive integer \( k \) such that \( \|a_k(x)\| < \delta \). There exists \( \epsilon > 0 \) such that if each of \( c \) and \( d \) is an element of \( T \) and \( \|c'a_k(x)\| > \delta \), then \( \|c'da_k(x)\| > \|a_k(x)\| + \epsilon \). There exists a positive integer \( i \) such that \( i \geq k \) and \( \|a_i(x)\| < L + \epsilon \). If \( j > i \), then \( a_j = a_j + a_j a_i \). Thus, \( \|a_j(x)\| = \|a_j + a_j a_i(x)\| < L + \epsilon \leq \|a_j(x)\| + \epsilon \) and, hence, \( \|a_j(x) - a_i(x)\| = \|a_j a_i(x)\| \leq \delta \).

Definition 2.4. If each of \( x \) and \( y \) is an element of \( G \) and \( t > 0 \), then

(1) \( Q(t, x) = \{a \in T; \|a(x)\| < t\} \), and
(2) \( r(t, x, y) = \sup \{\|a(y)\|; a \in Q(t, x)\} \).

Lemma 2.3.2. Suppose each of \( x \) and \( y \) is an element of \( G \), \( \|y\| < \infty \), \( r(t) = r(t, x, y) \), \( r = \lim_{t \to +\infty} r(t) < \infty \), and \( \epsilon > 0 \). Then there exists a sequence \( \{b_i\} \downarrow \) in \( T \) such that

(1) \( \lim_i b_i(x) = 0 \),
(2) \( \lim_i \|b_i(y)\| > r - \epsilon \), and
(3) \( \lim_i b_i(y) \) exists.

Proof. If \( r = 0 \), it is sufficient to let \( b_i = 0 \) for \( i \geq 1 \). Suppose \( r > 0 \) and \( m \) is a positive integer such that \( 2^{-m} < \epsilon \). Let \( t_1 = 1 \). There exists \( \epsilon_1 > 0 \) such that

(1) \( \epsilon_1 < 2^{-1} \) and
(2) if \( a, b \in T \) and \( a'b(y) > 2^{-1} \), then \( \|(a + a'b)(y)\| > \|a(y)\| + \epsilon_1 \). There exists \( a_1 \in Q(t_1, x) \) such that \( r(t_1) = \|a_1(y)\| < \epsilon_1 \). Let \( t_2 = 2^{-1}(t_1 - \|a_1(x)\|) \).

If \( a \in Q(t_2, x) \), then \( \|a_1 + a'(a)(x)\| \leq \|a_1(x)\| + \|a(x)\| < t_2 \) and, hence, \( \|a_1 + a'(a)(y)\| \leq r(t_2) < \|a_1(y)\| + \epsilon_1 \). Thus, \( \|a'(a)(y)\| \leq 2^{-1} \). There exists \( \epsilon_2 > 0 \) such that if \( \|a'(a)(y)\| > 2^{-1} \), then \( \|(a + a'a')(y)\| > \|a(y)\| + \epsilon_2 \). There exists \( a_2 \in Q(t_2, x) \) such that \( r(t_2) = \|a_2(y)\| < \epsilon_2 \). If we repeat the preceding process inductively, we obtain a sequence \( \{a_i\} \) of elements of \( T \), a sequence \( \{\epsilon_i\} \) of positive numbers, and a sequence \( \{t_i\} \) of positive numbers such that

(1) \( t_1 = 1 \) and \( t_{i+1} = 2^{-1}(t_i - \|a_i(x)\|) \) for \( i > 1 \),
(2) \( 0 < \epsilon_i < 2^{-1} \),
(3) if \( a, b \in T \) and \( a'b(y) > 2^{-1} \), then \( \|(a + a'b)(y)\| > \|a(y)\| + \epsilon_i \),
(4) \( a_i \in Q(t_i, x) \),
(5) \( r(t_i) < \|a_i(y)\| + \epsilon_i \), and
(6) if \( a \in Q(t_{i+1}, x) \), then \((a_i + a_j')a \in Q(t_i, x)\) which implies \( \| (a_i + a_j')a(y) \| \leq r(t_i) \). Hence, \( \| a_j'(y) \| \leq 2^{-(m+i)} \). For each positive integer \( i \), \( a_i = a_i' + a_i'a_i'a_i'a_i'a_i' + \cdots + \prod_{j<i} a_j \). Let \( b_i = \prod_{j<i} a_j \). Then \( \{ b_i \} \uparrow \) in \( T \). Moreover,

\[
\begin{align*}
(1) & \quad \| b_i(x) \| \leq \| a_i(x) \| \leq 2^{-(i-1)}, \\
(2) & \quad \| (a_i - b_i)(y) \| \leq \| a_i a_i'(y) \| + \| a_i a_i'(y) \| \\
&\quad + \cdots + \left\| \prod_{j<i} a_j \right\| a_i'(y) \| \leq \sum_{j<i} 2^{-(m+j)}, \text{ and} \\
(3) & \quad r(t_i) - \| b_i(y) \| \leq r(t_i) - \| a_i(y) \| + \| (a_i - b_i)(y) \| \\
&\quad < e_i + \sum_{j<i} 2^{-(m+j)} < 2^{-(m+i)} + \sum_{j<i} 2^{-(m+j)} < 2^{-m} < \epsilon.
\end{align*}
\]

Hence, \( \lim_{i} \| b_i(y) \| \geq r - \epsilon \). However, \( \lim_{i} b_i(y) \leq \lim_{i} r(t_i) < \infty \) which implies (Lemma 2.3.1) that \( \lim_{i} b_i \) exists.

**Definition 2.5.** If each of \( x \) and \( y \) is an element of \( G \), then \( y \) is said to be

(1) absolutely continuous with respect to \( x \) (mod \( T \)) if for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( a \) is an element of \( T \) and \( \| a(x) \| < \delta \), then \( \| a(y) \| < \epsilon \), and

(2) singular with respect to \( x \) (mod \( T \)) if for each \( \epsilon > 0 \), there exists an element \( a \) of \( T \) such that \( \| a(x) \| < \epsilon \) and \( \| a'(y) \| < \epsilon \). Moreover, we denote by \( G_a(x, T) \) the set of elements \( h \) of \( G \) which are absolutely continuous with respect to \( x \) (mod \( T \)) and we denote by \( G_s(x, T) \) the set of elements \( u \) of \( G \) which are singular with respect to \( x \) (mod \( T \)).

**Lemma 2.3.3.** If \( x \in G \), then each of \( G_a(x, T) \) and \( G_s(x, T) \) is a subgroup of \( G \) and \( G_a(x, T) \cap G_s(x, T) = 0 \). Moreover, if \( h \in G_a(x, T) \), then \( G_a(h, T) \supseteq G_a(x, T) \).

**Proof.** Suppose each of \( y \) and \( z \) is an element of \( G_a(x, T) \) and \( \epsilon > 0 \), then there exists \( \delta > 0 \) such that if \( a \in T \) and \( \| a(x) \| < \delta \), then each of \( \| a(y) \| \) and \( \| a(z) \| < \epsilon/2 \) and, hence, \( \| a(y+z) \| < \epsilon \). Thus, \( G_a(x, T) \) is an algebraic subgroup of \( G \). Suppose \( \{ y_i \} \) is a sequence of elements of \( G_a(x, T) \), \( \lim_{i} y_i = y \), and \( \epsilon > 0 \). Then there exists a positive integer \( i \) such that \( \| y_i - y \| < \epsilon/2 \) and there exists \( \delta > 0 \) such that if \( a \in T \) and \( \| a(x) \| < \delta \), then \( \| a(y) \| < \epsilon/2 \) and, hence, \( \| a(y_i) \| = \| a(y) \| + \| a(y - y_i) \| < \epsilon \). Thus, \( y \in G_a(x, T) \). Suppose each of \( y \) and \( z \) is an element of \( G_a(x, T) \) and \( \epsilon > 0 \), then there exists \( \delta > 0 \) such that if \( a \in T \) and \( \| a(x) \| < \delta \), then \( \| a(y) \| < \epsilon/2 \) and, hence, \( \| a(y - z) \| = \| a(y) \| + \| a(z) \| < \epsilon \). Suppose \( \{ y_i \} \) is a sequence of elements of \( G_a(x, T) \), \( \lim_{i} y_i = y \), and \( \epsilon > 0 \). Then there exists a positive integer \( i \) such that \( \| y_i - y \| < \epsilon/2 \) and there exists \( a \in T \) such that \( \| a(x) \| < \epsilon/2 \) and, hence, \( \| a'(y) \| < \epsilon/2 \) and, hence, \( \| a'(y_i) \| = \| a'(y) \| + \| a'(y - y_i) \| < \epsilon \). Therefore, \( G_a(x, T) \) is a subgroup of \( G \). Suppose \( h \in G_a(x, T) \), \( s \in G_s(x, T) \), and \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that if \( a \in T \) and \( \| a(x) \| < \delta \), then
\|
a(h)\| < \epsilon \quad \text{and, since } \ s \in G_\epsilon(x, T), \text{ there exists } a \in T \text{ such that } \|
a(x)\| < \min[\epsilon, \delta] \text{ and } \|
a'(s)\| < \min[\epsilon, \delta]. \text{ Thus, } \|
a(h)\| < \epsilon \text{ and } \|
a'(s)\| < \epsilon. \text{ Hence, } 
\|s\| \in G_\epsilon(h, T).

Remark. We shall give two examples (Examples 3.1 and 3.2) to show that, in general, one cannot assert that $G$ is the direct sum of $G_\epsilon(x, T)$ and $G_\delta(x, T)$; however, Theorem 2.3 shows that \[ y \in G; \quad \|y\| < \infty \subset G_\epsilon(x, T) \oplus G_\delta(x, T) \] for each $x \in G$.

Lemma 2.3.4. If each of $x$ and $y$ is a nonzero element of $G$, each of $\|x\|$ and $\|y\|$ is finite, and $y$ is singular with respect to $x$, then $\|x+y\| = \max[\|x\|, \|y\|]$. 

Proof. Since the relation of being singular is symmetric, it is sufficient to show that $\|x+y\| > \|x\|$. There exists a sequence \{a_i\} of elements of $T$ such that $a_i(x) = x$ and $a_i(y) = y$. There exists $\epsilon > 0$ such that if $a \in T$ and $\|a'(x+y)\| > \|y\|/2$, then $\|x+y\| = \|a(x+y) + a'(x+y)\| > \|a(x+y)\| + \epsilon$. Thus, since $a_i(x+y) = x$ and $a_i'(x+y) = y$, $\|x+y\| = \lim_i \|a_i(x+y)y + a_i'(x+y)\| > \|x\| + \epsilon$.

Lemma 2.3.5. Suppose each of $x$ and $y$ is an element of $G$, $\{a_i\} \in T$, $z = \lim_i a_i(y) < \infty$, and $y = \lim_i a_i(x) = 0$. Then $\|z\| < r$.

Proof. It is sufficient to suppose $r < \infty$. If $\epsilon > 0$, then there exists $t > 0$ such that if $a \in T$ and $\|a(x)\| < t$, then $\|a(y)\| < r + \epsilon/2$ and there exists a positive integer $i$ such that $\|a_i(x)\| < t$ and $\|a_i(y) - z\| < \epsilon/2$. Thus, $\|z\| < \|z - a_i(y)\| + \|a_i(y)\| < r + \epsilon$.

Theorem 2.3. Suppose each of $x$ and $y$ is an element of $G$ and $\|y\| < \infty$. Then there exists uniquely an element $h$ of $G$ and an element $s$ of $G$ such that

(1) \[ y = h + s, \]

(2) \[ h \text{ is absolutely continuous with respect to } x \text{ (mod } T), \]

(3) \[ s \text{ is singular with respect to } x \text{ (mod } T). \]

Proof. Uniqueness follows from Lemma 2.3.3; the problem is to show existence. Let $r = \lim_i r(x, y) \in T$. If $r = 0$, let $h = y$ and $s = 0$ ($y \in G_\epsilon(x, T)$ if and only if $r = 0$). For each positive integer $i$, there exists $\epsilon_i > 0$ such that if each of $a$ and $b$ is an element of $T$ and $\|a'(b)\| > 2^{-i}$, then $\|a + a'b(y)\| > \|a(y)\| + \epsilon_i$. There exists (Lemma 2.3.2) a sequence \{a_i(y)\} in $T$ and an element $z_1$ of $G$ such that (1) $\lim_i a_i(x) = 0$, (2) $z_1 = \lim_i a_i(y)$, and (3) $r - \|z_1\| < \epsilon_i$. Let $y_i = \lim_i a_i(x)'(y) = y - z_1$ and let $r_i = \lim_i r(x, y_i)$. We assert that $r_i \leq 2^{-1}$. Suppose, on the contrary, that $r_i > 2^{-1}$. Then there exists a sequence \{b_i\} in $T$ and an element $w$ of $G$ such that (1) $\lim_i b_i(x) = 0$, (2) $w = \lim_i b_i(y_i)$ and (3) $\|w\| > 2^{-1}$ (Lemma 2.3.2 again). However, $\lim_i \|b_i(y_i)\| = \lim_i \|b_i a_i(x)'(y)\|$ and, hence,

\[ \|z_1 + w\| = \lim_i \|a_i(y) + a_i(x)'b_i(y)\| \geq \lim_i \|a_i(x)'(y)\| + \epsilon_i \]

but,

\[ \|z_1\| + \epsilon_i > r; \]
This contradicts the supposition that $r_1 > 2^{-1}$. There exists a sequence 
\{a(2,i)\} in $T$ and an element $z_2$ of $G$ such that (1) $\lim_i a(2,i)(x) = 0$, 
(2) $z_2 = \lim_i a(2,i)(y_i) = \lim_i a(2,i)a(1,i)'(y)$, and (3) $r_1 - ||z_2|| < \epsilon_2$. Let 
$y_2 = \lim_i a(2,i)'(y_i) = \lim_i a(2,i)'a(1,i)'(y)$ and let $r_2 = \lim_{t \rightarrow 0^+} r(t, x, y_2)$. Then $r_2 \leq 2^{-2}$. Proceeding by induction, either there exists a smallest positive 
integer $i$ such that $r_i = 0$ or $r_i > 0$ for each positive integer $i$. In the former 
case we let $h = y_i$ and $s = \sum j \leq j \in S_j$ while in the latter case we let $h = \lim_i y_i$ 
and $s = \sum j \leq j \in S$. We must first show that each of $\lim_i y_i$ and $\sum j \leq j \in S$ 
exists. Since $y_i = y - \sum j \leq j \in S_j$ it is sufficient to show that $\sum j \leq j \in S_j$ exists and this 
is done as follows. Let $s_i = \sum j \leq j \in S_j$. If $j > i$, then $||s_j - s_i|| = \sum k \leq k \in S_j ||s_k - s_i|| = \sum k \leq k \in S_j ||s_k|| \leq (Lemma \ 2.3.5) \sum k \leq k \in S_j 2^{-(i-k)} < 2^{-(i-1)}$ and hence, $\lim_i s_i = \sum j \leq j \in S_j$ exists. By our construction, each $z_i \in G_s(x)$ 
and, by Lemma 2.3.3, $G_s(x, T)$ is a subgroup of $G$. Thus, $s \in G_s(x, T)$. In 
order to complete a proof of Theorem 2.3, it is sufficient to show that 
$h \in G_s(x, T)$. To this end, suppose $\epsilon > 0$ and $2^{-(i-1)} < \epsilon/2$. Then $||h - y_i||$ 
$= ||s - s_i|| \leq 2^{-i} < \epsilon/2$ and $r_i = \lim_{t \rightarrow 0^+} r(t, x, y_i) \leq 2^{-i} < \epsilon/2$ which implies that 
there exists $t > 0$ such that $r(t, x, y_i) < \epsilon/2$. If $a \in T$ and $||a(x)|| < t$, then 
$||a(h)|| = ||a(h - y_i) + a(y_i)|| \leq ||h - y_i|| + ||a(y_i)|| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, 
h \in G_s(x, T).

Definition 2.6. The statement that a finite subset, \([a_i; j \leq n]\) of $T$ is a 
finite partition of $e$ in $T$ means that $a_i a_j = 0$ if $i \neq j$ and $\sum i \leq j a_i = e$.

Theorem 2.4. Suppose $x \in G$, $||x|| < \infty$, and $\epsilon > 0$. Then there exists a finite 
partition $P = [a_i; i \leq n]$ of $e$ in $T$ such that if $a \in T$ and $i \leq n$, then at least one 
of $||a a_i(x)||$ and $||a' a_i(x)|| < \epsilon$.

Proof. Suppose, on the contrary, that Theorem 2.4 is false. Then there 
extists a pair $(x, \epsilon)$ which contradicts Theorem 2.4: $||x|| < \infty$, $\epsilon > 0$, and if 
\([a_i; i \leq n]\) is a finite partition of $e$ in $T$ then there exists an element $a$ of $T$ 
and a positive integer $i \leq n$ such that each of $||a a_i(x)||$ and $||a' a_i(x)|| \geq \epsilon$.

Moreover, since the pair $(x, \epsilon)$ contradicts Theorem 2.4, for each element $a$ of $T$ 
at least one of the pairs $(a(x), \epsilon)$ and $(a'(x), \epsilon)$ contradicts Theorem 2.4, 
i.e., if $P = [a_i; i \leq m]$ works for $a(x)$ (i.e., if $b \in T$ and $i \leq m$ imply at least one 
of $||b a_i a_i(x)||$ and $||b' a_i a_i(x)|| < \epsilon$) and $Q = [b_j; j \leq n]$ works for $a'(x)$, then 
$R = [a_i; i \leq m] \cup [b a_j; j \leq n]$ works for $x$. Hence, there exists $a_\in T$ such that 
(1) $||a_0(x)|| \geq \epsilon$ and (2) the pair $(a_\epsilon(x), \epsilon)$ contradicts Theorem 2.4; · · · ; 
there exists $a_{i+1} \in T$ such that (1) $||a_{i+1} \prod j \leq j a_j(x)|| \geq \epsilon$ and (2) the pair 
$(\prod j \leq j a_j(x), \epsilon)$ contradicts Theorem 2.4. Let $b_i = \sum j \leq j a_j$. But, by Lemma 
2.3.1, $\lim_i b_i(x)$ exists and, hence, $\lim_i b_i(x)$ exists. This contradiction 
($||x|| < \infty$) establishes Theorem 2.4.

We shall apply Theorem 2.4 in §3. However, we shall first conclude this
section by giving four examples. Our first example sheds some light on the
question: How strong an analytic condition is needed on an algebra \(U\) of
endomorphisms on \(G\) in order to assure that Theorem 2.3 will hold (mod \(U\))? 

**Example 2.1.** In this example, \(T\) will be an algebra of endomorphisms on
\(G\) for which Theorem 2.3 does not hold; however, \(T\) will have the property
that if \(a, b \in T\), then \(a \leq b\) if and only if \(\|a(x)\| \leq \|b(x)\|\) for all \(x \in G\).

Let \(S\) be an algebra of subsets of a set \(X\), \(S\) contain an infinite number
of elements, \(G = \{x; x\) is a real valued function on \(X, \|x\| = \sup \{ |x(t)|; t \in X \}\),
and \(T = \{P_E; P_E(x) = x \cdot C(E)\) where \(C(E)(t) = 1\) if \(t \in E\) and \(C(E)(t) = 0\) if \(t \notin E\}. Then there exist bounded elements \(x\) and \(y\) of \(G\) such that if each of
\(h\) and \(s\) is an element of \(G\), \(y = h + s\), and \(h\) is absolutely continuous with re-
spect to \(x\) (mod \(T\)), then \(s\) is not singular with respect to \(x\) (mod \(T\)).

**Proof.** Since \(S\) is infinite, there exists a sequence \(\{E_i\}\) of non-null pair-
wise disjoint elements of \(S\). Let \(y = C(X)\) and \(x = \sum 2^{-i} C(E_i)\). Suppose
\(y = h + s\) and \(h \in G_\delta(x, T)\). Then there exists \(\delta > 0\) such that if \(E \in S\) and
\(\|P_E(x)\| < \delta\), then \(\|P_E(h)\| < 2^{-1}\) and, hence, there exists a positive integer \(i\)
such that

\[
\left\| h \cdot C \left( \bigcup_{i \geq i} E_i \right) \right\| < 2^{-1} \text{. Thus, for all } j > i, \inf \{s(t); t \in E_j\} \geq 2^{-1}
\]

and, hence, \(s\) is not singular with respect to \(x\) (mod \(T\)).

**Example 2.2.** Let \(X, S, G,\) and \(T\) be defined as in Example 2.1 except that
if \(x \in G\), then \(\|x\| = (\sum_{t \in X} |x(t)|^Q)^{1/Q}\), where \(Q\) is a real number \(\geq 1\). Then
\(T\) is an admissible algebra of endomorphisms on \(G\).

**Example 2.3.** Let \(G\) be a Hilbert space, let \([E_\lambda; - \infty \leq \lambda \leq \infty]\) be a resolu-
tion of the identity, and let \(T\) be the algebra of projection operators gener-
ated by projections of the form \(E_\lambda + \mu - E_\lambda, \mu \geq 0\). Then \(T\) is an admissible
algebra of endomorphisms on \(G\).

**Example 2.4.** Let \(X, S, G,\) and \(T\) be defined as in Example 2.1 except that
\(X\) is the set of positive integers, if \(x \in G\), then \(\|x\| = |x(1)| + \sum_{i=1}^\infty (|x(2i)|^{1/4} + |x(2i+1)|^{1/4})\), and each one element subset \([i]\) of \(X\) is an element of \(S\).

For each positive integer \(i\) we let \(x_i = C([2i, 2i+1])\) and we let \(a_i = P_{[2i]}\). Then
\(\|x_i\| = 2^{1/4}, \|a_i(x_i)\| = 1\) and \(\|a'_i(x_i)\| = 1\). Thus, in this example, while
\(T\) is admissible, the \(\varepsilon\) we get in satisfying Property A depends not only on \(\delta\)
and \(\|x\|\) but also on \(x\).

3. Spaces of finitely additive set functions. Throughout this section, \(X\)
will denote a set, \(S\) will denote an algebra of subsets of \(X, G\) will denote the
generalized complete normed abelian group of finitely additive set functions
on \(S\) where the norm \(\|x\|\) of the elements \(x\) of \(G\) is the total variation
\((V(x, X))\) of \(x\) on \(X,\) and \(T\) will denote the admissible algebra of projection
operators induced by \(S,\) i.e., \(T = \{P_E; P_E(F) = x(E \cap F)\) for \(E, F \subseteq S\) and
\(x \in G\).
Let us recall that if $S$ is an infinite set, then there exist unbounded finitely additive set functions $x$ on $S$ (i.e., elements $x$ of $G$ such that $\|x\| = \infty$).

We shall extend the Lebesgue type decomposition for bounded and finitely additive set functions on a set algebra $S$ which was presented in [2]. The definitions of absolute continuity and singularity which we use here are equivalent to those which were used in [2]. In order to make this paper self-contained with respect to notation and terminology, it is necessary to observe the following:

1. $\|P_E(x)\| = V(x, E)$ for $E \in S$ and $x \in G$,
2. $P_E P_F = P_{E \cap F}$,
3. $P_E' = P_{(E')}$, where $E' = X - E$,
4. if $E \cap F = \emptyset$: $P_E P_F = 0$, then $\|P_E(x) + P_F(x)\| = \|P_{E \cup F}(x)\| = \|P_E(x)\| + \|P_F(x)\|$ for all $x \in G$, and
5. $P_E \leq P_F$ if and only if $E \subset F$. Our first extension is the following consequence of Theorem 2.3.

**Theorem 3.1.** If $x$ is a finitely additive set function on $S$ and $y$ is a bounded and finitely additive set function on $S$, then there exists uniquely an element $h$ of $G$ and an element $s$ of $G$ such that

1. $y = h + s$,
2. $h$ is absolutely continuous with respect to $x$ (mod $T$), and
3. $s$ is singular with respect to $x$ (mod $T$).

**Theorem 3.2.** If $x$ is a bounded finitely additive set function on $S$ and $y$ is absolutely continuous with respect to $x$ (mod $T$), then $y$ is bounded.

**Proof.** Since $y$ is absolutely continuous with respect to $x$ (mod $T$), there exists $\delta > 0$ such that if $E \in S$ and $V(x, E) < \delta$, then $V(y, E) < 1$. By Theorem 2.4, there exists a finite partition $\{P_i; i \leq n\}$ of $P_S$ in $T$ such that if $E \in S$ and $i \leq n$, then at least one of $V(x, E \cap E_i)$ and $V(x, E' \cap E_i) < \delta$ and, hence, at least one of $V(y, E \cap E_i)$ and $V(y, E' \cap E_i) < 1$. For each positive integer $i \leq n$, $|y(E \cap E_i) - y(E_i)| = |y(E' \cap E_i)|$. Hence $|y(E \cap E_i)| < |y(E_i)| + 1$ for all $E \in S$. Thus, $V(y, E_i) \leq 2 (\sup\{|y(E \cap E_i); E \in S\}) \leq 2(|y(E_i)| + 1) < \infty$ for $i \leq n$ and, hence, $\|y\| = V(y, X) = \sum_{i=n} V(y, E_i) < \infty$.

In the general setting of §2, the analog of Theorem 3.2 is not, in general, true. For example, let $S$ be infinite and let $T'$ be the subalgebra of $T$ which consists of 0 and $e$. Then any two nonzero elements of $G$ are absolutely continuous with respect to each other (mod $T'$); but, there exist unbounded, as well as nonzero bounded, elements of $G$.

**Theorem 3.3.** If each of $x$ and $y$ is a finitely additive set function on $S$ and at least one of $x$ and $y$ is bounded, then $y$ is decomposable with respect to $x$ (mod $T$) if and only if there exists a sequence $\{E_i\} \downarrow$ in $S$ such that $\lim_i V(x, E_i) = 0$ and $\lim_i V(y, E_i) < \infty$. 

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Proof. If \( \|y\| < \infty \) a decomposition exists; moreover, it is sufficient to let \( E_i = \emptyset \) for \( i \geq 1 \). Suppose \( \|y\| = \infty \) and \( \|x\| < \infty \).

Necessity. Suppose \( y = h + s, h \in G_a(x, T), \) and \( s \in G_a(x, T) \). Then, by Theorem 3.3, \( \|h\| < \infty \) and, by the definition of singularity, there exists a sequence \( \{F_i\} \) of elements of \( S \) such that \( V(x, F_i) < 2^{-i} \) and \( V(y, F_i) < 2^{-i} \). Let \( E_i = \bigcap_{j=i} F_i \). Then \( \{E_i\} \subseteq \mathcal{S} \), \( V(x, E_i) < 2^{-i} \), and \( V(y, E_i) = V(h, E_i) + V(s, E_i) \leq V(h, X) + \sum_{j=i} V(s, F_j) < \|h\| + 1 < \infty \).

Sufficiency. Let \( y_i = P_{E_i} y \). Then \( \|y_{i+1} - y_i\| = \|V(y, E_i) - V(y, E_i')\| = V(y, E_i \cap E_i') = \|y_{i+1}\| - \|y_i\| \). Hence \( z = \lim_{i} y_i \) exists and \( \|z\| < \infty \); moreover, \( y - z \in G_a(x; T) \). By Theorem 3.1, there exist \( h \) and \( s \) such that \( z = h + s, h \in G_a(x, T) \), and \( s \in G_a(x, T) \). Finally, \( s = y - h = (y - z) + s \in G_a(x, T) \).

Example 3.1. Let \( X \) be the set of positive integers and let \( \mathcal{S} \) be the algebra of all subsets of \( X \). Let \( x \in G \) such that if \( E \in \mathcal{X} \) and \( E \notin \mathcal{S} \), then \( x(E) = \sum_{k \leq 2^n} 2^{-i} \). Let \( y \in \mathcal{G} \) such that \( y(X) = 0 \) and \( y([i]) = 1 \) for all \( i \in X \). Then \( y \) is not decomposable with respect to \( x \) (mod \( T \)).

Example 3.2. Let \( \mathcal{X} \) be the half open interval \([0, 1)\). Let \( \mathcal{S} \) be the algebra of subsets of \( \mathcal{X} \) generated by elements of the form \( E(m, n) = \left[ \frac{m}{2^n}, \frac{m+1}{2^n} \right], 0 \leq m < 2^n \), i.e., \( \mathcal{S} = \bigcup_{i \leq k} E(m_i, n_i); 0 \leq m_i < 2^n \). We shall define \( y \) inductively as follows. Let \( y(X) = 1 \), \( y(E(2m, n + 1)) = 2y(E(m, n)) \), and \( y(E(2m + 1, n + 1)) = -y(E(m, n)) \). Then \( y \) is unbounded on each nonempty element of \( S \). Hence \( y \) is decomposable with respect to no bounded finitely additive set function on \( S \) except the constant function \( 0 \); but, every finitely additive set function on \( S \) is absolutely continuous with respect to \( y \). Cameron (cf. [1]) has shown that a complex Wiener measure is unbounded on every nonempty set of the algebra on which it is defined.

Corollary 3.3.1. If \( y \) is an unbounded finitely additive set function on \( S \) (i.e., \( y \in \mathcal{G} \) and \( \|y\| = \infty \)), then there exists a bounded finitely additive set function \( x \) on \( S \) such that \( y \) is not decomposable with respect to \( x \).

Proof. Let \( K = \{E \in \mathcal{S}; V(y, E) < \infty \} \). Then \( K \) is a proper ideal in \( S \). There exists a maximal proper ideal \( J \) in \( S \) such that \( K \subset J \). There exists, uniquely, \( x \in G \) such that \( x(E) = 0 \) if \( E \in J \) and \( x(E) = 1 \) if \( E \notin J \). It is impossible to decompose \( y \) with respect to \( x \): if \( V(x, E) < 1 \), then \( E' \in K \) and, hence, \( V(y, E') = \infty \).

Theorem 3.4. Suppose \( S \) is a sigma algebra, \( y \) is a countably additive set function on \( S \), and \( x \) is a finitely additive set function on \( S \). Then \( y \) is \( (\varepsilon - \delta) \) absolutely continuous with respect to \( x \) (mod \( T \)) if and only if \( y \) is \( 0 - 0 \) absolutely continuous with respect to \( x \), i.e., if and only if \( E \in S \) and \( V(x, E) = 0 \) imply \( V(y, E) = 0 \).

Proof. Sufficiency. Suppose \( y \in G_a(x, T) \). Then \( \lim_{t \to 0^+} r(t, x, y) > 0 \), and, by Lemma 3.2, there exists a sequence \( \{E_i\} \subseteq \mathcal{S} \) such that \( \lim_{t \to 0^+} V(x, E_i) = 0 \).
and \( \lim_i V(y, E_i) > 0 \). Since \( S \) is a sigma algebra, \( E = \bigcap E_i \subseteq S \); moreover, 
\( V(x, E) \leq \lim_i V(x, E_i) = 0 \). Finally, since \( y \) is countably additive on \( S \), 
\( V(y, E) = \lim_i V(y, E_i) > 0 \).

**Bibliography**


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**ERRATA TO VOLUME 98**

C. C. Elgot. *Decision problems of finite automata design and related arithmetics*

Page 23, Lines 10, 11. Replace each \( f \) by \( \hat{f} \).

Page 23, 3.6(b), Line 2. The words “by a finite number . . . ” should start a new line.

Page 24, Line 9 (second display formula). Replace “(a, b)” by “(b, a)”.

Page 46, 8.6.2, Line 5. Replace “let \( n \) be the maximum” by “let \( n \) be one more than the maximum”.

Line 7. Replace “for some \( n \)-ary \( R \)” by “for some \( R \) which is \( n \)-ary”.

The third sentence (beginning on the sixth line) of §8.6.2 on page 46 is in error but is readily correctable. “It may be seen that \( T_{n+m+n}'(\Lambda z M) = S_1 \cup S_2 \cup \cdots \cup S_k \), where \( S_j, j = 1, 2, \cdots, k \), is the set of all infinite \( R \)-sequences \( f \) such that \( (f \upharpoonright n) \subseteq E_j \), for appropriate \( R_j \), \( E_j \), and that \( k \) need not be 1. For example, let \( M \) be

\[
0 \in F_1 \land 0 \in F_2 \land (x \in F_1 \land x \in F_2 \land \forall x \in F_1 \land x \in F_2) : \forall \cdot 0 \in F_1 \land 0 \in F_2 \land (x \in F_1 \land x \in F_2 \land \forall x \in F_1 \land x \in F_2).
\]

Then \( T_{n}^{*}(\Lambda z M) \) is the union of the set of all infinite sequences in \( \langle 1, 0 \rangle \) and \( \langle 1, 1 \rangle \) which begin with \( \langle 0, 1 \rangle \) and the set of all infinite sequences in \( \langle 0, 1 \rangle \) and \( \langle 1, 1 \rangle \) which begin with \( \langle 0, 1 \rangle \). Thus, in this case, \( k = 2 \). Let \( Q \) be

\[
(0 \in F_1 \land 0 \in F_2 \land \forall \cdot 0 \in F_1 \land 0 \in F_2) : \forall \cdot (x \in F_1 \land x \in F_2 \land x \in F_3 \land x' \in F_2) : \forall \cdot x \in F_1 \land x \in F_2 \land \forall x \in F_3 \land x' \in F_3 \land \forall x \in F_1 \land x \in F_2 \land x \in F_3 \land x' \in F_3).
\]

Then \( \Lambda z M = \forall F_1 \land \forall Q \), and \( T_{n}^{*}(\Lambda z Q) \) is a set of \( R \)-sequences, for the binary \( R \) indicated by the formula, beginning in a designated way and \( T_{n}^{*}(\Lambda z M) \) is a projection of \( T_{n}^{*}(\Lambda z Q) \). Quite generally it is the case that \( S_1 \cup S_2 \cup \cdots \cup S_k \) is the projection of a set of \( R \)-sequences beginning in a designated way so