A DECOMPOSITION FOR COMPLETE NORMED ABELIAN GROUPS WITH APPLICATIONS TO SPACES OF ADDITIVE SET FUNCTIONS

BY

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1. Introduction. The purpose of this paper is twofold. Our principal objective is to present a Lebesgue type decomposition Theorem (Theorem 2.3) for a generalized complete normed abelian group G, where generalized means (1) that the norm ($\|x\|$) of the nonzero elements x of G may be infinite (i.e. if $x\in G$ and $x\neq 0$, then $0<\|x\|\leqq\infty$) and (2) that only the subgroup of bounded elements x (i.e. $[x; \|x\|<\infty]$) is required to be complete. In §3, we apply this decomposition theorem to the space of finitely additive set functions on an algebra S of subsets of a set X in order to generalize the Lebesgue decomposition for bounded and finitely set functions on S (cf. [2]).

The basic form of our decomposition depends on what we call an admissible algebra T of endomorphisms on G (Definition 2.3). It will be seen that T is a Boolean algebra of projection operators with a condition on the manner in which projection on disjoint subgroups effects the norm. It is this latter condition which will provide our principal analytic tool.

Throughout this paper, G will denote a generalized complete normed abelian group.

2. Decompositions and examples. We shall develop the notion of an admissible algebra T of endomorphisms on G in two stages: the first algebraic and the second analytic.

Definition 2.1. A set T of endomorphisms on G is said to be an algebra of endomorphisms on G if whenever each of a and b is an element of T, then

(1) $ab=ba\in T$ where $ab(x)=a(b(x))$ for $x\in G$,

(2) $aa=a$, and

(3) $a'=e-a\in T$ where $e(x)=x$ for $x\in G$.

Moreover, for each element a of T we let $P(a)=[x\in G; a(x)=x]$.

We shall see that the mapping $a\to P(a)$ is an isomorphism of T onto a Boolean algebra of subgroups of G. We have, from (2), that $\|a\|=\|a^n\|\leqq\|a\|^n$ and, hence, if $a\neq 0$ then $\|a\|\geqq 1$ ($\|a\|$ may be infinite). Moreover, T has the following properties:

(i) $0\in T$ $(aa'=0)$,

(ii) $e\in T$ $(e=0')$, and

(iii) $a+b-ab=(a'b')'\in T$ [note that $a'b'(a+b-ab)=0$ and $a'b'+(a+b-ab)=a'b'+(ab+ab')+b-ab=(a'b'+ab')+b=b'+b=e$].

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Definition 2.2. If T is an algebra of endomorphisms on G and each of a and b is an element of T, then $a \leqq b$ means $ab = a$.

Theorem 2.1. *If each of a and b is an element of an algebra T of endomorphisms on G, then*

(i) $0 \leqq a \leqq e$,

(ii) $a \leqq b \Leftrightarrow a = ab \Leftrightarrow ab' = 0 \Leftrightarrow b' = a'b' \Leftrightarrow a' \geqq b' \Leftrightarrow b = a + a'b \Leftrightarrow$ *there exists an element c of T such that* $a = bc$,

(iii) $ab = 0 \Leftrightarrow a = ab' \Leftrightarrow a \leqq b'$,

(iv) *if* $ab = 0$, $c \leqq a$, *and* $d \leqq b$, *then* $cd = 0$,

(v) $a \cap b = ab$ *where* $a \cap b = \sup[c \in T;\ c \leqq a,\ c \leqq b]$,

(vi) $a \cup b = a + b - ab$ *where* $a \cup b = \inf[c \in T;\ c \geqq a,\ c \geqq b]$,

(vii) $a \leqq b \Leftrightarrow P(a) \subset P(b)$,

(viii) *if* $a \leqq b$, *then* $P(b) = P(a) \oplus P(a'b)$, *and*

(ix) $P(ab) = P(a) \cap P(b)$.

Proof. Parts (i), (ii), (iii), (iv), (v), and (vi) follow readily from our definitions. (vii) If $a \leqq b$, then $b'a = ab' = 0$ and, hence, if $ax = x$, then $b'(x) = b'a(x) = 0$. (viii) It follows from (vii) that $P(a) \oplus P(a'b) \subset P(b)$. Suppose $x \in P(b)$. Then $x = b(x) = (a + a'b)(x) = ax + a'b(x)$; however, $a(x) \in P(a)\,(aa = a)$ and $a'b(x) \in P(a'b)$. Thus, $x \in P(a) \oplus P(a'b)$. (ix) We have, by (vii), that $P(ab) \subset P(a) \cap P(b)$. Suppose $x \in P(a) \cap P(b)$. Then $ab(x) = a(b(x)) = a(x) = x$ and, hence, $x \in P(ab)$.

We shall now introduce our analytic tool which we shall denote by Property A.

Definition 2.3. If T is an algebra of endomorphisms on G, then T is said to be an admissible algebra of endomorphisms on G, if T has Property A: If $x \in G$, $\|x\| < \infty$, and $\delta > 0$, then there exists $\epsilon > 0$ such that if each of a and b is an element of T and $\|a'b(x)\| > \delta$, then $\|(a + a'b)(x)\| > \|a(x)\| + \epsilon$.

Remark. We note that Property A is a condition only on the bounded elements of G. At the end of this section, we shall give examples to show (1) that ϵ may depend only on δ (Example 2.2 with $Q = 1$), (2) that ϵ may depend on δ and $\|x\|$ but not on x (Example 2.2 with $Q > 1$), and (3) that ϵ may depend not only on δ and $\|x\|$ but also on x (Example 2.4).

Henceforth T shall denote an admissible algebra of endomorphisms on G.

Theorem 2.2. *Suppose each of a and b is an element of T, then* $a \leqq b$ *if and only if* $\|a(x)\| \leqq \|b(x)\|$ *for all* $x \in G$.

Proof. If $a \leqq b$, then $b = a + a'b$ and, hence, if $x \in G$, then $\|b(x)\| = \|(a + a'b)(x)\| \geqq \|a(x)\|$; in fact, inequality holds unless $\|a'b(x)\| = 0$. If $a \nleqq b$, then $ab' \neq 0$ and, hence, there exists an element x of G such that $\|ab'(x)\| \neq 0$. Thus, $a(ab'(x)) = ab'(x) \neq 0$ while $b(ab'(x)) = 0$.

Corollary 2.2.1. *If a is an element of T and* $a \neq 0$, *then* $\|a\| = 1$.

Proof. We have remarked earlier that $\|a\| = \|a^n\| \leqq \|a\|^n$ and, hence, $\|a\| \geqq 1$. By Theorem 2.2, we have that $\|a\| \leqq \|e\| = 1$. Thus, $\|a\| = 1$.

REMARK. Later we shall give an example (Example 2.1) to show that the condition: $a \leqq b$ if and only if $\|a(x)\| \leqq \|b(x)\|$ for each $x \in G$ is not sufficient to insure a decomposition. Property A is equivalent to: if $x \in G$, $\|x\| < \infty$, and $\delta > 0$, then there exists $\epsilon > 0$ such that if each of a and b is an element of T, $ab = 0$, and $\|b(x)\| > \delta$, then $\|(a+b)(x)\| > \|a(x)\| + \epsilon$.

LEMMA 2.3.1. *If* $x \in G$, $\{a_i\} \downarrow$ *in* T, *and* $\lim_i \|a_i(x)\| < \infty$, *then* $\lim_i a_i(x)$ *exists.*

Proof. Let $L = \lim_i \|a_i(x)\|$ and let $\delta > 0$. There exists a positive integer k such that $\|a_k(x)\| < \infty$. There exists $\epsilon > 0$ such that if each of c and d is an element of T and $\|c'da_k(x)\| > \delta$, then $\|(c+c'd)a_k(x)\| > \|ca_k(x)\| + \epsilon$. There exists a positive integer i such that $i \geqq k$ and $\|a_i(x)\| < L + \epsilon$. If $j > i$, then $a_i = a_j + a_j' a_i$. Thus, $\|a_i(x)\| = \|a_j + a_j' a_i(x)\| < L + \epsilon \leqq \|a_j(x)\| + \epsilon$ and, hence, $\|a_i(x) - a_j(x)\| = \|a_j' a_i(x)\| \leqq \delta$.

DEFINITION 2.4. If each of x and y is an element of G and $t > 0$, then

(1) $Q(t, x) = [a \in T; \|a(x)\| < t]$, and

(2) $r(t, x, y) = \sup [\|a(y)\|; a \in Q(t, x)]$.

LEMMA 2.3.2. *Suppose each of x and y is an element of G, $\|y\| < \infty$, $r(t) = r(t, x, y)$, $r = \lim_{t \to 0+} r(t) < \infty$, and $\epsilon > 0$. Then there exists a sequence $\{b_i\} \downarrow$ in T such that*

(1) $\lim_i b_i(x) = 0$,

(2) $\lim_i \|b_i(y)\| > r - \epsilon$, *and*

(3) $\lim_i b_i(y)$ *exists.*

Proof. If $r = 0$, it is sufficient to let $b_i = 0$ for $i \geqq 1$. Suppose $r > 0$ and m is a positive integer such that $2^{-m} < \epsilon$. Let $t_1 = 1$. There exists $\epsilon_1 > 0$ such that

(1) $\epsilon_1 < 2^{-(m+1)}$ and

(2) if $a, b \in T$ and $\|a'b(y)\| > 2^{-(m+1)}$, then $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_1$.

There exists $a_1 \in Q(t_1, x)$ such that $r(t_1) - \|a_1(y)\| < \epsilon_1$. Let $t_2 = 2^{-1}(t_1 - \|a_1(x)\|)$. If $a \in Q(t_2, x)$, then $\|(a_1 + a_1' a)(x)\| \leqq \|a_1(x)\| + \|a(x)\| < t_1$ and, hence, $\|(a_1 + a_1' a)(y)\| \leqq r(t_1) < \|a_1(y)\| + \epsilon_1$. Thus, $\|a_1' a(y)\| \leqq 2^{-(m+1)}$. There exists $\epsilon_2 > 0$ such that if $\|a'b(y)\| > 2^{-(m+2)}$, then $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_2$. There exists $a_2 \in Q(t_2, x)$ such that $r(t_2) < \|a_2(y)\| + \epsilon_2$. If we repeat the preceding process inductively, we obtain a sequence $\{a_i\}$ of elements of T, a sequence $\{e_i\}$ of positive numbers, and a sequence $\{t_i\}$ of positive numbers such that

(1) $t_1 = 1$ and $t_{i+1} = 2^{-1}(t_i - \|a_i(x)\|)$ for $i > 1$,

(2) $0 < \epsilon_i < 2^{-(m+i)}$,

(3) if $a, b \in T$ and $\|a'b(y)\| > 2^{-(m+i)}$, then $\|(a+a'b)(y)\| > \|a(y)\| + \epsilon_i$,

(4) $a_i \in Q(t_i, x)$,

(5) $r(t_i) < \|a_i(y)\| + \epsilon_i$, and

(6) if $a\in Q(t_{i+1}, x)$, then $(a_i+a_i'a)\in Q(t_i, x)$ which implies $\|(a_i+a_i'a)(y)\| \leqq r(t_i) < \|a_i(y)\|+\epsilon_i$ and hence, $\|a_i'a(y)\| \leqq 2^{-(m+i)}$.

For each positive integer i, $a_i = a_ia_{i-1}'+a_ia_{i-1}a_{i-2}'+ \cdots +\prod_{j\leq i} a_j$. Let $b_i = \prod_{j\leq i} a_j$. Then $\{b_i\}\downarrow$ in T. Moreover,

$$(1) \qquad \|b_i(x)\| \leqq \|a_i(x)\| \leqq 2^{-(i-1)},$$

$$(2) \qquad \begin{aligned} \|(a_i - b_i)(y)\| &\leqq \|a_ia_{i-1}'(y)\| + \|a_ia_{i-1}a_{i-2}'(y)\| \\ &\quad + \cdots + \left\|\left(\prod_{1<j\leq i} a_j\right)a_1'(y)\right\| \leqq \sum_{j<i} 2^{-(m+j)}, \quad \text{and} \end{aligned}$$

$$(3) \qquad \begin{aligned} r(t_i) - \|b_i(y)\| &\leqq r(t_i) - \|a_i(y)\| + \|(a_i - b_i)(y)\| \\ &< \epsilon_i + \sum_{j<i} 2^{-(m+j)} < 2^{-(m+i)} + \sum_{j<i} 2^{-(m+j)} < 2^{-(m)} < \epsilon. \end{aligned}$$

Hence, $\lim_i \|b_i(y)\| \geqq r-\epsilon$. However, $\lim_i b_i(y) \leqq \lim_i r(t_i) < \infty$ which implies (Lemma 2.3.1) that $\lim_i b_i$ exists.

DEFINITION 2.5. If each of x and y is an element of G, then y is said to be

(1) absolutely continuous with respect to x (mod T) if for each $\epsilon>0$, there exists $\delta>0$ such that if a is an element of T and $\|a(x)\|<\delta$, then $\|a(y)\|<\epsilon$, and

(2) singular with respect to x (mod T) if for each $\epsilon>0$, there exists an element a of T such that $\|a(x)\|<\epsilon$ and $\|a'(y)\|<\epsilon$. Moreover, we denote by $G_a(x, T)$ the set of elements h of G which are absolutely continuous with respect to x (mod T) and we denote by $G_s(x, T)$ the set of elements u of G which are singular with respect to x (mod T).

LEMMA 2.3.3. *If $x\in G$, then each of $G_a(x, T)$ and $G_s(x, T)$ is a subgroup of G and $G_a(x, T)\cap G_s(x, T)=0$. Moreover, if $h\in G_a(x, T)$, then $G_s(h, T)\supset G_s(x, T)$.*

Proof. Suppose each of y and z is an element of $G_a(x, T)$ and $\epsilon>0$, then there exists $\delta>0$ such that if $a\in T$ and $\|a(x)\|<\delta$, then each of $\|a(y)\|$ and $\|a(z)\|<\epsilon/2$ and, hence, $\|a(y+z)\|<\epsilon$. Thus, $G_a(x, T)$ is an algebraic subgroup of G. Suppose $\{y_i\}$ is a sequence of elements of $G_a(x, T)$, $\lim_i y_i = y$, and $\epsilon>0$. Then there exists a positive integer i such that $\|y_i-y\|<\epsilon/2$ and there exists $\delta>0$ such that if $a\in T$ and $\|a(x)\|<\delta$, then $\|a(y_i)\|<\epsilon/2$ and, hence, $\|a(y)\| \leqq \|a(y_i)\|+\|a(y-y_i)\| \leqq \|a(y_i)\|+\|y-y_i\|<\epsilon$. Thus, $y\in G_a(x, T)$. Suppose each of y and z is an element of $G_s(x, T)$ and $\epsilon>0$, then there exists a and $b\in T$ such that $\|a(x)\|<\epsilon/2$, $\|b(x)\|<\epsilon/2$, $\|a'(y)\|<\epsilon/2$ and $\|b'(z)\|<\epsilon/2$ and, hence, $\|(a+b-ab)(x)\| \leqq \|a(x)\|+\|(b-ab)(x)\| \leqq \|a(x)\|+\|b(x)\|<\epsilon$ and $\|(a+b-ab)'(y+z)\| = \|a'b'(y+z)\| \leqq \|a'(y)\|+\|b'(z)\|<\epsilon$. Suppose $\{y_i\}$ is a sequence of elements of $G_s(x, T)$, $\lim_i y_i = y$, and $\epsilon>0$. Then there exists a positive integer i such that $\|y_i-y\|<\epsilon/2$ and there exists $a\in T$ such that $\|a(x)\|<\epsilon/2$ and $\|a'(y_i)\|<\epsilon/2$ and, hence, $\|a'(y)\| \leqq \|a'(y_i)\|+\|a'(y-y_i)\| <\epsilon$. Therefore, $G_s(x, T)$ is a subgroup of G. Suppose $h\in G_a(x, T)$, $s\in G_s(x, T)$, and $\epsilon>0$. Then there exists $\delta>0$ such that if $a\in T$ and $\|a(x)\|<\delta$, then

$\|a(h)\| < \epsilon$ and, since $s \in G_s(x, T)$, there exists $a \in T$ such that $\|a(x)\| < \min[\epsilon, \delta]$ and $\|a'(s)\| < \min[\epsilon, \delta]$. Thus, $\|a(h)\| < \epsilon$ and $\|a'(s)\| < \epsilon$. Hence, $s \in G_s(h, T)$.

REMARK. We shall give two examples (Examples 3.1 and 3.2) to show that, in general, one can not assert that G is the direct sum of $G_a(x, T)$ and $G_s(x, T)$; however, Theorem 2.3 shows that $[y \in G; \|y\| < \infty] \subset G_a(x, T) \oplus G_s(x, T)$ for each $x \in G$.

LEMMA 2.3.4. *If each of x and y is a nonzero element of G, each of $\|x\|$ and $\|y\|$ is finite, and y is singular with respect to x, then* $\|x+y\| > \max[\|x\|, \|y\|]$.

Proof. Since the relation of being singular is symmetric, it is sufficient to show that $\|x+y\| > \|x\|$. There exists a sequence $\{a_i\}$ of elements of T such that $a_i x \to x$ and $a_i'(y) \to y$. There exists $\epsilon > 0$ such that if $a \in T$ and $\|a'(x+y)\| > \|y\|/2$, then $\|x+y\| = \|a(x+y) + a'(x+y)\| > \|a(x+y)\| + \epsilon$. Thus, since $a_i(x+y) \to x$ and $\|a_i'(x+y)\| \to y$, $\|x+y\| = \lim_i \|a_i(x+y) + a_i'(x+y)\| \geqq \|x\| + \epsilon$.

LEMMA 2.3.5. *Suppose each of x and y is an element of G, $\{a_i\} \downarrow$ in T, $z = \lim_i a_i(y)$, $r = \lim_{t \to 0+} r(t, x, y)$, and $\lim_i a_i(x) = 0$. Then $\|z\| \leqq r$.*

Proof. It is sufficient to suppose $r < \infty$. If $\epsilon > 0$, then there exists $t > 0$ such that if $a \in T$ and $\|a(x)\| < t$, then $\|a(y)\| < r + \epsilon/2$ and there exists a positive integer i such that $\|a_i(x)\| < t$ and $\|a_i(y) - z\| < \epsilon/2$. Thus, $\|z\| \leqq \|z - a_i(y)\| + \|a_i(y)\| < r + \epsilon$.

THEOREM 2.3. *Suppose each of x and y is an element of G and $\|y\| < \infty$. Then there exists uniquely an element h of G and an element s of G such that*

(1) $y = h + s$,

(2) *h is absolutely continuous with respect to x* (mod T), *and*

(3) *s is singular with respect to x* (mod T).

Proof. Uniqueness follows from Lemma 2.3.3; the problem is to show existence. Let $r = \lim_{t \to 0+} r(t, x, y)$. If $r = 0$, let $h = y$ and $s = 0$ ($y \in G_a(x, T)$ if and only if $r = 0$). Suppose $r > 0$. For each positive integer i, there exists $\epsilon_i > 0$ such that if each of a and b is an element of T and $\|a'b(y)\| > 2^{-i}$, then $\|(a + a'b)(y)\| > \|a(y)\| + \epsilon_i$. There exists (Lemma 2.3.2) a sequence $\{a(1, i)\} \downarrow$ in T and an element z_1 of G such that (1) $\lim_i a(1, i)(x) = 0$, (2) $z_1 = \lim_i a(1, i)(y)$, and (3) $r - \|z_1\| < \epsilon_1$. Let $y_1 = \lim_i a(1, i)'(y) = y - z_1$ and let $r_1 = \lim_{t \to 0+} r(t, x, y_1)$. We assert that $r_1 \leqq 2^{-1}$. Suppose, on the contrary, that $r_1 > 2^{-1}$. Then there exists a sequence $\{b_i\} \downarrow$ in T and an element w of G such that (1) $\lim_i b_i(x) = 0$, (2) $w = \lim_i b_i(y_1)$ and (3) $\|w\| > 2^{-1}$ (Lemma 2.3.2 again). However, $\lim_i \|b_i(y_1)\| = \lim_i \|b_i a(1, i)'(y)\|$ and, hence,

$$\|z_1 + w\| = \lim_i \|a(1, i)(y) + a(1, i)' b_i(y)\| \geqq \lim_i \|a(1, i)(y)\| + \epsilon_1$$

$$= \|z_1\| + \epsilon_1 > r;$$

but,

$$\lim_i \|(a(1,i)+a(1,i)'b_i)(x)\| \leqq \lim_i \|a(1,i)(x)\| + \|b_i(x)\| = 0.$$

This contradicts the supposition that $r_1 > 2^{-1}$. There exists a sequence $\{a(2,i)\}\downarrow$ in T and an element z_2 of G such that (1) $\lim_i a(2,i)(x)=0$, (2) $z_2=\lim_i a(2,i)(y_1)=\lim_i a(2,i)a(1,i)'(y)$, and (3) $r_1-\|z_2\|<\epsilon_2$. Let $y_2=\lim_i a(2,i)'(y_1)=\lim_i a(2,i)'a(1,i)'(y)$ and let $r_2=\lim_{t\to 0+} r(t, x, y_2)$. Then $r_2 \leqq 2^{-2}$. Proceeding by induction, either there exists a smallest positive integer i such that $r_i=0$ or $r_i>0$ for each positive integer i. In the former case we let $h=y_i$ and $s=\sum_{j\leqq i} z_j$ while in the latter case we let $h=\lim_i y_i$ and $s=\sum z_i$—of course, we must first show that each of $\lim_i y_i$ and $\sum z_i$ exists. Since $y_i=y-\sum_{j\leqq i} z_j$, it is sufficient to show that $\sum z_i$ exists and this is done as follows. Let $s_i=\sum_{j\leqq i} z_j$. If $j>i$, then $\|s_j-s_i\|=\|\sum_{k\leqq j} z_k-\sum_{k\leqq i} z_k\| =\|\sum_{i<k\leqq j} z_k\| \leqq \sum_{i<k\leqq j}\|z_k\| \leqq$ (Lemma 2.3.5) $\sum_{i<k\leqq j} r_{k-1} \leqq \sum_{i<k\leqq j} 2^{-(k-1)} < 2^{-(i-1)}$ and hence, $\lim_i s_i=\sum z_i$ exists. By our construction, each $z_i\in G_s(x)$ and, by Lemma 2.3.3, $G_s(x, T)$ is a subgroup of G. Thus, $s\in G_s(x, T)$. In order to complete a proof of Theorem 2.3, it is sufficient to show that $h\in G_a(x, T)$. To this end, suppose $\epsilon>0$ and $2^{-(i-1)}<\epsilon/2$. Then $\|h-y_i\| =\|s-s_i\| \leqq 2^{-(i-1)} < \epsilon/2$ and $r_i=\lim_{t\to 0+} r(t, x, y_i) \leqq 2^{-i} < \epsilon/2$ which implies that there exists $t>0$ such that $r(t, x, y_i)<\epsilon/2$. If $a\in T$ and $\|a(x)\|<t$, then $\|a(h)\| = \|a(h-y_i)+a(y_i)\| \leqq \|h-y_i\| + \|a(y_i)\| < \epsilon/2+\epsilon/2=\epsilon$. Therefore, $h\in G_a(x, T)$.

DEFINITION 2.6. The statement that a finite subset $[a_i;\ i\leqq n]$ of T is a finite partition of e in T means that $a_ia_j=0$ if $i\neq j$ and $\sum_{i\leqq n} a_i=e$.

THEOREM 2.4. *Suppose $x\in G$, $\|x\|<\infty$, and $\epsilon>0$. Then there exists a finite partition $P=[a_i;\ i\leqq n]$ of e in T such that if $a\in T$ and $i\leqq n$, then at least one of $\|a\ a_i(x)\|$ and $\|a'\ a_i(x)\|<\epsilon$.*

Proof. Suppose, on the contrary, that Theorem 2.4 is false. Then there exists a pair (x, ϵ) which contradicts Theorem 2.4: $\|x\|<\infty$, $\epsilon>0$, and if $[a_i;\ i\leqq n]$ is a finite partition of e in T then there exists an element a of T and a positive integer $i\leqq n$ such that each of $\|a\ a_i(x)\|$ and $\|a'\ a_i(x)\|\geqq\epsilon$. Moreover, since the pair (x, ϵ) contradicts Theorem 2.4, for each element a of T at least one of the pairs $(a(x), \epsilon)$ and $(a'(x), \epsilon)$ contradicts Theorem 2.4, i.e., if $P=[a_i;\ i\leqq m]$ works for $a(x)$ (i.e., if $b\in T$ and $i\leqq m$ imply at least one of $\|b\ a_i\ a(x)\|$ and $\|b'\ a_ia(x)\|<\epsilon$) and $Q=[b_j;\ j\leqq n]$ works for $a'(x)$, then $R=[a_ia;\ i\leqq m]\cup[b_ja';\ j\leqq n]$ works for x. Hence, there exists $a_1\in T$ such that (1) $\|a_1(x)\|\geqq\epsilon$ and (2) the pair $(a_1'(x), \epsilon)$ contradicts Theorem 2.4; $\cdots$; there exists $a_{i+1}\in T$ such that (1) $\|a_{i+1}\prod_{j\leqq i} a_j'(x)\|\geqq\epsilon$ and (2) the pair $(\prod_{j\leqq i+1} a_j'(x), \epsilon)$ contradicts Theorem 2.4. Let $b_i=\sum_{j\leqq i} a_j$. But, by Lemma 2.3.1, $\lim_i b_i'(x)$ exists and, hence, $\lim_i b_i(x)$ exists. This contradiction $(\|x\|<\infty)$ establishes Theorem 2.4.

We shall apply Theorem 2.4 in §3. However, we shall first conclude this section by giving four examples. Our first example sheds some light on the question: How strong an analytic condition is needed on an algebra U of endomorphisms on G in order to assure that Theorem 2.3 will hold (mod U)?

EXAMPLE 2.1. In this example, T will be an algebra of endomorphisms on G for which Theorem 2.3 does not hold; however, T will have the property that if $a, b \in T$, then $a \leqq b$ if and only if $\|a(x)\| \leqq \|b(x)\|$ for all $x \in G$.

Let S be an algebra of subsets of a set X, S contain an infinite number of elements, $G = [x; x$ is a real valued function on X, $\|x\| = \sup [|x(t)|; t \in X]]$, and $T = [P_E; P_E(x) = x \cdot C(E)$ where $C(E)(t) = 1$ if $t \in E$ and $C(E)(t) = 0$ if $t \notin E]$. Then there exist bounded elements x and y of G such that if each of h and s is an element of G, $y = h + s$, and h is absolutely continuous with respect to x (mod T), then s is not singular with respect to x (mod T).

Proof. Since S is infinite, there exists a sequence $\{E_i\}$ of non-null pairwise disjoint elements of S. Let $y = C(X)$ and $x = \sum 2^{-i} C(E_i)$. Suppose $y = h + s$ and $h \in G_a(x, T)$. Then there exists $\delta > 0$ such that if $E \in S$ and $\|P_E(x)\| < \delta$, then $\|P_E(h)\| < 2^{-1}$ and, hence, there exists a positive integer i such that

$$\left\| h \cdot C\left(\bigcup_{j>i} E_j \right) \right\| < 2^{-1}. \text{ Thus, for all } j > i,\ \inf[s(t); t \in E_j] \geqq 2^{-1}$$

and, hence, s is not singular with respect to x(mod T).

EXAMPLE 2.2. Let X, S, G, and T be defined as in Example 2.1 except that if $x \in G$, then $\|x\| = (\sum_{t \in X} |x(t)|^Q)^{1/Q}$, where Q is a real number $\geqq 1$. Then T is an admissible algebra of endomorphisms on G.

EXAMPLE 2.3. Let G be a Hilbert space, let $[E_\lambda; -\infty \leqq \lambda \leqq \infty]$ be a resolution of the identity, and let T be the algebra of projection operators generated by projections of the form $E_{\lambda+\mu} - E_\lambda$, $\mu \geqq 0$. Then T is an admissible algebra of endomorphisms on G.

EXAMPLE 2.4. Let X, S, G, and T be defined as in Example 2.1 except that X is the set of positive integers, if $x \in G$, then $\|x\| = |x(1)| + \sum_{i \geq 1} (|x(2i)|^i + |x(2i+1)|^i)^{1/i}$, and each one element subset $[i]$ of X is an element of S. For each positive integer i we let $x_i = C([2i, 2i+1])$ and we let $a_i = P_{[2i]}$. Then $\|x_i\| = 2^{1/i}$, $\|a_i(x_i)\| = 1$ and $\|a_i'(x_i)\| = 1$. Thus, in this example, while T is admissible, the ϵ we get in satisfying Property A depends not only on δ and $\|x\|$ but also on x.

3. **Spaces of finitely additive set functions.** Throughout this section, X will denote a set, S will denote an algebra of subsets of X, G will denote the generalized complete normed abelian group of finitely additive set functions on S where the norm ($\|x\|$) of the elements x of G is the total variation ($V(x, X)$) of x on X, and T will denote the admissible algebra of projection operators induced by S, i.e., $T = [P_E; P_E(x)(F) = x(E \cap F)$ for $E, F \in S$ and $x \in G]$.

Let us recall that if S is an infinite set, then there exist unbounded finitely additive set functions x on S (i.e., elements x of G such that $\|x\| = \infty$).

We shall extend the Lebesgue type decomposition for bounded and finitely additive set functions on a set algebra S which was presented in [2]. The definitions of absolute continuity and singularity which we use here are equivalent to those which were used in [2]. In order to make this paper self-contained with respect to notation and terminology, it is necessary to observe the following:

(1) $\|P_E(x)\| = V(x, E)$ for $E \in S$ and $x \in G$,

(2) $P_E P_F = P_{E \cap F}$,

(3) $P_E' = P_{(E')}$, where $E' = X - E$,

(4) if $E \cap F = \theta$: $P_E P_F = 0$, then $\|P_E(x) + P_F(x)\| = \|P_{E \cup F}(x)\| = \|P_E(x)\| + \|P_F(x)\|$ for all $x \in G$, and

(5) $P_E \leqq P_F$ if and only if $E \subset F$. Our first extension is the following consequence of Theorem 2.3.

THEOREM 3.1. *If x is a finitely additive set function on S and y is a bounded and finitely additive set function on S, then there exists uniquely an element h of G and an element s of G such that*

(1) $y = h + s$,

(2) *h is absolutely continuous with respect to x* (mod T), *and*

(3) *s is singular with respect to x* (mod T).

THEOREM 3.2. *If x is a bounded finitely additive set function on S and y is absolutely continuous with respect to x* (mod T), *then y is bounded.*

Proof. Since y is absolutely continuous with respect to x (mod T), there exists $\delta > 0$ such that if $E \in S$ and $V(x, E) < \delta$, then $V(y, E) < 1$. By Theorem 2.4, there exists a finite partition $[P_{E_i}; i \leqq n]$ of P_X in T such that if $E \in S$ and $i \leqq n$, then at least one of $V(x, E \cap E_i)$ and $V(x, E' \cap E_i) < \delta$ and, hence, at least one of $V(y, E \cap E_i)$ and $V(y, E' \cap E_i) < 1$. For each positive integer $i \leqq n$, $|y(E \cap E_i) - y(E_i)| = |y(E' \cap E_i)|$. Hence $|y(E \cap E_i)| < |y(E_i)| + 1$ for all $E \in S$. Thus, $V(y, E_i) \leqq 2$ $(\sup[|y(E \cap E_i)|; E \in S]) \leqq 2(|y(E_i)| + 1) < \infty$ for $i \leqq n$ and, hence, $\|y\| = V(y, X) = \sum_{i \leqq n} V(y, E_i) < \infty$.

In the general setting of §2, the analog of Theorem 3.2 is not, in general, true. For example, let S be infinite and let T' be the subalgebra of T which consists of 0 and e. Then any two nonzero elements of G are absolutely continuous with respect to each other (mod T'); but, there exist unbounded, as well as nonzero bounded, elements of G.

THEOREM 3.3. *If each of x and y is a finitely additive set function on S and at least one of x and y is bounded, then y is decomposable with respect to x* (mod T) *if and only if there exists a sequence $\{E_i\} \downarrow$ in S such that* $\lim_i V(x, E_i) = 0$ *and* $\lim_i V(y, E_i') < \infty$.

Proof. If $\|y\| < \infty$ a decomposition exists; moreover, it is sufficient to let $E_i = \theta$ for $i \geqq 1$. Suppose $\|y\| = \infty$ and $\|x\| < \infty$.

Necessity. Suppose $y = h + s$, $h \in G_a(x, T)$, and $s \in G_s(x, T)$. Then, by Theorem 3.3, $\|h\| < \infty$ and, by the definition of singularity, there exists a sequence $\{F_i\}$ of elements of S such that $V(x, F_i) < 2^{-i}$ and $V(y, F_i') < 2^{-i}$. Let $E_i = \prod_{j \leq i} F_j$. Then $\{E_i\} \downarrow$ in S, $V(x, E_i) < 2^{-i}$, and $V(y, E_i') = V(h, E_i') + V(s, E_i') \leqq V(h, X) + \sum_{j \leq i} V(s, F_j') < \|h\| + 1 < \infty$.

Sufficiency. Let $y_i = P'_{E_i}(y)$. Then $\|y_{i+j} - y_i\| = | V(y, E'_{i+j}) - V(y, E_i') | = V(y, E_i \cap E'_{i+j}) = \|y_{i+j}\| - \|y_i\|$. Hence $z = \lim_i y_i$ exists and $\|z\| < \infty$; moreover, $y - z \in G_s(x; T)$. By Theorem 3.1, there exist h and s_1 such that $z = h + s_1$, $h \in G_a(x, T)$, and $s_1 \in G_s(x, T)$. Finally, $s = y - h = (y - z) + s_1 \in G_s(x, T)$.

EXAMPLE 3.1. Let X be the set of positive integers and let S be the algebra of all subsets of X. Let $x \in G$ such that if $E \subset X$ and $E \neq \theta$, then $x(E) = \sum_{i \in E} 2^{-i}$. Let $y \in G$ such that $y(X) = 0$ and $y([i]) = 1$ for all $i \in X$. Then y is not decomposable with respect to x (mod T).

EXAMPLE 3.2. Let X be the half open interval $[0, 1)$. Let S be the algebra of subsets of X generated by elements of the form $E(m, n) = [m/2^n, (m+1)/2^n$, $0 \leqq m < 2^n]$, i.e., $S = [\bigcup_{i \leq k} E(m_i, n_i);\ 0 \leqq m_i < 2^{n_i}]$. We shall define y inductively as follows. Let $y(X) = 1$, $y(E(2m, n+1)) = 2y(E(m, n))$, and $y(E(2m+1, n+1)) = -y(E(m, n))$. Then y is unbounded on each nonempty element of S. Hence y is decomposable with respect to no bounded finitely additive set function on S except the constant function 0; but, every finitely additive set function on S is absolutely continuous with respect to y. Cameron (cf. [1]) has shown that a complex Wiener measure is unbounded on every nonempty set of the algebra on which it is defined.

COROLLARY 3.3.1. *If y is an unbounded finitely additive set function on S (i.e., $y \in G$ and $\|y\| = \infty$), then there exists a bounded finitely additive set function x on S such that y is not decomposable with respect to x.*

Proof. Let $K = [E \in S;\ V(y, E) < \infty]$. Then K is a proper ideal in S. There exists a maximal proper ideal J in S such that $K \subset J$. There exists, uniquely, $x \in G$ such that $x(E) = 0$ if $E \in J$ and $x(E) = 1$ if $E \notin J$. It is impossible to decompose y with respect to x: if $V(x, E) < 1$, then $E' \notin K$ and, hence, $V(y, E') = \infty$.

THEOREM 3.4. *Suppose S is a sigma algebra, y is a countably additive set function on S, and x is a finitely additive set function on S. Then y is $(\epsilon - \delta)$ absolutely continuous with respect to x* (mod T) *if and only if y is $0 - 0$ absolutely continuous with respect to x, i.e., if and only if $E \in S$ and $V(x, E) = 0$ imply $V(y, E) = 0$.*

Proof. *Sufficiency.* Suppose $y \notin G_a(x, T)$. Then $\lim_{t \to 0+} r(t, x, y) > 0$, and, by Lemma 3.2, there exists a sequence $\{E_i\} \downarrow$ in S such that $\lim_i V(x, E_i) = 0$

and $\lim_i V(y, E_i) > 0$. Since S is a sigma algebra, $E = \bigcap E_i \in S$; moreover, $V(x, E) \leqq \lim_i V(x, E_i) = 0$. Finally, since y is countably additive on S, $V(y, E) = \lim_i V(y, E_i) > 0$.

Bibliography

1. R. H. Cameron, *A family of integrals serving to connect the Wiener and Feynman integrals*, J. Math. and Phys. **39** (1960), 126–140.

2. R. B. Darst, *A decomposition of finitely additive set functions*, J. Math. Reine Angew. **210** (1962), 31–37.

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ERRATA TO VOLUME 98

C. C. Elgot. *Decision problems of finite automata design and related arithmetics*

Page 23, Lines 10, 11. Replace each $\hat{f}$ by $\hat{p}$.

Page 23, 3.6(b), Line 2. The words "by a finite number . . . " should start a new line.

Page 24, Line 9 (second display formula). Replace "(a, b)" by "(b, a)".

Page 46, 8.6.2, Line 5. Replace "let n be the maximum" by "let n be one more than the maximum".

Line 7. Replace "for some n-ary R" by "for some R which is n-ary".

The third sentence (beginning on the sixth line) of §8.6.2 on page 46 is in error but is readily correctable. "It may be seen that $T^{\infty}_{m+m'+r}(\bigwedge_x M) = S_1 \cup S_2 \cup \cdots \cup S_k$, where S_j, $j = 1, 2, \cdots, k$, is the set of all infinite R_j-sequences f such that $(f \restriction n) \in E_j$, for appropriate R_j, E_j, and that k need not be 1. For example, let M be

$$0 \in F_1 \wedge 0 \notin F_2 \wedge (x \in F_1 \wedge x \notin F_2 \cdot \vee \cdot x \in F_1 \wedge x \in F_2) : \vee :$$
$$0 \notin F_1 \wedge 0 \in F_2 \wedge (x \in F_1 \wedge x \in F_2 \cdot \vee \cdot x \notin F_1 \wedge x \in F_2).$$

Then $T^{\infty}_2(\bigwedge_x M)$ is the union of the set of all infinite sequences in $\langle 1, 0\rangle$ and $\langle 1, 1\rangle$ which begin with $\langle 1, 0\rangle$ and the set of all infinite sequences in $\langle 0, 1\rangle$ and $\langle 1, 1\rangle$ which begin with $\langle 0, 1\rangle$. Thus, in this case, $k = 2$. Let Q be

$$(0 \in F_1 \wedge 0 \notin F_2 \cdot \vee \cdot 0 \notin F_1 \wedge 0 \in F_2)$$
$$: \wedge : (x \in F_1 \wedge x \notin F_2 \wedge x \in F_3 \wedge x' \in F_3 \cdot \vee \cdot x \in F_1 \wedge x \in F_2 \wedge x \in F_3 \wedge x' \in F_3$$
$$\cdot \vee \cdot x \in F_1 \wedge x \in F_2 \wedge x \notin F_3 \wedge x' \notin F_3 \cdot \vee \cdot x \notin F_1 \wedge x \in F_2 \wedge x \notin F_3 \wedge x' \notin F_3).$$

Then $\bigwedge_x M \equiv \bigvee_{F_3} \bigwedge_x Q$ and $T^{\infty}_3 \bigwedge_x Q$ is a set of R-sequences, for the binary R indicated by the formula, beginning in a designated way and $T^{\infty}_2(\bigwedge_x M)$ is a projection of $T^{\infty}_3(\bigwedge_x Q)$. Quite generally it is the case that $S_1 \cup S_2 \cup \cdots \cup S_k$ is the projection of a set of R-sequences beginning in a designated way so