TENSOR PRODUCTS AND RELATED QUESTIONS

BY

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0. **Introduction.** The subject of tensor products of linear spaces has received attention from authors with varying points of view [2; 3; 4; 5; 8; 10]. It is the purpose of this paper to offer some contributions to the systematic development of this area, especially as it is concerned with tensor products of Banach algebras. There are points at which digressions from the main theme appear to be in order, and the results of these digressions seem to have independent interest.

1. **Tensor products.** If L_1 and L_2 are two modules over a ring R with identity (L_1 a right R-module, L_2 a left R-module) then $L_1 \otimes_R L_2$ is nothing more nor less than the quotient of the free abelian group (L_1, L_2) based on $L_1 \times L_2$ by the subgroup $N_R(L_1, L_2)$ generated by elements of the form $(l_i, l_i', l_i'' \in L_i, r \in R)$:

$$(l_1' + l_1'', l_2) - (l_1', l_2) - (l_1'', l_2),$$
$$(l_1, l_2' + l_2'') - (l_1, l_2') - (l_1, l_2''),$$
$$(l_1 r, l_2) - (l_1, r l_2);$$

i.e., $L_1 \otimes_R L_2 = (L_1, L_2)/N_R(L_1, L_2)$ [1]. When R is the real or complex number field and L_1 and L_2 are Banach spaces E_1 and E_2 or Banach algebras A_1 and A_2, the above definition takes no account of the essential aspects of these structures, that is of their norms. In [2; 3; 5· 8; 10] are given various modifications of the above definition for Banach spaces and algebras. We summarize in

Definition 1. If E_1 and E_2 are Banach spaces, let, in general, X^Y denote the set of all mappings from Y to X, and let

$$F_\gamma(E_1, E_2) = \Big\{ f \mid f \in C^{E_1 \times E_2}, f(0, x_2) = f(x_1, 0) = 0,$$
$$\|f\| \equiv \sum_{(x_1, x_2)} |f(x_1, x_2)| \, \|x_1\| \, \|x_2\| < \infty \Big\}$$

where C is the system of complex numbers, $x_i \in E_i$, $i = 1, 2$.

Regarded as an $L_1(\mu)$ space over $E_1 \times E_2$, where $\mu(x_1, x_2) = \|x_1\| \, \|x_2\|$, $F_\gamma(E_1, E_2)$ is a Banach space.

If E_i is a Banach algebra A_i, $i = 1, 2$, then $F_\gamma(A_1, A_2)$ may be made into

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a Banach algebra by defining multiplication of two functions $f, g \in F_\gamma(A_1, A_2)$ as follows:

$$f * g(x_1, x_2) = 0, \qquad \text{if } x_1 = 0 \text{ or } x_2 = 0,$$
$$= \sum_{a_1 b_1 = x_1;\, a_2 b_2 = x_2} f(a_1, a_2) g(b_1, b_2), \quad \text{otherwise.}$$

As indicated in [3] this is a valid definition (the series above converges and $f * g \in F_\gamma(A_1, A_2)$).

To produce a normed structure akin to $L_1 \otimes_R L_2$, we let I be the closed linear subspace generated by functions f satisfying one of the four conditions ($\alpha \in C$):

(i) $f(x_1, x_2') = f(x_1, x_2'') = -f(x_1, x_2' + x_2'')$, $f = 0$ otherwise;
(ii) $f(x_1', x_2) = f(x_1'', x_2) = -f(x_1' + x_1'', x_2)$, $f = 0$ otherwise;
(iii) $f(x_1, x_2) = -\alpha f(\alpha x_1, x_2)$, $f = 0$ otherwise;
(iv) $f(x_1, x_2) = -\alpha f(x_1, \alpha x_2)$, $f = 0$ otherwise;

and then define $E_3 \equiv E_1 \otimes_\gamma E_2$ as $F_\gamma(E_1, E_2)/I$. (Conditions (i)–(iv) correspond to the descriptions given earlier of the generators of $N_R(L_1, L_2)$.)

A simple computation, exploiting the fact that "finite" functions (i.e., functions vanishing at all but a finite number of points) are dense in both $F_\gamma(E_1, E_2)$ and I, reveals that when E_i is a Banach algebra A_i, $i = 1, 2$, then I is an *ideal* in the *algebra* $F_\gamma(A_1, A_2)$ whence $A_3 \equiv A_1 \otimes_\gamma A_2$ is a Banach algebra. The norm γ on E_3 (quotient norm for the quotient space) satisfies the equation $\gamma(x_1 \otimes x_2) = \|x_1\| \, \|x_2\|$.

As opposed to the usual constructions, the one described above produces $E_1 \otimes_\gamma E_2$ as a (complete) quotient space of a complete space. (Customarily one norms equivalence classes of finite expressions [5, p. 28; 8] and then completes the resulting structure.)

There is another important tensor product for Banach spaces, viz., $E_1 \otimes_\lambda E_2$ [8]. We shall define it by a direct construction:

Let

$$F_\lambda(E_1, E_2) = \Big\{ f \,\Big|\, f \in C^{E_1 \times E_2}, \ \sum_{(x_1, x_2)} f(x_1, x_2) \lambda_1(x_1) \lambda_2(x_2)$$
$$\text{convergent for all } (\lambda_1, \lambda_2) \in E_1^* \times E_2^* \Big\}.$$

In the above

$$\sum_{(x_1, x_2)} \cdots$$

is taken to mean the unordered sum and convergence of

$$\sum_{(x_1, x_2)} \cdots$$

means convergence of the finite partial sums regarded as a net over the directed set of finite subsets of $E_1 \times E_2$. A corollary to the Banach-Steinhaus theorem is the fact that for $f \in F_\lambda(E_1, E_2)$,

$$\|f\| \equiv \sup_{\|\lambda_1\|, \|\lambda_2\| \leq 1} \left| \sum_{(x_1, x_2)} f(x_1, x_2)\lambda_1(x_1)\lambda_2(x_2) \right| < \infty$$

and with this *semi*norm $F_\lambda(E_1, E_2)$ is a (non-T_2) topological vector space. If J is the closed linear hull of the generators used for I before, then $F_\lambda(E_1, E_2)/J$ is precisely $E_1 \otimes_\lambda E_2$, a Banach space. Again $\lambda(x_1 \otimes x_2) = \|x_1\| \, \|x_2\|$. There seems to be no clear way of making $A_1 \otimes_\lambda A_2$ (A_1, A_2 Banach algebras) into a Banach algebra. This last difficulty explains in part the favored role played by γ.

Grothendieck [5, p. 51] shows that $E_1 \otimes_\gamma E_2$ may also be regarded as $F_{\gamma 0}(E_1, E_2)/I_0$, where

$$F_{\gamma 0}(E_1, E_2) = \left\{ f \,\middle|\, f \in F_\gamma(E_1, E_2), f(x_1, x_2) = 0 \text{ if } \|x_1\| \text{ or } \|x_2\| \text{ is greater than } 1, \sum_{(x_1, x_2)} |f(x_1, x_2)| < \infty \right\},$$

and $I_0 = F_{\gamma 0} \cap I$. We note that when E_1 and E_2 are Banach algebras A_1 and A_2, then $F_{\gamma 0}(A_1, A_2)$ is again an algebra and I_0 is a closed ideal in this algebra. We shall return to this point later.

Thus far we have offered no detailed proofs because these can be found in the literature [3; 5, loc. cit.]. There is however one remark that we give as a proposition together with its proof.

PROPOSITION 1. *Let A_1 and A_2 be Banach algebras. Then $F_\gamma(A_1, A_2)$ has an identity if and only if both A_1 and A_2 have identities.*

Proof. The symbol $\delta_{(x_1, x_2)}$ will denote that member of $F_\gamma(E_1, E_2)$ equal to 1 at (x_1, x_2) and 0 otherwise. It is clear that if e_i is the identity of A_i, $i = 1, 2$, then $\delta_{(e_1, e_2)}$ is the identity in $F_\gamma(A_1, A_2)$.

On the other hand, let f be the identity of $F_\gamma(A_1, A_2)$. For any $g \in F_\gamma(A_1, A_2)$, $\bar{f} * g = (f * \bar{g})^- = \bar{\bar{g}} = g$, whence $f = \bar{f}$, i.e., f is real. We show next that $f \geqq 0$.

First we note that neither A_1 nor A_2 has an annihilator other than 0. For if $x_1 A_1 = 0$, and $x_1 \neq 0 \neq x_2 \in A_2$, then for

$$\begin{aligned} g = \delta_{(x_1, x_2)}, \qquad g * f(x_1, x_2) &= 1, \\ &= \sum_{x_1 y_1 = x_1;\, x_2 y_2 = x_2} f(y_1, y_2). \end{aligned}$$

But $x_1 y_1 = 0 \neq x_1$, whence the right sum is empty and cannot equal 1.

Now let $S = \{(c_n, d_n) \mid f(c_n, d_n) \neq 0\}$. S is a countable set since

$$\sum_{(x_1,x_2)} |f(x_1, x_2)| \, \|x_1\| \, \|x_2\| < \infty.$$

Let $\{t_n\}$ be an enumeration of all *nonzero* differences $c_p - c_q$, $\{s_n\}$ an enumeration of all *nonzero* differences $d_p - d_q$. (If either $\{t_n\}$ or $\{s_n\}$ is empty an argument given below applies. This is a less interesting and less involved situation.)

Choose $k_1 \in A_1$ such that $k_1 t_1 \neq 0$. If $k_1 t_2 \neq 0$, let $k_2 = k_1$. If $k_1 t_2 = 0$ let $\epsilon \in A_1$ be such that $\epsilon t_2 \neq 0$. Then for $\|\epsilon\|$ small enough we can be certain that $k_1 + \epsilon \equiv k_2$ satisfies:

(a) $\|k_2 t_1\| > (1/2)\|k_1 t_1\|$;

(b) $k_2 t_2 \neq 0$;

(c) $\|k_2 - k_1\| < 1/2$.

Having chosen $k_1, k_2, \cdots, k_n$ satisfying:

(a_1) $\|k_n t_j\| > (1/2)\|k_j t_j\|, j = 1, 2, \cdots, n-1$;

(b_1) $k_n t_n \neq 0$;

(c_1) $\|k_{j+1} - k_j\| < 1/2^j, j = 1, 2, \cdots, n-1$;

choose k_{n+1} to be k_n if $k_n t_{n+1} \neq 0$. Otherwise let $\epsilon_1 t_{n+1} \neq 0$ and then choose $\|\epsilon_1\|$ so small that $k_{n+1} \equiv k_n + \epsilon_1$ satisfies

(a_2) $\|k_{n+1} t_j\| > (1/2)\|k_j t_j\|, j = 1, 2, \cdots, n$;

(b_2) $k_{n+1} t_{n+1} \neq 0$;

(c_2) $\|k_{n+1} - k_n\| < 1/2^n$.

Then $\lim_{n\to\infty} k_n = k$ exists, $\|k t_j\| \geqq (1/2)\|k_j t_j\| > 0$, all j.

Similarly choose $l \in A_2$, $\|l s_j\| > 0$, all j, and let $g = \delta_{(k,l)}$. Since $f * g = g$, we see

$$\sum_{ck=k;\, dl=l} f(c, d) = 1.$$

Our choice of k and l assures us that for only one pair (c_{n_0}, d_{n_0}) in the range of summation is the term $f(c, d) \neq 0$, whence $f(c_{n_0}, d_{n_0}) = 1$.

Assume $f(c_{n_1}, d_{n_1}) < 0$ and let $h = M\delta_{(k,l)} + \delta_{(kc_{n_1}, ld_{n_1})}$. Then

$$\begin{aligned} 1 = h(kc_{n_1}, ld_{n_1}) &= \sum_{xc=kc_{n_1};\, yd=ld_{n_1}} h(x, y) f(c, d) \\ &= M \sum_{kc=kc_{n_1};\, ld=ld_{n_1}} f(c, d) + \sum_{kc_{n_1}c=kc_{n_1};\, ld_{n_1}d=ld_{n_1}} f(c, d). \end{aligned}$$

Our choice of k, l insures that the first sum consists of one term $Mf(c_{n_1}, d_{n_1})$. Hence the right member can be made arbitrarily large (negatively) since the second sum is finite and independent of M. This contradiction shows $f \geqq 0$ in this case.

In case one of the sets, say $\{t_n\}$, is empty, then all c_n are equal, say to c_0. Thus for any $x_1 \neq 0$, $x_2 \neq 0$, $g = \delta_{(x_1, x_2)}$ we find

$$\sum_{c_0 y_1 = x_1;\, d_n y_2 = x_2} f(c_0, d_n) g(y_1, y_2) = \sum_{c_0 x_1 = x_1;\, d_n x_2 = x_2} f(c_0, d_n) = 1.$$

This means the sum is not empty and $c_0x_1=x_1$ for all x_1, and similarly $x_1c_0=x_1$ for all x_1, whence c_0 is an identity for A_1.

Again using $g=\delta_{(x_1,x_2)}$, we find from the equation $f * g=g$, that

$$\sum_{cx_1=t_1;\, dx_2=t_2} f(c, d) = \begin{cases} 1, & \text{if } t_1 = x_1,\ t_2 = x_2, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f\geqq 0$, we see $f(c, d)=0$ if $cx_1\neq x_1$ or $dx_2\neq x_2$. Thus if $f(c, d)>0$, then $cx_1=x_1$, $dx_2=x_2$, any x_1, x_2. Similarly, from the equation $g * f=g$ we conclude $x_1c=x_1$; $x_2d=x_2$ if $f(c, d)>0$. Thus if $f(c, d)>0$, c and d are (unique) identities for A_1 and A_2. In other words for only one pair (c, d) is $f(c, d)>0$, c is the identity of A_1, d the identity of A_2 and $f(c, d)=\delta_{(c,d)}$.

A corollary to the proof is the fact that $F_{\gamma 0}(A_1, A_2)$ has an identity if and only if both A_1 and A_2 have identities.

PROPOSITION 2.
(a) $(F_\gamma(E_1, E_2))^*=\{\lambda \mid \lambda\in C^{E_1\times E_2},\ |\lambda(x_1, x_2)| \leqq K\|x_1\|\ \|x_2\|\}$.
(b) $(E_1 \otimes_\gamma E_2)^*=\mathrm{Hom}_b(E_1, E_2^*)=\mathrm{Hom}_b(E_2, E_1^*)$.
(c) *For A_1, A_2 commutative Banach algebras,*

$$\mathfrak{M}_{F_\gamma(A_1,A_2)} = \{\chi \mid \chi \in C^{A_1\times A_2},\ \chi(a_1, a_2)\chi(b_1, b_2) = \chi(a_1b_1, a_2b_2),\ |\chi(a_1, a_2)| \leqq \|a_1\|\,\|a_2\|\}.$$

(d) *For A_1, A_2 commutative Banach algebras,*

$$\mathfrak{M}_{A_1\otimes_\gamma A_2} = \{\chi \mid \chi \in \mathfrak{M}_{F_\gamma(A_1,A_2)},\ \chi \textit{ bilinear}\}$$

[2; 3; 5, p. 30; 6] ($\mathrm{Hom}_b(\cdots)$ *and* $\mathfrak{M}\cdots$ *are explained in the proof*).

Proof. We note first that the set $\{\delta_{(x_1,x_2)}\}$ is a basis for $F_\gamma(E_1, E_2)$ (and for $F_\lambda(E_1, E_2)$). Furthermore, for $f\in F_\gamma(E_1, E_2)$ (or $F_\lambda(E_1, E_2)$),

$$f = \sum_{(x_1,x_2)} f(x_1, x_2)\delta_{(x_1,x_2)}.$$

(a) If $\Lambda\in(F_\gamma(E_1, E_2))^*$, let $\Lambda(\delta_{(x_1,x_2)})=\lambda(x_1, x_2)$. Then $|\lambda(x_1, x_2)| \leqq K\|\delta_{(x_1,x_2)}\|=K\|x_1\|\ \|x_2\|$ where $\|\Lambda\|=K$, and

$$\Lambda(f) = \sum_{(x_1,x_2)} f(x_1, x_2)\lambda(x_1, x_2).$$

It is clear also that $K=\sup\{|\lambda(x_1, x_2)|/\|x_1\|\ \|x_2\| \mid x_1, x_2\neq 0\}$. Conversely, if $|\lambda(x_1, x_2)| \leqq K\|x_1\|\ \|x_2\|$, then Λ defined by

$$\Lambda(f) = \sum_{(x_1,x_2)} f(x_1, x_2)\lambda(x_1, x_2)$$

is in $(F_\gamma(E_1, E_2))^*$ and $\|\Lambda\|=\sup\{|\lambda(x_1, x_2)|/\|x_1\|\ \|x_2\| \mid x_1, x_2\neq 0\}$.

(b) Since $E_1 \otimes_\gamma E_2 = F_\gamma(E_1,E_2)/I$, $(E_1 \otimes_\gamma E_2)^*$ is precisely $I^\perp\subset (F_\gamma(E_1, E_2))^*$, i.e., the continuous linear functionals on $F_\gamma(E_1, E_2)$ that annihilate I. But $\Lambda \in I^{\perp}$ if and only if Λ annihilates the generators of I. This implies that $\lambda(x_1, x_2)$ is a bilinear functional which means that $\lambda(x_1, x_2)$ for fixed x_2 may be regarded as an element of E_1^* and that this association between x_2 and elements of E_1^* is a bounded linear mapping. The set of such bounded linear mappings is denoted by $\mathrm{Hom}_b(E_2, E_1^*)$. It is clear that each element of $\mathrm{Hom}_b(E_2, E_1^*)$ corresponds to a $\Lambda \in I^{\perp}$ and that the correspondence is 1-1. By reversing the roles of x_1 and x_2 the equation $(E_1 \otimes_\gamma E_2)^* = \mathrm{Hom}_b(E_1, E_2)^*$ is proved.

(c) $\mathfrak{M} \cdots$ denotes the space of regular maximal ideals of $\cdots$. The proofs of (c) and (d) are straight-forward extensions of the proofs of (a) and (b) when notice is taken of the *multiplicative* nature of the functionals involved [6].

2. **Heredity.** Before we pursue further developments of the ideas just presented, we shall, within the limited context thus far available, discuss the problem of heredity and a related problem.

Definition 1. Let P be a property of Banach spaces in a certain class. (If the Banach space E has property P we write: $P(E)$.) P is called *hereditary under* $\otimes_\gamma$ (or simply *hereditary*) if:

$$P(E_1) \quad \text{and} \quad P(E_2) \Rightarrow P(E_1 \otimes_\gamma E_2).$$

A simple example of an hereditary property P is that of commutativity in the class of Banach algebras. We shall find below a number of hereditary properties P together with a property P for which the problem of heredity leads to extremely general and central problems in functional analysis.

Theorem 1. *The following properties are hereditary*:

P_1: *regularity* (*in the class of commutative Banach algebras*);

P_2: *density of* $\mathfrak{F}^{-1}(C_0(\mathfrak{M}))$ (*same*); (*see proof below for explanation of symbols*);

P_3: *presence of a basis* (*in the class of Banach spaces*);

P_4: *presence of a continuous involution* (*in the class of Banach algebras*);

P_5: *analyticity* (*in the class of commutative Banach algebras*);

P_6: *self-adjointness* (*in the class of commutative Banach algebras*);

P_7: *antisymmetry* (*in the class of commutative Banach algebras*);

P_8: *simplicity, i.e., absence of regular ideals* (*in the class of Banach algebras*).

Proof. P_1. Let A_1 and A_2 be regular (commutative) Banach algebras. From [2; 3] we know that $\mathfrak{M}_{A_1 \otimes_\gamma A_2}$ and $\mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}$ are homeomorphic in a natural way when weak* topologies are used throughout. To prove $P_1(A_1 \otimes_\gamma A_2)$ we choose a point $(M_{10}, M_{20}) \in \mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}$ and a neighborhood of the form $U_1(M_{10}) \times U_2(M_{20})$. In A_i we choose an x_i such that $\hat{x}_i(M_{i0}) = 1$, $\hat{x}_i(M_i) = 0$, $M_i \notin U_i(M_{i0})$, $i = 1, 2$. Then clearly in $A_1 \otimes_\gamma A_2$ the element z represented by $x_1 \otimes x_2$ has the property: $\hat{z}(M_{10}, M_{20}) = 1$, $\hat{z}(M_1, M_2) = 0$ if $(M_1, M_2) \notin U_1(M_{10}) \times U_2(M_{20})$.

P_2. [10]. If A is a commutative Banach algebra with maximal ideal space $\mathfrak{M}$, let $\mathfrak{F}: A \to C(\mathfrak{M})$ be the canonical map taking $x \in A$ into $\hat{x}(M)$ in $C(\mathfrak{M})$. Let $C_0(\mathfrak{M})$ be the set of functions of $C(\mathfrak{M})$ with compact support. Then $\mathfrak{F}^{-1}(C_0(\mathfrak{M}))$ is the set of elements x in A such that $\hat{x}(M)$ has compact support. The denseness of this set in A is critical in the Wiener Tauberian theorem.

We now assume P_2 obtains for A_1 and A_2 and begin with

LEMMA 1. *If z in $A_1 \otimes_\gamma A_2$ is represented by $x_1 \otimes x_2$, then support $\hat{z} = ($support $\hat{x}_1) \times ($support $\hat{x}_2)$.*

Proof. Let $(M_{10}, M_{20}) \in ($support $\hat{x}_1) \times ($support $\hat{x}_2)$. If $(M_{10}, M_{20}) \notin$ support $\hat{z}$, there are neighborhoods $U_i(M_{i0})$, $i=1, 2$ such that $\hat{z}(M_1, M_2) \equiv \hat{x}_1(M_1)\hat{x}_2(M_2) = 0$ for $(M_1, M_2) \in U_1(M_{10}) \times U_2(M_{20})$. Hence either $\hat{x}_1(M_1) \equiv 0$ in $U_1(M_{10})$ (which contradicts: $M_{10} \in$ support $\hat{x}_1$) or there is an $M_{11} \in U_1(M_{10})$ such that $\hat{x}_1(M_{11}) \neq 0$. In the latter case $\hat{x}_2(M_2) \equiv 0$ in $U_1(M_{10})$ and this second contradiction forces the conclusion: $($support $\hat{x}_1) \times ($support $\hat{x}_2) \subset$ support $\hat{z}$.

On the other hand, if $(M_{10}, M_{20}) \in$ support $\hat{z}$ and, say, $M_{10} \notin$ support $\hat{x}_1$, then there is a neighborhood $U_1(M_{10})$ such that $\hat{x}_1(M_1) = 0$ in $U_1(M_{10})$. Then for arbitrary M_2, $\hat{z}(M_1, M_2) = 0$, $(M_1 \in U_1(M_{10}))$, and thus $(M_{10}, M_{20}) \notin$ support $\hat{z}$.

We return to the proof proper. The essence of the idea is to show that elements in $A_1 \otimes_\gamma A_2$ represented by *finite* sums of the form

$$\sum_{(x_1, x_2)} f(x_1, x_2)\delta_{(x_1, x_2)},$$

where $\hat{x}_i$ has compact support, $i=1, 2$, whenever $f(x_1, x_2) \neq 0$, are dense in $A_1 \otimes_\gamma A_2$. Thus let z be arbitrary in $A_1 \otimes_\gamma A_2$ and let it be represented by f. Then we may use the formal sum

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n}),$$

where $\{(x_{1n}, x_{2n})\}_{n=1}^{\infty}$ is the set where f is not zero and $a_n = f(x_{1n}, x_{2n})$, to represent z.

For $\epsilon > 0$, choose u_{1n}, u_{2n} such that the supports of $\hat{u}_{1n}$, $\hat{u}_{2n}$ are compact and such that

$$\|x_{1n} - u_{1n}\| < \epsilon/6 \cdot 2^n K, \qquad \|x_{2n} - u_{2n}\| < \epsilon/6 \cdot 2^n K,$$

where $K = \|f\|$. Set $x_{in} = u_{in} + v_{in}$, $i=1, 2$, $n = 1, 2, \cdots$. Then the formal sum

$$\sum_{n=1}^{\infty} a_n(u_{1n} \otimes u_{2n}) + \sum_{n=1}^{\infty} a_n(u_{1n} \otimes v_{2n})$$
$$+ \sum_{n=1}^{\infty} a_n(v_{1n} \otimes u_{2n}) + \sum_{n=1}^{\infty} a_n(v_{1n} \otimes v_{2n})$$

also corresponds to a representative of z. Let the first group of terms of the formal sum represent z_ϵ. Clearly $\gamma(z-z_\epsilon)<\epsilon/2$. But for large enough N,

$$\sum_{n=1}^{N} a_n(u_{1n} \otimes u_{2n})$$

represents a $z_{\epsilon N}$ with the property $\gamma(z-z_{\epsilon N})<\epsilon$ and clearly $z_{\epsilon N}$ has compact support.

P_3. This follows from results in [4].

P_4. If * represents a continuous involution in each of A_1 and A_2, we can define an involution, again denoted by *, in $F_\gamma(A_1, A_2)$ by the formula:

$$f^*(x_1, x_2) = (f(x_1^*, x_2^*))^-.$$

Then

$$\begin{aligned}\|f^*\| &= \sum_{(x_1,x_2)} \left| (f(x_1^*, x_2^*))^- \right| \|x_1\| \, \|x_2\| \\ &\leqq K \sum_{(x_1,x_2)} \left| f(x_1^*, x_2^*) \right| \|x_1^*\| \, \|x_2^*\|\end{aligned}$$

where, if $\|x_i^*\| \leqq K_i\|x_i\|$, $i=1, 2$, $K\geqq K_1, K_2$. Furthermore for $f\in I$, $f^*\in I$. Thus for $z\in A_1 \otimes_\gamma A_2$ and f representing z, the formula $z^*=(f^*/I)$ defines z^* unambiguously. If $g_n\in I$, $\|f-g_n\|\to\gamma(z)$ as $n\to\infty$, then $\|f^*-g_n^*\|\leqq K\|f-g_n\|$, whence $\gamma(z^*)\leqq\|f^*-g_n^*\|\leqq K\gamma(z)$ and the involution is continuous.

P_5. Assume A_1 and A_2 are analytic (commutative) Banach algebras. This means that if $x_i\in A_i$ and $\hat{x}_i(M_i)=0$ for all M_i in an open set of $\mathfrak{M}_{A_i}$, then $\hat{x}_i\equiv 0$. We must show that if $z\in A_1 \otimes_\gamma A_2$ and $\hat{z}(M_1, M_2)$ is zero for all (M_1, M_2) in an open set of $\mathfrak{M}_{A_1\otimes_\gamma A_2}$, then $\hat{z}\equiv 0$. Thus if $\hat{z}=0$ on an open set, there are open sets $U_i\subset\mathfrak{M}_{A_i}$, $i=1, 2$, such that $\hat{z}=0$ on $U_1\times U_2$. Fix $M_{20}\in U_2$. Then $\hat{z}(M_1, M_{20})$, regarded as a function on $\mathfrak{M}_{A_1}$, actually arises from an element of A_1. Indeed if z is represented by

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n}),$$

then the element

$$x_1 = \sum_{n=1}^{\infty} a_n\hat{x}_{2n}(M_{20})x_{1n}$$

of A_1 serves (the series converges absolutely). Hence $\hat{x}_1\equiv 0$ since A_1 is analytic. Similarly for M_{10} fixed in U_1, $\hat{z}(M_{10}, M_2)$ regarded as a function on $\mathfrak{M}_{A_2}$ arises from the element

$$x_2 = \sum_{n=1}^{\infty} a_n\hat{x}_{1n}(M_{10})x_{2n}$$

of A_2 and thus $\hat{x}_2 \equiv 0$. For arbitrary $M_{11} \in \mathfrak{M}_{A_1}$ we find then $\hat{z}(M_{11}, M_2) = 0$ for $M_2 \in U_2$, whence $\hat{z}(M_{11}, M_2) = 0$ for arbitrary M_2 and the result follows.

P_6. The proof of this resembles that for P_4 and is omitted.

P_7. A commutative Banach algebra A is called antisymmetric if the existence of a pair x, y in A such that $\hat{x} = \hat{y}^-$ implies either that $x = y = 0$ (modulo the radical) or that there is an identity e in A and that for some constant α, $x = \alpha e$, $y = \bar{\alpha} e$. Now assume z and w in $A_1 \otimes_\gamma A_2$ satisfy: $\hat{z} = \hat{w}^-$. As in the proof of P_5, and in the notation of that proof, the elements x_1 and x_2 exist, and corresponding to w there are elements y_1 and y_2. It follows that $\hat{x}_1 = \hat{y}_1^-$, $\hat{x}_2 = \hat{y}_2^-$, and a repetition of the argument in the proof of P_5 then shows that either $z = w = 0$ (modulo the radical) or that $A_1 \otimes_\gamma A_2$ has an identity e and that for some constant α, $z = \alpha e$, $w = \bar{\alpha} e$.

P_8. Assume A_1 and A_2 contain no regular ideals. Let R be a regular ideal in $A_1 \otimes_\gamma A_2$. Then there is a regular maximal ideal $M \supset R$ and M engenders [3] regular maximal ideals M_1 and M_2 in A_1 and A_2. However, this contradicts the assumption of simplicity of A_1 and A_2.

THEOREM 2. *Let A_1 and A_2 be commutative Banach algebras with Šilov boundaries Γ_{A_1} and Γ_{A_2}. When the maximal ideal space of $A_1 \otimes_\gamma A_2$ is identified with $\mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}$, the Šilov boundary of $A_1 \otimes_\gamma A_2$ is mapped onto $\Gamma_{A_1} \times \Gamma_{A_2}$.*

Proof. (a) $\Gamma_{A_1} \times \Gamma_{A_2} \subset \Gamma_{A_1 \otimes_\gamma A_2}$. Let $(M_{10}, M_{20}) \in \Gamma_{A_1} \times \Gamma_{A_2}$ and let $U(M_{10}, M_{20})$ be a neighborhood of (M_{10}, M_{20}). We can find neighborhoods $U_1(M_{10})$ and $U_2(M_{20})$ such that $U_1(M_{10}) \times U_2(M_{20}) \subset U(M_{10}, M_{20})$. Then we can find $x_i \in A_i$, $i = 1, 2$, such that $|\hat{x}_i|$ reaches its maximum in $U_i(M_{i0})$ and is strictly smaller outside $U_i(M_{i0})$. Then z represented by $x_1 \otimes x_2$ is such that $|\hat{z}|$ reaches its maximum in $U_1(M_{10}) \times U_2(M_{20})$ and is strictly smaller outside $U_1(M_{10}) \times U_2(M_{20})$, *a fortiori* strictly smaller outside $U(M_{10}, M_{20})$. Hence $(M_{10}, M_{20}) \in \Gamma_{A_1 \otimes_\gamma A_2}$.

(b) $\Gamma_{A_1 \otimes_\gamma A_2} \subset \Gamma_{A_1} \times \Gamma_{A_2}$. Let $z \in A_1 \otimes_\gamma A_2$. Then for fixed M_2, $\hat{z}(M_1, M_2) \in \hat{A}_1$ and thus

$$|\hat{z}(M_1, M_2)| \leqq \sup\{ |\hat{z}(\gamma_1, M_2)| \, \gamma_1 \in \Gamma_{A_1} \}.$$

Let $|\hat{z}(\gamma_{10}, M_2)| = \sup |\hat{z}(M_1, M_2)|$. Then

$$\begin{aligned} |\hat{z}(\gamma_{10}, M_2)| &\leqq \sup\{ |\hat{z}(\gamma_{10}, \gamma_2)| \mid \gamma_2 \in \Gamma_{A_2} \} \\ &\leqq \sup\{ |\hat{z}(\gamma_1, \gamma_2)| \mid (\gamma_1, \gamma_2) \in \Gamma_{A_1} \times \Gamma_{A_2} \}. \end{aligned}$$

Hence for arbitrary (M_1, M_2), we find

$$|\hat{z}(M_1, M_2)| \leqq \sup\{ |\hat{z}(\gamma_1, \gamma_2)| \mid (\gamma_1, \gamma_2) \in \Gamma_{A_1} \times \Gamma_{A_2} \},$$

and this shows that $\Gamma_{A_1 \otimes_\gamma A_2} \subset \Gamma_{A_1} \times \Gamma_{A_2}$.

Conspicuously absent from the list of properties in Theorem 1 is the property of semisimplicity. Its omission is due to our inability to prove that semisimplicity is hereditary and not to the "fact" that semisimplicity is known not to be hereditary. Because of the ramifications of the problem, we shall treat it in a separate section.

3. **Semisimplicity.** By definition, a Banach algebra A is semisimple if and only if its radical is $\{0\}$. The radical, for our purposes, is not the Jacobson radical but the intersection of all regular maximal two-sided ideals of A. Thus our basic question is: If A_1 and A_2 are semisimple, is $A_1 \otimes_\gamma A_2$ semisimple? A partial solution to this problem has been found by Tomiyama [10] and was discovered independently by the present writer in the course of the research presented herein. (Note: If $A_1 \otimes_\gamma A_2$ is semisimple, clearly so are A_1 and A_2.)

We saw earlier (§1) that $A_1 \otimes_\gamma A_2$ deserves special attention because it seems to be *the* tensor product that is also a nontrivial Banach algebra. When A_1 and A_2 are semisimple (*which we shall assume hereafter in this section*) the following theorem holds.

THEOREM 1. *There is a tensor product $A_1 \otimes_\nu A_2$ that is a semisimple Banach algebra. The norm ν is a cross-norm and the elements corresponding to representatives that are finite formal sums (not in I)*

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n}),$$

$K=1, 2, \cdots$, are all nonzero and are dense in $A_1 \otimes_\nu A_2$. The norm is of "general character" [8], $\lambda \leqq \nu \leqq \gamma$ and $\nu=\gamma$ if and only if $A_1 \otimes_\gamma A_2$ is semisimple.

Proof. As we know from Proposition 2, §1 and its extension to the noncommutative case [3], $\mathfrak{M}_{A_1 \otimes_\gamma A_2}$ may be identified with a subset of $\mathfrak{M}_{F_\gamma(A_1,A_2)}$, namely the maximal ideals annihilating I, i.e., hull $(I) \equiv h(I)$ [7]. In the context of this identification, it is clear that

$$I \subset kh(I) = \bigcap_{\mathfrak{M}_{A_1 \otimes_\gamma A_2}} M \equiv N.$$

The semisimplicity of $A_1 \otimes_\gamma A_2$ is thus equivalent to the statement $I=N$. In any event $F_\gamma(A_1, A_2)/N$ is a Banach algebra whose norm ν is the usual quotient norm. We denote the result by $A_1 \otimes_\nu A_2$.

Since the finite functions of $F_\gamma(A_1, A_2)$ are dense therein, it is clear that the elements of $A_1 \otimes_\nu A_2$ with representatives that are finite formal sums are dense in $A_1 \otimes_\nu A_2$. Furthermore, if z has a finite representative not in I, we may choose it [8] to be

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n})$$

where the sets $\{x_{1n}\}_1^K$, $\{x_{2n}\}_1^K$ are both linearly independent. We show

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n}) \notin N.$$

If it were in N, then we could read off the following conclusions when A_1 and A_2 are commutative:

$$\sum_{n=1}^{K} a_n\hat{x}_{1n}(M_1)\hat{x}_{2n}(M_2) = 0, \qquad \text{all } (M_1, M_2) \in \mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2};$$

$$\sum_{n=1}^{K} a_n\hat{x}_{1n}(M_1)x_{2n} = 0, \qquad \text{all } M_1 \in \mathfrak{M}_{A_1}$$

(since A_2 is semisimple);

$$a_n\hat{x}_{1n}(M_1) = 0, \qquad \text{all } M_1 \in \mathfrak{M}_{A_1}, n = 1, 2, \cdots, K$$

(since $\{x_{2n}\}_1^K$ is a linearly independent set). We may assume all $a_n \neq 0$ whence $x_{1n}=0$, $n=1, 2, \cdots, K$, since A_1 is semisimple and thus a contradiction results.

When A_1 and A_2 are not commutative, the reasoning runs this way:

$$\sum_{n=1}^{K} a_n((x_{1n}/M_1) \otimes (x_{2n}/M_2))$$

represents 0 in $(A_1/M_1) \otimes_\gamma (A_2/M_2)$ for all (M_1, M_2). Hence

$$\sum_{n=1}^{K} a_n\phi_2(x_{2n}/M_2)x_{1n}/M_1 = 0$$

for all $\phi_2 \in (A_2/M_2)^*$, (M_2 fixed) and all M_1. Hence

$$\sum_{n=1}^{K} a_n\phi_2(x_{2n}/M_2)x_{1n} = 0$$

since A_1 is semisimple. Hence $x_{2n}/M_2=0$, all M_2, whence $x_{2n}=0$ since A_2 is semisimple, and a contradiction results.

To prove ν is a cross-norm we proceed as follows.

LEMMA 1. *If*

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n}) \sim f \in N \subset F_\gamma(A_1, A_2)$$

and if $\phi_1 \in A_1^*$, *then*

$$\sum_{n=1}^{\infty} a_n\phi_1(x_{1n})x_{2n} = 0.$$

Proof. The series clearly converges. If the sum is not zero, then, by virtue of the semisimplicity of A_2, there is an $M_2 \in \mathfrak{M}_{A_2}$ such that

$$\sum_{n=1}^{\infty} a_n \phi_1(x_{1n}) x_{2n}/M_2 \neq 0.$$

Hence there is a $\Psi_2 \in (A_2/M_2)^*$ such that

$$\sum_{n=1}^{\infty} a_n \phi_1(x_{1n}) \Psi_2(x_{2n}/M_2) \neq 0.$$

Thus

$$\sum_{n=1}^{\infty} a_n \Psi_2(x_{2n}/M_2) x_{1n} \neq 0$$

and since A_1 is semisimple, there is an $M_1 \in \mathfrak{M}_{A_1}$ such that

$$\sum_{n=1}^{\infty} a_n \Psi_2(x_{2n}/M_2) x_{1n}/M_1 \neq 0.$$

For some $\Psi_1 \in (A_1/M_1)^*$ then

$$(\Psi_1 \otimes \Psi_2) \sum_{n=1}^{\infty} a_n((x_{1n}/M_1) \otimes (x_{2n}/M_2)) \neq 0.$$

This implies that if w in $(A_1/M_1) \otimes_\lambda (A_2/M_2)$ is represented by

$$\sum_{n=1}^{\infty} a_n((x_{1n}/M_1) \otimes (x_{2n}/M_2)),$$

then $\lambda(w) > 0$ and hence $\gamma(w) > 0$ when we regard w as a member of $(A_1/M_1) \otimes_\gamma (A_2/M_2)$ [8]. Thus $f \notin N$ and the contradiction yields the result.

If now z is represented by $x_1 \otimes x_2$, we know $\nu(z) \leqq \|x_1\| \, \|x_2\|$. On the other hand if

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n})$$

is in N and if $\phi_1 \in A_1^*$ satisfies $\|\phi_1\| = 1$, $\phi_1(x_1) = \|x_1\|$, then setting

$$x_1 \otimes x_2 + \sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n}) = \sum_{n=1}^{\infty} b_n(y_{1n} \otimes y_{2n})$$

we find, using Lemma 1,

$$\|x_1\| \, \|x_2\| = \|\phi_1(x_1)x_2\| = \left\| \sum_{n=1}^{\infty} b_n \phi_1(y_{1n}) y_{2n} \right\| \leqq \sum_{n=1}^{\infty} |b_n| \, \|y_{1n}\| \, \|y_{2n}\|.$$

Thus $\|x_1\| \, \|x_2\| \leqq \nu(z)$, i.e., $\|x_1\| \, \|x_2\| = \nu(z)$, and $\nu \leqq \gamma$ [10].

Furthermore, if z is represented by the finite formal sum

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n})$$

and if $\phi_i \in A_i^*$, $\|\phi_i\| = 1$, $i = 1, 2$, we have

$$\begin{aligned} |(\phi_1 \otimes \phi_2)(z)| &= |(\phi_1 \otimes \phi_2)(z + w)| = \left|(\phi_1 \otimes \phi_2) \sum_{n=1}^{\infty} b_n(y_{1n} \otimes y_{2n})\right| \\ &\leqq \sum_{n=1}^{\infty} |b_n| \, \|y_{1n}\| \, \|y_{2n}\| \end{aligned}$$

where

$$w \sim \sum_{n=1}^{\infty} c_n(w_{1n} \otimes w_{2n}) \in N$$

and

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n}) + \sum_{n=1}^{\infty} c_n(w_{1n} \otimes w_{2n}) = \sum_{n=1}^{\infty} b_n(y_{1n} \otimes y_{2n}).$$

Hence $|(\phi_1 \otimes \phi_2)(z)| \leqq \nu(z)$, i.e., $\lambda(z) \leqq \nu(z)$.

If $A_1 \otimes_\gamma A_2$ is semisimple, $I = N$ and $\nu = \gamma$. If $\nu = \gamma$, then for $f \in N$, $\nu(f/N) = \gamma(f/N) = 0$, i.e., $f \in I$, and hence $N = I$.

REMARKS. 1. For arbitrary Banach spaces E_1, E_2 the above procedure can be modified to produce an $E_1 \otimes_{\tilde{\nu}} E_2$. In this case we consider $\tilde{N} = \{f \mid f \in F_\gamma(E_1, E_2),\ (\phi_1 \otimes \phi_2)(f) = 0,\ \text{all } (\phi_1, \phi_2) \in E_1^* \times E_2^*\}$, and define $E_1 \otimes_\nu E_2$ to be $F_\gamma(E_1, E_2)/\tilde{N}$. We shall return to this point later.

2. Reflection on Theorem 1 suggests the following procedure when A_1 and A_2 are commutative: For z with a finite representative

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n}),$$

let

$$\omega(z) = \sup\left\{\left|\sum_{n=1}^{K} a_n \hat{x}_{1n}(M_1)\hat{x}_{2n}(M_2)\right| \,\middle|\, (M_1, M_2) \in \mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}\right\}.$$

Then $\omega(z)$ is independent of the representative used for z. Furthermore, we may repeat an earlier argument to conclude that if z has a representative

$$\sum_{n=1}^{K} a_n(x_{1n} \otimes x_{2n}) \notin I$$

then $\omega(z) > 0$ (we assume A_1 and A_2 semisimple). ω is then a true norm on the *algebraic* tensor product $A_1 \otimes A_2$ and the completion $A_1 \otimes_\omega A_2$ can be shown to be a semisimple algebra with maximal ideal space $\mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}$. However, ω is not in general a cross-norm. This follows from:

$$\omega(z) = \|x_1\|_\infty \|x_2\|_\infty \qquad (z \sim (x_1 \otimes x_2)),$$

which is not in general $\|x_1\| \, \|x_2\|$, but $\leqq \|x_1\| \, \|x_2\|$.

The partial solution to the semisimplicity problem mentioned before is: $A_1 \otimes_\gamma A_2$ is semisimple if either one of A_1 and A_2 satisfies the *approximation condition* [5, pp. 164–191; 10], viz., (for A_1) for K_1 compact in A_1 and $\epsilon > 0$, there is a finite dimensional linear transformation T_1 defined on A_1 such that $\|x_1 - T_1 x_1\| < \epsilon$ for $x_1 \in K_1$. In [5, pp. 164–191] there is an exhaustive discussion of equivalences and logical relations among the approximation condition and other phenomena. Particularly relevant is the equivalent *condition of monomorphy*: The natural mapping $\tau: A_1 \otimes_\gamma E_2 \rightarrow A_1 \otimes_\lambda E_2$ is 1-1, for all Banach *spaces* E_2. (For each element z of $A_1 \otimes_\gamma E_2$ let $f \in F_\gamma(A_1, E_2)$ be a representative of z. Then clearly $f \in F_\lambda(A_1, E_2)$ and represents an element w in $A_1 \otimes_\lambda E_2$. It is clear that w is independent of the choice of f, i.e., w is uniquely determined by z, whence we write $w = \tau(z)$, $\tau: A_1 \otimes_\gamma E_2 \rightarrow A_1 \otimes_\lambda E_2$.) Our next result bears on this point. It was also observed by Robert Bonic in a letter to the writer.

THEOREM 2. *When A_1 and A_2 are commutative, $A_1 \otimes_\gamma A_2$ is semisimple if and only if $\tau: A_1 \otimes_\gamma A_2 \rightarrow A_1 \otimes_\lambda A_2$ is* 1-1.

Proof. Assume $A_1 \otimes_\gamma A_2$ is semisimple and assume that $\tau(z) = 0$. This means that for all $(\phi_1, \phi_2) \in A_1^* \times A_2^*$, and for any representative

$$\sum_{n=1}^\infty a_n (x_{1n} \otimes x_{2n})$$

of z,

$$\sum_{n=1}^\infty a_n \phi_1(x_{1n}) \phi_2(x_{2n}) = 0.$$

In particular

$$\sum_{n=1}^\infty a_n \hat{x}_{1n}(M_1) \hat{x}_{2n}(M_2) = 0$$

for all M_1, M_2, and hence $z = 0$.

If τ is 1-1, and even if A_1 and A_2 are not commutative, we shall show that $A_1 \otimes_\gamma A_2$ is semisimple. For if

$$\sum_{n=1}^\infty a_n ((x_{1n}/M_1) \otimes (x_{2n}/M_2))$$

represents 0 for all M_1, M_2 and if

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n})$$

does not represent 0, there is a $\phi_2 \in A_2^*$ such that

$$\sum_{n=1}^{\infty} a_n \phi_2(x_{2n}) x_{1n} \neq 0.$$

Then there is an M_1 such that

$$\sum_{n=1}^{\infty} a_n \phi_2(x_{2n})(x_{1n}/M_1) \neq 0,$$

and a $\Psi_1 \in (A_1/M_1)^*$ such that

$$\sum_{n=1}^{\infty} a_n \phi_2(x_{2n}) \Psi_1(x_{1n}/M_1) \neq 0.$$

Hence

$$\sum_{n=1}^{\infty} a_n x_{2n} \Psi_1(x_{1n}/M_1) \neq 0,$$

and for some M_2

$$\sum_{n=1}^{\infty} a_n (x_{2n}/M_2) \Psi_1(x_{1n}/M_1) \neq 0.$$

Consequently there is a $\Psi_2 \in (A_2/M_2)^*$ such that

$$\sum_{n=1}^{\infty} a_n \Psi_2(x_{2n}/M_2) \Psi_1(x_{1n}/M_1) \neq 0.$$

Thus in $(A_1/M_1) \otimes_\lambda (A_2/M_2)$, the element 0 is represented by

$$\sum_{n=1}^{\infty} a_n((x_{1n}/M_1) \times (x_{2n}/M_2))$$

and yet $\lambda(0) > 0$. The contradiction yields the result.

THEOREM 3. *When A_1 and A_2 are commutative semisimple Banach algebras, $A_1 \otimes_\gamma A_2$ is both the completion of a tractable (i.e., not necessarily complete but semisimple) normed algebra and the quotient algebra of a semisimple Banach algebra.*

Proof. Let B be the image in $A_1 \otimes_\gamma A_2$ of the finite functions of $F_\gamma(A_1, A_2)$. Then B is dense in $A_1 \otimes_\gamma A_2$ and, as we saw in the proof of Theorem 1, B is semisimple. (It is an easy matter to verify that $\mathfrak{M}_B$ is also $\mathfrak{M}_{A_1} \times \mathfrak{M}_{A_2}$.)

Let S denote the semigroup consisting of the pairs

$$\{(x_1, x_2) \mid x_i \in A_i, 0 \leqq \|x_i\| \leqq 1, i = 1, 2\}$$

where $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$. Then in the notation of [6] let

$$l_1(S) = \left\{ f \mid f \in C^S, \|f\| \equiv \sum_{(x_1, x_2)} |f(x_1, x_2)| < \infty \right\}.$$

In [6] it is shown that $l_1(S)$ is semisimple if and only if:

$$(x_1, x_2)^2 = (y_1, y_2)^2 = (x_1, x_2)(y_1, y_2)$$

implies $(x_1, x_2) = (y_1, y_2)$. In our case, when the displayed equations hold, we find $x_1^2 = y_1^2 = x_1y_1$ and thus $(x_1 - y_1)^2 = 0$. Similarly $(x_2 - y_2)^2 = 0$. Since A_1 and A_2 are semisimple, $x_1 = y_1$, $x_2 = y_2$ and we conclude $l_1(S)$ is semisimple.

We define a surjection $\sigma: l_1(S) \to F_{\gamma 0}(A_1, A_2)$ as follows: For $f \in l_1(S)$, let $\sigma f = g$ where

$$\begin{aligned} g(0, x_2) &= g(x_1, 0) = 0, \\ g(x_1, x_2) &= f(x_1, x_2), \qquad x_1, x_2 \neq 0. \end{aligned}$$

We see at once that σ is a norm-decreasing homomorphic surjection. Since there is a homomorphism of $F_{\gamma 0}(A_1, A_2)$ onto $A_1 \otimes_\gamma A_2$ the result follows.

Remarks. 1. A simple rewording of definitions shows that if B_1 is a Banach algebra, I a closed ideal in B_1, and if $B_2 = B_1/I$, then the inverse image of the radical of B_2 is $kh(I)$. Thus the semisimplicity of B_2 is equivalent to the "spectral synthesis" of I: $I = kh(I)$ [7]. Since spectral synthesis is known to fail [P. Malliavin, Inst. des Hautes Etudes Scientifiques, Publications Math., No. 2, 61–68] in many Banach algebras, it follows that there are semisimple Banach algebras A such that for some ideal $I \subset A$, A/I is not semisimple.

2. The closure of a tractable algebra need not be semisimple.

For example, if

$$A = \left\{ \{a_n\} \mid a_n \in C, \|\{a_n\}\| \equiv \sum_{n=1}^{\infty} |a_n| 2^{-n^2} < \infty \right\},$$

then relative to convolution as multiplication A is a commutative Banach algebra. The element $x = \{0, 1, 0, 0, \cdots\}$ in A is such that $\|x^n\| = 2^{-n^2}$ and thus $\|x^n\|^{1/n} = 2^{-n} \to 0$. Hence A is not semisimple. On the other hand, the elements of A with finite support form a dense subalgebra isomorphic to the algebra of polynomials in one variable. This algebra is tractable and thus we have a nonsemisimple algebra A that is the closure of a tractable algebra. Dr. John Lindberg in a paper not yet published has given another example of this phenomenon in the area of algebraic extensions of Banach algebras.

(Note: If $x=\{a_n\}\neq 0$ in A, and if a_{n_1} is the first nonzero component then

$$\|x^2\| \geqq |a_{n_1}^2| 2^{-n_1^2} > 0.$$

Hence A is free of nonzero elements x such that $x^2=0$. In analogy with the proof of the second statement of Theorem 3, $A = l(S_1)/N$ where $S_1=\{x \mid 0\leqq\|x\|\leqq 1\}$ and N is a closed ideal in $l_1(S)$. Thus A is also a non-semisimple quotient algebra of a semisimple algebra $l_1(S)$.)

THEOREM 4. *When A_1 and A_2 are commutative and semisimple, $A_1 \otimes_\gamma A_2$ has an identity if and only if both A_1 and A_2 have identities.*

Proof. It is clear from direct inspection that $A_1 \otimes_\gamma A_2$ has an identity if both A_1 and A_2 have identities.

On the other hand, if

$$\sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n})$$

represents an identity in $A_1 \otimes_\gamma A_2$, then

$$\sum_{n=1}^{\infty} a_n\hat{x}_{1n}(M_1)\hat{x}_{2n}(M_2) \equiv 1.$$

Hence

$$x_2 \equiv \sum_{n=1}^{\infty} a_n\hat{x}_{1n}(M_1)x_{2n}$$

for any M_1 is an identity modulo any M_2. Thus for any $y_2\in A_2$, $x_2y_2-y_2 \in\bigcap_{\mathfrak{M}_{A_2}} M_2=\{0\}$. Hence x_2 is an identity for A_2 and is independent of M_1. Similarly

$$\sum_{n=1}^{\infty} a_n\hat{x}_{2n}(M_2)x_{1n}$$

is an identity for A_1 and is independent of M_2.

Apropos of the discussion of the existence of an identity in $A_1 \otimes_\gamma A_2$, the following may be observed: If A_1 and A_2 are commutative Banach algebras with identities, then $A_1 \otimes_\gamma A_2$ is the direct sum of two ideals if and only if one of A_1 and A_2 is the direct sum of two ideals. This follows from Šilov's theorem [9], from the equation $\mathfrak{M}_{A_1\otimes_\gamma A_2}=\mathfrak{M}_{A_1}\times\mathfrak{M}_{A_2}$ and from the fact that $\mathfrak{M}_{A_1}\times\mathfrak{M}_{A_2}$ is connected if and only if both $\mathfrak{M}_{A_1}$ and $\mathfrak{M}_{A_2}$ are connected.

We shall continue a discussion of the central problem of semisimplicity. However, at this point, the algebraic aspects of A_1 and A_2 will no longer intervene and we shall be concentrating on the purely "Banach-space" sides of the problem.

4. **Monomorphy.** In the preceding section we defined the condition of monomorphy for a Banach E_1 [5, pp. 36, 164–191].

THEOREM 1. *Each of the following statements implies that E_1 satisfies the condition of monomorphy*:

(a) *There is a constant K with the following property: If $C(S_1^*)$ denotes the space of continuous functions on the unit ball S_1^* of E_1^*, and if E_1 is regarded as isometrically embedded in $C(S_1^*)$, then for any finite set $\{x_i\}_{i=1}^m \subset E_1$ and functions $\{f_j\}_{j=1}^n \subset C(S_1^*)$ for which*

$$x_i = \sum_{j=1}^n a_{ij} f_j, \qquad i = 1, 2, \cdots, m$$

(*a_{ij} constants*), *there exist elements $\{y_j\}_{j=1}^n \subset E_1$ such that $\|y_j\| \leqq K\|f_j\|$ and*

$$x_i = \sum_{j=1}^n a_{ij} y_j, \qquad i = 1, 2, \cdots, m.$$

(b) *There is a bounded projection P of $F(E_1)$ onto N_1 where $F(E_1)$ $= \{f | f \in C^{E_1}, \|f\| \equiv \sum_x |f(x)| \|x\| < \infty\}$, $N_1 = \{f | f \in F(E_1), \sum_x f(x)x = 0\}$.*

(c) *There is no matrix $C = \{c_{ij}\}_{i,j=1}^\infty$ satisfying*:

(i) $C^2 = 0$;

(ii) $|c_{ij}| \leqq a_i a_j$, $a_i \geqq 0$, $\sum_{i=1}^\infty a_i^2 < \infty$;

(iii) $\sum_{i=1}^\infty c_{ii} = 1$.

((c) *actually implies the condition of monomorphy for all Banach spaces.*)

Proof. Assume (a) is true and let E_2 be an arbitrary Banach space for which $\tau: E_1 \otimes_\gamma E_2 \to E_1 \otimes_\lambda E_2$ is not 1-1. This means there is an $f \in F_\gamma(E_1, E_2)$,

$$f \sim \sum_{n=1}^\infty a_n(x_{1n} \otimes x_{2n})$$

such that for any pair $(\phi_1, \phi_2) \in E_1^* \times E_2^*$,

$$\sum_{n=1}^\infty a_n \phi_1(x_{1n}) \phi_2(x_{2n}) = 0,$$

and yet $f \notin I$.

Now regard E_1 as isometrically embedded in $C(S_1^*) \equiv F_1$. It is then clear that for $(\Psi_1, \phi_2) \in F_1^* \times E_2^*$,

$$\sum_{n=1}^\infty a_n \Psi_1(x_{1n}) \phi_2(x_{2n}) = 0.$$

This means that f, regarded as a member of $F_\gamma(F_1, E_2)$ via the obvious embedding, must be in J, the closed linear subspace of $F_\gamma(F_1, E_2)$ used to define $F_1 \otimes_\gamma E_2$, $(F_\gamma(F_1, E_2)/J = F_1 \otimes_\gamma E_2)$, because F_1 [5, pp. 185–191] fulfills the condition of monomorphy. This in turn means, that for $\epsilon > 0$, there are generators $g_1, g_2, \cdots, g_{n(\epsilon)}$ of J and coefficients $b_1(\epsilon), \cdots, b_{n(\epsilon)}^{(\epsilon)}$ such that

$$\left\| f - \sum_{i=1}^{n(\epsilon)} b_i(\epsilon) g_i \right\| < \epsilon/K.$$

We may write

$$f - \sum_{i=1}^{n(\epsilon)} b_i(\epsilon) g_i \sim \sum_{n=1}^{\infty} c_n(y_{1n} \otimes y_{2n}) + \sum_{m=1}^{n(\epsilon)} d_m(z_{1m} \otimes z_{2m})$$

where $y_{1n} \in E_1$, $z_{1m} \in F_1 \backslash E_1$. Clearly for finitely many y_{1n_i} there are relations of the form

$$y_{1n_i} = \sum_{j=1}^{n(\epsilon)} a_{ij} z_{1j}, \qquad i = 1, 2, \cdots, p.$$

If, in accordance with the condition, we replace each z_{1j} by some $w_{1j} \in E_1$ such that

$$y_{1n_i} = \sum_{j=1}^{n(\epsilon)} a_{ij} w_{1j}, \qquad \|w_{1j}\| \leqq K\|z_{1j}\|,$$

then the $g_1, \cdots, g_{n(\epsilon)}$ will be replaced by $h_1, \cdots, h_{n(\epsilon)}$ in I and

$$\begin{aligned}\left\| f - \sum_{i=1}^{n(\epsilon)} b_i(\epsilon) h_i \right\| &= \sum_{n=1}^{\infty} |c_n| \, \|y_{1n}\| \, \|y_{2n}\| + \sum_{m=1}^{n(\epsilon)} |d_m| \, \|w_{1m}\| \, \|z_{2m}\| \\ &\leqq K \left\| f - \sum_{i=1}^{n(\epsilon)} b_i(\epsilon) g_i \right\| < \epsilon.\end{aligned}$$

(We may and do assume $K \geqq 1$.) Thus f is in fact in I, and a contradiction yields the result.

Assume (b) is true. Since $F(E_1)$ is essentially $L_1(\mu)$ where μ is discrete measure on E_1 ($\mu(x_1) = \|x_1\|$), $F(E_1)$ satisfies the condition of monomorphy [5, loc. cit.]. Clearly the mapping $T\colon F(E_1) \to E_1$ defined by the formula $T(f) = \sum_x f(x)x$ is a continuous surjection and $T^{-1}(0) = N_1$. If $P(F(E_1)) = N_1$, then $(I-P)(F(E_1))$ is isomorphic to E_1 (I is the identity operator). Since $(I-P)$ is a bounded projection we shall prove our result by establishing the following

LEMMA 1. *If E is a Banach space satisfying the condition of approximation, and if P is a bounded projection of E onto a closed linear subspace N, then N, qua Banach space, satisfies the condition of approximation.*

Proof. Let K be a compact set in N and let $\epsilon > 0$. Then K is compact in E and there is a finite dimensional linear transformation S such that for $x \in K$, $\|x - Sx\| < \epsilon/\|P\|$. Let $S' = PS$. Then S' is finite dimensional, $S'(N) \subset N$, and for $x \in K$, $\|x - S'x\| = \|Px - PSx\| \leqq \|P\| \, \|x - Sx\| < \epsilon$.

Next assume (c) is true and let E_1, E_2 be arbitrary Banach spaces. If τ is not 1-1 for the pair in question there is an

$$f \sim \sum_{n=1}^{\infty} b_n(x_{1n} \otimes x_{2n})$$

in $F_\gamma(E_1, E_2)$ such that $f \notin I$ and

$$\sum_{n=1}^{\infty} b_n\phi_1(x_{1n})\phi_2(x_{2n}) = 0,$$

all $(\phi_1, \phi_2) \in E_1^* \times E_2^*$. Since $f \notin I$, from the Hahn-Banach theorem we see that there is (Proposition 2, §1) a linear mapping $\lambda: E_2 \to E_1^*$ such that

$$\sum_{n=1}^{\infty} b_m\langle x_{1n}, \lambda(x_{2n})\rangle = 1.$$

We may represent f by

$$\sum_{n=1}^{\infty} (b_1 x_{1n} \otimes x_{2n}) \equiv \sum_{n=1}^{\infty} (y_{1n} \otimes x_{2n})$$

and then again by

$$\sum_{n=1}^{\infty} (\alpha_n y_{1n} \otimes (1/\alpha_n)x_{2n}), \qquad \alpha_n \neq 0.$$

If

$$\alpha_n = (\|x_{2n}\|/\|y_{1n}\|)^{1/2},$$

then

$$\|\alpha_n y_{1n}\| = (\|x_{2n}\| \, \|y_{1n}\|)^{1/2}$$

and

$$\|(1/\alpha_n)x_{2n}\| = (\|x_{2n}\| \, \|y_{1n}\|)^{1/2}.$$

Setting $u_{1n} = \alpha_n y_{1n}$, $v_{1n} = (1/\alpha_n)x_{2n}$, we see $\|u_{1n}\| = \|v_{1n}\|$ and

$$\sum_{n=1}^{\infty} \|u_{1n}\|^2 = \sum_{n=1}^{\infty} \|v_{1n}\|^2 < \infty.$$

Clearly we may assume

$$\sum_{n=1}^{\infty} \|u_{1n}\|^2 = 1.$$

Thus we may and do assume $b_n = 1$,

$$\sum_{n=1}^{\infty} \|x_{1n}\|^2 = \sum_{n=1}^{\infty} \|x_{2n}\|^2 = 1, \qquad \|x_{1n}\| = \|x_{2n}\|.$$

Set $c_{mn} = \langle x_{1m}, \lambda(x_{2n})\rangle$. Then $|c_{mn}| \leqq \|x_{1m}\| \, \|\lambda\| \, \|x_{2n}\|$. Thus if

$$a_n = (\|\lambda\|)^{1/2}\|x_{1n}\|,$$

we see

$$\sum_{n=1}^{\infty} a_n^2 < \infty, \qquad |c_{mn}| < a_m a_n.$$

We see too that

$$\sum_{n=1}^{\infty} \phi_1(x_{1n})x_{2n} = 0,$$

all $\phi_1 \in E_1^*$. Since λ is a linear mapping we find

$$\sum_{n=1}^{\infty} \phi_1(x_{1n})\langle x_{1m}, \lambda(x_{2n})\rangle = 0, \qquad \text{all } \phi_1 \in E_1^*,$$

and thus

$$\sum_{n=1}^{\infty} c_{mn}x_{1n} = 0, \qquad m = 1, 2, \cdots.$$

Finally

$$0 = \left\langle \sum_{n=1}^{\infty} c_{mn}x_{1n}, \lambda(x_{2k})\right\rangle = \sum_{n=1}^{\infty} c_{mn}c_{nk}.$$

Thus the conditions (i)–(iii) are fulfilled and we have a contradiction.

REMARKS. 1. For a given Banach space E_1, (a) is implied by the existence of a bounded projection $P: C(S_1^*) \to E_1$. Such a projection exists when $E_1 = C(X)$ for some topological space X. (Embed X in S_1^* and let P be the restriction mapping.) Of course, when $E_1 = C(X)$, it is hardly necessary to find P in order to establish the condition of monomorphy.

2. In [5, p. 171, (f'')] a condition analogous to (c) is discussed. We note that such a C can not be in trace class since such a C is unitarily equivalent to a matrix $D = (d_{mn})$ with diagonal elements 0. If C were in the trace class,

$$0 = \sum_{n=1}^{\infty} d_{nn} = \sum_{n=1}^{\infty} c_{nn} = 1.$$

C is dominated, entry by entry, by the matrix $A = (a_m a_n)$ and A *is* in the trace class ($A = BB^t$ where B is the Hilbert-Schmidt matrix (b_{ij}); ($b_{i1} = a_i$, $b_{ij} = 0$, $j > 1$)). Furthermore, it is possible to have: $|c_{mn}| \leqq t_{mn}$, $(t_{mn}) = T$ in the trace class, C not in the trace class. This is implicit in [4]. There it is shown that if $\{x_i\}$ is an orthonormal basis for l_2, then $\{x_i \otimes x_j\}$ is ***not*** an unconditional basis for $l_2 \otimes_\gamma l_2$ which is known to be the set of trace class operators in l_2. It is also shown that $\{x_i \otimes x_j\}$ *is* an unconditional basis for $l_2 \otimes_\gamma l_2$ if and only if for every sequence $\{a_i\} \in l_2$, the matrix $(\epsilon_{ij} a_i a_j)$ is in the trace class for all matrices (ϵ_{ij}), $|\epsilon_{ij}| \leqq 1$.

5. $E_1 \otimes_{\tilde{\nu}} E_2$. Let α be a cross-norm and consider the following situation: For each z in $E_1 \otimes_\alpha E_2$ such that z has a finite representative, say

$$\sum_{n=1}^{m} a_n(x_{1n} \otimes x_{2n}),$$

let w be the element of $E_1 \otimes_\lambda E_2$ with the same representative. As in an earlier discussion, we note that w is independent of the choice of representative, i.e., $w = \tau_0(z)$. If τ_0 (obviously a linear transformation) is bounded, it can be extended to a mapping $\tau_\alpha : E_1 \otimes_\alpha E_2 \to E_1 \otimes_\lambda E_2$. *When we write τ_α in the theorem below, we assume that τ_0 is bounded.*

THEOREM 1. *If τ_α is* 1-1, *then $\alpha \leqq \tilde{\nu}$; $\tilde{\nu} = \gamma$ if and only if $\tau : E_1 \otimes_\gamma E_2 \to E_1 \otimes_\lambda E_2$ is* 1-1.

Proof. If f is a finite function in $F_\gamma(E_1, E_2)$, let $T_0 f$ be the element of $E_1 \otimes_\alpha E_2$ represented by f. Since α is a cross-norm, $\alpha(T_0 f) \leqq \|f\|$, i.e., T_0 is a bounded linear transformation from the (incomplete) normed space of finite functions in $F_\gamma(E_1, E_2)$ to $E_1 \otimes_\alpha E_2$, and $\|T_0\| \leqq 1$. (Since α is a cross-norm, $\|T_0\| = 1$.) Let T be the extension of T_0 to all $F_\gamma(E_1, E_2)$. Again $\alpha(Tf) \leqq \|f\|$. Let $\hat{N} = T^{-1}(0)$, and, as in Remark 1 following Theorem 1, §3, let $\tilde{N} = \{f \mid f \in F_\gamma(E_1, E_2),\ (\phi_1 \otimes \phi_2)(f) = 0,\ \text{all } (\phi_1, \phi_2) \in E_1^* \times E_2^*\}$.

We show $\tilde{N} \subset \hat{N}$. For if $f \in \tilde{N}$, then

$$\begin{aligned} \|\tau_\alpha(Tf)\| &= \lim_{n\to\infty} \|\tau_\alpha(Tf_n)\| \\ &= \lim_{n\to\infty} \sup \left\{ \left| \sum_{m=1}^{n} a_m \phi_1(x_{1m}) \phi_2(x_{2m}) \right| \,\middle|\, (\phi_1, \phi_2) \in S_1^* \times S_2^* \right\} \end{aligned}$$

where

$$f \sim \sum_{m=1}^{\infty} a_m(x_{1m} \otimes x_{2m}), \qquad f_n \sim \sum_{m=1}^{n} a_m(x_{1m} \otimes x_{2m}).$$

But for $(\phi_1, \phi_2) \in S_1^* \times S_2^*$, since $f \in \tilde{N}$

$$\begin{aligned} \left| \sum_{m=1}^{n} a_m \phi_1(x_{1m}) \phi_2(x_{2m}) \right| &\leqq \left| \sum_{m=1}^{\infty} a_m \phi_1(x_{1m}) \phi_2(x_{2m}) \right| + \left| \sum_{m=n+1}^{\infty} a_m \phi_1(x_{1m}) \phi_2(x_{2m}) \right| \\ &\leqq 0 + \sum_{m=n+1}^{\infty} |a_m| \, \|x_{1m}\| \, \|x_{2m}\|. \end{aligned}$$

Since the last sum approaches 0, independently of (ϕ_1, ϕ_2) as $n \to \infty$ we see

$\|\tau_\alpha(Tf)\|=0$, i.e., $\tau_\alpha(Tf)=0$, whence $Tf=0$ since τ_α is assumed to be 1-1.

We now resort to the following elementary result, which we prove for the sake of completeness.

LEMMA 1. *If S is a bounded linear transformation $S: F_1 \to F_2$ where F_i are Banach spaces, $i=1, 2$, and if $K_1=S^{-1}(0)$, then for any $x_1 \in F_1$, $\|S(x_1)\| \leqq \|S\|\rho(x_1, K_1)$, where $\rho(x_1, K_1)$ denotes the distance of x_1 from K_1.*

Proof. For any $k_1 \in K_1$, $S(x_1+k_1)=S(x_1)$, whence

$$\|S(x_1)\| = \|S(x_1 + k_1)\| \leqq \|S\| \, \|x_1 + k_1\|.$$

Hence $\|S(x_1)\| \leqq \|S\|\rho(x_1, K_1)$.

In our situation we find $\alpha(Tf) \leqq \rho(f, \hat{N}) \leqq \rho(f, \tilde{N})$. This means $\alpha(Tf) \leqq \tilde{\nu}(Tf)$, and this is the desired result.

Otherwise stated: $\tilde{\nu}$ is the greatest of cross-norms α for which τ_α is definable and 1-1. Clearly $\tilde{\nu}=\gamma$ is the necessary and sufficient condition that τ be 1-1.

THEOREM 2. *Let A_1 and A_2 be semisimple commutative Banach algebras. Then $\tau_\nu: A_1 \otimes_\nu A_2 \to A_1 \otimes_\lambda A_2$ is 1-1 and $\nu \leqq \tilde{\nu}$. Finally, there is an injection $\eta: A_1 \otimes_\nu A_2 \to A_1 \otimes_\omega A_2$, and $A_1 \otimes_\omega A_2$ is semisimple.*

Proof. For $z \in A_1 \otimes_\nu A_2$ such that z has a finite representative f in $F_\gamma(A_1, A_2)$, let w be the element of $A_1 \otimes_\lambda A_2$ with the same representative. Then w is uniquely defined by z, $w=\tau_0(z)$ and we know $\lambda(w) \leqq \nu(z)$. Thus τ_0 may be extended to τ_ν defined on all $A_1 \otimes_\nu A_2$.

Now assume z in $A_1 \otimes_\nu A_2$ has representative

$$f \sim \sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n})$$

in $F_\gamma(A_1, A_2)$, and let $\tau_\nu(z)=0$. This means that

$$\left| \sum_{n=1}^{N} a_n\phi_1(x_{1n})\phi_2(x_{2n}) \right| \to 0$$

uniformly for (ϕ_1, ϕ_2) in $S_1^* \times S_2^*$, and this implies $f \in N$, $z=0$. From Theorem 1 we see $\nu \leqq \tilde{\nu}$.

Similarly we define η_0 on those members z of $A_1 \otimes_\nu A_2$ with finite representatives in $F_\gamma(A_1, A_2)$: $\eta_0(z)$ is that element w of $A_1 \otimes_\omega A_2$ with the same representative as that of z (w is independent of the choice of representative). η_0 is bounded and serves to define an $\eta: A_1 \otimes_\nu A_2 \to A_1 \otimes_\omega A_2$. If $\eta(z)=0$, let

$$f \sim \sum_{n=1}^{\infty} a_n(x_{1n} \otimes x_{2n})$$

represent z and let

$$\sum_{n=1}^{N} a_n(x_{1n} \otimes x_{2n})$$

represent z_N. Then $z_N \to z$ and $\hat{z}_N \to 0$ uniformly on $\mathfrak{M}_{A_1 \otimes_\gamma A_2}$. Thus $\nu(z) = 0$, i.e., $z = 0$, η is an injection. The proof that $A_1 \otimes_\omega A_2$ is semisimple proceeds similarly.

PROBLEM. If $\nu = \tilde{\nu}$ is $A_1 \otimes_\gamma A_2$ semisimple?

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