THEORY OF COVERING SPACES

BY

SAUL LUBKIN

Introduction. In this paper I study covering spaces in which the base space is an arbitrary topological space. No use is made of arcs and no assumptions of a global or local nature are required.

In consequence, the fundamental group $\pi(X, x_0)$ which we obtain frequently reflects local properties which are missed by the usual arcwise group $\pi_1(X, x_0)$.

We obtain a perfect Galois theory of covering spaces over an arbitrary topological space. One runs into difficulties in the non-locally connected case unless one introduces the concept of *space* [§1], a common generalization of topological and uniform spaces. The theory of covering spaces (as well as other branches of algebraic topology) is better done in the more general domain of spaces than in that of topological spaces.

The Poincaré (or deck translation) group plays the role in the theory of covering spaces analogous to the role of the Galois group in Galois theory. However, in order to obtain a perfect Galois theory in the non-locally connected case one must define the *Poincaré filtered group* [§§6, 7, 8] of a covering space. In §10 we prove that every filtered group is isomorphic to the Poincaré filtered group of some regular covering space of a connected topological space [Theorem 2]. We use this result to translate topological questions on covering spaces into purely algebraic questions on filtered groups. In consequence we construct examples and counterexamples answering various questions about covering spaces [§11].

In §11 we also discuss other applications of the general theory of covering spaces—e.g., a theory of covering spaces with singularities [§2, Example 1; §11, Example 1]; the defining of a "relative fundamental group" $\pi(X, A)$ [§2, Example 2; §11, Example 3]; the interpretation of the étale coverings in characteristic 0 of algebraic geometry as covering spaces [§11, Example 2]; and the resolution of the classical problem of the classification of the covering spaces of the topological space: $\{(0, 0)\} \cup \mathrm{Graph}[\sin(1/x), x>0]$ [§11, Example 4].

The following table summarizes some of the similarities between Galois theory and the theory of covering spaces:

Galois theory	Covering Spaces
finite separable algebraic extension	covering space
finite Galoisian extension	regular covering space
Galois group	Poincaré filtered group
separable algebraic closure	universal covering space
Galois group of separable algebraic closure	fundamental filtered group

Received by the editors February 22, 1961.

This paper started as a term paper for a course in algebraic topology taught by Professor James Eells at Columbia. I am indebted to Professor Eells for his interest and enthusiasm.

0. **Motivation.** If X is a topological space then the most obvious definition of a covering space of X is a pair (C, p) where C is a topological space and $p: C \to X$ is a continuous function such that, for every $x \in X$, there exists an open neighborhood V_x of x such that $p^{-1}(V_x)$ may be written as a disjoint union $p^{-1}(V_x) = \bigcup_{i \in I_x} U_i$ of open subsets of C each of which is mapped homeomorphically onto V_x(1). In fact if X is *locally connected* this is essentially the usual definition [2].

However the above definition is not adequate in the non-locally connected case. In the locally connected case we may assume that each V_x is connected —or, equivalently, that the decomposition into disjoint open sets: $p^{-1}(V_x) = \bigcup_{i \in I_x} U_i$ is *uniquely determined* for all x. This is no longer true in the non-locally connected case.

Hence, in the general case, let us fix a *distinguished decomposition* $p^{-1}(V_x) = \bigcup_{i \in I_x} U_i$ for each $x \in X$. If V_x', $p^{-1}(V_x') = \bigcup_{i \in I_x'} U_i'$ is another such collection of choices, consider that this set defines the same covering space of X as the earlier set if they agree on some $V_x'' \subset V_x, V_x'$. One obtains a definition of covering space over an arbitrary topological space for which all the classical theorems go through. To formalize this procedure, we introduce the concept of *space*.

1. **Spaces.** An *entourage* for a set X is a subset V of $X \times X$ containing the diagonal; equivalently, a function $V: X \to 2^X$ such that $x \in V_x$, all $x \in X$. A *contiguity* for the set X is a filter of entourages on $X \times X$; a *space* is a set with contiguity. A *base* for the space X is a base for the contiguity of X considered as a filter.

If X is a topological space, then we regard X as a space with contiguity $\mathfrak{C}$ by taking $V \in \mathfrak{C}$ *iff* V_x is a neighborhood of x, all $x \in X$. Thus, every topological space is in a natural way a space.

Clearly a uniformity is a contiguity, so that every uniform space is a space.

If X and Y are spaces, and $f: X \to Y$ a function, then f is *contiguous iff* $V \in \text{contig}(Y)$ implies there exists $U \in \text{contig}(X)$ such that $fU \subset Vf$ (i.e., $f(U_x) \subset V_{f(x)}$, all $x \in X$). In particular, if X and Y are topological (resp.: uniform) spaces, then f is contiguous iff f is continuous (resp.: uniformly continuous).

If V is an entourage for the set X, then let $\overline{V}$ be the equivalence relation generated by the relation V. Then the *V-components* of X are the $\overline{V}$ equivalence classes of X. Thus, x and y belong to the same V-component of X *iff* there exists a finite sequence $x_1, \cdots, x_n \in X$ such that $V_{x_i} \cap V_{x_{i+1}} \neq \emptyset$,

(1) Such a pair (C, p) is called a *locally trivial sheaf*.

$1 \leqq i \leqq n-1$, and $x \in V_{x_1}$, $y \in V_{x_n}$. Such a sequence is called a *V-chain* from x to y. X is *V-connected iff* X has at most one V-component.

If X is a space and $x \in X$ then the *quasicomponent* of the element x of X is the intersection of the V-components of x for all $V \in \text{contig}(X)$. (This coincides with the usual definition when X is a topological space.) X is *connected* if it has at most one quasicomponent; equivalently, *iff* every contiguous function mapping X into a discrete topological space is constant. The contiguous image of a connected space is connected.

REMARK. A topological space is topologically connected *iff* it is connected as a space; but the *uniform space* of rational numbers is connected, although the underlying topological space is not.

As in the case of topological spaces, one can define connected components of a space. Moreover, a subset A of a space X has a natural contiguity (the coarsest contiguity rendering the inclusion contiguous); similarly, a quotient-set of a space is a space. The Cartesian product of spaces has a natural contiguity, that which makes it into the category-theoretic direct product; similarly for direct sum, projective limit, and inductive limit.

If $f_i: X \to Y$ are functions, $i \in I$, then f_i are *equicontiguous iff* $V \in \text{contig}(Y)$ implies there exists $U \in \text{contig}(X)$ such that $f_i U \subset V f_i$, all $i \in I$.

If $(A_i)_{i \in I}$ is a family of subspaces of the space X, and $f_i: A_i \to Y$ are functions, then f_i are *equi-open* if $U \in \text{contig}(X)$ implies there exists $V \in \text{contig}(Y)$ such that $f_i(U_x \cap A_i) \supset V_{f_i(x)} \cap f_i(A_i)$, all $x \in A_i$, all $i \in I$(2).

THEOREM 1. *If C is a connected space, $U \in \text{contig}(C)$, $\emptyset \neq S \subset C$, and $[c \in C,\ U_c \cap S \neq \emptyset$ implies $U_c \subset S]$, then $S = C$.*

Proof. S is a nonempty union of U-components of X. Since X is U-connected, $S = X$.

2. **Covering spaces.** If $p: C \to X$ is a surjective contiguous function, then (C, p) is a *covering space* of X *iff* C has a base consisting of entourages U such that there exists $V \in \text{contig}(X)$ such that:

(C1) $p \mid U_c$ is an isomorphism of spaces from U_c onto $V_{p(c)}$, all $c \in C$.

(C2) $p^{-1}(V_x) = \text{disjoint} \bigcup_{p(c)=x} U_c$, all $x \in X$.

(C3) For each $x \in X$, the family of functions $(p \mid U_c)_{c \in p^{-1}(x)}$ is equi-open.

REMARK. Condition (C3) is the necessary and sufficient condition that a covering space be smaller than a regular covering space. It therefore corresponds to the condition of separability in the Galois theory of fields. In the locally connected case, it is a consequence of (C1) and (C2).

If (C, p) is a covering space of X, then we say that $p: C \to X$ is a *covering map*, and write $C \to^p X \to 0$. We then denote the set of U obeying (C1), (C2), and (C3) by $\text{ad}(C)$; $\text{ad}(C)$ is thus a base for $\text{contig}(C)$.

If C is a set, X a space, and $p: C \to X$ a surjective function, then an entourage U for C is a *p-adaptation* if there exists $V \in \text{contig}(X)$ such that:

(2) If we put $U^A = U \cap (A \times A)$ and $V^B = V \cap (B \times B)$, all $A \subset X$, $B \subset Y$, then this condition may be rewritten: $f_i \circ U^{A_i} \supset V^{f_i(A_i)} \circ (f_i \mid A_i)$, all $i \in I$.

(1) $pU = Vp$; (2) $p|U_c$ is injective, all $c \in C$; (3) $p^{-1}(V_x) =$ disjoint $\bigcup_{p(c)=x} U_c$; (4) $x \in X$ implies there exists $W \in \text{contig}(V_x)$ such that $p(U_d \cap U_c) \supset W_{p(d)}$, all $d \in U_c$, all $c \in p^{-1}(x)$.

THEOREM 1. *If C is a set, X a space, and U a p-adaptation for C, then there exists a unique contiguity $\mathcal{C}$ for C containing U rendering p: $(C, \mathcal{C}) \to X$ a covering map. If U^1, U^2 are p-adaptations for C, then they define the same contiguity* iff *there exists a p-adaptation $U^3 \subset U^1$, U^2.*

Proof. If U is a p-adaptation and $V = pUp^{-1}$, then the mapping which to each p-adaptation $U^1 \subset U$ associates the entourage $W = pU^1p^{-1} \in \text{contig}(X)$ defines a lattice isomorphism between the p-adaptations contained in U and the entourages $W \in \text{contig}(X)$ with $W \subset V$. Let $\mathcal{C}$ be the contiguity for C with base the p-adaptations contained in U. Then p: $(C, \mathcal{C}) \to X$ is a covering map with $U \in \text{ad}(C, \mathcal{C}, p)$ [(1), (2), and (4) imply (C1); (4) implies (C3); and (3) implies (C2)]. The other assertions follow readily from the stated lattice isomorphism.

A covering space of a *uniform space* is a uniform space, the covering map being uniformly continuous. However, a covering space C of a *topological space* X (unless finite-to-one) is rarely a topological space. Nevertheless, it does possess a natural topology (the neighborhood system of the point $c \in C$ consisting of all V_c with $V \in \text{contig}(C)$) making the projection into a locally trivial sheaf.

In particular, if X is a *connected and locally connected topological space*, we have a one-to-one correspondence between the connected covering spaces of X as defined above and the covering spaces of X in the sense of Chevalley [2].

EXAMPLE 1. Let X be a space and $\mathcal{S}$ an ideal in the Boolean ring 2^X (= ring of all subsets of X). Then let $X^{\mathcal{S}}$ be the space whose contiguity is generated by the contiguity of X together with the entourages V^S defined by:

$$V_x^S = \begin{cases} \{x\}, & x \in S, \\ X, & x \notin S, \end{cases}$$

for all $S \in \mathcal{S}$. Then a covering space of $X^{\mathcal{S}}$ is called a *covering space of X with singularities over an element of* $\mathcal{S}$. Several special cases of interest are: (a) $\mathcal{S}$ = all finite subsets of X; (b) $\mathcal{S}$ = ideal generated by $\{A\}$, $A \subset X$; (c) If X is an n-dimensional variety (differentiable, analytic, or algebraic) and $0 \leqq r \leqq n$, then let $\mathcal{S}$ be the ideal generated by the set of all closed subvarieties of dimension $\leqq r$.

The covering spaces described in each of these special cases are covering spaces of X: (a) with finitely many singularities; (b) with singularities over A; (c) with singularities over a closed subvariety of dimension $\leqq r$.

EXAMPLE 2. Let X be a space and A a subspace. Then for $V \in \text{contig}(X)$, put

$$V_x' = \begin{cases} V_x, & x \notin A, \\ V_x \cup A, & x \in A. \end{cases}$$

Then we obtain a new space (X, A). A covering space of (X, A) is a *covering space of X such that A is evenly covered.*

A space X is *semi-simple* if for every covering space (C, p) of X there exists $U \in \mathrm{contig}(C)$ such that the U-components of C (with the induced contiguity) are *connected* covering spaces of X. Thus every semi-simple space is connected.

A space X is *locally connected* if it has a base consisting of entourages V such that V_x is connected, all $x \in X$. (This coincides with the usual definition for topological spaces.) The quasicomponents of a locally connected space are semi-simple.

Hereafter, an arrow $(X \to Y)$ will signify a contiguous function, and a notation like: $X \to Y \to 0$ will signify a covering map. If $X \to^f Y \to^g Z$ and $V \in \mathrm{contig}(Y)$, then $g \circ V \circ f$ or gVf denotes the composite relation: $X \to Z$ defined by: $(g \circ V \circ f)_x = g(V_{f(x)})$.

3. Elementary properties of covering spaces.

THEOREM 1. *The composite of two covering maps is a covering map.*

[REMARK. The analogous theorem about locally trivial sheafs, even in the case of locally connected topological spaces, is false (Example 12, §11).]

Proof. Let $q: B \to C$ and $r: C \to X$ be covering maps. Then given $U^1 \in \mathrm{contig}(B)$, choose $U^2 \in \mathrm{ad}(B, q)$ with $U^2 \subset U^1$. Let $W^2 = qU^2q^{-1}$. Choose $W \in \mathrm{ad}(C, r)$ such that $W \subset W^2$. Let U be the unique q-adaptation contained in U^2 such that $qUq^{-1} = W$. Then $U \subset U^1$ and obeys Axioms (C1)–(C3) for the composite map $p = rq: B \to X$.

If (C, p) and (D, q) are covering spaces of X, then a *map* $f: (C, p) \geqq (D, q)$ is a contiguous function $f: C \to D$ such that $p = qf$.

THEOREM 2. *If (C, p) and (D, q) are covering spaces of X, D connected, and f is a map: $(C, p) \geqq (D, q)$, then (C, f) is a covering space of D.*

$$\begin{tikzcd}
C \arrow[dr, "f"] \arrow[dd, "p"'] & \\
& D \arrow[dl, "q"] \\
X \arrow[dl] \arrow[dd] & \\
0 & \\
& 0
\end{tikzcd}$$

Proof. Let $V \in \mathrm{ad}(D, q)$ and $U^1 \in \mathrm{ad}(C, p)$ be such that $fU^1 \subset Vf$. Then all $U \in \mathrm{ad}(C, p)$ with $U \subset U^1$ obey Axioms (C1)–(C3) for the map $f: C \to f(C)$. It is left to check that f is an epimorphism.

If D is empty there is nothing to prove. Otherwise put $S = f(C)$. Then S

obeys the hypotheses of Theorem 1 of §1 relative to the entourage $W^1 = fU^1f^{-1}$ so that $f(C) = D$.

[REMARK. Note that, whether D is connected or not, f is a covering map onto a union of W'-components of D for some $W' \in \text{contig}(D)$.]

The cardinal number of each fiber of a covering space (C, p) of a connected space X (being a contiguous function into a discrete space) is constant; it is called the *covering number* of the covering space (C, p).

THEOREM 3 (UNIQUENESS). *If (C, q) is a covering space of X, D a connected space, $f, g: D \to C$ contiguous functions agreeing at a point of D such that $qf = qg$, then $f = g$.*

$$\begin{tikzcd} D, d_0 \arrow[r, "f, g"] & C, c_0 \arrow[d, "q"] \\ & X, x_0 \arrow[d] \\ & 0 \end{tikzcd}$$

Proof. Let $V \in \text{ad}(C)$. Choose $U \in \text{contig}(D)$ such that $fU \subset Vf$ and $gU \subset Vg$. Let $S =$ all points $d \in D$ such that $f(d) = g(d)$. Then $d_0 \in S$, $S \neq \emptyset$, so that by Theorem 1 of §1 we have $S = D$.

REMARK. The assumption that D be connected can be weekened to: "There exists $U \in \text{contig}(D)$ and $V \in \text{ad}(C)$ such that $fU \subset Vf$, $gU \subset Vg$, and D is U-connected."

It follows from the above theorem that, if D is connected and we have a diagram:

$$\begin{tikzcd} D, d_0 \arrow[d] & C, c_0 \arrow[d] \\ Y, y_0 \arrow[r, "f"] \arrow[d] & X, x_0 \arrow[d] \\ 0 & 0 \end{tikzcd}$$

then there exists at most one contiguous function $\bar{f}: D \to C$ completing the diagram to a commutative rectangle. Such a map $\bar{f}$, if it exists, is called a *map over* f.

COROLLARY 3.1. *If* (B, b, p) *is a connected covering space and* (C, c, q) *a covering space of a space* X *with base point* x, *then there exists at most one map* $f: (B, b, p) \geqq (C, c, q)$.

In particular, two connected covering spaces (with base point) of a space (with base point), each of which is larger than the other, are canonically isomorphic.

4. **Regular covering spaces.** If $p: B \to X$ is a function of sets and G is a group of permutations of the set B such that $pg = p$, all $g \in G$, then G is *p-transitive iff* the induced map on the coset space: $B|G \to X$ is injective.

If $p: B \to X$ is a contiguous function, then let $P(B, X)$ or $P(p)$ denote the group of all automorphisms n of the space B such that $pn = p$. The function p is *regular* if there exists an equicontiguous subgroup of $P(B, X)$ that acts p-transitively.

If (B, p) is a covering space of X, then $P(B, X)$ is called the *Poincaré* (or *deck translation*) *group* of the covering space (B, p). By Corollary 3.1 of the last section, if (B, p) is a connected covering space of X then $P(B, X)$ acts freely on B. A covering space is *regular* if the associated covering map is regular. Hence if (C, p) is a connected regular covering space of X then the Poincaré group acts freely and p-transitively on C. A regular covering space of X remains regular over any intermediate covering space.

REMARK. If (C, p) is a covering space of X and G a group of permutations of C such that $pg = p$, all $g \in G$, then G is equicontiguous *iff* G is equi-open *iff* there exists $U \in \mathrm{ad}(C)$ such that $gU = Ug$, all $g \in G$. If, under these circumstances, C is U-connected, then (by the remark following Theorem 3 of §3) G acts freely on C.

THEOREM 1. *If* (C, p) *is a covering space of the space* X, *then there exists a regular covering space* (D, q) *of* X *and a covering map* $f: (D, q) \geqq (C, p)$. *Moreover,* (D, q) *can be chosen minimal with this property.*

[*I.e., if* (R, ρ) *is any regular covering space of* X *such that there exists a covering map* $\eta: (R, \rho) \geqq (C, p)$, *then* $(R, \rho) \geqq (D, q)$.]

REMARK. The fiber of D over $x \in X$ has the same cardinality as the full group of permutations of the fiber of C over x.

The proof of Theorem 1 is delayed to the end of §5.

COROLLARY 1.1. *If* (C, p) *is a connected covering space of* X, *then there exists a smallest regular covering space* $(D, q) \geqq (C, p)$.

Proof. Let (D, q) be taken as in Theorem 1. Then if (R, ρ) is any regular covering space of $X \geqq (C, p)$, then by Theorem 2 of §3 any map $f: (R, \rho) \geqq (C, p)$ is a covering map. Hence by Theorem 1 $(R, \rho) \geqq (D, q)$.

REMARK. Later [§11, Example 20] we will give an algebraic proof of Corollary 1.1, and an algebraic proof [§11, Example 21] of the minimality condition of Theorem 1.

PROPOSITION 2. *If (B, p) is a covering space of $X \neq \emptyset$ and (C, q) a connected covering space of Y, $f: X \to Y$ is a contiguous function, $\tilde{f}: B \to C$ a map over f, and N the set of all $g \in P(B, X)$ such that there exists $\alpha \in P(C, Y)$ with $\alpha\tilde{f} = \tilde{f}g$, then N is a subgroup of $G = P(B, X)$, and there exists a unique homomorphism $\phi: N \to P(C, Y)$ such that $\phi(g)\tilde{f} = \tilde{f}g$, all $g \in N$. The sequence of groups: $0 \to G \cap P(\tilde{f}) \to N \to^{\phi} P(C, Y)$ is exact.*

$$\begin{tikzcd}
B \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & C \arrow[d, "q"] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
0 & 0
\end{tikzcd}$$

Proof. If $g \in N$, $\alpha, \beta \in P(C, Y)$ and $\alpha\tilde{f} = \tilde{f}g = \beta\tilde{f}$, then let $b_0 \in B$. We have $\alpha\tilde{f}(b_0) = \beta\tilde{f}(b_0)$, $q\alpha\tilde{f} = q\tilde{f} = q\beta\tilde{f}$, so that by Theorem 3 of §3, $\alpha = \beta$. Hence ϕ is a well-defined function, $N \to P(C, Y)$.

If $g, h \in G$, $\alpha, \beta \in P(C, Y)$ and $\alpha\tilde{f} = \tilde{f}g$, $\beta\tilde{f} = \tilde{f}h$, then $\alpha\beta\tilde{f} = \alpha\tilde{f}h = \tilde{f}gh$. Hence N is a submonoid of G and ϕ is multiplicative.

If $g \in N$, say $\alpha\tilde{f} = \tilde{f}g$, then $\alpha\alpha^{-1}\tilde{f} = \tilde{f} = \tilde{f}gg^{-1} = \alpha\tilde{f}g^{-1}$ so that $\alpha^{-1}\tilde{f} = \tilde{f}g^{-1}$, and $g^{-1} \in N$. Hence N is a subgroup of G and $\phi: N \to P(C, Y)$ is a homomorphism.

$$\begin{tikzcd}
B \arrow[dr, "f"] \arrow[dd, "p"'] & \\
& C \arrow[dl, "q"] \\
X \arrow[dl] \arrow[dd] & \\
0 & \\
& 0
\end{tikzcd}$$

Finally, if $g \in G$ then $g \in \ker(\phi)$ *iff* $\tilde{f} = \tilde{f}g$, i.e., *iff* $g \in P(\tilde{f})$.

A special case of interest is the one in which f is the identity:

THEOREM 3. *If $f: (B, p) \geqq (C, q)$ are connected covering spaces of X, the former regular, and N is the normalizer of $P(B, C)$ in $P(B, X)$, then there exists a unique homomorphism $\phi: N \to P(C, X)$ such that $\phi(g)f = fg$, all $g \in N$. The sequence of groups: $0 \to P(B, C) \to N \to^{\phi} P(C, X) \to 0$ is exact.*

Proof. We sketch the proof.

Let N be as in Proposition 2. Then N is the normalizer of $P(B, C)$ in $P(B, X)$: in fact, by Proposition 2, N is contained in said normalizer. Conversely, if g is in the normalizer, g is well-defined modulo the equivalence relation on B: $b \sim b'$ *iff* $f(b) = f(b')$, so that there exists α such that $fg = \alpha f$.

ϕ is an epimorphism: If B is nonempty choose $b \in B$. Given $\alpha \in P(C, X)$ choose $b' \in B$ such that $f(b') = \alpha f(b)$. Choose $g \in P(B, X)$ such that $g(b) = b'$. Then $\phi(g) = \alpha$.

COROLLARY 3.1. *If (C, p) is a connected covering space of X smaller than a connected regular covering space of X, then the Poincaré group $P(C, X)$ of C over X is equicontiguous.*

Proof. Let $f: (R, \rho) \geqq (C, p)$ where (R, ρ) is a connected regular covering space of X. By Theorem 2 of §3, f is a covering map from R onto C. Let $U \in \mathrm{ad}(R, \rho)$, $\overline{U} \in \mathrm{ad}(C, p)$ be chosen so that $fU = \overline{U}f$ and $gU = Ug$, all $g \in P(R, X)$. Then if $\alpha \in P(C, X)$, by Theorem 3 there exists $g \in P(R, X)$ such that $fg = \alpha f$. Then $\alpha \overline{U} f = \alpha f U = fgU = fUg = \overline{U} fg = \overline{U} \alpha f$, so that $\alpha \overline{U} = \overline{U} \alpha$. $\alpha \in P(C, X)$ being arbitrary, it follows that $P(C, X)$ is equicontiguous.

REMARK. We shall give an algebraic proof of Corollary 3.1 later [Example 20, §11].

COROLLARY 3.2. *The Poincaré group of a connected covering space of a semisimple space is always equicontiguous.*

THEOREM 4 (GALOIS THEOREM). *If (R, r_0, ρ) is a connected regular covering space of the space (X, x_0), then the mapping which associates to every connected covering space (C, c_0, q) of (X, x_0) smaller than (R, r_0, ρ) the Poincaré group $P(R, C)$ of R over C defines an order-reversing isomorphism from the lattice of isomorphism classes of connected covering spaces smaller than (R, r_0, ρ) with the lattice of subgroups of $P(R, X)$; a covering space is regular* iff *the associated subgroup is normal.*

If N is the normalizer of $P(R, C)$ in $P(R, X)$, then $P(C, X) \approx N/P(R, C)$. Hence, if C is regular, then $P(C, X) \approx P(R, X)/P(R, C)$.

Proof. If $f: (R, r_0, \rho) \geqq (C, c_0, q)$, then by Theorem 2 of §3 (R, f) is a covering space of C, so that (C, c_0) is the quotient space of (R, r_0) by the relation: "$f(r) = f(r')$"; i.e., by the relation: "There exists $h \in P(R, C)$ such that $h(r) = r'$." Thus (C, c_0) is the orbit space of (R, r_0) by the subgroup $P(R, C)$ of

$P(R, X)$, the map q: $(C, c_0) \to (X, x_0)$ being deduced from p: $(R, r_0) \to (X, x_0)$ by "passage to the quotient."

Conversely, if G is any subgroup of $P(R, X)$, then let C be the orbit space of R by the group G and f: $R \to C$ be the natural map. Let c_0 be the image of r_0, and q: $C \to X$ be deduced from ρ: $R \to X$ by "passage to the quotient" (i.e., by requiring that $qf = \rho$). By the remark preceding Theorem 1, there exists $U \in \mathrm{ad}(R, \rho)$ such that $gU = Ug$, all $g \in P(R, X)$, and in particular, all $g \in G$. Hence U defines a unique entourage $\overline{U} \in \mathrm{contig}(C)$ such that $\overline{U}f = fU$. $\overline{U}$ clearly obeys Axioms (C1)–(C3) of §2 for the map q: $C \to X$; since the $\overline{U}$ thus obtained are a base for $\mathrm{contig}(C)$, (C, c_0, q) is a covering space of X with f: $(R, r_0, \rho) \geqq (C, c_0, q)$ and $G = P(R, C)$.

By Corollary 3.1, the Poincaré group $P(C, X)$ of C over X is equicontiguous. If $c \in C$, say $c = f(r)$, then $P(C, X)$ is transitive on the fiber over $q(c)$ *iff* $[c' \in C, q(c) = q(c')$ implies there exists $\alpha \in P(C, X)$ such that $\alpha(c) = c']$. Let N be the normalizer of $P(R, C)$ in $P(R, X)$ and ϕ be as in Theorem 3; choose $r' \in R$ and $g, g' \in P(R, X)$ such that $c' = f(r')$, $r' = g'(r)$, $\alpha = \phi(g)$. Then this condition becomes: "For all $g' \in P(R, X)$, there exists $g \in N$ such that $fg(r) = fg'(r)$." But by Theorem 3 of §3, this implies that $fg = fg'$, $fg'g^{-1} = f$, $g'g^{-1} \in P(R, C)$, $g' \in N$. Hence the condition becomes: $N = P(R, X)$, i.e., $P(R, C)$ is normal in $P(R, X)$.

Remark. In the course of the proof of Theorem 4 we have observed that a connected covering space smaller than a connected regular covering space is regular *iff* its Poincaré group is transitive on some fiber. Actually, a trivial application of Theorem 1 of §1 shows that a covering space of a connected space is regular *iff* there exists an equicontiguous subgroup of the Poincaré group that is transitive on a single fiber. [More generally, it is true that, if (C, p) is a covering space of X, $U \in \mathrm{ad}(C, p)$, $V = pUp^{-1}$, and there is a group of automorphisms of (C, p) commuting with U that is transitive on some fiber $p^{-1}(x)$, then, *if X is V-connected*, the covering space (C, p) is regular.]

5. The Lifting Theorem.

Theorem 1 (Lifting Theorem). *Every diagram*:

$$\begin{tikzcd} & D \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \arrow[d] \\ & 0 \end{tikzcd}$$

can be imbedded in a commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d, "p"] & D \arrow[d, "q"] \\ X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 & 0 \end{tikzcd}$$

If f (resp.: q) is regular, then so is $\tilde{f}$ (resp.: p). If both f and q are regular then so is $fp = q\tilde{f}$.

The proof is obvious.

REMARK. (1) The construction is the usual universal lift, or "pull-back."

REMARK. (2) If X is semi-simple, then we can require that C be connected by passing to a quasicomponent of C.

COROLLARY 1.1. *If (C, p) is a covering space of X, Y a subspace of X, $D = p^{-1}(Y)$, and $p_D = p|D$, then (D, p_D) is a covering space of Y.*

COROLLARY 1.2. *If (C, p) is a covering space of X, and $\mathfrak{C}$ is a contiguity for the underlying set of X finer than* contig(X), *then there exists a unique contiguity $\mathfrak{A}$ for the underlying set of C finer than* contig(C) *such that p: $(C, \mathfrak{A}) \to (X, \mathfrak{C})$ is a covering map.*

COROLLARY 1.3. *If (Y, y) is a space with base point, then any two covering spaces (with base points) admit a supremum. The supremum of two regular covering spaces (with base points) is regular.*

COROLLARY 1.4. *The isomorphism classes of connected covering spaces with base point of a semi-simple space with base point form a lattice in which the regular connected covering spaces are cofinal. The supremum of two regular elements is regular.*

REMARK. Corollaries 1.1 through 1.4 follow trivially from Theorem 1 and the Remark immediately following.

COROLLARY 1.5. *The image of a semi-simple space by a contiguous and open function is semi-simple.*

Proof. Let $f: X \to Y$ be a contiguous and open function from the semi-simple space X onto the space Y; let (D, q) be a covering space of Y. Let

C, p, $\tilde{f}$ be as in Theorem 1. Then since f is an epimorphism, so is $\tilde{f}$. Choose $U \in \mathrm{ad}(D, q)$ and $T \in \mathrm{ad}(C, p)$ such that $\tilde{f}T \subset U\tilde{f}$ and the T-components of C are connected. Let $W = pTp^{-1}$, $V = qUq^{-1}$. Then $fW \subset Vf$. Since f is open, we can choose $V' \subset V$, $V' \in \mathrm{contig}(Y)$ such that $fW \supset V'f$. Let U' be the unique element of $\mathrm{ad}(D, q)$ contained in U such that $qU' = V'q$. Then $\tilde{f}T \supset U'\tilde{f}$. If now $c \in C$ and $d = \tilde{f}(c)$ then since $\tilde{f}$ is an epimorphism, we have $\tilde{f}(T\text{-component of } c) \supset (U'\text{-component of } d)$. Since the T-component of c is connected, so is its image under $\tilde{f}$; hence the U'-component of d, being contained in a connected set, is itself connected.

COROLLARY 1.6 (MONODRONOMY THEOREM). *Let X be a semi-simple space and $x_0 \in X$. Then the following conditions are equivalent*:

(1) *Every diagram*

$$\begin{tikzcd} & C, c_0 \arrow[d] \\ X, x_0 \arrow[r] & Y, y_0 \arrow[d] \\ & 0 \end{tikzcd}$$

can be imbedded in a commutative diagram

$$\begin{tikzcd} & C, c_0 \arrow[d] \\ X, x_0 \arrow[ur] \arrow[r] & Y, y_0 \arrow[d] \\ & 0 \end{tikzcd}$$

(2) *If $p: (C, c_0) \to (X, x_0)$ is a covering map, then there exists $f: (X, x_0) \to (C, c_0)$ such that $pf = 1_X$.*

(3) *Every connected covering space of X is isomorphic to the trivial covering space.*

Proof. (3)$\Rightarrow$(1) follows immediately from Theorem 1 and Remark (2) immediately following. Taking the map $X \to Y$ to be the identity, we see that

(1)$\Rightarrow$(2). Finally, (2)$\Rightarrow$(3). In fact, assume (2). Then f: $(X, x_0, 1_X) \geqq (C, c_0, p)$ and p: $(C, c_0, p) \geqq (X, x_0, 1_X)$, so that by Corollary 3.1 of §3 we have $(C, c_0, p) \approx (X, x_0, 1_X)$.

A semi-simple space is called *simply connected* if it obeys one of these three equivalent conditions. (In §7, the concept of simple connectedness will be extended to the nonsemi-simple case.)

EXAMPLE 1. The unit interval is simply connected.

EXAMPLE 2. A connected and locally arcwise connected topological space such that every loop is null-homotopic is simply-connected. Conversely, an lc^1 topological space that is simply connected has the property that every loop is null-homotopic.

EXAMPLE 3. The *uniform space* of rational numbers is connected, locally connected, and simply connected.

By Corollaries 1.4 and 1.6, a connected covering space of a semi-simple space X is simply connected if and only if it is maximal, and, when it exists, it is unique up to isomorphism—canonical if one fixes base points.

PROPOSITION 2. *Let $(C_i, p_i)_{i\in I}$ be a family of covering spaces with base points of the space with base point X. Then they admit an upper bound* iff *there exists $U^i \in \mathrm{contig}(C_i)$ such that the restriction of U^i to the U^i-component D_i of the base point is in* $\mathrm{ad}(D_i)$, *all $i\in I$, and $V\in \mathrm{contig}(X)$ such that*:

(S1) $p_iU_i = Vp_i$, *all $i\in I$.*

(S2) *If $x\in X$, then there exists $W^x \in \mathrm{contig}(V_x)$ such that $p_i(U^i_c \cap U^i_d) \supset W^x_{p_i(d)}$, all $d\in U^i_c$, $c\in p_i^{-1}(x)$, $i\in I$.*

Proof. We prove sufficiency, and defer the proof of necessity until the end of this section.

Assume we are given U_i, V, and W^x with the stated properties. Passing to U^i-components if necessary, we may assume that C_i is U^i-connected. Then $U^i \in \mathrm{ad}(C_i, p_i)$. Let C be the set of all families $(c_i)_{i\in I} \in \prod_{i\in I} C_i$ such that $p_i(c_i) = p_j(c_j)$, all i, $j\in I$. Let p: $C\to X$ be the function such that $p(c_i)_{i\in I} = p_i(c_i)$, all $i\in I$. For each $c = (c_i)_{i\in I} \in C$, let $U_c = C \cap \prod_{i\in I} U^i_{c_i}$; then U is a p-adaptation with $pU = Vp$. Let C be endowed with the unique contiguity containing U rendering (C, p) a covering space of X [Theorem 1, §2]. Then if f_i: $C\to C_i$ is the restricted projection (all $i\in I$), we have f_i: $(C, p) \geqq (C_i, p_i)$, all $i\in I$.

REMARK. Consider the category $\mathfrak{A}_{X,x}$ of quadruples: (C, c, p, U) where (C, c, p) is a covering space of the fixed space with base point (X, x) and $U\in \mathrm{ad}(C, p)$ [*equivalently*: C is a set, $c\in C$, p: $C\to X$ is a function such that $p(c) = x$, and U is a p-adaptation], the maps f: $(C, c, p, U) \to (D, d, q, U')$ being the maps of covering spaces f: $(C, c, p) \geqq (D, d, q)$ such that $fU \subset U'f$ [*equivalently*: the functions f: $C\to D$ such that $qf = p$, $f(c) = d$, $fU \subset U'f$]. Then the upper bound (C, c, p, U) for the family $(C_i, c_i, p_i, U^i)_{i\in I}$ constructed in Proposition 2 is the *direct product* [4] $\times_{i\in I} (C_i, c_i, p_i, U^i)$ of the given family in this category, the maps f_i being the canonical projections. In par-

ticular, for each permutation π of the set of indices I, there exists a unique isomorphism α_π: $\times_{i\in I}(C_{\pi(i)}, c_{\pi(i)}, p_{\pi(i)}, U^{\pi(i)}) \approx \times_{i\in I}(C_i, c_i, p_i, U^i)$ such that $f_i\alpha_\pi = g_\pi^{-1}{}_{(i)}$ all $i\in I$, where f_i, g_i are the canonical projections.

Proof of Theorem 1 of §4. Let $U\in \mathrm{ad}(C, p)$ and $V=pUp^{-1}$. Then working separately over each V-component of X if necessary, we may assume that X is V-connected. Choose a base point $x\in X$ and let $I=p^{-1}(x)$. Then the conditions (S1) and (S2) of Proposition 2 hold for the family $(C, i, p, U)_{i\in I}$, so that the direct product: $(\overline{C}, \bar{c}_0, \bar{p}, \overline{U}) = \times_{i\in I}(C, i, p, U)$ in the category $\mathfrak{A}_{X,x}$ exists: let f_i denote the projections. Then $\overline{C}$ is the set of all families $(c_i)_{i\in I}$ of elements of C indexed by the set I such that $p(c_i)=p(c_j)$ all $i, j\in I$, $\bar{c}_0$ is the identity family: $(i)_{i\in I}$, and f_i: $\overline{C}\to C$ is the function $f_i(c_j)_{j\in I}=c_i$. Let D be the set of all elements c of $\overline{C}$ such that the mapping: $i\to f_i(c)$ is a bijection from $I=p^{-1}(x)$ onto $p^{-1}\bar{p}(c)$. Then $\bar{c}_0\in D$, and whenever $d\in D$, $c\in\overline{C}$ and $d\in\overline{U}_c$ we have $\overline{U}_c\subset D$; hence D is a union of $\overline{U}$-components of $\overline{C}$. Since X is $V=pUp^{-1}$-connected, if $U'=\overline{U}|D$, $q=\bar{p}|D$, then $(D, \bar{c}_0, q)$ is a covering space of X with $U'\in\mathrm{ad}(D, q)$, and $(f_i|D)$ is a map of quadruples: $(D, \bar{c}_0, q, U')\to(C, i, p, U)$, all $i\in I$.

Now if π is a permutation of I, then the direct product $\times_{i\in I}(C, \pi(i), p, U)$ has the same underlying space, adaptation, and projections as $(\overline{C}, \bar{c}_0, \bar{p}, \overline{U})$ but has the point $(\pi(i))_{i\in I}$ as base point. Hence the isomorphism α_π: $\times_{i\in I}(C, \pi(i), p, U)\approx\times_{i\in I}(C, i, p, U)$ is an automorphism $\alpha_\pi\in P(\overline{C}, X)$ of the covering space $(\overline{C}, \bar{p})$ of X such that $\alpha_\pi\overline{U}=\overline{U}\alpha_\pi$, $f_i\alpha_\pi=f_{\pi^{-1}(i)}$, and $\alpha_\pi(\pi(i)_{i\in I})=\bar{c}_0$. If $c\in\overline{C}$, then $c\in D$ *iff* the mapping $i\to f_i(c)$ is a bijection from I onto $p^{-1}\bar{p}(c)$ *iff* the mapping $i\to f_{\pi(i)}(c)=f_i(\alpha_\pi(c))$ is a bijection from I onto $p^{-1}\bar{p}(c)=p^{-1}\bar{p}(\alpha_\pi(c))$ *iff* $\alpha_\pi(c)\in D$; hence the restriction β_π of α_π to D is an automorphism of the covering space (D, q) of X such that $\beta_\pi U'=U'\beta_\pi$, $\beta_\pi(\pi(i)_{i\in I})=\bar{c}_0$. But if $d\in D$ and $q(d)=x$, then $d=(\pi(i)_{i\in I})$, where π is the unique permutation of I such that $f_i(d)=\pi(i)$. Hence the group G of all automorphisms of D commuting with U' acts transitively on the fiber $q^{-1}(x)$ over x; by the Remark following Theorem 4 of §4, (D, q) is a regular covering space of X.

Now let (R, ρ) be an arbitrary regular covering space of X and η a covering map: $(R, \rho)\geqq(C, p)$. Fix $r_0\in p^{-1}(x)$. Choose $U''\in\mathrm{ad}(R)$ such that $\eta U''\subset U\eta$, and such that the group of automorphisms G of the covering space (R, ρ) of X commuting with U'' is ρ-transitive. Let R' be the U''-component of r_0 in R. Working separately over each $\rho U''\rho^{-1}$-component of X if necessary, we may assume that X is $\rho U''\rho^{-1}$-connected; i.e., that $\rho(R')=X$. For each point $i\in p^{-1}(x)$, choose $g_i\in G$ such that $\eta g_i(r_0)=i$. Then ηg_i: $(R, r_0, \rho, U'')\to(C, i, p, U)$ is a map of quadruples, all $i\in I$; hence [by definition of the category-theoretic direct product] there exists a unique map of quadruples τ: $(R, r_0, \rho, U'')\to\times_{i\in I}(C, i, p, U)=(\overline{C}, \bar{c}_0, \bar{p}, \overline{U})$ such that $\eta g_i=f_i\tau$, all $i\in I$. If $r\in R$, $s\in U_r''$ and $\tau(s)\in D$, then $\tau(U_r'')\subset D$; hence $\tau^{-1}(D)$ is a union of U''-components of R. Since $\tau(r_0)=\bar{c}_0\in D$, $r_0\in\tau^{-1}(D)$, so

that $\tau^{-1}(D)\supset R'$, and the restriction $\sigma_{R'}$ of τ to R' is a map of quadruples: $(R', r_0, \rho|R', U''|R')\to(D, \bar{c}_0, q, U')$. Fixing base points in the other U''-components of R, we obtain similarly maps $\sigma_H: H\to D$, all U'-components H of D. Let σ be their common extension to R. Then σ is a map of quadruples: $(R, r_0, \rho, U'')\to(D, \bar{c}_0, q, U')$; in particular, σ is a map: $(R, \rho)\geqq(D, q)$.

Completion of the proof of Proposition 2. Assume an upper bound (C, p) for the given family exists. Passing to a larger covering space if necessary [§4, Theorem 1], we may assume that (C, p) is regular; passing to a suitable union of U'^i-components of C_i, we may assume [by the Remark following Theorem 2 of §3] that all of the maps $f_i: C\to C_i$ are covering maps. Choose $U\in\mathrm{ad}(C, p)$ and a p-transitive group G of automorphisms of (C, p) such that $gU=Ug$, all $g\in G$; let $G_i=P(C, C_i)$. Then C_i is the orbit space $C|G_i$ and $f_i: C\to C_i$ is the natural map; hence there exists a unique element U^i of $\mathrm{contig}(C_i)$ such that $f_iU=U^if_i$. Put $V=pUp^{-1}$, and let $W^x\in\mathrm{contig}(V_x)$ be chosen obeying condition (4) in the definition of p-adaptation [§2], all $x\in X$. Then U^i, V, and W^x obey (S1) and (S2).

6. **Quasispaces and quasimorphisms.** Let X be a space which is not necessarily semi-simple and $x_0\in X$. If (C, c_0) is a covering space of X then the quasicomponent of c_0 need not be a covering space of X [Example 4, §11]. Call two covering spaces *equivalent* if there exists $U\in\mathrm{contig}(C)$ and $U'\in\mathrm{contig}(C')$ such that the U-component of C is isomorphic to the U'-component of C'. Then the concept of "equivalence class of covering spaces of X" enjoys all the properties of that of "isomorphism class of connected coverings of a semi-simple space X." In fact, if X is semi-simple, the two concepts coincide—that is, the equivalence classes of coverings are in one-to-one correspondence with the isomorphism classes of connected coverings.

In this section and in §§7 and 8, we shall develop this "trick" into a general theory. We shall obtain the results of §§3–5 without any connectedness assumptions.

A *quasispace* is a pair (X, A) where X is a space and A is a quasicomponent of X. If (X, A) is a quasispace and $U\in\mathrm{contig}(X)$, then the *U-component* of (X, A) is the quasispace (Y, A) where Y is the U-component of X containing A.

If $f, g: (X, A)\to(Y, B)$ are functions, then f and g are *equivalent* if they agree on the U-component of (X, A) for some $U\in\mathrm{contig}(X)$. A *quasimorphism* $\{f\}: (X, A)\to(Y, B)$ is an equivalence class of contiguous functions $f: (X, A)\to(Y, B)$.

Remark. If (X, A) and (Y, B) are quasispaces, and $f: X\to Y$ is a contiguous function, then in order that $f(A)\subset B$ it suffices that f apply a single point of A into B.

Example. Every connected space X defines a quasispace (X, X). Moreover, if X and Y are connected spaces, then the notion of "contiguous function from X into Y" coincides with that of "quasimorphism from (X, X)

into (Y, Y)." Hence quasispaces may be thought of as a generalization of connected spaces.

Quasispaces and quasimorphisms form a *category*; in particular, we have the notions of quasimonomorphism and quasiisomorphism. "Quasicomponent" is a functor from this category into the category of spaces and contiguous functions.

EXAMPLE. The inclusion from the U-component of the quasispace (X, A) into (X, A) is a quasiisomorphism.

A *pointed space* is a pair (X, x) where X is a space and $x \in X$. If A is the quasicomponent of x in X, then (X, A) is a quasispace. A *morphism* $\{f\}: (X, x) \to (Y, y)$ is a quasimorphism $\{f\}$ of the underlying quasispaces such that $f(x) = y$.

A quasimorphism $\{p\}: (X, A) \to (Y, B)$ is a *covering morphism* if there exists $U \in \text{contig}(X)$ and $V \in \text{contig}(Y)$ such that the restriction of p is a covering map from the U-component of (X, A) onto the V-component of (Y, B). A *covering morphism* of pointed spaces is a morphism $\{p\}: (X, x) \to (Y, y)$ of pointed spaces such that the underlying quasimorphism is a covering morphism of the underlying quasispaces. A *covering space* of a quasispace or pointed space is defined in the obvious way.

The definitions and theorems of §3 can be rewritten or generalized as follows for *pointed spaces* (all maps are morphisms of pointed spaces, and all spaces are pointed).

THEOREM 1.1. *The composite of covering morphisms is a covering morphism.*

THEOREM 3.1 (UNIQUENESS). *A covering morphism is a (category-theoretic) monomorphism.*

If (C, p) and (D, q) are covering spaces of X, then $(C, p) \geqq (D, q)$ *iff* there exists $f: C \to D$ such that $p = qf$; by Theorem 3.1, such an f is necessarily unique.

REMARK. Thus, $(C, p) \geqq (D, q)$ *iff* $(C, p) \subset (D, q)$ as category-theoretic subobjects of X.

Note that two covering spaces C and D of the pointed space X are category-theoretically isomorphic *iff* each is larger than the other *iff* there exist $U \in \text{contig}(C)$ and $U' \in \text{contig}(D)$ such that the U-component of C and the U'-component of D are isomorphic considered as covering spaces of the underlying space of some V-component of X through an isomorphism carrying the base point of C into that of D.

THEOREM 2.1 AND COROLLARY 3.1.1. *If (C, p) and (D, q) are covering spaces of the pointed space X and $(C, p) \geqq (D, q)$ then the unique morphism f such that $p = qf$ is a covering morphism.*

A *morphism* over a diagram

$$\begin{tikzcd} C \arrow[d] & D \arrow[d] \\ X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 & 0 \end{tikzcd}$$

is a morphism $\tilde{f}$: $C \to D$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d] & D \arrow[d] \\ X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 & 0 \end{tikzcd}$$

is commutative. There exists at most one morphism over f.

We remark that all the special results on connected covering spaces and/or semi-simple base spaces obtained previously follow from the quasitheory—in fact, a space is semi-simple *iff* it is connected and all its covering spaces are quasiisomorphic to connected spaces.

7. **Filtered groups.** A *filtered group* is a group G together with a filter on the underlying set of G having a base consisting of subgroups of G. These subgroups are called the *disks* of the filtered group G.

Let G and H be filtered groups and D a disk in G. Let B_D denote the set of all homomorphisms f from D into H such that the inverse image of every

disk in H is a disk in G. Then if $D \subset E$ are disks in G, we have a *restriction map*: $B_E \to B_D$. Let $\mathrm{Hom}(G, H)$ be the direct limit. The elements of $\mathrm{Hom}(G, H)$ are called *homomorphisms* from G into H. With this definition of homomorphism filtered groups form a category.

If G is an abstract group then we regard G as a filtered group by taking G to be the only disk (indiscrete filtration). The *intersection* $\bigcap H$ of a filtered group H is the intersection of the disks of H. A filtered group H *reduces to a group* if it is isomorphic to a group; equivalently, if the canonical homomorphism: $\bigcap H \to H$ is an isomorphism. Alternatively, *iff* it has a smallest disk.

If G is an abstract group and H is a filtered group then a homomorphism of filtered groups from G into H is simply a homomorphism of abstract groups from G into the intersection of H.

Recall [4] that a *subobject* of an object A of a category is an equivalence class $\{f\}$ of monomorphisms with range A, two such monomorphisms f_1 and f_2 being equivalent *iff* there exists an isomorphism n such that $f_1 = f_2 n$. The identity map of A defines a subobject of A which we identify with A.

If G is a filtered group and H a subgroup of the underlying group of G, then we regard H as a filtered group by taking the coarsest filter rendering the inclusion a morphism. The inclusion from the filtered group H into the filtered group G is then a monomorphism, and defines a subobject of G. We call such a subobject a *filtered subgroup*. Hence, each filtered subgroup of the filtered group G is given by an ordinary subgroup H of the abstract group G, two such subgroups defining the same filtered subgroup $\{H\}$ *iff* their intersections with a "sufficiently small" disk of G are identical.

In particular, every disk of G defines "the whole filtered group."

A filtered subgroup $\{H\}$ of the filtered group G is *normal* if its intersection $H \cap D$ with a "sufficiently small" disk D of G is normal in D. If N is a normal filtered subgroup of G, then one defines the *quotient filtered group* G/N.

8. **Galois theory and the Lifting Theorem (non-connected case).** For the remainder of this paper, unless otherwise stated, all spaces are pointed and all maps are morphisms of pointed spaces.

A covering space C of a pointed space X is *regular* if some U-component is regular (in the sense of spaces and contiguous functions).

If (C, p) is a covering space of the pointed space X and $U \in \mathrm{ad}(C)$ then let $P_U(C, X)$ denote the group of automorphisms α of the underlying space D of the U-component of C such that $p\alpha = p$ and $\alpha U = U\alpha$. Then by the Remark preceding Theorem 1 of §4, $P_U(C, X)$ acts freely on the U-component of C. If (C, p) is a *regular covering space* of X, then there exists $U \in \mathrm{ad}(C)$ such that $P_U(C, X)$ acts p-transitively on the U-component of C. Then for $U' \subset U$ every $g \in P_{U'}(C, X)$ extends uniquely to an element of $P_U(C, X)$, so that $P_{U'}(C, X) \subset P_U(C, X)$. Let $\mathcal{P}(C, X)$ be the filtered group whose underlying group is $P_U(C, X)$ and has for base the disks $P_{U'}(C, X)$ for $U' \subset U$, U'

$\in$contig(C). Then up to a canonical isomorphism the filtered group $\mathcal{P}(C, X)$ is independent of the choice of $U \in \mathrm{ad}(C)$. $\mathcal{P}(C, X)$ is called the *Poincaré filtered group* of the covering space (C, p) of X; its intersection is denoted $P(C, X)$ and is called the *Poincaré group* [it is the group of quasiautomorphisms α of the underlying quasispace of C such that $p\alpha = p$].

REMARK. For covering spaces which are not necessarily regular there is an invariant analogous to the Poincaré filtered group.

A *quasigroup* is a direct family $(G_i, \alpha_{ij})_{i,j \in I}$ of filtered groups with each $\alpha_{ij}: G_i \to G_j$ a monomorphism making G_i into a filtered subgroup of G_j. A *morphism* of quasigroups: $(G_i) \to (H_j)$ is a morphism of "formal direct limits": $(\lim_{\to i} G_i) \to (\lim_{\to j} H_j)$ (3). With this notion of morphism each quasigroup $(G_i)_{i \in I}$ is the direct limit of the filtered groups G_i. In particular, a "constant family" $(G)_{i \in I}$ is isomorphic to the filtered group G.

A *subquasigroup* of the quasigroup $(G_i)_{i \in I}$ is a subobject $(F_k)_{k \in K}$ such that for each $i \in I$ there exists $k \in K$ such that the homomorphism: $F_k \to G_i$ makes F_k into a filtered subgroup of G_i.

If (C, p) is an arbitrary (not necessarily regular) covering space of the pointed space X, then for $U \in \mathrm{ad}(C)$ let $\mathcal{P}_U(C, X)$ be the filtered group whose underlying group is the group $P_U(C, X)$ described above, the filtration being given by the disks $P_U(C, X) \cap P_{U'}(C, X)$ for $U' \subset U$, $U' \in \mathrm{ad}(C)$. Then, for $U^1 \subset U^2$ in $\mathrm{ad}(C)$, $\mathcal{P}_{U^2}(C, X)$ is a filtered subgroup of $\mathcal{P}_{U^1}(C, X)$. We therefore have a quasigroup $\mathcal{P}(C, X)$ called the *Poincaré quasigroup of C over X*. (If C is regular, this is isomorphic to the Poincaré filtered group.)

If H is a filtered subgroup of the filtered group G, then for each disk D in G let N_D be the normalizer of the group $H \cap D$ in D. Then N_D defines a filtered subgroup of G. If $D_1 \subset D_2$ are disks in G, then *as filtered subgroups of G $N_{D_1} \supset N_{D_2}$*; hence, we have a quasigroup N, called the *normalizer of H in G*. (One can extend this definition to the case in which G and H are quasigroups.) Then H is a normal subquasigroup of N, and one can form the quotient quasigroup N/H.

The theorems of §§4 and 5 all generalize to the quasitheory (all spaces are pointed, all morphisms are morphisms of pointed spaces):

THEOREM 1.1. *If (C, p) is a covering space of X, then there exists a regular covering space (D, q) of X such that $(D, q) \geqq (C, p)$.*

REMARK. The analogue of Corollary 1.1 of §4 to the quasitheory is *false*; cf. Example 20 of §11. See, however, Example 21 of §11.

Note that the assignment $(C, X) \to \mathcal{P}(C, X)$ is *functorial* for regular covering spaces: i.e., given a diagram:

---

(3) In other words, a *morphism*: $(G_i) \to (H_j)$ of quasigroups is a morphism from (H_j) to (G_i) considered as pro-objects in the dual category of the category of filtered groups [3].

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 & 0 \end{tikzcd}$$

with the vertical arrows regular covering morphisms one obtains a morphism: $\mathcal{P}(B, X) \to \mathcal{P}(C, Y)$ of filtered groups.

(Moreover, the generalization of Proposition 2 of §4 to the quasitheory describes functorality of the assignment $(C, X) \to \mathcal{P}(C, X)$ where C is an arbitrary covering space of X and $\mathcal{P}(C, X)$ is the Poincaré quasigroup. The induced morphism (as in Proposition 2 of §4) is defined on a subquasigroup of $\mathcal{P}(B, X)$.)

THEOREM 3.1.

$$\begin{tikzcd} B \arrow[dr, "f"] \arrow[dd, "p"'] & \\ & C \arrow[dl, "q"] \\ X \arrow[dl] \arrow[dd] & \\ 0 & \\ & 0 \end{tikzcd}$$

If f: $(B, p) \geqq (C, q)$ are covering spaces of X, the former regular, and $\mathfrak{N}$ is the normalizer of the filtered group $\mathcal{P}(B, C)$ in $\mathcal{P}(B, X)$, then there exists a unique morphism of quasigroups ϕ: $\mathfrak{N} \to \mathcal{P}(C, X)$ such that $\phi(g)f = fg$, all $g \in \mathfrak{N}$. The sequence of quasigroups: $0 \to \mathcal{P}(B, C) \to \mathfrak{N} \to^{\phi} \mathcal{P}(C, X) \to 0$ *is exact. Otherwise*

stated, ϕ induces an isomorphism from the quotient quasigroup $\mathfrak{N}/\mathcal{P}(B, C)$ onto $\mathcal{P}(C, X)$.

REMARK. (B, f) is a regular covering space of C; and (C, q) is regular *iff* the filtered group $\mathcal{P}(B, C)$ is normal in the filtered group $\mathcal{P}(B, X)$, in which case the quotient filtered group $\mathcal{P}(B, X)/\mathcal{P}(B, C)$ is isomorphic to $\mathcal{P}(C, X)$.

THEOREM 4.1 (GALOIS THEOREM).

$$\begin{tikzcd}
R \arrow[dr, "f"] \arrow[dd, "\rho"'] & \\
& C \arrow[dl, "q"] \\
X \arrow[dl] \arrow[dd] & \\
0 & \\
& 0
\end{tikzcd}$$

If (R, ρ) is a regular covering space of X, then the mapping which associates to every covering space (C, q) of X smaller than (R, ρ) the Poincaré filtered group $\mathcal{P}(R, C)$ of R over C defines an order-reversing isomorphism from the ordered set of isomorphism classes of covering spaces smaller than (R, ρ) with the ordered set of filtered subgroups of the Poincaré filtered group $\mathcal{P}(R, X)$; a covering space is regular iff *the associated filtered subgroup is normal.*

REMARK. A covering space of a pointed space some U-component of which has an equicontiguous Poincaré group is regular *iff* the Poincaré group is transitive on the fiber of the base point of some V-component.

The Lifting Theorem (Theorem 1, §5) and Corollaries 1.1 and 1.2 of §5 hold without change for pointed spaces.

COROLLARIES 1.3.2 AND 1.4.2. *The isomorphism classes of covering spaces of a pointed space form an ordered set closed under finite suprema in which the regular covering spaces are cofinal. The supremum of two regular covering spaces is regular.*

REMARK. Said ordered set is *not*, in general, a lattice—cf. Example 17, §11.

COROLLARY 1.6.2 (MONODRONOMY THEOREM). *If X is a pointed space, then the following three conditions are equivalent:*

(1) *Every diagram*

$$\begin{tikzcd} & C \arrow[d] \\ X \arrow[r] & Y \arrow[d] \\ & 0 \end{tikzcd}$$

can be imbedded in a commutative diagram

$$\begin{tikzcd} & C \arrow[d] \\ X \arrow[ur] \arrow[r] & Y \arrow[d] \\ & 0 \end{tikzcd}$$

(2) *If* $p: C \to X$ *is a covering morphism, then there exists a morphism* $f: X \to C$ *such that* $pf = 1_X$.

(3) *Every covering space of* X *is isomorphic to the trivial covering space.*

A pointed space is called *simply connected* if it obeys one of these three equivalent conditions. Clearly this definition depends only on the underlying quasispace (and, for semi-simple spaces, coincides with the definition given earlier).

A covering space of a pointed space X is simply connected *iff* it is maximal; hence, any two simply connected covering spaces of X are isomorphic, and a simply connected covering space is regular. Therefore if X admits a simply connected covering space $(\tilde{X}, p)$ then the Galois Theorem (4.1) applies, so that the isomorphism classes of covering spaces of X are in one-to-one correspondence with the filtered subgroups of $\mathcal{P}(\tilde{X}, X)$.

Proposition 2 of §5 yields:

PROPOSITION 2.2. *Let* $(C_i, p_i)_{i\in I}$ *be a family of covering spaces of the pointed space* X. *Then they admit an upper bound* iff *there exist covering spaces* $(C_i', p_i') \approx (C_i, p_i)$, *all* $i\in I$, $U^i\in \mathrm{ad}(C_i')$, *and* $V\in \mathrm{contig}(X)$ *such that*:

(S1) $p_i' U^i = V p_i'$, *all* $i\in I$.

(S2) *If* $x\in X$, *then there exists* $W^x\in \mathrm{contig}(V_x)$ *such that* $p_i'(U_c^i \cap U_d^i) \supset W^x_{p'_i(d)}$, *all* $i\in I$, $c\in p_i'^{-1}(x)$, $d\in U_c^i$.

COROLLARY 2.1.2. *The pointed space* X *admits a simply connected covering space* iff *there exists* $V\in \mathrm{contig}(X)$ *and, for each* $x\in X$, $W^x\in \mathrm{contig}(V_x)$ *such that, whenever* (C, p) *is a covering space of* X, *there exists a covering space* (C', p') *of* X *isomorphic to* (C, p) *and* $U\in \mathrm{ad}(C')$ *such that* $p'U = Vp'$, $p'(U_c \cap U_d) \supset W^x_{p'(d)}$ *all* $x\in X$, $c\in p'^{-1}(x)$, $d\in U_c$.

REMARK. A connected and locally connected topological space which is locally simply connected in the sense of Chevalley obviously obeys the conditions of the preceding corollary, and therefore admits a simply connected covering space.

Note that a connected and simply connected space is semi-simple; although we shall give many examples of connected spaces which admit simply connected covering spaces, yet are not semi-simple [§§10, 11].

9. **Fundamental group, universal covering space.** If X is a pointed space then by Corollary 1.4.2 of §8, the covering spaces of X form an ordered set closed under finite suprema, and the regular covering spaces a cofinal subset $\mathfrak{R}$. The *fundamental group* $\pi(X)$ is the projective limit of the Poincaré groups of regular covering spaces of X, $\pi(X) = \lim_{\leftarrow} P(C, X)$. Hence $\pi(X)$ is a Hausdorff topological group with the inverse limit topology. By the Lifting Theorem (Theorem 1, §5) and the observation immediately following Theorem 1.1 of §8, π is a functor. The topology of $\pi(X)$ may be given as follows: if (C, p) is a covering space (resp.: regular covering space) of X, then $\pi(p): \pi(C) \to \pi(X)$ is an injection the image of which is an open (resp.: open normal) subgroup of $\pi(X)$. These subgroups form a base for the neighborhood system at the identity, and $\pi(X)$ is totally disconnected. If X admits a simply connected covering space, then $\pi(X)$ is discrete. Also, if X is simply connected, then $\pi(X) = 1$.

If X is a semi-simple pointed space and the projections: $\pi(X) \to P(C, X)$ are epimorphisms for all regular covering spaces C of X, then the correspondence between isomorphism classes of covering spaces of X and open subgroups of $\pi(X)$ is one-to-one. This is always the case when X is *locally arcwise connected.*

If X is a semi-simple space, then one can define a "fundamental group tower" in much the same manner as is usually done in the algebraic case [1]. Then one obtains a perfect correspondence between isomorphism classes of connected covering spaces and "subtowers."

There is a natural homomorphism from the arcwise fundamental group $\pi_1(X)$ of a topological space X into the fundamental group $\pi(X)$ which we have

just defined. This mapping is an isomorphism if X is lc^1; but is neither injective nor surjective in general [Example 22, §11].

If X is a semi-simple pointed space, then let $\tilde{X}$ be the inverse limit of the connected covering spaces of X. If the natural maps from $\tilde{X}$ into each connected covering space of X are epimorphisms, then the natural homomorphisms from $\pi(X)$ into the Poincaré groups of the regular covering spaces of X are likewise epimorphisms, so that the open subgroups of $\pi(X)$ classify the isomorphism classes of connected covering spaces of X in one-to-one fashion. [This is always the case when X is *locally arcwise connected.*] Moreover, $\pi(X)$ is a topological transformation group of $\tilde{X}$, and the connected covering spaces of X are the orbit spaces of $\tilde{X}$ by the open subgroups of $\pi(X)$. Under these hypotheses, the closed subgroups of $\pi(X)$ correspond in one-to-one fashion to certain inverse limits of covering spaces of X, which we may call *pro-covering spaces* of X. If this notation is adopted, then $\tilde{X}$ is the *universal* (largest) pro-covering space of X. Under these conditions, X admits a simply connected covering space *iff* $\pi(X)$ is discrete *iff* $\tilde{X}$ is a covering space of X; in which case, $\tilde{X}$ is the universal covering space of X.

10. **Fundamental filtered group.** If X is a pointed space admitting a simply connected covering space $(\tilde{X}, p)$, then the filtered group $\mathcal{P}(\tilde{X}, X)$ is called the *fundamental filtered group* $\boldsymbol{\pi}(X)$ of X (its intersection is the fundamental group $\pi(X)$). By our observations immediately before Proposition 2.2 of §8, we have:

THEOREM 1. *The isomorphism classes of covering spaces of the pointed space X are in one-to-one correspondence with the filtered subgroups of the fundamental filtered group $\boldsymbol{\pi}(X)$; a covering space is regular* iff *the corresponding filtered subgroup is normal.*

REMARK. Even if a simply connected covering space does not exist, we may make use of a "fundamental tower of filtered groups" to get a perfect correspondence between "subtowers" and isomorphism classes of covering spaces. An equivalent approach is obtained by adjoining inverse limits to the category of filtered groups.

EXAMPLE. Let S be a set and $\mathfrak{F}$ a filter on S. Then let $X = X(S, \mathfrak{F})$ be the connected topological space defined as follows. Regard S as a discrete topological space, and let X' be the quotient space of $S \times [0, 1]$ formed by identifying $S \times \{1\}$ to a point. Let X be the disjoint union of X' and a point P. Topologize X by requiring that a subset of X be open *iff* it is an open subset of X' or if its intersection with X' is an open subset of X' containing $F \times \{0\}$ for some $F \in \mathfrak{F}$.

Then X is a connected topological space; fix $x_0 = P$ as base point. By Corollary 2.1.2 of §8, X admits a simply connected covering space so that the fundamental filtered group $\boldsymbol{\pi}(X, x_0)$ is defined. Explicitly, $\boldsymbol{\pi}(X, x_0)$ is the free group on a family $(z_i)_{i \in S}$ of elements indexed by S, a base for the disks being

the disks D_F = subgroup generated by the elements $z_i z_j^{-1}$ for $i, j \in F$ (all $F \in \mathfrak{F}$).

To prove this, let us compute the simply connected covering space (C, p) of X. Let G be the group generated by a family $(y_i)_{i \in S}$ indexed by S with relations $y_i^2 = 1$, all $i \in S$. Then every element y of G may be written uniquely in the form: $y_{i_1} \cdots y_{i_n}$ where $i_j \neq i_{j+1}$, $1 \leqq j \leqq n-1$; call n the *length* of y. Let H (resp.: K) denote the set of words y of G of even (resp.: odd) length; then H is the kernel of the unique homomorphism from G into $\mathbf{Z}_2$ which is non-trivial on the y_i (all $i \in S$), and $K = G - H$. Let $C = (H \times \{P\}) \cup (K \times X')$, and $p: C \to X$ be the projection on the second coordinate. Let $V \in \text{contig}(X)$ be defined by:

$$V_P = \{P\} \cup S \times \left[0, \frac{1}{2}\right); \qquad x \neq P,\ V_x = X'.$$

Let the entourage U for C be defined by:

$$\begin{aligned} &y \in H, && U_{(y,P)} = (\{(yy_i, i) : i \in S\} \times [0, 1/2)) \cup \{(y, P)\}, \\ &x \in X', \quad y \in K, && U_{(y,x)} = \{y\} \times X'. \end{aligned}$$

Then U is a p-adaptation with $pU = Vp$; regard (C, p) as a covering space of X by endowing it with the contiguity induced by U [Theorem 1, §2]. Fix $c = (1, P)$ as a base point of C. Then (C, c, p) is the simply connected covering space of the pointed space (X, x_0). C is U-connected; for each $y \in H$, the transformation: $(y', x) \to (yy', x)$ is an automorphism of C commuting with U. Hence $H \approx P_U(C, X)$.

For each $F \in \mathfrak{F}$, let G_F denote the subgroup of G generated by the elements y_i for $i \in F$. Let $H_F = H \cap G_F$. Then for each $F \in \mathfrak{F}$, $C_F = C \cap (G_F \times X')$ is a U'-component of C for some $U' \in \text{ad}(C)$, and every U'-component for some $U' \in \text{ad}(C)$ contains a C_F.

Hence the Poincaré filtered group $\mathcal{P}(C, X) = \pi(X, x_0)$ is the group H with the H_F $(F \in \mathfrak{F})$ as a base for the disks. Fix $i_0 \in S$, and put $z_j = y_j y_{i_0}$, all $j \in S$. Then H is the free group on the family $(z_j)_{j \in S}$, and H_F is the subgroup generated by the elements $z_i z_j^{-1} = y_i y_j$ for $i, j \in F$, so that $\pi(X, x_0)$ is as described above.

Remark. By a different construction [pasting the intervals $\{i\} \times [0, 1]$ together at alternate end points and letting $1/2$ play the role of 0], we may construct $X = X(S, \mathfrak{F})$ having the above stated properties such that X minus a point is a "long line" of Kelley [5]; if S is countable, then we may take X minus a point to be the real line.

Theorem 2. *Every filtered group is isomorphic to the Poincaré filtered group of some regular covering space of a connected pointed topological space.*

[Remark. The base space X may be chosen such that X minus the base point is a "long line" of Kelley—the real line if G is countably generated.]

Proof. Let G be a filtered group. Let S be the underlying set of G, $\mathfrak{F}$ the underlying filter of G and $X=X(S, \mathfrak{F})$ the connected topological space with base point x_0 constructed in the preceding example. Then $\pi(X, x_0)$ is the free group on a family $(z_g)_{g\in G}$ of elements indexed by G, with base for the disks the subgroups $D_F=$subgroup generated by the elements $z_g z_h^{-1}$ for $g, h\in F$, all disks F of G. Let $\sigma: \pi(X, x_0)\to G$ be the unique homomorphism of the underlying groups such that $\sigma(z_g)=g$, all $g\in G$. Then $\sigma(D_F)=F$, all disks F of G. Hence G is isomorphic to the quotient of the filtered group $\pi(X, x_0)$ by the normal filtered subgroup $K=\text{kernel}(\sigma)$. The desired conclusion follows from Theorem 1 and from Theorem 4.1 of §8.

11. **Applications, examples and counterexamples.**

EXAMPLE 1. If X is a pointed space and $\mathcal{S}$ is a Boolean ideal in 2^X then form the space $X^{\mathcal{S}}$ described in Example 1 of §2. Then the open subgroups of $\pi(X^{\mathcal{S}})$ classify the covering spaces of X with singularities over elements of $\mathcal{S}$ (although not, in general, in a one-to-one fashion).

In particular, if X is an n-dimensional irreducible complex analytic (resp.: real differentiable) variety and $r\leqq n-1$ (resp.: $n-2$), take $\mathcal{S}$ as in (c) of Example 1 of §2. Then $X^{\mathcal{S}}$ obeys the hypotheses of the second paragraph of §9, so that the open subgroups of the topological group $\pi(X^{\mathcal{S}})$ classify in one-to-one fashion the isomorphism classes of covering spaces of X with singularities over a closed $\leqq r$-dimensional subvariety.

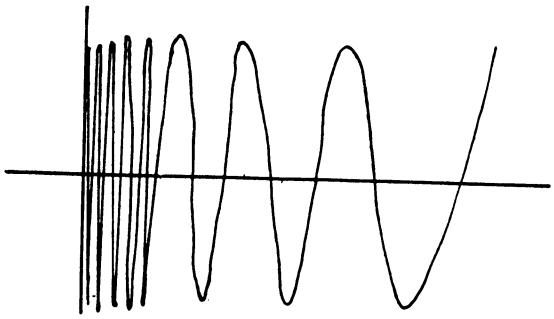
EXAMPLE 2. Let X be the rational points of an n-dimensional complete variety over an algebraically closed subfield k of the field of complex numbers. Then $X\times_k \mathbf{C}$ is a complete complex algebraic variety, and the points of $X\times_k\mathbf{C}$ rational/$\mathbf{C}$ therefore form a compact topological (and hence also uniform) space with the "usual" topology. Let X be endowed with the induced uniform structure. Then if X is connected in the Zariski topology, X is connected as a uniform space. The correspondence: $C\to$ (points of C rational/k) defines a one-to-one correspondence between the biregular equivalence classes of pairs (C, p) where C is a variety/k and p is a finite étale k-morphism: $C\to X$ and the isomorphism classes of finite-to-one covering spaces of the uniform space X. The "algebraic fundamental group" [1] therefore coincides with the completion relative to subgroups of finite index of our fundamental group $\pi(X)$, so that its open subgroups of finite index classify algebraic covering spaces in one-to-one fashion.

EXAMPLE 3. Let V be an entourage in the contiguity of the pointed space X. Then the inverse limit of the Poincaré groups of covering spaces of X having an adaptation U with $pU=Vp$ is denoted $\pi(X, V)$. It is a totally disconnected topological group, and, if X is semi-simple, then its open subgroups classify in one-to-one fashion the covering spaces of X "lying over V." The fundamental group $\pi(X)$ is the inverse limit of the topological groups $\pi(X, V)$ as V ranges through contig(X).

In the special case that X is locally connected, by Proposition 2.2 of

§8 there is a largest covering space (C, p) lying over V. By the construction in Theorem 1 of §4, this covering space is regular. Hence, the group $\pi(X, V)$ is discrete, and is simply the Poincaré group $P(C, X)$ of this covering space (C, p).

EXAMPLE 4.



Let X be the graph of the function $\sin(1/x)$ $(x>0)$ together with the origin $(0, 0)=x_0$. Then (X, x_0) is a pointed topological space. X is connected, but neither locally connected nor arcwise connected. X is not simply connected. Nevertheless, the fundamental group and arcwise fundamental group are trivial. X admits a connected regular two-to-one covering space, and a non-connected non-regular infinite-to-one covering space. The quasicomponent of this latter covering space (as well as the universal covering space) reduces to the base point. Hence X is not semi-simple.

Nevertheless, X obeys the hypotheses of Corollary 2.1.2 of §8, so that X admits a simply connected covering space. Computing this covering space turns out to be a combinatorial problem which is easily resolved. The fundamental filtered group $\pi(X, x_0)$ is the free group on $\aleph_0$ generators x_i $(i \geqq 1)$. The filtration is given by the disks D_n = subgroup generated by x_i, $i \geqq n$. By Theorem 1 of §9, the filtered subgroups of this filtered group are in one-to-one correspondence with the isomorphism classes of covering spaces of (X, x_0). We thus have an example in which the fundamental filtered group is needed to classify covering spaces.

EXAMPLE 5. If X is a space and $A \subset X$, then form the space (X, A) described in Example 2 of §2 (keep the same base point as that of X). Then the group $\pi(X, A)$ (or the filtered group $\pi(X, A)$) is called the *relative fundamental group of X modulo A* (or *relative fundamental filtered group of X modulo A*). We have a natural function from the usual arcwise set $\pi_1(X, A)$ into $\pi(X, A)$. (Note that $\pi(X, A)$ is a group although $\pi_1(X, A)$ is not.)

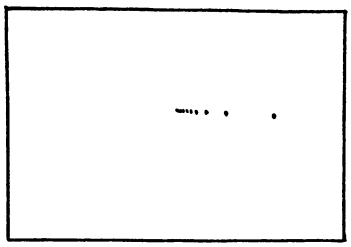
EXAMPLE 6. The topological space of rational numbers (with arbitrary base point), although neither connected nor locally connected, is simply connected.

EXAMPLE 7. The *uniform space* of rational numbers is the simply con-

nected covering space of the "rational circle" (uniform space of rationals mod 1).

EXAMPLE 8. A topological space that is irreducible in the sense of Serre is simply connected [6]. A discrete or indiscrete topological space (with base point) is simply connected.

EXAMPLE 9. Let X be the plane with an infinite sequence of points $(1/2^n, 0)$, $n \geqq 1$ deleted. Fix a base point, say at the origin.



Then X is a connected and locally arcwise connected pointed topological space; yet X does *not* admit a simply connected covering space. Since X is locally arcwise connected, the hypotheses of the last paragraph of §9 hold, and there is a "universal pro-covering space." The open (resp.: closed) subgroups of the fundamental group $\pi(X)$ classify the isomorphism classes of covering spaces (resp.: pro-covering spaces) in one-to-one fashion. $\pi(X)$ is the completion of the free group on $\aleph_0$ generators with open subgroups those containing all but finitely many of the generators.

EXAMPLE 10. A regular covering space is isomorphic to a connected covering space *iff* its Poincaré filtered group reduces to a group. The corresponding statement about the Poincaré quasigroup of arbitrary covering spaces is *false*: in fact, there exists a nontrivial regular covering space of the pointed space $X = \text{graph } (\sin 1/x) \cup \{(0, 0)\}$ discussed in Example 4 whose quasicomponent reduces to the base point and whose Poincaré quasigroup/X is trivial.

EXAMPLE 11. If (C, p) is a regular covering space of the *topological space* X, and G is an equicontiguous p-transitive group of automorphisms of C, then the contiguity for C is completely determined by the topology of C and the group of transformations G.

EXAMPLE 12. Let X be the topological space described in Example 9 and (C, p) be the smallest covering space of X that "unravels the hole" at $(0, 1/2)$. Let $(c_i)_{1 \leq i < \infty}$ be a one-to-one indexing of the fiber over the origin, and $U \in \text{ad}(C, p)$ be such that $p(U_{c_i}) =$ open sphere of radius $1/3$ about the origin, all i. Let (D, q) be the smallest covering space *of the underlying topological space* C' of C that, for each $i \geqq 2$, unravels the hole in U_{c_i} lying over $(0, 1/2^i)$, and let D' be the underlying topological space of D. Then $q: D' \to C'$ and $p: C' \to C$ are Chevalley covering spaces, yet the compositum $pq: D' \to C$ is not, since

the origin is not "evenly covered." Compare Theorems 1 of §3 and 1.2 of §6.

EXAMPLE 13. The following example from group theory is due to Gorenstein:

Let Q_n be the additive abelian group of rational numbers with denominators a power of n, and $\mathbf{Z}$ the additive group of integers. Then the function: $x \to nx$ is an automorphism of Q_n. Let G be the extension of Q_n by this automorphism; then $Q_n \subset G$, and there exists $g \in G$ such that $gxg^{-1} = nx$, all $x \in Q_n$. Then $g\mathbf{Z}g^{-1} = n\mathbf{Z} \subset \mathbf{Z}$, yet $g^{-1}\mathbf{Z}g = n^{-1}\mathbf{Z} \not\subset \mathbf{Z}$.

Translating this example into the language of covering spaces by means of Theorem 2 of §10, Theorem 1 of §4, Theorem 4.1 of §8, and the analogue of Theorem 3.1 of §8 for monoids, we see that there exists a connected covering space (C, p) of a connected locally connected and separable topological space X and a map $\tau: (C, p) \geqq (C, p)$ of covering spaces that is an n-to-one covering map $\tau: C \to C$.

$$\begin{tikzcd}
C \arrow[dr, "\tau"] \arrow[dd, "p"'] & & \\
& C \arrow[dl, "p"] \arrow[dr] & \\
X \arrow[dl] \arrow[dd] & & 0 \\
0 & & \\
0 & &
\end{tikzcd}$$

[REMARK. Using a more classical construction than that of Theorem 2 of §10, we may take X to be lc^1 and T_3.]

EXAMPLE 14. If (C, p) is a covering space of the pointed space X, then the Poincaré quasigroup $\mathcal{P}(C, X)$ is *transitive on the fiber of the base point iff* there exists $U \in \text{contig}(C)$ such that, for c, d in the U-component of X and in the fiber of the base point, the covering spaces (C, c) and (C, d) of the pointed space X are isomorphic.

An easy application of Theorem 4.1 of §8 shows that: If C is a covering space of the pointed space X, R a regular covering space $\geqq C$, $G = \mathcal{P}(R, X)$ and $H = \mathcal{P}(R, C)$, then the Poincaré quasigroup $\mathcal{P}(C, X)$ is transitive on the fiber of the base point *iff* there exists a disk D of G such that, given $g \in D$, there exists a disk $D' \subset D$ such that $g(H \cap D)^{g^{-1}} \cap D' = H \cap D'$.

The question arises whether such a covering space (C, p) is always regular. This is always the case if the base space is semi-simple (Corollary 3.2 of

§4), or, more generally, if C is smaller than some connected regular covering space (Corollary 3.1 of §4). However it is false in the general case, as is shown by the following counterexample:

Let G_i be the free group on denumerably many generators: $x_{i1}, x_{i2}, \cdots$, and let G be the "weak direct product"(4) $G=\prod_{i\geq 1}^{w} G_i$. Let D_i^n be the subgroup of G_i generated by x_{ij} for $j\geqq n$, and $D^n=\prod_{i\geq 1}^{w} D_i^n$. Then G is a filtered group with the D^n $(n\geqq 1)$ as a base for the disks. Let H be the filtered subgroup generated by the elements: $x_{11}, x_{22}, x_{33}, \cdots$. Then if $g\in G$, choose n so that $g\notin D^n$. Then $(gHg^{-1})\cap D^n=H\cap D^n$. However, the normalizer of $H\cap D^n$ in D^n is the group generated by the x_{ij} for $i<n$, $j\geqq n$, together with $H\cap D^n$, all n, so that H is *not* a normal filtered subgroup of G. Letting R be a regular covering space of a pointed space X with Poincaré filtered group G [Theorem 2 of §10], and letting C be the intermediate covering space corresponding to H [Theorem 4.1 of §8], we have the desired counterexample.

EXAMPLE 15. The quasitheory gives information even for connected covering spaces. Thus if R is a regular covering space of the connected pointed space X with Poincaré filtered group $G=\mathcal{P}(R, X)$ and C is an intermediate covering space with filtered subgroup $H=\mathcal{P}(R, C)$, then C is isomorphic to a connected pointed space *iff* there exists a disk D such that:

(1) $(H\cap D)\cdot D'=D$, all disks $D'\subset D$.

[*Note.* If $A, B\subset G$, then $A\cdot B$ denotes $\{a\cdot b: a\in A, b\in B\}$.]

EXAMPLE 16. If (C, p) is a covering space of the pointed space X, then the plausible analogue of equicontiguity is that there exist $U\in \mathrm{ad}(C, p)$ such that for $V\subset U$ every automorphism α of the V-component of C with $\alpha V=V\alpha$ extends (necessarily uniquely) to an automorphism β of the U-component of C such that $\beta U=U\beta$. An equivalent condition is: "The Poincaré quasigroup $\mathcal{P}(C, X)$ is isomorphic to a filtered group."

If G is the Poincaré filtered group of a regular covering space R larger than C and H is the Poincaré filtered group of R/C, and if N_D denotes the normalizer of $H\cap D$ in D, all disks D, then (C, p) is equicontiguous *iff* there exists a disk D such that:

(2) $N_{D'}=N_D\cap D'$, all disks $D'\subset D$.

The counterexample constructed in Example 14 shows that this is not always the case, even when $\mathcal{P}(C, X)$ is transitive on the fiber of the base point. Nevertheless Corollary 3.2 of §4 shows that this is always the case when the base space is *semi-simple*, or, more generally, (Corollary 3.1 of §4) when the given covering space (C, p) is smaller than a connected regular covering space [see, however, Example 14](5). From an algebraic point of view, this fact is

(4) I.e., the subgroup $\prod_{i\geq 1}^{w} G_i$ of the Cartesian direct product $\prod_{i\geq 1} G_i$ consisting of all elements all but finitely many of whose coordinates are trivial.

(5) However, a careful analysis of the construction in Theorem 1 of §4 reveals that if (C, q) is a covering space of the *space* (non-pointed) X, $U\in \mathrm{ad}(C, q)$ and α is an automorphism of C such that $q\alpha=q$, then there exists $U'\in \mathrm{ad}(C, q)$ such that U_c' is a union of W^c-components of U_c for some $W^c\in \mathrm{contig}(U_c)$, all $c\in C$, and such that $\alpha U'=U'\alpha$.

obvious; in fact, the equivalent statement about filtered groups reads: "If G is a group regarded as a filtered group with indiscrete filtration, then the quotient quasigroup N/H of the normalizer N of any filtered subgroup H of G is a group."

For a connected covering space (C, p) of X, the Poincaré quasigroup is simply an increasing family of groups whose union is the Poincaré group $P(C, X)$. The equicontiguity condition here signifies that the filtration be trivial. I do not know whether a *connected* covering space always has equicontiguous Poincaré group. Conditions (1) and (2) in this and the preceding example, and Theorem 2 of §10 show that the construction of a counterexample is equivalent to the following purely algebraic problem on filtered groups:

"Does there exist a filtered group G with a filtered subgroup H such that, if N_D is the normalizer of $H \cap D$ in D, all disks D, then:

(1′) $H \cdot D = G$, all disks D.

(2′) For all disks D, there exists a disk $D' \subset D$ such that $N_{D'} \neq N_D \cap D'$?"

EXAMPLE 17. The isomorphism classes of connected covering spaces of a pointed semi-simple space form a lattice (Corollary 1.4, §5); moreover, the isomorphism classes of covering spaces of a pointed space smaller than some *connected* regular covering space form a lattice (Theorem 4, §4).

In general, it is true that the ordered set of isomorphism classes of covering spaces of a pointed space is closed under finite suprema (Corollary 1.4.2, §8)—this result is equivalent to the easily observable fact that the filtered subgroups of a filtered group are closed under finite infima.

However, the following statements are all *false*:

(a) The infimum of two covering spaces of a pointed space always exists.

(b) Given a family of covering spaces of a pointed space that admit an upper bound, they admit a supremum.

(a′) The supremum of two filtered subgroups of a filtered group always exists.

(b′) The infimum of an arbitrary collection of filtered subgroups of a filtered group always exists.

From the quasitheory, we know that (a)$\Leftrightarrow$(a′), (b)$\Leftrightarrow$(b′); and from the theory of lattices, (b)$\Rightarrow$(a), (b′)$\Rightarrow$(a′). Hence, to disprove all assertions, it suffices to give a counterexample to (a′).

In fact, let G be the free abelian group on the symbols x_i, y_i, $1 \leqq i < \infty$. Let D_n be the subgroup generated by $x_1+y_1, \cdots, x_{n-1}+y_{n-1}; x_n, y_n, x_{n+1}, y_{n+1}, \cdots$. Regard G as a filtered group with base for disks the D_n $(n \geqq 1)$. Let H (resp.: K) be the filtered subgroup generated by the x_i (resp.: the y_i), $1 \leqq i < \infty$. Then $H \cap D_n$ (resp.: $K \cap D_n$) is the subgroup generated by x_i (resp.: y_i) for $i \geqq n$. Let $L_n = (H \cap D_n) + (K \cap D_n)$; then L_n is the filtered subgroup of G generated by x_i, y_i for $i \geqq n$. Then *as filtered subgroups of* G, $L_n \supset H, K$. Since $L_n \cap \bigcap_{i=1}^{\infty} D_i$ contains the element $x_n + y_n \notin L_{n+1}$, the L_n form a strictly decreasing sequence of filtered subgroups of G each of which is bigger than

H, K. If B is any filtered subgroup of G containing H and K (as filtered groups), then for some n we must have $B \supset (H \cap D_n) + (K \cap D_n) = L_n$ (as groups), so that $B \supset L_n$ for some n. Hence, H and K do not admit a supremum.

REMARK. The above example from filtered groups actually shows more: There exists a connected pointed topological space X having two *connected regular* covering spaces (C, p) and (D, q) which do *not* admit an infimum. The sequence $(L_n)_{n \geq 1}$ defines a strictly increasing sequence $(A_i)_{i \geq 1}$ of connected regular covering spaces of X each smaller than both (C, p) and (D, q) which do *not* admit a supremum. Their supremum *in the lattice of connected covering spaces smaller than* (C, p) [resp.: (D, q)] is (C, p) [resp.: (D, q)].

Moreover, (C, p) and (D, q) are connected covering spaces of X which do *not* admit a *connected* upper bound. This follows from the fact that the supremum of (C, p) and (D, q) is regular with abelian Poincaré filtered group.

EXAMPLE 18. Although two regular covering spaces of the pointed space X need not admit an infimum even if connected (Example 17), nevertheless the following is true:

Two regular covering spaces $\geqq$ the covering space (C, p) of the pointed space X admit a regular lower bound $\geqq (C, p)$.

Taking a regular covering space larger than the two in question, we see that the corresponding statement about filtered groups is:

If N_1 and N_2 are normal filtered subgroups of the filtered group G and H a filtered subgroup of G containing (*as filtered subgroups*) N_1 and N_2, then there exists a normal filtered subgroup N of G with $H \supset N \supset N_1, N_2$ (as filtered subgroups).

[In fact, take a disk D such that *as groups* $H \cap D \supset N_1 \cap D$, $N_2 \cap D$, and put $N = (N_1 \cap D) \cdot (N_2 \cap D)$.]

In particular, if (C, p) is a covering space of X smaller then some connected regular covering space, then for every regular covering space (R, ρ) with $(R, \rho) \geqq (C, p)$ there exists a connected regular covering space (R', ρ') with $(R, \rho) \geqq (R', \rho') \geqq (C, p)$.

It follows that, if $(C, p) \leqq (R, \rho)$ are covering spaces of X, the latter regular, and $G = \mathcal{P}(R, X)$, $H = \mathcal{P}(R, C)$, then (C, p) is smaller than some connected regular covering space *iff* there exists a normal subgroup N of some disk D of G with $N \subset H$ such that $N \cdot D' = D$, all disks $D' \subset D$.

EXAMPLE 19. Need a connected covering space (C, p) of a connected space X always be smaller than some *connected* regular covering space?

In the language of filtered groups: Does there exist a filtered group G and a filtered subgroup H such that:

(1) $H \cdot D = G$, all disks D; and

(2) if D is a disk and N a normal subgroup of D contained in H then there exists a disk $D' \subset D$ such that $N \cdot D' \neq D$?

By Corollary 3.1 of §4, a counterexample to the question raised in Example 16 would be a counterexample to this problem.

REMARK. If H is a subgroup of the group G, then there is a largest normal subgroup H^0 of G contained in H; explicitly, $H^0 = \bigcap_{g \in G} gHg^{-1}$.

EXAMPLE 20. Corollary 1.1 of §4 tells us that if C is a *connected* covering space of the pointed space X, then there exists a smallest regular covering space of X larger than C.

Translated into the language of groups by means of the quasitheory, this reads: "If H and D are subgroups of a group G such that $G = H \cdot D$, then $D \cap \bigcap_{g \in G} (gHg^{-1}) = \bigcap_{d \in D} d(H \cap D)d^{-1}$"—a fact that is easily seen directly.

However, if we drop the restriction of connectedness, then this result no longer holds. In fact, there exists a nonregular covering space C [necessarily not connected] of a connected pointed topological space X and a strictly decreasing sequence R_i $(i \geqq 1)$ of regular covering spaces $\geqq C$ admitting C as an infimum.

The corresponding statement about filtered groups is: There exists a filtered group G having a non-normal filtered subgroup H and a strictly increasing sequence L_i $(i \geqq 1)$ of normal filtered subgroups admitting H as a supremum.

Let G_i be the free group on the two generators x_i, y_i and let G be the weak direct product $\prod_{i \in I}^{w} G_i$. Let G be filtered by the disks D^n = subgroup generated by all the x_i, and the y_j for $j \geqq n$. Let H be the subgroup generated by the x_i $(i \geqq 1)$, and L_n the subgroup generated by $x_1, \cdots, x_{n-1}$. Then $L_n \cap D_n$ is contained in the center of D_n, so that the filtered subgroups determined by the L_n are normal in G. On the other hand, $y_n \in D_n$, $x_n \in H \cap D_n$, and $y_n x_n y_n^{-1} \notin H$, so that $H \cap D_n$ is *not* normal in D_n, all $n \geqq 1$; hence the filtered subgroup H of G is not normal. However since L_n $(n \geqq 1)$ and H are contained in all disks, and *as groups* $H = \sup_{n \geqq 1} (L_n)$, it follows that $H = \sup_{n \geqq 1} (L_n)$ as filtered groups.

EXAMPLE 21. Let (C, c_0, p) be a covering space of the pointed space (X, x_0). Then [Example 20] we have seen that there need not exist a smallest regular covering space larger than (C, c_0, p). However, the "minimality" condition on the regular covering space $(D, q) \geqq (C, p)$ constructed in Theorem 1 of §4 can be interpreted in the language of the quasitheory. Namely: "The family of covering spaces (C, c, p) of the pointed space (X, x_0) for $c \in p^{-1}(x_0)$ admits a supremum (D, d, q). The supremum is regular."

The corresponding statement about filtered groups is perfectly obvious: "If G is a filtered group and $\mathfrak{H}$ a complete family of conjugate subgroups of the group G, then the induced family of filtered subgroups admits an infimum. The infimum is normal."

EXAMPLE 22. In §9, we have remarked that the canonical homomorphism ϕ from the arcwise group: $\pi_1(X, x_0) \to \pi(X, x_0)$ is in general neither injective

nor surjective. In fact, the map is injective *iff* every loop not homotopic to a point can be lifted to an arc with distinct end points in some covering space; and Fox has constructed an example where this is not the case.

The following is an example of a connected and locally connected topological space X for which ϕ is not surjective.

Let S be a denumerable set and $\mathfrak{F}$ the trivial filter $\mathfrak{F}=\{S\}$. Then the topological space $X=X(S, \mathfrak{F})$ is connected and locally connected; fix a base point x_0. Then by the Example preceding Theorem 2 of §10, $\pi(X, x_0)$ is the free group on denumerably many generators [since X is locally connected, $\pi(X, x_0)$ reduces to $\pi(X, x_0)$]. However, $\pi_1(X, x_0)=\{1\}$.

EXAMPLE 23. Let $(s_{ij})_{i,j\in N}$ be a one-to-one indexing of a denumerable set S by the pairs of natural numbers. Let $F_n=\{s_{0j}: j\geqq 0\}\cup\{s_{ij}: i\geqq 1, j\geqq n\}$, all $n\geqq 0$, and let $\mathfrak{F}$ be the filter on the set S with base the F_n. Then all the counterexamples constructed in this section can be taken to be covering spaces of the connected pointed topological space $X=X(S, \mathfrak{F})$ (6) constructed in the Example preceding Theorem 2 of §10. Here is a truly pathological topological space!

BIBLIOGRAPHY

1. Shreeram Abhyanker, *Tame coverings and fundamental groups of algebraic varieties. I. Branch loci with normal crossings*, Amer. J. Math. **81** (1959), 46–94.

2. Claude Chevalley, *Theory of Lie groups*. I, Chapter 2, pp. 40–61, Princeton Univ. Press, Princeton, N. J., 1946.

3. Alexandre Grothendieck, *Techniques de descente et théorèmes d'existence en géométrie algébrique*. II, Seminaire Bourbaki 195, February, 1960.

4. ———, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2)9 (1957), 119–221.

5. John Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

6. Jean Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2)61 (1955), 223, 229, Lemme 2.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS

(6) X may be taken so that X minus the base point is the real line.