ON RIESZ AND RIEMANN SUMMABILITY

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This paper investigates an inclusion relation between summability of a series of real or complex terms by Riesz typical means and by a generalised form of Riemann summability. We begin by defining the two summability methods.

Riesz' typical means. Let $\kappa \geq 0$, $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$, and write

$$A^\kappa_\lambda(\omega) = \sum_{\lambda_r < \omega} (\omega - \lambda_r)^\kappa a_r \quad \text{for } \omega > \lambda_0,$$

$$A^\kappa_\lambda(\omega) = 0 \quad \text{for } \omega \leq \lambda_0.$$

If $\omega^{-\kappa}A^\kappa_\lambda(\omega) \to s$ as $\omega \to \infty$ then we write

$$\sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa);$$

if $A^\kappa_\lambda(\omega) = O(\omega^\kappa)$ then $\sum a_n$ is bounded $(R, \lambda_n, \kappa)$. In the case $\kappa = 0$ we note that

$$A^0_\lambda(\omega) = A^0_{\lambda}(\omega) = \sum_{\lambda_r < \omega} a_r = a_0 + \cdots + a_n = A_n$$

for $\lambda_n < \omega \leq \lambda_{n+1}$ ($n = 0, 1, \cdots$). It is well-known that $A^\kappa_\lambda(\omega)$ is absolutely continuous in any finite interval of values of $\omega$, for $0 < \kappa \leq 1$, and differentiable with continuous derivative if $\kappa > 1$; in fact,

$$\frac{d}{d\omega} A^\kappa_\lambda(\omega) = \kappa A^{\kappa-1}_\lambda(\omega) \quad (\kappa > 1), \quad \frac{d}{d\omega} A^1_\lambda(\omega) = A^0_\lambda(\omega) \quad (\omega \neq \lambda_n).$$

As shown in Hardy and Riesz [9] or Chandresekharan and Minakshisundaram [5], we also have, for $\kappa \geq 0$, $\rho > 0$,

$$A^{\kappa+\rho}_\lambda(\omega) = \frac{\Gamma(\kappa + \rho + 1)}{\Gamma(\kappa + 1)\Gamma(\rho)} \int_0^\omega (\omega - t)^{\kappa-1} A^\rho(t)dt.$$

We shall employ the limitation theorem for Riesz means:

If $A^\kappa_\lambda(\omega) = O(\omega^\kappa)$, $\kappa \geq 0$, then, for $r = 0, 1, \cdots, [\kappa]$,

Received by the editors August 17, 1961.

(1) This paper was written while the author was a Fellow at the 1961 Summer Research Institute of the Canadian Mathematical Congress.

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where $\lambda_n < \omega \leq \lambda_{n+1}$ and $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$.

The form of this theorem stated in [9, Theorem 22] and [5, Theorem 1.62] (we use $O$ in place of $o$) is $A_\lambda^r(\omega) = O(\lambda_n^{s-r})$; the stronger form (3) is a special case of a result of Borwein [1, Lemma 2].

Finally, we need the "consistency theorem" for Riesz means:

(4) \[ \text{If } A_\lambda^r(\omega) = O(\omega^s), \quad \kappa \geq 0, \quad \text{then } A_\lambda^s(\omega) = O(\omega^p) \quad \text{for } p \geq \kappa. \]

Riemann summability. Let $\mu > 0$, $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$,

\[ R_\lambda^\mu(h) = \sum_{n=0}^{\infty} a_n f_\mu(\lambda_n h) \]

if the series converges for each $h$ in a deleted neighbourhood of the origin, and if $R_\lambda^\mu(h) \to s$ as $h \to 0$, then we write

\[ \sum_{n=0}^{\infty} a_n = s \quad (R, \lambda_n, \mu). \]

The case where $\lambda_n = n$ and $\mu$ is a positive integer is usually known as Riemann summability. The more general definition above has been given by Burkill [2] for $\mu = 1, 2$, and by Burkill and Petersen [4] for $\mu$ rational with odd denominator (which ensures that $f_\mu(x)$ is real); alternatively, for any $\mu > 0$ we may define $(\sin x)^{\omega} = e^{i\omega \pi} (-\sin x)^{\omega}$ when $x > 0$, $\sin x < 0$, and $f_\mu(-x) = f_\mu(x)$. In fact, any definition is suitable for our purpose, which ensures that

\[ d \quad (\sin x)^{\omega} = \mu (\sin x)^{\mu - 1} \cos x, \quad |(\sin x)^{\mu}| \leq 1 \quad (\mu > 0), \]

and since $f_\mu(x)$ is an even function we may suppose throughout, in the definition of $(R, \lambda_n, \mu)$ summability, that $h > 0$.

Burkill [3] has shown that if $\lambda_0 = 0$, $0 < \rho \leq \lambda_{n+1} - \lambda_n$, and $\kappa$ is a positive integer, then summability $(R, \lambda_n, \kappa)$ implies summability $(R, \lambda_n, \mu)$ for $\mu > \kappa + 1$ (and $\mu$ rational with odd denominator). Burkill and Petersen [4] have proved this for $\kappa = 1$, remarking that from the point of view of applications (for instance, to the theory of almost periodic functions—see, for example, [2] and [11]) it would be desirable to proceed from a nonintegral Riesz means.

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mean to an integral Riemann mean. The present paper furnishes such a result, which also contains the theorem referred to above; we prove, more generally, the following

**Theorem.** If \( \sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa) \), \( \kappa \geq 0 \), and if \( \sum_{n=1}^{\infty} \Delta_n \lambda_n^{-\mu} \) converges, where \( \Delta_n = \lambda_{n+1} / (\lambda_{n+1} - \lambda_n) \) and \( \mu > \kappa + 1 \), then \( \sum_{n=0}^{\infty} a_n = s(\mathcal{R}, \lambda_n, \mu) \).

In the special case \( \lambda_n = n \), \( (R, \lambda_n, \kappa) \) is equivalent to Cesàro summability \((C, \kappa)\), and \((\mathcal{R}, \lambda_n, \mu)\) becomes ordinary Riemann summability, which will be denoted by \((\mathcal{R}, \mu)\); if, in addition, \( \mu \) is a positive integer greater than 1, we obtain a result of Verblunsky [12] that \((C, \kappa) \subseteq (\mathcal{R}, \mu) \) for \( 0 \leq \kappa < \mu - 1 \), \( \mu = 2, 3, \ldots \); Hardy and Littlewood [7; 8] had proved earlier that \((C, \kappa) \subseteq (\mathcal{R}, 1) \) for \( 0 \leq \kappa < 0 \). Kuttner [10] has proved that \((\mathcal{R}, \mu) \subseteq (C, \mu + \delta) \) for \( \delta > 0 \), \( \mu = 1, 2 \), and that the result is false for \( \mu = 3 \); and he has shown that \((\mathcal{R}, \mu) = (\mathcal{R}, n, \mu) \subseteq (R, \log n, \mu) \) for \( \mu = 1, 2 \). See also Hardy [6, Appendix III].

Some lemmas are needed. We remark that in general throughout this paper \( K \) will denote a positive quantity independent of the particular variables under consideration, and not necessarily the same at each occurrence; thus, for example, in the first lemma the constants \( K \) may depend on \( \mu \) or \( p \), but are independent of \( x \) or \( n \).

**Lemma 1.** Let \( p \) be a non-negative integer, and define \( f_0(x) = 1 \).

(a) For any \( \mu \geq p \), \( f_{\mu}^{(p)}(x) \) is continuous everywhere, and

\[
| f_{\mu}^{(r)}(x) | \leq K (0 < x < 1), \quad | f_{\mu}^{(r)}(x) | \leq K x^{-n} (x \geq 1), \quad r = 0, 1, \ldots, p.
\]

(b) If \( \mu > p \) then \( f_{\mu}^{(r)}(n\pi) = 0 \) \( (n = 1, 2, \ldots; \ r = 0, 1, \ldots, p) \). Also \( f_{\mu}^{(p+1)}(x) \) is continuous in \( (n-1)\pi < x < n\pi \) \( (n = 1, 2, \ldots) \) and, in each such interval, satisfies the inequality

\[
| f_{\mu}^{(p+1)}(x) | \leq K n^{-p} \{ (n\pi - x)^{\mu-p-1} + [x - (n-1)\pi]^{\mu-p-1} \}.
\]

**Proof.** We first note that, for each non-negative integer \( s \),

\[
| f_{1}^{(s)}(x) | \leq K (0 < x < 1), \quad | f_{1}^{(s)}(x) | \leq K x^{-1} (x \geq 1);
\]

the first of these inequalities is an immediate consequence of the fact that \( f_1(x) \) has a power series expansion with infinite radius of convergence, while the second follows from the formula

\[
f_{1}^{(s)}(x) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k k! x^{-k-1} \sin \left[ x + \frac{1}{2} (s - k)\pi \right].
\]

It is clear that \( f_{\mu}(x) \) is differentiable as often as we please, except perhaps at \( x = \pm \pi, \pm 2\pi, \ldots \); also \( f_{\mu}^{(p+1)}(x) = (\mu+1) f_\mu'(x) f_\mu'(x) \), and on differentiating \( p \) times this gives

\[\text{(*) This inequality is also given (for } \mu \text{ rational with odd denominator) in [3, Lemma 2].}\]
which enables us to proceed by induction on $p$. We shall merely verify the inequalities (5) and (6).

(a) Suppose that, for some fixed non-negative integer $p$ and for any $\mu \geq p$, (5) holds; then since $\mu \geq p$ implies $\mu + 1 \geq p$, (5) also holds with $\mu + 1$ in place of $\mu$ (and $r = 0, 1, \ldots, p$). Further, (8) shows, by (7) and the inductive hypothesis, that $f_{\mu+1}^{(p+1)}(x)$ is bounded in $(0, 1)$ and is $O(x^{-p-1})$ as $x \to \infty$. Since (5) may be verified directly from the definition of $f_{\mu}(x)$ in the case $p = 0$, it follows that (5) is true for any non-negative integer $p$ and any $\mu \geq p$.

(b) If $\mu \geq p + 1$ then (6) is equivalent to $|f_{\mu+1}^{(p+1)}(x)| \leq Kn^{-\mu}$ for $0 \leq (n-1)\pi < x < n\pi$, which has already been proved in part (a) of the lemma. Suppose, therefore, that for some fixed non-negative integer $p$ and $0 < |n\pi - x| \leq \pi/2$ ($n = 0, 1, \ldots$),

\begin{equation}
|f_{\mu+1}^{(p+1)}(x)| \leq Kn^{-\mu} \quad \text{for } p < \mu < p + 1;
\end{equation}

in addition, we already know from (5) and (7) that

\begin{equation}
|f_{\mu}^{(r)}(x)| \leq Kn^{1-\mu} \quad (r = 0, 1, \ldots, p),
\end{equation}

\begin{equation}
|f_{1}^{(s)}(x)| \leq Kn^{1-\mu} \quad (s = 0, 1, \ldots).
\end{equation}

Now use (8) with $p + 1$ in place of $p$, together with (9) and (10), and we get

\begin{equation}
|f_{\mu+1}^{(p+2)}(x)| \leq K(n + 1)^{-\mu} |n\pi - x|^{-p-1} + Kn^{1-\mu};
\end{equation}

or, writing $v$ for $p + 1$,

\begin{equation}
|f_{\mu+1}^{(v+2)}(x)| \leq K(n + 1)^{-\mu} |n\pi - x|^{-v-2} \quad \text{for } p + 1 < v < p + 2.
\end{equation}

Since we may verify (9) directly for $p = 0$, (9) therefore follows, by induction, for any non-negative integer $p$; and by combining the results for the two halves of the interval $(n-1)\pi < x < n\pi$, we obtain (6).

Defining $A_{n+1}(A_{-1} = 0)$ and $A_{1}(\tau)$ as before, we now prove

**Lemma 2.** If $\mu \geq 1, \lambda_{n} < \Omega \leq \lambda_{n+1}$ ($n = 0, 1, \ldots$), then

\begin{equation}
\sum_{r=0}^{n} a_{rf_{\mu}}(\lambda, h) = A_{1}(\Omega)f_{\mu}(\Omega h) - h \int_{0}^{a} f_{r}'(rh)A_{1}(\tau)d\tau.
\end{equation}

**Proof.** Since $f_{r}'(x)$ is continuous for any $x$, when $\mu \geq 1$, and $A_{1}(\tau) = A$, for $\lambda_{r} < \tau \leq \lambda_{r+1}$ we have, for $\lambda_{n} < \Omega \leq \lambda_{n+1}$,
\[ h \int_0^a f_\mu^j (\tau h) A_\lambda (\tau) d\tau = h \left\{ \sum_{r=0}^{n-1} \int_{\lambda r}^{\lambda r+1} \frac{1}{h} f_\mu (\tau h) + h A_\mu \left[ \frac{1}{h} f_\mu (\tau h) \right]_{\lambda r}^a \right\} \]

= \left\{ \sum_{r=0}^{n-1} A_\mu \left[ \frac{1}{h} f_\mu (\tau h) \right]_{\lambda r}^a \right\} + h A_\mu \left[ \frac{1}{h} f_\mu (\tau h) \right]_{\lambda n}^a \]

= A_n f_\mu (\Omega h) - \sum_{r=0}^n (A_r - A_{r-1}) f_\mu (\lambda_r h),

by partial summation; and this gives (11).

Now to obtain \( R^n_\lambda (h) \) we must let \( n \to \infty \) in (11); the following lemma gives sufficient conditions for the existence of \( R^n_\lambda (h) \).

**Lemma 3.** If \( \sum a_n \) is bounded (or summable) \((R, \lambda_n, \kappa)\), \( \kappa \geq 0 \), and if \( \sum \lambda_n^{-\mu} \) converges, then \( \sum a_n f_\mu (\lambda_r h) \) converges (absolutely) for each fixed \( h > 0 \).

**Proof.** If \( \sum a_n \) is bounded \((R, \lambda_n, \kappa)\) then by (3) (with \( r = 0 \)), \( A_n = O(\lambda_n^\kappa) \); moreover, for any fixed \( h > 0 \), \( f_\mu (\lambda_n h) = O(\lambda_n^{-\mu}) \) as \( n \to \infty \). Hence

\[ a_n f_\mu (\lambda_n h) = (A_n - A_{n-1}) f_\mu (\lambda_n h) \]

\[ = \{ O(\lambda_n^\kappa) + O(\lambda_{n-1}^\kappa) \} O(\lambda_n^{-\mu}) \]

\[ = O(\lambda_n^\kappa \lambda_n^{-\mu}) + O(\lambda_{n-1}^\kappa \lambda_n^{-\mu}) \],

and the lemma follows.

**Lemma 4.** Let \( p \) be a positive integer, \( 0 \leq \sigma < 1, \mu > \rho \), and

\[ I(\alpha) = \int_{a}^{\infty} (x - \alpha)^{-\sigma} d\mu^{(p)} (x) \]

Then

\[ |I(\alpha)| \leq Kn^{-\nu} \{(n\pi - \alpha)^{-\sigma} + [\alpha - (n - 1)\pi]^\mu - \nu - 1\} \]

when \((n - 1)\pi < \alpha < n\pi, n = 1, 2, \ldots\).

**Proof.** Let \((n - 1)\pi < \alpha < n\pi\); then

\[ I(\alpha) = \left\{ \int_{a}^{n\pi} + \int_{n\pi}^{\infty} \right\} (x - \alpha)^{-\sigma} d\mu^{(p)} (x) = J_1 + J_2, \]

say.

Since, by Lemma 1, \( f_\mu^{(p)} (n\pi) = 0 \) and \( |f_\mu^{(p)} (x)| \leq Kx^{-\mu} (x \geq 1) \), we have, for \( \sigma \geq 0, \mu > \rho \), on integrating by parts,

\[ |J_2| = |\sigma \int_{n\pi}^{\infty} (x - \alpha)^{-\sigma - 1} f_\mu^{(p)} (x) dx| \]

\[ \leq Kn^{-\nu}(n\pi - \alpha)^{-\sigma}. \]
Noting that $0 \leq \sigma < 1$, $\mu > p$, $0 < n\pi - \alpha < \pi$, we now use (6), together with the formula
\[
\int_a^b (x - a)^{q-1}(b - x)^{r-1}dx = (b - a)^{q+r-1}B(q, r) \quad (q, r > 0);
\]
then
\[
|J_1| = \left| \int_a^b (x - a)^{-\sigma}f_a^{(p+1)}(x)dx \right|
\leq Kn^{-\sigma}\left\{ \int_a^b (x - a)^{-\sigma}(n\pi - x)^{\mu-p-1}dx + \int_a^b (x - a)^{-\sigma}[x - (n - 1)\pi]^{\mu-p-1}dx \right\}
\leq Kn^{-\sigma}\left\{ (n\pi - \alpha)^{\mu-p-\sigma} + (n\pi - \alpha)^{1-\sigma}\left[ \pi^{\mu-p-1} + (\alpha - n\pi - \pi)^{\mu-p-1} \right] \right\}
\leq Kn^{-\sigma}\left\{ (n\pi - \alpha)^{1-\sigma} + (\alpha - (n - 1)\pi)^{\mu-p-1} \right\}.
\]
Since $|I(\alpha)| \leq |J_1| + |J_2|$, the lemma now follows from (12) and (13).

**Proof of the Theorem.** We may suppose that $\kappa = \sigma + p - 1$, where $0 \leq \sigma < 1$ and $p$ is a positive integer. By (1) and Lemma 2 we have, for $\mu > p$ and $\lambda_n < \Omega \leq \lambda_{n+1},$
\[
\sum_{n=0}^n a_n f_{\mu}'(\lambda, h) = A_\lambda(\Omega)f_{\mu}(\Omega h) - h \int_0^\infty f_{\mu}'(\tau h)dA_\lambda(\tau)
\tag{14}
= \sum_{r=0}^p \frac{(-h)^r}{r!} A_\lambda(\Omega)f_{\mu}'(\tau h) + \frac{(-1)^{p+1}h^p}{p!} \int_0^\infty A_\lambda(\tau)dA_{\mu}^{(p)}(\tau h),
\]
after $p$ integrations by parts ($A_{\lambda}(0) = 0$). Using (2) with $p = 1 - \sigma$, $\kappa = \sigma + p - 1$, and writing $C = \{\Gamma(\sigma+p)/\Gamma(1-\sigma)\}^{-1}$,
\[
\frac{1}{p!} \int_0^\infty A_\lambda(\tau)dA_{\mu}^{(p)}(\tau h) = C \int_0^\infty dA_{\mu}^{(p)}(\tau h) \int_0^\tau (\tau - t)^{-\sigma}A_\lambda^{p+1}(t)dt
\tag{15}
= C \int_0^\infty A_\lambda^{p+1}(t)dt \int_t^\infty (\tau - t)^{-\sigma}dA_{\mu}^{(p)}(\tau h)
= C \int_0^\infty A_\lambda^{p+1}(t)dt \left\{ \int_t^\infty - \int_\Omega^\infty \right\} (\tau - t)^{-\sigma}dA_{\mu}^{(p)}(\tau h)
= I_1 - I_2, \text{ say.}
\]
For each fixed $h > 0$, $\sigma \geq 0$, $\mu > p \geq 1$, for $t < \Omega$, and for all $\Omega \geq h^{-1}$ we have, on integrating by parts and using $|f_{\mu}^{(p)}(x)| \leq Kx^{-\sigma}(x \geq 1),$
\[
\int_\Omega^\infty (\tau - t)^{-\sigma}dA_{\mu}^{(p)}(\tau h) \leq K\Omega^{-\sigma}(\Omega - t)^{-\sigma},
\]
where $K$ is independent of $\Omega$ and $t$. Since, by hypothesis, $A^r_{\alpha}(t) = O(t^{p+\rho})$, it then follows that, as $\Omega \to \infty$,

$$| I_1 | \leq K \int_0^{\infty} t^{p+\rho-\mu} \Omega^{-\mu} (\Omega - t) d\Omega \leq K \Omega^{p-\rho} \to 0.$$  

We now observe that, for $r = 0, 1, \cdots, \rho - 1$, (3) and (5) give

$$A_i^r(\Omega) f^{(r)}_\mu(\Omega h) = O\left( \frac{\Omega^{r-\mu}}{\sum A_n^{\alpha_n} \Omega^{-\mu}} \right) = O\left( \frac{\Omega^{r-\mu}}{\Omega_n^{\alpha_n} \Omega^{-\mu}} \right) = O\left( \frac{\Omega^{r-\mu}}{\Omega_n^{\alpha_n} \Omega^{-\mu}} \right) = o(1),$$

since $\mu > \rho > \kappa \geq r$ and $\sum A_n^{\alpha_n} \Omega^{-\mu}$ converges; while, by (4) and (5),

$$A_i^p(\Omega) f^{(p)}_\mu(\Omega h) = O\left( \frac{\Omega^{p-\mu}}{\Omega_n^{\alpha_n} \Omega^{-\mu}} \right) = o(1).$$

Thus the series on the right of (14) tends to zero as $\Omega \to \infty$, while (by Lemma 3) the series on the left tends to a limit $R_\alpha^b(h)$. Hence the integral on the right of (14) tends to a limit; then, since $I_2 \to 0$, we may let $\Omega \to \infty$ in (15) and substitute the result into (14) to give, for $h > 0$,

$$R_\alpha^b(h) = C(-1)^{p+1} \int_0^{\infty} \phi(h, t) t^{-\rho} A_i^1(t) dt,$$

where

$$\phi(h, t) = h^p t^{-\rho} \int_t^{\infty} (\tau - t)^{-\rho} d\mu^{(p)}(\tau h).$$

The theorem will then follow if we can show that $t^{-\rho} A_i^1(t) \to s$ as $t \to \infty$ implies $R_\alpha^b(h) \to s$ as $h \to 0+$; by Hardy [6, Theorem 6], sufficient conditions for this are:

$$\int_0^{\infty} | \phi(h, t) | dt \leq M \text{ independently of } h > 0,$$

$$\lim_{h \to 0+} \int_0^{T} | \phi(h, t) | dt = 0 \text{ for every finite } T > 0,$$

$$\lim_{h \to 0+} C(-1)^{p+1} \int_0^{\infty} \phi(h, t) dt = 1.$$  

For (20) we can apply (16) to sequences $\{ \lambda_n \}, \{ a_n \}$ satisfying $\lambda_0 = 0, a_0 = 1, a_n = 0$ ($n \geq 1$) to obtain at once
for any $h > 0$, since for the sequences in question
\[ A_h^+(t) = t^\sigma (t > 0), \quad \Re_h^+(k) = 1. \]

Now the substitution $x = \tau h, \alpha = \theta h$ in (17) gives
\[
\int_0^T |\phi(h, t)| \, dt = \int_0^{Th} \alpha^\sigma |I(\alpha)| \, d\alpha,
\]
where
\[
I(\alpha) = \int_0^\infty (x - \alpha)^{-\tau} \delta f_\mu(x).
\]

Thus both (18) and (19) will follow if we can show that
\[
\int_0^\infty \alpha^\sigma |I(\alpha)| \, d\alpha < \infty.
\]

But by Lemma 4,
\[
\int_0^\infty \alpha^\sigma |I(\alpha)| \, d\alpha = \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \alpha^\sigma |I(\alpha)| \, d\alpha
\leq K \sum_{n=1}^\infty n^\sigma \int_{(n-1)\pi}^{n\pi} (n\pi - \alpha)^{-\sigma} + [\alpha - (n - 1)\pi]^{\mu - \sigma - 1} \, d\alpha
\leq K \sum_{n=1}^\infty n^{\sigma - \mu} \text{ since } \sigma < 1 \text{ and } \mu > \rho
< \infty \text{ when } \mu > \kappa + 1,
\]

so that (21) holds and the proof is complete.

References


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