ON RIESZ AND RIEMANN SUMMABILITY

BY

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This paper investigates an inclusion relation between summability of a series of real or complex terms by Riesz typical means and by a generalised form of Riemann summability. We begin by defining the two summability methods.

Riesz’ typical means. Let \( \kappa \geq 0 \), \( 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty \), and write

\[
A^\kappa_\lambda(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu) \nu a_\nu \quad \text{for } \omega > \lambda_0,
\]

\[
A^\kappa_\lambda(\omega) = 0 \quad \text{for } \omega \leq \lambda_0.
\]

If \( \omega^{-\kappa} A^\kappa_\lambda(\omega) \to s \) as \( \omega \to \infty \) then we write

\[
\sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa);
\]

if \( A^\kappa_\lambda(\omega) = O(\omega^\kappa) \) then \( \sum a_n \) is bounded \( (R, \lambda_n, \kappa) \). In the case \( \kappa = 0 \) we note that

\[
A^\kappa_\lambda(\omega) = A^0_\lambda(\omega) = \sum_{\lambda_\nu < \omega} a_\nu = a_0 + \cdots + a_n = A_n
\]

for \( \lambda_n < \omega \leq \lambda_{n+1} \) \((n = 0, 1, \cdots)\). It is well-known that \( A^\kappa_\lambda(\omega) \) is absolutely continuous in any finite interval of values of \( \omega \), for \( 0 < \kappa \leq 1 \), and differentiable with continuous derivative if \( \kappa > 1 \); in fact,

\[
(1) \quad \frac{d}{d\omega} A^\kappa_\lambda(\omega) = \kappa A^{\kappa-1}_\lambda(\omega) \quad (\kappa > 1), \quad \frac{d}{d\omega} A^1_\lambda(\omega) = A_\lambda(\omega) \quad (\omega \neq \lambda_n).
\]

As shown in Hardy and Riesz [9] or Chandrasekharan and Minakshisundaram [5], we also have, for \( \kappa \geq 0, \rho > 0 \),

\[
(2) \quad A^{\kappa+\rho}_\lambda(\omega) = \frac{\Gamma(\kappa + \rho + 1)}{\Gamma(\kappa + 1)\Gamma(\rho)} \int_0^\omega (\omega - t)^{\kappa-1} A^\rho(t) \, dt.
\]

We shall employ the limitation theorem for Riesz means:

If \( A^\kappa_\lambda(\omega) = O(\omega^\kappa) \), \( \kappa \geq 0 \), then, for \( r = 0, 1, \cdots, [\kappa] \),

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\( A_\lambda^*(\omega) = O(\omega_\lambda^{\Delta_{\lambda}}), \)

where \( \lambda_n < \omega \leq \lambda_{n+1} \) and \( \Delta_n = \lambda_{n+1} / (\lambda_{n+1} - \lambda_n) \).

The form of this theorem stated in [9, Theorem 22] and [5, Theorem 1.62] (we use \( O \) in place of \( o \)) is \( A_\lambda^*(\omega) = O(\lambda_{n+1}^{\lambda_*}) \); the stronger form (3) is a special case of a result of Borwein [1, Lemma 2].

Finally, we need the “consistency theorem” for Riesz means:

(4) \( \text{If } A_\lambda^*(\omega) = O(\omega^\kappa), \quad \kappa \geq 0, \text{ then } A_\lambda^*(\omega) = O(\omega^p) \quad \text{for } p \geq \kappa. \)

**Riemann summability.** Let \( \mu > 0, 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty, \)

\[ f_\mu(x) = \left( \frac{\sin x}{x} \right)^\mu (x \neq 0), \quad f_\mu(0) = 1; \]

if the series

\[ R_\lambda^\mu(h) = \sum_{n=0}^{\infty} a_n f_\mu(\lambda_n h) \]

converges for each \( h \) in a deleted neighbourhood of the origin, and if \( R_\lambda^\mu(h) \to s \) as \( h \to 0 \), then we write

\[ \sum_{n=0}^{\infty} a_n = s \quad (R, \lambda_n, \mu). \]

The case where \( \lambda_n = n \) and \( \mu \) is a positive integer is usually known as Riemann summability. The more general definition above has been given by Burkill [2] for \( \mu = 1, 2 \), and by Burkill and Petersen [4] for \( \mu \) rational with odd denominator (which ensures that \( f_\mu(x) \) is real); alternatively, for any \( \mu > 0 \) we may define \( (\sin x)^\mu = e^{\mu x}(\sin x)^\mu \) when \( x > 0, \sin x < 0, \) and \( f_\mu(-x) = f_\mu(x) \). In fact, any definition is suitable for our purpose, which ensures that

\[ \frac{d}{dx} (\sin x)^\mu = \mu(\sin x)^{\mu-1} \cos x, \quad (\sin x)^\mu \leq 1 \quad (\mu > 0), \]

\[ (\sin x)^\mu \sim |x - \pi n|^\mu \quad (x \to \pi n); \]

and since \( f_\mu(x) \) is an even function we may suppose throughout, in the definition of \( (R, \lambda_n, \mu) \) summability, that \( h > 0 \).

Burkill [3] has shown that if \( \lambda_0 = 0, 0 < \mu \leq \lambda_{n+1} - \lambda_n \leq q, \) and \( \kappa \) is a positive integer, then summability \( (R, \lambda_n, \kappa) \) implies summability \( (R, \lambda_n, \mu) \) for \( \mu > \kappa + 1 \) (and \( \mu \) rational with odd denominator). Burkill and Petersen [4] have proved this for \( \kappa = 1 \), remarking that from the point of view of applications (for instance, to the theory of almost periodic functions—see, for example, [2] and [11]) it would be desirable to proceed from a nonintegral Riesz
mean to an integral Riemann mean. The present paper furnishes such a result, which also contains the theorem referred to above; we prove, more generally, the following

**Theorem.** If \( \sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa) \), \( \kappa \geq 0 \), and if \( \sum_{n=1}^{\infty} \lambda_n^\mu \lambda_n^{-\mu} \) converges, where \( \lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n) \) and \( \mu > \kappa + 1 \), then \( \sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \mu) \).

In the special case \( \lambda_n = n \), \( (R, \lambda_n, \kappa) \) is equivalent to Cesàro summability \((C, k)\), and \( (R, \lambda_n, \mu) \) becomes ordinary Riemann summability, which will be denoted by \((R, \mu)\); if, in addition, \( \mu \) is a positive integer greater than 1, we obtain a result of Verblunsky [12] that \((C, k) \subseteq (R, \mu)\) for \( 0 \leq k < \mu - 1 \), \( \mu = 2, 3, \ldots \); Hardy and Littlewood [7, 8] had proved earlier that \((C, k) \subseteq (R, 1)\) for \(-1 \leq k < 0\). Kuttner [10] has proved that \((R, \mu) \subseteq (C, \mu + \delta)\) for \( \delta > 0 \), \( \mu = 1, 2 \), and that the result is false for \( \mu = 3 \); and he has shown that \((R, \mu) = (R, n, \mu) \subseteq (R, \log n, \mu)\) for \( \mu = 1, 2 \). See also Hardy [6, Appendix III].

Some lemmas are needed. We remark that in general throughout this paper \( K \) will denote a positive quantity independent of the particular variables under consideration, and not necessarily the same at each occurrence; thus, for example, in the first lemma the constants \( K \) may depend on \( \mu \) or \( p \), but are independent of \( x \) or \( n \).

**Lemma 1.** Let \( p \) be a non-negative integer, and define \( f_0(x) = 1 \).

(a) For any \( \mu \geq p \), \( f_\mu^{(p)}(x) \) is continuous everywhere, and

\[
| f_\mu^{(r)}(x) | \leq K (0 < x < 1), \quad | f_\mu^{(r)}(x) | \leq K x^{-p} (x \geq 1), \quad r = 0, 1, \ldots, p.
\]

(b) If \( \mu > p \) then \( f_\mu^{(p)}(n\pi) = 0 \) \( (n = 1, 2, \ldots) ; r = 0, 1, \ldots, p \)\). Also \( f_\mu^{(p+1)}(x) \) is continuous in \( (n - 1)\pi < x < n\pi \ (n = 1, 2, \ldots) \) and, in each such interval, satisfies the inequality

\[
| f_\mu^{(p+1)}(x) | \leq K n^{-p} \left\{ (n\pi - x)^{\mu-p-1} + [x - (n - 1)\pi]^{\mu-p-1} \right\}.
\]

**Proof.** We first note that, for each non-negative integer \( s \),

\[
| f_1^{(s)}(x) | \leq K (0 < x < 1), \quad | f_1^{(s)}(x) | \leq K x^{-1} (x \geq 1);
\]

the first of these inequalities is an immediate consequence of the fact that \( f_1(x) \) has a power series expansion with infinite radius of convergence, while the second follows from the formula

\[
f_1^{(s)}(x) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k k! x^{s-k} \sin \left[ x + \frac{1}{2} (s - k)\pi \right].
\]

It is clear that \( f_\mu(x) \) is differentiable as often as we please, except perhaps at \( x = \pm \pi, \pm 2\pi, \ldots \); also \( f_\mu^{(p+1)}(x) = (\mu + 1) f_\mu(x) f_\mu^{(p)}(x) \), and on differentiating \( p \) times this gives

\[
(\text{f)} \text{ This inequality is also given (for } \mu \text{ rational with odd denominator) in [3, Lemma 2].}
\]
which enables us to proceed by induction on $p$. We shall merely verify the inequalities (5) and (6).

(a) Suppose that, for some fixed non-negative integer $p$ and for any $\mu \geq p$, (5) holds; then since $\mu \geq p$ implies $\mu + 1 \geq p$, (5) also holds with $\mu + 1$ in place of $\mu$ (and $\tau = 0, 1, \ldots, p$). Further, (8) shows, by (7) and the inductive hypothesis, that $f_{\mu+1}(x)$ is bounded in $(0, 1)$ and is $O(x^{-\mu})$ as $x \to \infty$. Since (5) may be verified directly from the definition of $f_{\mu}(x)$ in the case $p = 0$, it follows that (5) is true for any non-negative integer $p$ and any $\mu \geq p$.

(b) If $\mu \geq p + 1$ then (6) is equivalent to $|f_{\mu+1}(x)| \leq Kn^{-\mu}$ for $0 \leq (n - 1)\pi < x < n\pi$, which has already been proved in part (a) of the lemma. Suppose, therefore, that for some fixed non-negative integer $p$ and $0 < |n\pi - x| \leq \pi/2$ ($n = 0, 1, \ldots$),

$$|f_{\mu+1}(x)| \leq Kn^{-\mu} |n\pi - x|^{p+1}$$

for $p < \mu < p + 1$;

in addition, we already know from (5) and (7) that

$$|f_{\mu}^{(r)}(x)| \leq Kn^{1-\mu} (r = 0, 1, \ldots, p),$$

$$|f_{\mu}^{(s)}(x)| \leq Kn^{1-\mu} (s = 0, 1, \ldots).$$

Now use (8) with $p + 1$ in place of $p$, together with (9) and (10), and we get

$$|f_{\mu+1}(x)| \leq Kn^{1-\mu} |n\pi - x|^{p+1} + Kn^{1-\mu};$$

or, writing $\nu$ for $\mu + 1$,

$$|f_{\nu}(x)| \leq Kn^{1-\nu} |n\pi - x|^{p+2}$$

for $p + 1 < \nu < p + 2$.

Since we may verify (9) directly for $p = 0$, (9) therefore follows, by induction, for any non-negative integer $p$; and by combining the results for the two halves of the interval $(n - 1)\pi < x < n\pi$, we obtain (6).

Define $A_{n+1}(A_{p+1})$ and $A_{1}(A)$ as before, we now prove

**Lemma 2. If $\mu \geq 1$, $\lambda_n < \Omega \leq \lambda_{n+1}$ ($n = 0, 1, \ldots$), then**

$$\sum_{\nu=0}^{n} a_{\nu} f_{\nu}(\lambda, h) = A_{1}(\Omega)f_{\mu}(\Omega h) - h \int_{0}^{\nu} f_{\nu}'(\nu h)A_{1}(\nu h)\,d\nu.$$

**Proof.** Since $f_{\nu}'(x)$ is continuous for any $x$, when $\mu \geq 1$, and $A_{1}(\nu) = A$, for $\lambda < \tau \leq \lambda_{n+1}$ we have, for $\lambda_n < \Omega \leq \lambda_{n+1},$
\[ h \int_0^a f'_\mu(\tau h) A_\lambda(\tau) d\tau = h \left\{ \sum_{\substack{r=0 \atop \lambda_r \neq 0}}^{n-1} \int_{\lambda_r}^{\lambda_{r+1}} f'_\mu(\tau h) A_\lambda(\tau) d\tau + \int_{\lambda_n}^a f'_\mu(\tau h) A_\lambda(\tau) d\tau \right\} \\
= h \sum_{\substack{r=0 \atop \lambda_r \neq 0}}^{n-1} A_r \left[ \frac{1}{h} f_\mu(\tau h) \right]_{\lambda_r}^{\lambda_{r+1}} + h A_n \left[ \frac{1}{h} f_\mu(\tau h) \right]_{\lambda_n}^a \\
= A_n f_\mu(\Omega h) - \sum_{r=0}^n (A_r - A_{r-1}) f_\mu(\lambda_r h), \]

by partial summation; and this gives (11).

Now to obtain \( R^\mu_\lambda(h) \) we must let \( n \to \infty \) in (11); the following lemma gives sufficient conditions for the existence of \( R^\mu_\lambda(h) \).

**Lemma 3.** If \( \sum a_n \) is bounded (or summable) \((R, \lambda_n, \kappa)\), \( \kappa \geq 0 \), and if \( \sum \Lambda^\mu_\kappa \) converges, then \( \sum a_n f_\mu(\lambda_r h) \) converges (absolutely) for each fixed \( h > 0 \).

**Proof.** If \( \sum a_n \) is bounded \((R, \lambda_n, \kappa)\) then by (3) (with \( r = 0 \)), \( A_n = O(\lambda_n^\kappa) \); moreover, for any fixed \( h > 0 \), \( f_\mu(\lambda_r h) = O(\lambda_n^{-\kappa}) \) as \( n \to \infty \). Hence

\[ a_n f_\mu(\lambda_r h) = (A_n - A_{n-1}) f_\mu(\lambda_r h) \]

\[ = \{ O(\lambda_n^\kappa) + O(\lambda_{n-1}^\kappa) \} O(\lambda_n^{-\kappa}) \]

\[ = O(\lambda_n^\kappa + \lambda_{n-1}^{-\kappa}), \]

and the lemma follows.

**Lemma 4.** Let \( p \) be a positive integer, \( 0 \leq \sigma < 1 \), \( \mu > p \), and

\[ I(\alpha) = \int_a^\infty (x - \alpha)^{-\sigma} f_\mu^{(p)}(x). \]

Then

\[ |I(\alpha)| \leq Kn^{-\mu} \{(n\pi - \alpha)^{-\sigma} + [\alpha - (n - 1)\pi]^{p-\sigma-1}\} \]

when \((n-1)\pi \leq \alpha < n\pi\), \( n = 1, 2, \ldots \).

**Proof.** Let \((n-1)\pi \leq \alpha < n\pi\); then

\[ I(\alpha) = \left\{ \int_a^{n\pi} + \int_{n\pi}^\infty \right\} (x - \alpha)^{-\sigma} f_\mu^{(p)}(x) = J_1 + J_2, \]

say.

Since, by Lemma 1, \( f_\mu^{(p)}(n\pi) = 0 \) and \( |f_\mu^{(p)}(x)| \leq Kx^{-\mu}(x \geq 1) \), we have, for \( \sigma \geq 0 \), \( \mu > p \), on integrating by parts,

\[ |J_2| = |\sigma \int_{n\pi}^\infty (x - \alpha)^{-\sigma-1} f_\mu^{(p)}(x) dx| \]

\[ \leq Kn^{-\mu}(n\pi - \alpha)^{-\sigma}. \]
Noting that $0 \leq \sigma < 1$, $\mu > p$, $0 < n \pi - \alpha < \pi$, we now use (6), together with the formula

$$\int_a^b (x - a)^{\alpha - 1}(b - x)^{\gamma - 1} \, dx = (b - a)^{\alpha + \gamma - 1} B(\gamma, \alpha) \quad (\gamma, \alpha > 0);$$

then

$$|J_1| = \left| \int_a^b (x - a)^{-\sigma} f_{\mu}^{(p+1)}(x) \, dx \right|$$

$$\leq Kn^{-r} \left\{ \int_a^b (x - a)^{-\sigma} (n \pi - x)^{\mu - p - 1} \, dx \right\}$$

$$+ \int_a^b (x - a)^{-\rho} [x - (n - 1) \pi]^{\mu - p - 1} \, dx \right\}$$

$$\leq Kn^{-r} \left\{ (n \pi - \alpha)^{\mu - p - \sigma} + (n \pi - \alpha)^{1 - \sigma} [\pi^{\mu - p - 1} + (\alpha - n \pi - \pi)^{\mu - p - 1}] \right\}$$

$$\leq Kn^{-r} \left\{ (n \pi - \alpha)^{-\sigma} + [\alpha - (n - 1) \pi]^{\mu - p - 1} \right\}.$$  

Since $|I(\alpha)| \leq |J_1| + |J_2|$, the lemma now follows from (12) and (13).

**Proof of the Theorem.** We may suppose that $\kappa = \sigma + p - 1$, where $0 \leq \sigma < 1$ and $p$ is a positive integer. By (1) and Lemma 2 we have, for $\mu > p$ and $\lambda_n < \Omega \leq \lambda_{n+1}$,

$$\sum_{r=0}^n a_{\lambda_n}(\lambda, h) = A_\lambda(\Omega)f_\alpha(\Omega h) - h \int_0^\Omega f_\alpha'(\lambda h) \, dA_\lambda(\lambda)$$

$$= \sum_{r=0}^p \frac{(-h)^r}{r!} A_\lambda(\Omega) f_\alpha^{(r)}(\Omega h) + \frac{(-1)^{p+1} \kappa p}{p!} \int_0^\Omega A_\lambda^{(p)}(\tau) f_\alpha^{(p)}(\tau h),$$

after $p$ integrations by parts $(A_\lambda(0) = 0)$. Using (2) with $p = 1 - \sigma$, $\kappa = \sigma + p - 1$, and writing $C = \{\Gamma(\sigma + p) \Gamma(1 - \sigma)\}^{-1}$,

$$\frac{1}{p!} \int_0^\Omega A_\lambda^{(p)}(\tau) f_\alpha^{(p)}(\tau h) = C \int_0^\Omega f_\alpha^{(p)}(\tau h) \int_\tau^\tau (\tau - t)^{-\sigma} A_\lambda^{(p+1)}(t) \, dt$$

$$= C \int_0^\Omega A_\lambda^{(p+1)}(t) \, dt \int_\tau^\tau (\tau - t)^{-\sigma} f_\alpha^{(p)}(\tau h)$$

$$= C \int_0^\Omega A_\lambda^{(p+1)}(t) \, dt \{\int_\tau^\tau - \int_\Omega^\Omega (\tau - t)^{-\sigma} f_\alpha^{(p)}(\tau h)$$

$$= I_1 - I_2, \text{ say.}$$

For each fixed $h > 0$, $\sigma > 0$, $\mu > \rho > 1$, for $t < \Omega$, and for all $\Omega \geq h^{-1}$ we have, on integrating by parts and using $\left| f_\alpha^{(p)}(x) \right| \leq K x^{-\mu} (x \geq 1)$,

$$\left| \int_\Omega^\Omega (\tau - t)^{-\sigma} f_\alpha^{(p)}(\tau h) \right| \leq K \Omega^{-\mu}(\Omega - t)^{-\sigma},$$

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where $K$ is independent of $\Omega$ and $t$. Since, by hypothesis, $A^{x+p-1}_x(t) = O(t^{x+p-1})$, it then follows that, as $\Omega \to \infty$,

$$|I_1| \leq K \int_0^\Omega t^{x+p-1} \Omega^{-\mu}(\Omega - t)^{-\sigma} dt \leq K \Omega^{p-\mu} \to 0.$$

We now observe that, for $r = 0, 1, \ldots, p-1$, (3) and (5) give

$$A^r_x(\Omega) f_{(x)}^{(r)}(\Omega) = O\{|\Omega^r A^{x-r}_x(\Omega)|\} = O\{|\Delta^r_x \lambda^{-r}_x\}| = O\{|\Delta^r_x \lambda^{-r}_x\}| O\{1 + (\lambda_n/A_n)^r\} = O\{|\Delta^r_x \lambda^{-r}_x\}| + O\{|\lambda_n^{x-r}\| = o(1) + o(1),$$

since $\mu > p > \kappa \geq r$ and $\sum \Delta^r_x \lambda^{-r}_x$ converges; while, by (4) and (5),

$$A^p_x(\Omega) f_{(p)}^{(p)}(\Omega) = O\{|\Omega^p \Omega^{-p}\| = o(1).$$

Thus the series on the right of (14) tends to zero as $\Omega \to \infty$, while (by Lemma 3) the series on the left tends to a limit $\mathcal{R}_x(h)$. Hence the integral on the right of (14) tends to a limit; then, since $I_1 \to 0$, we may let $\Omega \to \infty$ in (15) and substitute the result into (14) to give, for $h > 0$,

$$\mathcal{R}_x(h) = C(-1)^{p+1} \int_0^\infty \phi(h, t) t^{-x} A^x(t) dt,$$

where

$$\phi(h, t) = h^p t^x \int_1^\infty (t - s)^{-\sigma} d_{(p)}(\sigma h).$$

The theorem will then follow if we can show that $t^{-x} A^x(t) \to s$ as $t \to \infty$ implies $\mathcal{R}_x(h) \to s$ as $h \to 0+$. By Hardy [6, Theorem 6], sufficient conditions for this are:

$$\int_0^\infty |\phi(h, t)| dt \leq M \text{ independently of } h > 0,$$

$$\lim_{h \to 0^+} \int_0^T |\phi(h, t)| dt = 0 \text{ for every finite } T > 0,$$

$$\lim_{h \to 0^+} C(-1)^{p+1} \int_0^\infty \phi(h, t) dt = 1.$$  

For (20) we can apply (16) to sequences $\{\lambda_n\}, \{a_n\}$ satisfying $\lambda_0 = 0, a_0 = 1, a_n = 0 \text{ for } n \geq 1$ to obtain at once
for any $h>0$, since for the sequences in question
\[ A_h^t(t) = t^s (t > 0), \quad \Re x(h) = 1. \]
Now the substitution $x = \tau h$, $\alpha = \tau h$ in (17) gives
\[
\int_0^T |\phi(h,t)| \, dt = \int_0^{Th} \alpha^s |I(\alpha)| \, d\alpha,
\]
where
\[ I(\alpha) = \int_\alpha^\infty (x - \alpha)^{-\sigma} df^{(p)}(x). \]
Thus both (18) and (19) will follow if we can show that
\[
\int_0^\infty \alpha^s |I(\alpha)| \, d\alpha < \infty.
\]
But by Lemma 4,
\[
\int_0^\infty \alpha^s |I(\alpha)| \, d\alpha = \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \alpha^s |I(\alpha)| \, d\alpha
\leq K \sum_{n=1}^\infty n^{-\sigma} \int_{(n-1)\pi}^{n\pi} \left( n\pi^\sigma - (n-1)\pi^\sigma + [\alpha - (n-1)\pi]^{s-\rho} \right) \, d\alpha
\leq K \sum_{n=1}^\infty n^{\sigma-\rho} \quad \text{since } \sigma < 1 \text{ and } \mu > \rho
< \infty \quad \text{when } \mu > \kappa + 1,
\]
so that (21) holds and the proof is complete.

References


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