THE COHOMOLOGY OF A SUBALGEBRA OF THE STEENROD ALGEBRA

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1. **Introduction.** Let A be the Steenrod algebra over Z_p, p a prime. It is well known [4] that it is an augmented Hopf algebra over Z_p. The groups $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ occur as the E_2 term of a spectral sequence which may be used to determine the p-primary components of stable homotopy groups of spheres [1]. Recent results of Adams [3] indicate that it is fruitful to study the groups $\mathrm{Ext}_B(Z_p, Z_p)$, where B is a Hopf subalgebra of A. In this paper we study a certain subalgebra of the Steenrod algebra. The results obtained here will be used in proving an Adams periodicity theorem for p an odd prime. The fundamental tool in this investigation is the twisted tensor product construction introduced by Wall [5].

2. **A Hopf algebra on two generators.** Let W be a graded, connected algebra with unit and augmentation, generated as an algebra over Z_p (p a prime) by two elements: x of grade q, z of grade pq, satisfying the following relations:

$$\begin{aligned} x[z, x] &= [z, x]x, \\ z[z, x] &= [z, x]z, \\ x^p &= 0, \\ z^p &= ([z, x])^{p-1}x, \end{aligned}$$

where $[z, x] = zx - xz$.

For $p=2$ we may interpret W as the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2, by letting $q=1$, $x=Sq^1$, $z=Sq^2$. For p odd, we let $x=P^1$, $z=P^p$, $q=2p-2$: in this case W is again the subalgebra of the Steenrod algebra generated by P^1 and P^p [4].

Let us write y for $[z, x]$. It is an easy consequence of the above relations that $y^p=0$.

Let V be the subalgebra of W generated by 1, x and y. Since x and y commute, and $x^p=y^p=0$, V is, as a (Hopf) algebra, just a tensor product of two truncated polynomial algebras. Thus a minimal resolution for Z_p over V can be taken to be a tensor product of two minimal resolutions of Z_p over truncated polynomial algebras. We thus have

$$\begin{aligned} \mathrm{Ext}_V^{*,*}(Z_p, Z_p) &\cong \mathfrak{L}\{h_0, \lambda_0, h_{2,0}, \lambda_{2,0}\} && \text{if } p \text{ odd}, \\ &\cong Z_2[h_0, h_{2,0}] && \text{if } p = 2, \end{aligned}$$

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where $\mathcal{L}\{\ \}$ denotes the free associative, commutative bigraded algebra over Z_p on generators

$$
\begin{array}{ll}
h_0 \text{ of bidegree} & (1, q), \\
h_{2,0} \text{ of bidegree} & (1, q(p+1)), \\
\lambda_0 \text{ of bidegree} & (2, pq), \\
\lambda_{2,0} \text{ of bidegree} & (2, pq(p+1)).
\end{array}
$$

A minimal resolution for Z_p over V is given by the following complex $\mathfrak{F}$:

$$0 \leftarrow Z_p \overset{\epsilon_F}{\leftarrow} F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_s \overset{d_F}{\leftarrow} F_{s+1} \leftarrow \cdots.$$

We take $F_s = V \otimes_{Z_p} \overline{F}_s$, where $\overline{F}_s \cong \mathrm{Ext}_V^{s,*}(Z_p, Z_p)$. We thus conveniently confuse the generators of F_s with elements of a Z_p basis of $\mathrm{Ext}_V^{s,*}(Z_p, Z_p)$.

We take as a basis for $\overline{F}_s$ the set of formal monomials $h_0^\epsilon \lambda_0^n h_{2,0}^\eta \lambda_{2,0}^r$, where $\epsilon, \eta = 1$ or 0, and $\epsilon + 2n + \eta + 2r = s$. If $p = 2$ we omit $\lambda_{2,0}$ and substitute $h_{2,0}^2$ for every occurrence of $\lambda_{2,0}$. We define the differential d_F in $\mathfrak{F}$ as follows:

$$
\begin{aligned}
d_F(\lambda_0^n \lambda_{2,0}^r) &= x^{p-1} h_0 \lambda_0^{n-1} \lambda_{2,0}^r + y^{p-1} \lambda_0^n h_{2,0} \lambda_{2,0}^{r-1}, \\
d_F(h_0 \lambda_0^n \lambda_{2,0}^r) &= x \lambda_0^n \lambda_{2,0}^r - y^{p-1} h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^{r-1}, \\
d_F(\lambda_0^n h_{2,0} \lambda_{2,0}^r) &= x^{p-1} h_0 \lambda_0^{n-1} h_{2,0} \lambda_{2,0}^r + y \lambda_0^n \lambda_{2,0}^r, \\
d_F(h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^r) &= x \lambda_0^n h_{2,0} \lambda_{2,0}^r - y h_0 \lambda_0^n \lambda_{2,0}^r,
\end{aligned}
$$

where we set $\alpha^k = 1$ if $k = 0$, $\alpha^k = 0$ if $k < 0$.

We notice that V is a *normal* subalgebra of W: $\overline{V}W = W\overline{V}$, where $\overline{V}$ is the augmentation ideal. This follows from the relations

$$
\begin{aligned}
zx &= xz + y, \\
zy &= yz.
\end{aligned}
$$

The normal quotient $U = W//V = W/W\overline{V}$ is clearly a truncated polynomial algebra on the residue class of z, denoted $\bar{z}$. We know that

$$\mathrm{Ext}_U^{*,*}(Z_p, Z_p) \cong \mathcal{L}\{h_1, \lambda_1\},$$

where h_1 is of bidegree $(1, pq)$ and λ_1 is of bidegree $(2, p^2q)$, and where (once more) if $p = 2$ we omit λ_1 and substitute h_1^2 for every occurrence of λ_1. We let $\mathfrak{B}$ be the following minimal resolution of Z_p over U:

$$0 \leftarrow Z_p \overset{\epsilon_B}{\leftarrow} B_0 \leftarrow B_1 \leftarrow \cdots \leftarrow B_r \overset{d_B}{\leftarrow} B_{r+1} \cdots,$$

where $B_r = U \otimes_{Z_p} \overline{B}_r$, and $\overline{B}_r$ is 1-dimensional over Z_p, with a basis $h_1^\epsilon \lambda_1^k$, where $\epsilon = 0$ or 1, and $\epsilon + 2k = r$.

We let

$$d_B(h_1\overset{k}{\lambda_1}) = \bar{z}\overset{k}{\lambda_1},$$
$$d_B(\overset{k+1}{\lambda_1}) = \bar{z}^{p-1}h_1\overset{k}{\lambda_1}.$$

Let $\mathcal{E}$ be the following (three-graded) free W-module:

$$\mathcal{E}_{r,s} = W \otimes_{Z_p} \overline{B}_r \otimes_{Z_p} \overline{F}_s,$$
$$\mathcal{E} = \sum_{r,s} \mathcal{E}_{r,s}.$$

We introduce an augmentation

$$\epsilon_s: \mathcal{E}_{s,*} \to B_s$$

by setting $\epsilon_s = \pi \otimes \epsilon_F$, where π is the projection

$$\pi: W \to U, \text{ and } \epsilon_F \text{ is the augmentation in } \mathfrak{F}.$$

3. **Twisted tensor product of resolutions.** Consider the following situation (we have described a special case of it above). Let W, U, V be augmented algebras over a field K. Let $\mathcal{B}$, $\mathfrak{F}$ be free resolutions of K over U, V, respectively. Let $\overline{\mathcal{B}}$, $\overline{\mathfrak{F}}$ be the K-complexes $K \otimes_U \mathcal{B}$, $K \otimes_V \mathfrak{F}$.

THEOREM 1 (WALL). *If* (1) *V is normal in W*, (2) *W is free as a right module over V*, (3) *$U = W//V = W/W\overline{V}$, then a free resolution $\mathcal{E}$ of K over W can be constructed as a twisted tensor product of $\overline{\mathcal{B}}$ and $\overline{\mathfrak{F}}$. That is, if*

$$\mathcal{E} = \sum_{r,s} \mathcal{E}_{r,s},$$
$$\mathcal{E}_{r,s} = W \otimes_K \overline{B}_r \otimes_K \overline{F}_s,$$

then there exists an augmentation

$$\epsilon: \mathcal{E} \to \mathcal{B}$$

and W-maps

$$d_k: \mathcal{E}_{r,s} \to \mathcal{E}_{r-k,s+k-1}$$

such that

$$d_0 = 1 \otimes 1 \otimes d_F,$$
$$\epsilon d_1 = d_B\epsilon,$$

and

$$\sum_{j=0}^{k} d_{k-j}d_j = 0 \qquad \textit{for } k = 0, 1, \cdots.$$

Proof. We take the proof of Wall [5] for the special case $W=Z(G)$, $U=Z(H)$, $V=Z(K)$, where G, H, K are finite groups, $K \triangleleft G$, $H=G/K$. If we substitute W, U, V for $Z(G)$, $Z(H)$, $Z(K)$, respectively, wherever they appear in Wall's proof, the result is a proof of Theorem 1.

We make $\mathcal{E}$ into a complex by setting $d_E = \sum_{j=0}^{\infty} d_j$.

We remark that the hypothesis (2) of the theorem is always satisfied if V is a Hopf subalgebra of the Hopf algebra W.

PROPOSITION 1. *If W, V, U, $\mathcal{E}$ are as in §2, we let γ_q be the chosen basis element of $\overline{B}_q$. We can define the maps d_k as follows:*

(1) *d_0 is induced by d_F,*

(2) *if q is an even integer,*

$$
\begin{aligned}
d_1(\gamma_{q+1} \otimes \lambda_0^n \lambda_{2,0}^r) &= z\gamma_q \otimes \lambda_0^n \lambda_{2,0}^r - x^{p-2}\gamma_q \otimes h_0 \lambda_0^{n-1} h_{2,0} \lambda_{2,0}^r, \\
d_1(\gamma_{q+1} \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^r) &= z\gamma_q \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^r, \\
d_1(\gamma_{q+1} \otimes h_0 \lambda_0^n \lambda_{2,0}^r) &= z\gamma_q \otimes h_0 \lambda_0^n \lambda_{2,0}^r - \gamma_q \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^r, \\
d_1(\gamma_{q+1} \otimes h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^r) &= z\gamma_q \otimes h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^r,
\end{aligned}
$$

(3) $d_2(\gamma_{q+2} \otimes \alpha) = (-1)^{q+1} y^{p-2} x \gamma_q \otimes h_{2,0}\alpha$, *where q is a non-negative integer, and α is a basic monomial,*

(4) $d_{2m} = 0$ *identically for $m \geqq 2$,*

(5) $d_3(\gamma_{q+3} \otimes \alpha) = (r+1)\gamma_q \otimes \alpha \lambda_{2,0}$, *where q is an even integer and α is a basic monomial of the form $h_0^{\epsilon} \lambda_0^n h_{2,0}^{\eta} \lambda_{2,0}^r$,*

(6) $d_{2m+1}(\gamma_{q+2m+1} \otimes \alpha)$ *for q odd is defined as follows:*

$$
\begin{aligned}
d_{2m+1}(\lambda_1^{k+m+1} \otimes \lambda_0^n \lambda_{2,0}^r) &= (-1)^m m! \binom{r+m}{r} z^{p-m-1} h_1 \lambda_1^k \otimes \lambda_0^n \lambda_{2,0}^{r+m} \\
&+ (-1)^{m+1} \binom{r+m}{r} \sum_{j=1}^{p-m-1} (-1)^j (j+m)! z^{p-m-1-j} y^{j-1} x^{p-1-j} h_1 \lambda_1^k \otimes h_0 \lambda_0^{n-1} h_{2,0} \lambda_{2,0}^{r+m},
\end{aligned}
$$

$$
d_{2m+1}(\lambda_1^{k+m+1} \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^r) = (-1)^m m! \binom{r+m}{r} z^{p-m-1} h_1 \lambda_1^k \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^{r+m},
$$

$$
\begin{aligned}
d_{2m+1}(\lambda_1^{k+m+1} \otimes h_0 \lambda_0^n \lambda_{2,0}^r) &= (-1)^m m! \binom{r+m}{r} z^{p-m-1} h_1 \lambda_1^k \otimes h_0 \lambda_0^n \lambda_{2,0}^{r+m} \\
&+ (-1)^m (m+1)! \binom{r+m}{r} z^{p-m-2} h_1 \lambda_1^k \otimes \lambda_0^n h_{2,0} \lambda_{2,0}^{r+m},
\end{aligned}
$$

$$
d_{2m+1}(\lambda_1^{k+m+1} \otimes h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^r) = (-1)^m m! \binom{r+m}{r} z^{p-m-1} h_1 \lambda_1^k \otimes h_0 \lambda_0^n h_{2,0} \lambda_{2,0}^{r+m},
$$

(7) $d_{2m+1}(\gamma_{q+2m+1} \otimes \alpha) = 0$ *if q is even and $m \geqq 2$.*

Proof. Easy, but tedious induction on the total degree. Special care must be given for d_j, $j \leqq 6$.

4. **Structure of** $\operatorname{Ext}_W(Z_p, Z_p)$. In order to determine the additive structure of $\operatorname{Ext}_W(Z_p, Z_p)$, it is sufficient to compute $\operatorname{Tor}^W(Z_p, Z_p)$, since the first is the graded dual of the second [2]. We exhibit certain elements of $\operatorname{Ext}_W(Z_p, Z_p)$ in the following table (here $p \neq 2$; the structure of $\operatorname{Ext}_W(Z_2, Z_2)$ is given in Theorem 3). The first column gives the element in Ext which is the dual of the corresponding element in the third column (this makes sense, for in each relevant grading $\operatorname{Tor}^W(Z_p, Z_p)$ turns out to be 1-dimensional).

TABLE 1 ($p \neq 2$)

Class in Ext	Bidegree	Representative in Tor
h_0	$(1, q)$	$1 \otimes h_0$
h_1	$(1, pq)$	$h_1 \otimes 1$
λ_0	$(2, pq)$	$1 \otimes \lambda_0$
λ_1	$(2, p^2q)$	$\lambda_1 \otimes 1$
μ_0	$(2, (p+2)q)$	$\frac{1}{2} 1 \otimes h_0 h_{2,0}$
ν_0	$(2, (2p+1)q)$	$\frac{1}{2} h_1 \otimes h_{2,0}$
χ	$(3, (p^2+p+1)q)$	$1 \otimes h_0 \lambda_{2,0}$
σ_{2r+1} $1 \leqq r < p-1$	$(2r+1, (rp^2+rp+p)q)$	$h_1 \otimes \lambda_{2,0}^r$
κ_{2s} $2 \leqq s < p-1$	$(2s, (sp^2-p^2+sp+p+1)q)$	$h_1 \otimes h_{2,0} \lambda_{2,0}^{s-1}$
ω	$(2p, p^2(p+1)q)$	$1 \otimes \lambda_{2,0}^p$

THEOREM 2 ($p \neq 2$). *The classes λ_0 and ω generate a free associative, commutative algebra L in* $\operatorname{Ext}_W(Z_p, Z_p)$. *The classes λ_1^k $0 \leqq k < p$, $h_0 \lambda_1^q$ $0 \leqq q < p-1$, $\mu_0 \lambda_1^k$ $0 \leqq k < p$, $\chi \lambda_1^k$ $0 \leqq k < p$, ν_0, $h_0 \nu_0$, σ_{2r+1} $1 \leqq r < p-1$, $h_0 \sigma_{2r+1}$ $1 \leqq r < p-1$, κ_{2s} $2 \leqq s < p-1$, $h_0 \kappa_{2s}$ $2 \leqq s < p-1$, $\mu_0 \sigma_{2r+1}$ $1 \leqq r < p-1$ is a system of generators for* $\operatorname{Ext}_W(Z_p, Z_p)$ *as a free left L-module.*

Proof. We first remark that the elements $1 \otimes \lambda_0$ and $1 \otimes \lambda_{2,0}^p$ indeed give rise to nonzero elements in $\operatorname{Ext}_W(Z_p, Z_p)$. Secondly, we notice that the resolution $\mathcal{E}$ of Z_p is honestly periodic with respect to the formal multiplication of the generators by λ_0 and $\lambda_{2,0}^p$. It is now clear that the formal multiplication by λ_0 or $\lambda_{2,0}^p$ corresponds to honest multiplication by λ_0 or ω in $\operatorname{Ext}_W(Z_p, Z_p)$: this is an immediate consequence of the preceding sentence and the Yoneda construction for the product (see, for example, p. 30 [2]). We now only have to verify that the listed classes indeed give an L-basis for $\operatorname{Ext}_W(Z_p, Z_p)$. This is indeed the case, for a basis for

$$\operatorname{Ker} \bar{d} / \operatorname{Im} \bar{d} = \operatorname{Tor}^W(Z_p, Z_p)$$

($\bar{d}$ denotes $1 \otimes d$ in $Z_p \otimes_W \mathcal{E}$) is given by the cosets of the following elements:

$$
\begin{array}{ll}
h_1 \otimes \lambda_{2,0}^{r} & 0 \leqq r < p-1, \\
h_1 \otimes h_0\lambda_{2,0}^{r} + (1/r)\, h_1\lambda_1 \otimes h_{2,0}\lambda_{2,0}^{r-1} & 1 \leqq r < p-1, \\
h_1 \otimes h_{2,0}\lambda_{2,0}^{r} & 0 \leqq r < p-2, \\
h_1 \otimes h_0h_{2,0}\lambda_{2,0}^{r} & 0 \leqq r \leqq p-2, \\
\lambda_1^{k} \otimes 1 & 0 \leqq k < p, \\
\lambda_1^{k} \otimes h_0 & 0 \leqq k < p-1, \\
\lambda_1^{k} \otimes h_0\lambda_{2,0} & 0 \leqq k < p-1, \\
\lambda_1^{k} \otimes h_0h_{2,0} & 0 \leqq k < p.
\end{array}
$$

For completeness, we give the result for $p=2$ also. This is actually very easy: a direct proof via a minimal resolution is here painless.

THEOREM 3 ($p=2$). *The algebra* $\mathrm{Ext}_W(Z_2, Z_2)$ *is isomorphic to the quotient of a polynomial algebra on classes*

$$
\begin{array}{ll}
h_0 & \textit{of bigrading } (1, q) \\
h_1 & \textit{of bigrading } (1, 2q), \\
u & \textit{of bigrading } (3, 7q), \\
\omega & \textit{of bigrading } (4, 12q),
\end{array}
$$

modulo the ideal generated by the classes h_0h_1, h_1^3, h_1u, $u^2+h_0^2\omega$.

Proof. The reader is invited to construct a minimal resolution.

To give an idea of the algebra structure of $\mathrm{Ext}_W(Z_p, Z_p)$, p odd, we give the result for $p=3$ (in a way this is unfair, because here we have many more relations than for the general $p\geqq 5$).

THEOREM 4 ($p=3$). *The indecomposable elements in* $\mathrm{Ext}_W(Z_3, Z_3)$ *have a basis consisting of the elements* $h_0, \lambda_0, h_1, \lambda_1, \mu_0, \nu_0, \chi, \omega, \sigma_3$. *The relations satisfied by these elements are* (*where* $\{\ ,\ ,\ \}$ *is the Massey triple product*):

$$
\begin{array}{ll}
h_1h_0 = 0, & h_1\chi = -h_0\sigma_3 = \lambda_1\nu_0, \\
\lambda_1h_1 = 0, & \lambda_1\sigma_3 = 0, \\
\lambda_1^2h_0 = 0, & h_1\lambda_0 = -h_0\mu_0, \\
\nu_0 = \{h_1, h_0, h_1\}, & \lambda_1h_0 = -h_1\nu_0, \\
\mu_0 = \{h_0, h_1, h_0\}, & h_1\sigma_3 = -\lambda_1^2, \\
\sigma_3 = \{h_1, \lambda_1, h_1\}, & h_0\chi = \lambda_1\mu_0, \\
\chi = \{h_0, h_1, \lambda_1\}, & \lambda_0\nu_0 = -\mu_0^2, \\
h_1\mu_0 = h_0\nu_0, & \lambda_1\chi = \sigma_3\nu_0.
\end{array}
$$

Proof. We can exhibit a minimal resolution for Z_3 over W and read off the above relations by means of the Yoneda constructions.

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